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Solvability near the characteristic set for a special class of complex vector fields

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This article deals with the solvability near the characteristic set $\Sigma = \{0\}\times S^1$ of operators of the form $L = \partial/\partial t + (x^n a(x) + ix^m b(x))\partial/\partial x$, $b \not\equiv 0$ and $a(0) \neq 0$, defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, where a and b are real-valued smooth functions in $(-\epsilon, \epsilon)$ and $m \geq 2n$. It is shown that given $f \in C^\infty(\Omega_\epsilon)$, satisfying a finite number of compatibility conditions, there is $u \in C^0$ solution of the equation $Lu = f$ in a neighborhood of Σ; moreover, the C^0 regularity is sharp.
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1. INTRODUCTION

Let $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, and let

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0, \tag{1}$$

be a complex vector field defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, where a and b are real-valued smooth functions in $(-\epsilon, \epsilon)$.

The characteristic set of L is the set of points $(x, t) \in \Omega_\epsilon$ where L fails to be elliptic, that is, the set

$$\Sigma = \{(x, t) \in \Omega_\epsilon;\ L_{(x,t)} \text{ and } \overline{L}_{(x,t)} \text{ are linearly dependent}\}.$$

It is easy to see that $(x, t) \in \Sigma$ if and only if $b(x) = 0$, that is,

$$\Sigma = \{(x, t) \in \Omega_\epsilon;\ b(x) = 0\}.$$

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In this article we will assume that $\Sigma = \{0\} \times S^1$; hence, L is elliptic on $\{x \neq 0\}$ and the well-known *Nirenberg-Treves* condition $(\mathcal{P})$ is satisfied.

Let us recall the definition of condition $(\mathcal{P})$: we say that L given by (1) satisfies $(\mathcal{P})$ if the function $b(x)$ does not change sign on any integral curve of the real vector field $\partial/\partial t + a(x)\partial/\partial x$ (see [13], theorem 3.7).

We are interested in solving the equation

$$Lu = f$$

near the characteristic set Σ, where $f \in C^\infty(\Omega_\epsilon)$, in sense of Hörmander (see [12]).

We say that L is solvable at Σ if given f belonging to a subspace of finite codimension of $C^\infty(\Omega_\epsilon)$ there exists $u \in \mathcal{D}'(\Omega_\epsilon)$ solving the equation $\mathsf{L}u = f$ in a neighborhood of Σ.

If $a(0) \neq 0$ then it follows from Hörmander's results (see [12]) that L is solvable at Σ; moreover, solutions can be obtained in C^∞ class.

A slight modification of the arguments in [6] shows that if the function $a + ib$ is flat at $x = 0$ then L is not solvable at Σ.

Hence, we can assume that we have $a(x) + ib(x) = x^n a_0(x) + ix^m b_0(x)$, where $(a_0 + ib_0)(0) \neq 0$; moreover, if the function a (resp. b) is not flat at $x = 0$ then n (resp. m) is the order of vanishing of a (resp. b) at $x = 0$; if the function a (resp. b) is flat at $x = 0$ then $n \geq m$ (resp. $m \geq n$). In this article we are interested in the case $m \geq 2$.

Let $r = \min\{m, n\}$. It is easy to see that if $f \in C^\infty(\Omega_\epsilon)$ is such that there is $u \in C^\ell$, $\ell \geq r$, solution of $Lu = f$ in a neighborhood of Σ then

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0, t)dt = 0, \quad j = 0, \ldots, r - 1. \tag{2}$$

Now, fixe $k \in \mathbb{Z}_+$. It follows from [3], [10] and [11] that given $f \in C^\infty(\Omega_\epsilon)$, satisfying (2), the equation $Lu = f$ has a C^k solution if and only if $2 \leq m \leq 2n - 1$; moreover, if $m > 2n - 1$ then there exists $f \in C^\infty(\Omega_\epsilon)$, flat along Σ, for which there is no $u \in C^1$ solution of the equation $Lu = f$ in any neighborhood of Σ. Thus, the solvability at Σ when $m \geq 2n$ is still an open problem.

Hence, assuming $a_0(0) \neq 0$ and $m \geq 2n$, a natural question appears: given $f \in C^\infty(\Omega_\epsilon)$, satisfying (2), is there C^0 solution of the equation $Lu = f$ in some neighborhood of Σ?

Note that when $u \in C^0 \setminus C^1$ the equation $Lu = f$ is to be understood in the distribution sense.

