CADERNOS DE MATEMÁTICA **12**, 77–83 May (2011)
ARTIGO NÚMERO SMA#346

Rigidity of bi-Lipschitz equivalence of weighted homogeneous function-germs in the plane

Alexandre Fernandes

Departamento de Matemática, Universidade Federal do Ceará, Av. Mister Hull s/n,Campus do PICI, Bloco 914,CEP: 60.455-760 - Fortaleza - CE - Brasil.
E-mail: alexandre.fernandes@ufc.br

Maria Ruas

Instituto de Ciências Matemáticas e Computação, Av. Trabalhador São-carlense 400, Centro Caixa Postal: 668 CEP 13560-970, São Carlos SP, Brasil
E-mail: maasruas@icmc.usp.br

The main goal of this work is to show that if two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are bi-Lipschitz equivalent, in the sense that these function-germs can be included in a strongly bi-Lipschitz trivial family of weighted homogeneous function-germs, then they are analytically equivalent. May, 2011 ICMC-USP

Key Words: bi-Lipschitz, isolated complex singularity

1. INTRODUCTION

$$f_t(x, y) = xy(x - y)(x - ty) \; ; \; 0 < |t| < 1$$

defines a family of function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with isolated singularity. In 1965, H. Whitney justified the rigidity of the analytic classification of function-germs by proving that: given $t \neq s$ there is no $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ germ of bi-analytic map such that $f_t = f_s \circ \phi$, i.e. f_t is not analytically equivalent to f_s. In another way, with respect to the topological point of view, this family is not so interesting, since for any $t \neq s$ there exists a germ of homeomorphism $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f_t = f_s \circ \phi$, i.e. f_t is topologically equivalent to f_s. In fact, the topological classification of reduced polynomial function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is well-understood, as it was shown in [4]. In the seminal paper [3], Henry and Parusinski considered the bi-Lipschitz equivalence, which is between the analytic and the topological equivalence, of function-germs. This paper motivated other papers about the problem of bi-Lipschitz classification of function-germs. For instance, [5] and [1] showed that, in some sense, for weighted homogeneous real function-germs

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

in two variables the problem of bi-Lipschitz classification is quite close to the problem of analytic classification. The results presented in [3] point out to a rigidity of the bi-Lipschitz classification of function-germs. More precisely, they considered the family

$$f_t(x,y) = x^3 + y^6 - 3t^2xy^4 \ ; \ 0 < |t| < \frac{1}{2}$$

and proved that: given $t \neq s$, there is no $\phi\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ germ of bi-Lipschitz map such that $f_t = f_s \circ \phi$, i.e. f_t is not bi-Lipschitz equivalent to f_s. The strategy used by them was to introduce a new invariant based on the observation that the bi-Lipschitz homeomorphism does not move much the regions around the relative polar curves. For a single germ f the invariant is given in terms of the leading coefficients of the asymptotic expansions of f along the branches of its generic polar curve. In the case that the bi-Lipschitz triviality of a family of function-germs comes by integrating a Lipschitz vector field, here called strong bi-Lipschitz triviality, the calculations are much easier, and very illustrative. The main goal of this work is to show that if two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2,0) \to (\mathbb{C},0)$ are bi-Lipschitz equivalent, in the sense that these function-germs can be included in a strongly bi-Lipschitz trivial family of weighted homogeneous function-germs, then they are analytically equivalent.

2. PRELIMINARIES

2.1. Analytic and bi-Lipschitz equivalences

Two function-germs $f, g\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ are called *analytically equivalent* if there exists a a germ of bi-analytic map $\phi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $f = g \circ \phi$.

Two function-germs $f, g\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ are called *bi-Lipschitz equivalent* if there exists a bi-Lipschitz map-germ $\phi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $f = g \circ \phi$.

Let f_t ($t \in U$ domain in $\mathbb{C}$) be a family of analytic function-germs. That is, there is a neighborhood W of 0 in $\mathbb{C}^n$ and an analytic function $F\colon W \times U \to \mathbb{C}$ such that $F(0,t) = 0$ and $f_t(x) = F(x,t)$ $\forall\, t \in U$, $\forall\, x \in W$. We call f_t *strongly bi-Lipschitz trivial* when there is a continuous family of Lipschitz vector fields $v_t\colon \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\frac{\partial f_t}{\partial t}(x) = (df_t)(x)(v_t(x))$$

$\forall\, t \in U$, $\forall\, x \in W$.

The following result shows that if we include two function-germs f and g into a strongly bi-Lipschitz trivial family of function-germs, then the two initial function-germs f and g are bi-Lipschitz equivalent.