This article concerns with the question above. We will show that for all $f \in C^\infty(\Omega_\epsilon)$, satisfying the compatibility conditions (2), there exist $u \in C^0$ solution of $Lu = f$ in a neighborhood of Σ; moreover, we will show that this result is sharp, that is, for each $0 < \alpha < 1$ there exists $f \in C^\infty(\Omega_\epsilon)$, flat along Σ, for which there is no $u \in C^\alpha$ solution of the equation $Lu = f$ in any neighborhood of Σ.

The question addressed here is related to those in [1], [2], [3], [4], [8], [9], [10], [11], and others.

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2. RESULTS

Let

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0, \tag{3}$$

be a complex vector field defined on Ω_ϵ, where a and b are C^∞ real-valued functions. Assume that we can write $(a+ib)(x) = x^n a_0(x) + ix^m b_0(x)$, with $n \geq 1$, $m \geq 2n$ and $a_0(x) \neq 0$ for all $x \in (-\epsilon, \epsilon)$. Assume that $\Sigma = \{0\} \times S^1$ is the characteristic set of L.

THEOREM 2.1. *Let L be given by (3). Given $f \in C^\infty(\Omega_\epsilon)$, satisfying*

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0,t)dt = 0, \quad j = 0, \ldots, n-1, \tag{4}$$

there exists $u \in C^0$ solution of the equation $Lu = f$ in a neighborhood of Σ.

Proof: As done in [3] (see also [6]), given $f \in C^\infty(\Omega_\epsilon)$, satisfying (4), there is $v \in C^\infty$ such that $Lv - f$ is flat along Σ. Hence, in order to find a C^0 solution to $Lu = f$ at Σ it is enough to consider f flat.

Let $f \in C^\infty(\Omega_\epsilon)$ be flat along Σ. Now, using partial Fourier series we can write $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ and $f(x,t) = \sum_{k\in\mathbb{Z}} \hat{f}_k(x)e^{ikt}$. We will use the fact that one has $Lu(x,t) = f(x,t)$ in a neighborhood of Σ if and only if the sequence $(\hat{u}_k)$ satisfies $L_k\hat{u}_k(x) = \hat{f}_k(x)$ in a neighborhood of $x = 0$, $\forall k \in \mathbb{Z}$, where $L_k\hat{u}_k(x) = ik\hat{u}_k(x) + (a+ib)(x)(\hat{u}_k)'(x)$; furthermore, the series $\sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ converges in the C^0 topology in a neighborhood of Σ.

For $k = 0$ we have to solve $(a+ib)\dfrac{d}{dx}\hat{u}_0(x) = \hat{f}_0(x)$. Since $\hat{f}_0$ is flat at $x = 0$ we have $\dfrac{\hat{f}_0}{a+ib} \in C^\infty$, hence

$$\hat{u}_0(x) = \int_0^x \frac{\hat{f}_0}{a+ib}(x)\mathrm{d}x$$

is a solution in a neighborhood of $x = 0$.

For $k \neq 0$ we will first solve $L_k\hat{u}_k(x) = \hat{f}_k(x)$ for $x > 0$. There is no loss of generality in assuming that $b \geq 0$ for such values of x. We find for $x > 0$ and for each $k > 0$,

$$\hat{u}_k(x) = \int_0^x e^{-ik(C(x)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y, \quad \text{where} \quad C(x) = -\int_x^\eta \frac{1}{a+ib}(y)\mathrm{d}y.$$

Hence,

$$|\hat{u}_k(x)| \leq \int_0^x e^{-k\int_y^x \frac{b}{a^2+b^2}(s)\ ds}\left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y \leq \int_0^x \left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y.$$

Now for $x > 0$ and for each $k < 0$ we find

$$\hat{u}_k(x) = -\int_x^\eta e^{-ik(C(x)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y + M,$$

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where

$$C(x) = -\int_x^\eta \frac{1}{a+ib}(y)\mathrm{d}y$$

and

$$M = \int_0^\eta e^{-ik(C(0)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y.$$

Thus, we have

$$|\hat{u}_k(x)| \le \int_x^\eta e^{k\int_x^y \frac{b}{a^2+b^2}(s)ds}\left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y + \int_0^\eta e^{k\int_0^y \frac{b}{a^2+b^2}(s)ds}\left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y$$

$$\le 2\int_0^\eta \left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y.$$

Note that in both cases ($k > 0$ and $k < 0$) $\hat{u}_k(0) = 0$.