THEOREM 2.1. *If f_t is strongly bi-Lipschitz trivial, then f_t is bi-Lipschitz equivalent to $f_{t'}$ for any $t \neq t' \in U$.*

The above theorem is known as a result of Thom-Levine type and its proof is immediate, since the flow of a Lipschitz vector fields defines a family of bi-Lipschitz homeomorphisms.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

2.2. Weighted homogeneous functions

Let $w = (w_1, \ldots, w_n)$ be an n-uple of positive integer numbers. We say that a polynomial function $f(x_1, \ldots, x_n)$ is *w-homogeneous* of degree d if $f(s^{w_1}x_1, \ldots, s^{w_n}x_n) = s^d f(x_1, \ldots, x_n)$ for all $s \in \mathbb{C}^*$. Let $H^d_w(n,1)$ denote the space of w–homogeneous n variables polynomials of degree d. Let $\mathcal{O}_n$ be the ring of analytic function-germs at the origin $0 \in \mathbb{C}^n$ and let $\mathcal{M}_n$ be the maximal ideal of $\mathcal{O}_n$.

Proposition 2.1. *Let $F(x_1, \ldots, x_n, t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, t)$ is w–homogeneous with an isolated singularity at $0 \in \mathbb{C}^n$, where U is a domain of $\mathbb{C}$. If, for any $t \in U$, $\frac{\partial F}{\partial t}$ belongs to the ideal of $\mathcal{O}_n$ generated by $\{x_i \dfrac{\partial F}{\partial x_j} \ : \ i,j = 1, \ldots, n\}$, then f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.*

Proof. Let us denote by TF the ideal of $\mathcal{O}_n$ generated by $\{x_i \dfrac{\partial F}{\partial x_j} \ : \ i,j = 1, \ldots, n\}$. It is clear that $H^d_w(n,1)$ can be considered a subset of the space of m–jets $J^m(n,1)$, for m large enough. The set

$$A^d_w(n,1) = \{f \in H^d_w(n,1) \ : \ f \text{ has an isolated singularity at origin } \}$$

is a Zariski open subset of $H^d_w(n,1)$. In particular, $A^d_w(n,1)$ can be seen as a connected submanifold of the m-jets space $J^m(n,1)$. Let $\mathcal{R}(n,n)$ be the group of analytic diffeomorphism-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. We consider the natural action of $G = j^m(\mathcal{R}(n,n))$ on the manifold $M = J^m(n,1)$ given by

$$(j^m(\phi), j^m(f)) \mapsto j^m(f \circ \phi).$$

So, given $f \in A$, we get:

$$\begin{aligned}
\operatorname{codim}(G \cdot f) &= \dim(M) - \dim(G \cdot f) \\
&= \dim(M) - \dim(T_f(G \cdot f)) \\
&= \dim(\frac{\mathcal{M}_n}{\mathcal{M}_n^{m+1}}) - \dim(\frac{TF}{\mathcal{M}_n^{m+1}}) \\
&= \dim(\frac{\mathcal{M}_n}{TF}) \\
&= n - 1 + \mu(f)
\end{aligned}$$

where $\mu(f)$ is the Milnor number of f. Since the Milnor number of $f \in A$ does not depend on f, we get that dimension of $T_f(G \cdot f)$ does not depend on $f \in A$. Let

$$P = \{f_t \ : \ t \in U\}.$$

Then,

1. by assumptions, $T_f P \subset T_f(G \cdot f)$, for any $f \in P$ and,
2. $\dim(T_f(G \cdot f))$ is constant for $f \in P$, because $P \subset A$.

Now we are ready to apply Mather's Lemma in order to know that P is included in a single G–orbit. ▮

3. RESULTS

Proposition 3.1. *Let $F(x,y,t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x,y) = F(x,y,t)$ is w–homogeneous ($w_2 > w_1$) with an isolated singularity at $(0,0) \in \mathbb{C}^2$, where U is an open subset of $\mathbb{C}$. If f_t defines a strongly bi-Lipschitz trivial family of function-germs at origin $(0,0) \in \mathbb{C}^2$, then there exists an algebraic continuous function $k \colon U \to \mathbb{C}$ such that*

$$\frac{\partial F}{\partial t}(x,y,t) - k(t) y \frac{\partial F}{\partial y}(x,y,t)$$

is identically null on the polar set $\{(x,y,t) \ : \ \frac{\partial F}{\partial x}(x,y,t) = 0\}$.

Proof. By assumptions, there exists a Lipschitz vector fields

$$v(x,y,t) = (v_1(x,y,t), v_2(x,y,t), 1)$$

such that $\dfrac{\partial F}{\partial v} = 0$. Let $a_1(t), ..., a_r(t)$ be continuous functions such that

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t) \ i = 1, \ldots, r$$

parameterize the branches of

$$\Gamma_t = \{(x,y,t) \ : \ \frac{\partial F}{\partial x}(x,y,t) = 0\}.$$

Let

$$k_i(t) = \frac{\frac{\partial F}{\partial t}(a_i(t),1,t)}{\frac{\partial F}{\partial y}(a_i(t),1,t)}.$$

We claim that $k_i(t) = k_j(t)$ for any $i \neq j$. In fact, since $v_2(x,y,t)$ is a Lipschitz function, we have that

$$|v_2(a_i(t)s^{w_1}, s^{w_2}) - v_2(a_j(t)s^{w_1}, s^{w_2})| \lesssim |s|^{w_1} \text{ as } s \to 0.$$

On the other hand,

$$\begin{aligned}|v_2(a_i(t)s^{w_1}, s^{w_2}, t) - v_2(a_j(t)s^{w_1}, s^{w_2}, t)| &= |\frac{\frac{\partial F}{\partial t}(a_i(t)s^{w_1}, s^{w_2}, t)}{\frac{\partial F}{\partial y}(a_i(t)s^{w_1}, s^{w_2}, t)} - \frac{\frac{\partial F}{\partial t}(a_j(t)s^{w_1}, s^{w_2})}{\frac{\partial F}{\partial y}(a_j(t)s^{w_1}, s^{w_2})}| \\ &= |k_i(t) - k_j(t)||s|^{w_2}.\end{aligned}$$

Since $w_1 > w_2$, we have $k_i(t) = k_j(t)$.

Let us denote $k(t) = k_1(t) = \cdots = k_r(t)$. Now, let us fix t. We see that the function

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y\frac{\partial F}{\partial y}(x, y, t)$$

is identically null on each branch of Γ_t at $(0, 0, t)$. ∎

THEOREM 3.1. *Let $F(x, y, t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x, y) = F(x, y, t)$ is w–homogeneous ($w_2 > w_1$) with an isolated singularity at $(0, 0) \in \mathbb{C}^2$, where $U \subset \mathbb{C}$ is a domain. If $\{f_t \ : \ t \in U\}$, as a family of function-germs at $(0, 0) \in \mathbb{C}^2$, is strongly bi-Lipschitz trivial , then f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.*

Proof. Let $k(t)$ be given by the above proposition, hence we have an algebraic continuous function $k \colon U \to \mathbb{C}$ such that

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y\frac{\partial F}{\partial y}(x, y, t)$$

is identically null on the polar set $\{(x, y, t) \ : \ \frac{\partial F}{\partial x}(x, y, t) = 0\}$.*

Let $t_0 \in U$. Using the Newton-Puiseux Parametrization Theorem, there exist two open $0 \in V \subset \mathbb{C}$ and $t_0 \in U' \subset U$ such that

$$s \mapsto s^N + t_0$$

maps V onto U' and the function

$$s \mapsto k(s^N + t_0)$$

is analytic for some $N \in \mathbb{N}$. Let us denote $G(x, y, s) = F(x, y, s^N + t_0)$, $g_s(x, y) = G(x, y, s)$ and $\tilde{k}(s) = k(s^n + t_0)$. Thus, we have an analytic funtion $\tilde{k} \colon V \to \mathbb{C}$ such that

$$\frac{\partial G}{\partial s}(x, y, s) - Ns^{n-1}\tilde{k}(s)y\frac{\partial G}{\partial y}(x, y, s)$$

is identically null on the polar set $\{(x, y, s) \ : \ \frac{\partial G}{\partial x}(x, y, s) = 0\}$. Let

$$P_1(x, y, s), \ldots, P_r(x, y, s)$$

be such that they define the analytic irreducible factors of $\frac{\partial G}{\partial x}(x, y, s)$ in $\mathcal{O}_3$. Let $j(s) = Ns^{n-1}\tilde{k}(s)$. Let

$$\alpha_i = \max\{\alpha \in \mathbb{N} \ : P_i^{\alpha_i} \text{ divides } \frac{\partial G}{\partial x} \text{ in } \mathcal{O}_3\}.$$

By hypothesis, we have an integer number $\beta_i \geq 1$ such that

$$\frac{\partial G}{\partial s}(x, y, s) = u(x, y, s)P_i^{\beta_i}(x, y, s) + j(s)y\frac{\partial F}{\partial y}(x, y, t)$$

with $u = u(x, y, s) \in \mathcal{O}_3$. Moreover, we can suppose that $P_i^{\beta_i}$ does not divide u in $\mathcal{O}_3$. We should show that $\beta_i \geq \alpha_i$. If $\alpha_i = 1$, we have nothing to do. Thus, let us consider $\alpha_i > 1$. It follows from

$$\frac{\partial G}{\partial s}(x, y, s) = u(x, y, s)P_i^{\beta_i}(x, y, s) + j(s)y\frac{\partial G}{\partial y}(x, y, s)$$

that

$$\frac{\partial^2 G}{\partial x \partial s} = \frac{\partial u}{\partial x}P_i^{\beta_i} + \beta_i u P_i^{\beta_i - 1}\frac{\partial P_i}{\partial x} + jy\frac{\partial^2 G}{\partial x \partial y}. \tag{1}$$