In an analogous way, for each $k \neq 0$, we can solve the equation $L_k\hat{u}_k = \hat{f}_k$ for $x < 0$ and the solution satisfies $\hat{u}_k(0) = 0$. Hence, for each $k \in \mathbb{Z}$, we obtain a C^0 solution to $L_k\hat{u}_k = \hat{f}_k$ in a neighborhood of $x = 0$.

Since $\dfrac{f}{a+ib} \in C^\infty(\Omega_\epsilon)$, we have that there exists $C > 0$ such that

$$\left|\frac{\hat{f}_k}{a+ib}(x)\right| \le \frac{C}{(1+|k|)^2}, \quad \text{for all } k \in \mathbb{Z} \text{ and for all } x \in (-\epsilon, \epsilon);$$

hence, the series $\sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ converges uniformely. Therefore, $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ is a continuous solution of the equation $Lu = f$ in a neighborhood of Σ. □

THEOREM 2.2. *Let L be given by* (3). *For each fixed $0 < \alpha < 1$, there exists $f \in C^\infty(\Omega_\epsilon)$, satisfying* (4), *for which the equation $Lu = f$ does not have a C^α solution in any neighborhood of Σ.*

Proof: Assume that for each function $f \in C^\infty(\Omega_\epsilon)$, satisfying (4), there exists $u \in C^\alpha$ solution of the equation $Lu = f$ in a neighborhood of Σ. As done in [2] (see also [5] and [11]), an argument using Baire's theorem and the open mapping theorem implies that there exists $\delta_0 > 0$ such that given $f \in C^\infty([-\delta_0, \delta_0] \times S^1)$, satisfying (4), there exists $u \in C^\alpha([-\delta_0, \delta_0] \times S^1)$ solution of $Lu = f$ in some neighborhood of $[-\delta_0, \delta_0] \times S^1$. In order to keep this work as self-contained as possible let us repeat the arguments here.

Let

$$\mathcal{S} = span < 1 \otimes \delta^{(j)} >, \quad j = 0, \dots, n-1,$$

and let $\mathcal{S}^\circ$ denote the annihilator of $\mathcal{S}$. For each $\ell \in \mathbb{Z}_+$, define

$$F_\ell = \{(f,u) \in (C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{S}^\circ) \times C^\alpha(\overline{\mathcal{U}}_\ell);\ Lu = f \text{ on } \mathcal{U}_\ell\},$$

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where $\mathcal{U}_\ell = (-\delta_\ell, \delta_\ell) \times S^1$ and $\delta_\ell \downarrow 0$ with $\delta_1 = \epsilon/2$. Note that, for each $\ell \in \mathbb{Z}_+$, F_ℓ is an Fréchet space. Consider

$$\begin{array}{rccl} \pi_\ell : & F_\ell & \to & C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{S}^\circ \\ & (f,u) & \mapsto & f \end{array},$$

the projection on the first factor.

Hence, under our assumptions, we have that

$$\bigcup_\ell \pi_\ell(F_\ell) = C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{S}^\circ.$$

Finally, from Baire's theorem and the open mapping theorem, we have that

$$\pi_{\ell_0}(F_{\ell_0}) = C^\infty([-\epsilon/2, \epsilon/2] \times S^1) \cap \mathcal{S}^\circ,$$

for some ℓ_0. This means that, for $\delta_0 = \delta_{\ell_0}$, the operator

$$L : C^\alpha([-\delta_0, \delta_0] \times S^1) \to C^\infty([-\delta_0, \delta_0] \times S^1) \cap \mathcal{S}^\circ \tag{5}$$

is surjective.

Next, we will contradict (5). The argument is similar that in [3], which we will recall.

There is no loss of generality in assuming that $b(x) > 0$ if $-\delta_0 < x < 0$. Define

$$g(x) = \begin{cases} -e^{\frac{1}{x}}, & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

For a fixed $x_0 \in (-\delta_0, 0)$ and for each $k \in \mathbb{Z}$, define $(\hat{f}_k)$ by

$$\hat{f}_k(x) = g(x) \cdot (a + ib)(x) \cdot c_k \cdot \ e^{-ik \int_{x_0}^{x} \frac{a}{a^2+b^2}(y) dy},$$

where (c_k) is a rapidly decreasing sequence of positive real numbers (to be chosen later).

It follows from [3] that the sequence $(\hat{f}_k)$ defines a function $f \in C^\infty([-\delta_0, \delta_0] \times S^1) \cap \mathcal{S}^\circ$, flat along Σ.

We claim that for f defined above there is no $u \in C^\alpha([-\delta_0, \delta_0] \times S^1)$ solving $Lu = f$ in a neighborhood of $[-\delta_0, \delta_0] \times S^1$. Recall that in [3] the authors exhibit a similar function f, for which the equation $Lu = f$ does not have C^∞ solution in any neighborhood of $[-\epsilon_0, \epsilon_0] \times S^1$, for some $0 < \epsilon_0 \leq \epsilon/2$.

Let f be the function defined above. From (5) there exists a C^α solution u to $Lu = f$ in a neighborhood of $[-\delta_0, \delta_0] \times S^1$. By using Fourier series, the equation $Lu = f$ can be rewritten as

$$\sum_{k \in \mathbb{Z}} (L_k \hat{u}_k) e^{ikt} = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ikt},$$

where $L_k\hat{u}_k(x) = ik\hat{u}_k(x) + (a+ib)(x)(\hat{u}_k)'(x)$. Hence, for each $k \in \mathbb{Z}$, we have $L_k\hat{u}_k(x) = \hat{f}_k(x)$ and we can write $\hat{u}_k(x) = v_k(x) + h_k(x)$, where h_k is an arbitrary solution of the homogeneous equation $L_k h_k = 0$ and v_k is a solution of the nonhomogeneous equation.

Define

$$C(x) = \int_{-\delta_0}^{x} \frac{1}{a+ib}(y)\mathrm{d}y.$$

It is easy to see that

$$v_k(x) = \begin{cases} 0, & x \geq 0 \\ -\int_x^0 e^{-ik(C(x)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y, & x < 0 \end{cases}$$

is a C^∞ solution to $L_k v_k = \hat{f}_k$ in a neighborhood of $[-\delta_0, \delta_0]$.

A simple calculation shows that for $k > 0$ we have

$$v_k(x_0) = -\int_{x_0}^{0} g(y)c_k e^{k\int_{x_0}^{y}\frac{b}{a^2+b^2}(s)ds}\mathrm{d}y$$

$$\geq e^{kM}M'c_k,$$

where $M = \displaystyle\int_{x_0}^{\frac{x_0}{2}} \frac{b}{a^2+b^2}(s)ds > 0$ and $M' = -\displaystyle\int_{\frac{x_0}{2}}^{0} g(y)\mathrm{d}y > 0$.

At this point, for each $k > 0$, we choose $c_k = e^{-Mk}$ to the definition of $\hat{f}_k$. Hence, we have that $v_k(x_0) \geq M'$; consequently, the sequence (v_k) cannot be the coefficients of a C^α function at $x = x_0$.

It is easy to see that, for $x < 0$ and for each $k \neq 0$, the function

$$h_k(x) = h_k(-\delta_0)e^{-ik\int_{-\delta_0}^{x}\frac{1}{a+ib}(y)dy}$$

$$= h_k(-\delta_0)e^{-ik\int_{-\delta_0}^{x}\frac{a}{a^2+b^2}(y)dy}e^{-k\int_{-\delta_0}^{x}\frac{b}{a^2+b^2}(y)dy}.$$

is the general solution to the homogeneous equation $L_k h_k(x) = 0$.

We claim that if $h_k(-\delta_0) \neq 0$ then h_k cannot be extended to a C^α function in any neighborhood of $x = 0$. Indeed, if $h_k \in C^\alpha$ in some neighborhood of $x = 0$ then

$$\lim_{x\to 0}\frac{h_k(x) - h_k(0)}{x^\alpha}$$

should exist. However,

$$\lim_{x\to 0^-}\frac{h_k(x) - h_k(0)}{x^\alpha} = \lim_{x\to 0^-}\frac{h_k'(x)}{\alpha x^{\alpha-1}} = \lim_{x\to 0^-}\frac{-ikh_k(x)}{\alpha(a_0(x) + ix^{m-n}b_0(x))}\cdot\frac{1}{x^{n+\alpha-1}},$$

which, of course, does not exist.

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Hence, for each $k > 0$, $\hat{u}_k(x) = v_k(x)$ is the only C^α solution to the equation $L_k \hat{u}_k(x) = \hat{f}_k(x)$ in a neighborhood of $[-\delta_0, \delta_0]$. Therefore, $(\hat{u}_k)$ cannot be the sequence of Fourier coefficients of any C^α periodic function, which contradicts (5). The proof is complete. $\square$

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