Now, since $P_i^{\alpha_i}$ divides $\frac{\partial G}{\partial x}$, we have that $P_i^{\alpha_i - 1}$ divides $\frac{\partial^2 G}{\partial s \partial x}$ and $\frac{\partial^2 G}{\partial y \partial x}$, hence, by using eq. 1, we get that

$$P_i^{\alpha_i - 1} \text{ divides } P^{\beta_i - 1}(\frac{\partial u}{\partial x}P_i + \beta_i u\frac{\partial P_i}{\partial x}). \tag{2}$$

Since P_i does not divide neither u and $\frac{\partial P_i}{\partial x}$, it follows from eq. 2 that $\beta_i \geq \alpha_i$.

Once $\beta_i \geq \alpha_i$, we have that $\frac{\partial G}{\partial s}$ belongs the ideal of $\mathcal{O}_3$ generated by

$$\{x\frac{\partial G}{\partial x}, x\frac{\partial G}{\partial y}, y\frac{\partial G}{\partial x}, y\frac{\partial G}{\partial y}\}.$$

Thus, it comes from Mather's Lemma or the analytic version of Thom-Levine result that g_{s_1} is analytically equivalent to g_{s_2} for any $s_1, s_2 \in V$. It means that, f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U'$. Finally, it follows from the connectivity of U that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$. ∎

In the above proof, from where is marked *, if $\frac{\partial F}{\partial x}$ were analytically reduced, this theorem could be proved in the following way. We fix $t \in U$ and, since $\frac{\partial F}{\partial x}(x, y, t)$ is analytically reduced, there exists an analytic function $u(x, y)$ such that

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y\frac{\partial F}{\partial y}(x, y, t) = u(x, y)\frac{\partial F}{\partial x}(x, y, t).$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

It comes from this equation that $u(x,y)$ is w–homogeneous of degree w_1, in particular $u(0,0)=0$. Thus, $\frac{\partial F}{\partial t}$ belongs to the ideal of $\mathcal{O}_2$ generated by

$$\{x\frac{\partial F}{\partial x}(x,y,t), x\frac{\partial F}{\partial y}(x,y,t), y\frac{\partial F}{\partial x}(x,y,t), y\frac{\partial F}{\partial y}(x,y,t)\}.$$

Finally, since $f_t(x,y)=F(x,y,t)$ is w–homogeneous for all $t\in U$, it follows from Proposition 2.1 that $\{f_t \ : \ t\in U\}$ defines a family of function-germs at origin $(0,0)\in\mathbb{C}^2$ such that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1,t_2\in U$.

The above argument allows us to extend the Theorem 3.1 to w-homogeneous polynomials in n variables ($w_1>\cdots>w_n$) with the additional hypothesis that the ideal

$$\{\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_{n-1}}\}$$

is radical. However, the following example shows that, for $n\geq 3$ variables, if we remove some hypothesis above this result is not true at all.

Example 3.1. $f_t(x,y,z)=x^4+y^4+z^k+tx^2y^2$ is strongly bi-Lipschitz trivial (for t close to 0), but f_{t_1} is not analytically equivalent to f_{t_2} when $t_1\neq t_2$.

Acknowledgements. This paper was written during the postdoctoral stage of ACGF at the Universidade de São Paulo which was partially supported by CNPq grant # 150578/2009-1. And MASR acknowledges the financial support from CNPq grant # 303774/2008-8 and FAPESP, grant # 08/54222-6.

REFERENCES

1. Lev Birbrair, Alexandre Fernandes and Daniel Panazzolo. *Lipschitz classification of functions on a Hölder triangle.* St. Petersburg Math. J. **20** (2009), 681-686. Original publication: Algebra i Analiz, tom **20** (2008), nomer 5.
2. Alexandre Fernandes and Maria Ruas. *Bilischitz determinacy of quasihomogeneous germs.* Glasgow Math. J. **46** (2004), pp 77–82.
3. Adam Parusinski and Jean-Pierre Henry. *Existence of moduli for bilipschitz equivalence of analytic functions.* Compositio Math. **136** (2003), pp 217–235.
4. Adam Parusinski. *A criterion for topological equivalence of two variable complex analytic function germs.* Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), no. 8, 147150.
5. Adam Parusinski and Satoshi Koike. *Equivalence relations for two variable real analytic functions Germs.* ArXiv:0801.2650

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC