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On the existence of G-equivariant maps

Denise de Mattos*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: deniseml@icmc.usp.br

Edivaldo L. dos Santos†

Departamento de Matemática, Universidade Federal de São Carlos-UFSCar, Caixa Postal 676, 13565-905, São Carlos-SP, Brazil
E-mail: edivaldo@dm.ufscar.br

Francielle R. de C. Coelho‡

Universidade Federal de Uberlândia, Faculdade de Matemática, 38408-100, Uberlândia MG, Brazil
E-mail: francielle@famat.ufu.br

Let G be a compact Lie group. Let X, Y be free G-spaces. In this paper, by using the numerical index $i(X; R)$, under cohomological conditions on the spaces X and Y, we consider the question of the existence of G-equivariant maps $f : X \to Y$. May, 2011 ICMC-USP

1. INTRODUCTION

Results on the absence of G-maps often lead to important consequences. For example, the classical Borsuk-Ulam theorem, which states that each continuous map of an n-dimensional sphere to n-dimensional Euclidean space takes the same values at pair of antipodal points, is equivalent to the assertion that there exists no $\mathbb{Z}_2$-map $S^n \to S^{n-1}$, where $\mathbb{Z}_2$ acts on spheres by antipodal involution. On the other hand, results of this kind can be interesting in themselves. We can prove that one space cannot be embedded in another it is sufficient to show that there exists no equivariant map between their deleted products considered with the natural action of a free involution; or one can consider in a more general manner other configuration spaces connected with the spaces in question and endowed with actions of suitable groups. One example of such a result is the well-known van Kampen-Flores

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theorem [7], which states that the k-dimensional skeleton of a $(2k+2)$-dimensional simplex cannot be embedded in $\mathbb{R}^{2k}$.

Let R be a PID and G a compact Lie group. We denote by $\beta_i(X;R)$ the i-th Betti number of X.

In [1, Theorem 1.1], it was proved that if X, Y are free G-spaces, Hausdorff, pathwise connected and paracompact such that for some natural $m \geq 1$, $H^q(X;R) = 0$ for $0 < q < m$, $H^{m+1}(Y/G;R) = 0$, where Y/G is the orbit space of Y by G and $\beta_m(X;R) < \beta_{m+1}(BG;R)$ then there is no G-equivariant map $f : X \to Y$.

In this paper, by using the numeral index $i(X;R)$ defined in [8], we prove the following theorem:

Theorem 1.1. *Let G be a compact Lie group and X, Y free G-spaces, Hausdorff, pathwise connected and paracompact. Suppose that for some natural $m \geq 1$, $i(X;R) \geq m+1$ and $H^{m+1}(Y/G;R) = 0$. Then, if $\beta_m(X;R) < \beta_{m+1}(BG;R)$, there is no G-equivariant map $f : X \to Y$.*

Remark 1. 1. In [8], Volovikov obtained the following result directly from definition of this index:"*If $H^q(X;R) = 0$ for $0 < q < m$, then $i(X;R) \geq m+1$*". Therefore, Theorem 1.1 generalizes the main theorem proved in [1]. Moreover, let $K = \Delta^{qs+q-2}_{s-1}$ be the $(s-1)$-dimensional skeleton of the $(qs+q-2)$-dimensional simplex Δ^{qs+q-2}, $q = p^n$, p prime and let $P^q_2(K) \subset K^q$ be the union of the subspaces os the form $\sigma_1 \times \sigma_2 \times \ldots \times \sigma_q$ in K^q, where $\sigma_1, \sigma_2, \ldots, \sigma_q$ are pairwise disjoint simplices of K. As shown in [7], we have that $i(P^q_2(K);\mathbb{Z}_p) = q(s-1)+1$ and $H^{s-1}(P^q_2(K);\mathbb{Z}_p) \neq 0$, which shows that exists a space X that satisfies the hypothesis $i(X;\mathbb{Z}_p) \geq m+1$ and does not satisfy the hypothesis $H^j(X;\mathbb{Z}_p) = 0$, for $0 < j < m$.

Note that if Y is a topological manifold with a free action of a compact Lie group G, then $\dim(Y/G) = \dim(Y) - \dim(G)$, where *dim* denote the usual topological dimension. Thus, if $\dim(G) \geq 1$, one has that $\dim(Y/G) < \dim(Y)$. We have the following Corollary of Theorem 1.1.

Corollary 1.1. *Let G be a compact Lie group of dimension p. Let X be a free G-space, Hausdorff, pathwise connected and paracompact such that $i(X;R) \geq m+1$, and let Y be a $(m+p)$-dimensional topological manifold with a free action of G. If $\beta_m(X;R) < \beta_{m+1}(BG;R)$, then there is no G-equivariant map $f : X \to Y$.*

Proof (Proof of Corollary 1.1). Since Y is a $(m+p)$-dimensional manifold with a free action of G, $\dim(Y/G) = m$ and therefore $H^{m+1}(Y/G;R) = 0$. It follows from Theorem 1.1 that there is no G-equivariant map $f : X \to Y$. ▮

Example 1.1. Let $G = S^1 \times S^1$, $X = S^5 \times S^5$ and $Y = S^3 \times S^3$, which admit free action of G. One has that $H^q(X;\mathbb{Z}) = 0$, for $0 < q < m = 5$ and thus $i(X) \geq 5+1$. Moreover, $H^6(Y/G;\mathbb{Z}) = 0$, since dim(Y/G) = 4, and $B(S^1 \times S^1) = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, which

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implies $\beta_5(X;\mathbb{Z}) = 2 < \beta_6(BG;\mathbb{Z}) = 4$. It follows from Corollary 1.1 that there is no G-equivariant map $f : X \to Y$.

2. PRELIMINARIES

We start by introducing some basic notions and notations. We assume that all spaces under consideration are Hausdorff and paracompact spaces. Throughout this paper, H^* will always denote the cohomology groups. For a given space B, let $\mathcal{G}$ be a system of local coefficients for B. We will denote by $H^*(B;\mathcal{G})$ the cohomology groups of B with local coefficients in $\mathcal{G}$. The symbol $\cong$ will denote an appropriate isomorphism between algebraic objects.

Suppose that G is a compact Lie group which acts freely on a Hausdorff and paracompact space X, then $X \to X/G$ is a principal G-bundle [2, Theorem II.5.8] and one can take

$$h : X/G \to BG$$

a classifying map for the G-bundle $X \to X/G$.

Remark 2. 1. Let us observe that if $\hat{h}$ is another classifying map for the principal G-bundle $X \to X/G$, then there is a homotopy between h and $\hat{h}$.

Given the G-space X, consider the product $EG \times X$ with the diagonal action given by $g(e, x) = (ge, gx)$ and let $EG \times_G X = (EG \times X)/G$ be its orbit space. The first projection $EG \times X \to EG$ induces a map

$$p_X : EG \times_G X \to (EG)/G = BG,$$

which is a fibration with fiber X and base space BG being the classifying space of G. This is called the *Borel construction*. It associates to each G-space X a space $EG \times_G X$, which will be denoted by X_G, over BG and to each G-map $X \to Y$ a fiber preserving map $EG \times_G X \to EG \times_G Y$ over BG.

Remark 2. 2. If G acts freely on X, then the map

$$X_G \to X/G$$

induced by the second projection $EG \times X \to X$ is a fibration with a contractible fibre EG and therefore a homotopy equivalence (for details, see[4]).

Now, let us recall the following theorem of Leray-Serre for fibrations, as given in [5, Theorem 5.2].

THEOREM 2.1. (The cohomology Leray-Serre Spectral Sequence) *Let R be a commutative ring with unit. Given a fibration $F \hookrightarrow E \xrightarrow{p} B$ where B is pathwise connected, there is*

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a first quadrant spectral sequence of algebras $\{E_r^{*,*}, d_r\}$*, with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; R)),$$

the cohomology of B *with local coefficients in the cohomology of* F*, the fibre of* p*, and coverging to* $H^*(E; R)$ *as an algebra. Furthermore, this spectral sequence is natural with respect to fibre-preserving maps of fibrations.*

Let us now recall one of the numerical indices defined by Volovikov in [8]. This is a function on G-spaces whose value is either a positive integer or ∞. The definition of $i(X; R)$ uses the spectral sequence of the bundle $p_X : X_G \to BG$ with fibre X (the Borel construction). The spectral sequence converges to the equivariant cohomology $H^*(X_G; R)$. Let Λ^* be the equivariant cohomology algebra $H^*(pt_G; R) = H^*(BG; R)$ of a point. Suppose that X is path connected. Then $E_2^{*,0} = \Lambda^*$. Assume that $E_2^{*,0} = \cdots = E_s^{*,0} \neq E_{s+1}^{*,0}$. Then, by definition, $i(X; R) = s$.

We state some properties of the index (see [8]).

Proposition 2.1. *Let* X*,* Y *and* Z *be* G*-spaces.*

(i)If there is a G*-equivariant map from* X *to* Y*, then* $i(X; R) \leq i(Y; R)$*.*
(ii)If $\tilde{H}^j(X; R) = 0$ *for all* $j \leq n-1$*, then* $i(X; R) \geq n+1$*.*
(iii)If $i(Z : R) < \infty$ *and* $H^j(Z; R) = 0$ *for all* $j \geq n+1$*, then* $i(Z : R) \leq n+1$*.*

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 will follow from the following lemmas

Lemma 3.1. *Let* R *be a PID,* G *a compact Lie group and* X *a free* G*-space Hausdorff, pathwise connected and paracompact. Suppose that for some natural* $m \geq 1$*,* $i(X; R) \geq m+1$*. Then, there exists the following exact sequence with coefficients in* R*,*

$$E_{m+1}^{0,m} \to H^{m+1}(BG; R) \overset{p_X^*}{\to} H^{m+1}(X_G; R).$$

Proof. Given the fibration $p_X : X_G = EG \times_G X \to BG$, it follows from Theorem 2.1 that there exists a first quadrant spectral sequence $\{E_r^{*,*}, d_r\}$, with

$$E_2^{p,q} \cong H^p(BG; \mathcal{H}^q(F)),$$

the cohomology of BG with local coefficients in the cohomology of X, the fibre of p, and coverging to $H^*(X_G; R)$.

Since X is pathwise connected the local coefficients system $\mathcal{H}^0(X)$ over BG is trivial and follows from [5, Proposition 5.18] that

$$E_2^{p,0} \cong H^p(BG; \mathcal{H}^0(X)) = H^p(BG; H^0(X)) = H^p(BG; R),\ \forall p.$$

On the other hand, since $i(X; R) \geq m+1$ follows that

$$E_2^{p,\,0} \cong E_3^{p,\,0} \cong \cdots \cong E_{m+1}^{p,\,0}, \forall\ p.$$

Thus

$$H^{m+1}(BG; R) = E_2^{m+1,\,0} = E_3^{m+1,\,0} = \cdots = E_{m+1}^{m+1,\,0}. \quad (1)$$

Now, we consider the exact sequence

$$0 \to Ker\, d_r \to E_r^{0,\,r-1} \xrightarrow{d_r} E_r^{r,\,0} \to \frac{E_r^{r,\,0}}{Im\, d_r} \to 0,$$

where $Im(d_r : E_r^{0,\,r-1} \to E_r^{r,\,0})$. We have that,

$$Ker\, d_r = E_\infty^{0,\,r-1} \quad and \quad \frac{E_r^{r,\,0}}{Im\, d_r} = E_\infty^{r,\,0}.$$

Therefore, we obtain the sequence,

$$0 \to E_\infty^{0,\,r-1} \to E_r^{0,\,r-1} \xrightarrow{d_r} E_r^{r,\,0} \to E_\infty^{r,\,0} \to 0 \quad (2)$$

Also, we prove that there exists a natural injection,

$$0 \to E_\infty^{m+1,\,0} \hookrightarrow H^{m+1}(X_G). \quad (3)$$

For this we consider the following decreasing filtration of $H^r(X_G, R)$,

$$0 = F^{r+1}(H^r(X_G)) \subset F^r(H^r(X_G)) \subset \ldots \subset F^1(H^r(X_G)) \subset F^0(H^r(X_G)) = H^r(X_G).$$

Since the spectral sequence $\{E_r^{*,\,*}; d_r\}$ converges to $H^*(X_G; R)$ as an algebra, we have that

$$E_\infty^{p,\,q} \cong E_0^{p,\,q}(H^*(X_G)) = F^p(H^{p+q}(X_G))/F^{p+1}(H^{p+q}(X_G)),$$

where $E_\infty^{*,\,*}$ is the limit term of the spectral sequence and $E_0^{*,\,*}(H^r(X_G))$ is the module associated bigraduade.

Since $F^{r+1}(H^r(X_G)) = \{0\}$, it follows that

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$$E_\infty^{r,0} \cong F^r(H^r(X_G))/F^{r+1}(H^r(X_G)) \cong F^r(H^r(X_G)) \subset H^r(X_G;R),$$

for any r, and this completes the existence of sequence in (3).

Taking $r = m+1$ and putting together (2) and (3), one obtains the exact sequence

$$E_{m+1}^{0,m} \xrightarrow{d_{m+1}} E_{m+1}^{m+1,0} \to H^{m+1}(X_G;R), \tag{4}$$

where d_{m+1} is the differential.

Since $H^{m+1}(BG;R) = E_{m+1}^{m+1,0}$ (eq. (1)), of the sequence (4) we obtain the desired sequence

$$E_{m+1}^{0,m} \xrightarrow{d_{m+1}} H^{m+1}(BG;R) \xrightarrow{p_X^*} H^{m+1}(X_G;R).$$

∎

Lemma 3.2. *Let X be a free G-space, Hausdorff, pathwise connected and paracompact. Suppose that for some natural $m \geq 1$, $i(X;R) \geq m+1$ and that $\beta_m(X;R) < \beta_{m+1}(BG;R)$ then the homomorphism $h^* : H^{m+1}(BG;R) \to H^{m+1}(X/G;R)$ is nontrivial, where $h : X/G \to BG$ is a classifying map for the principal G-bundle $X \to X/G$.*

Proof. Let $EG \to BG$ be the universal G-bundle and $h : X/G \to BG$ a classifying map for the principal G-bundle $X \to X/G$. Let $p_X : X_G \to BG$ the Borel-fibration associated to the G-space X, where X_G is the Borel space, as in (0). It follows from Remark (2.2) that the map $X_G \to X/G$ is a homotopy equivalence. Let $r : X/G \to X_G$ be its inverse homotopic. Then $p_X \circ r : X/G \to BG$ also classifies the principal G-bundle $X \to X/G$, and follows from Remark (2.1) that the map $(p_X \circ r)$ is homotopic to h. Since $r^* : H^{m+1}(X_G;R) \to H^{m+1}(X/G;R)$ is an isomorphism, it suffices to prove that $p_X^* : H^{m+1}(BG;R) \to H^{m+1}(X_G;R)$ is nontrivial. In fact, since $i(X;R) \geq m+1$, it follows from Lemma 3.1 that there exists an exact sequence with coefficients in R,

$$E_{m+1}^{0,m} \xrightarrow{d_{m+1}} H^{m+1}(BG;R) \xrightarrow{p_X^*} H^{m+1}(X_G;R). \tag{5}$$

Suppose that $p_X^* : H^{m+1}(BG;R) \to H^{m+1}(X_G;R)$ is the zero homomorphism. From (5), we have that $d_{m+1} : E_{m+1}^{0,m} \longrightarrow H^{m+1}(BG)$ is a surjective homomorphism, which implies that

$$\text{rank } (E_{m+1}^{0,m}) \geq \text{rank } (H^{m+1}(BG;R)) = \beta_{m+1}(BG;R). \tag{6}$$

On the other hand, since $E_{m+1}^{0,m}$ is isomorphic to a submodule of $H^0(BG;\mathcal{H}^m(X))$ and $H^0(BG;\mathcal{H}^m(X))$ is isomorphic to a submodule of $H^m(X;R)$ [9, theorem 3.2] then $E_{m+1}^{0,m}$ is isomorphic to a submodule of $H^m(X;R)$. Therefore,

$$\beta_m(X;R) = \text{rank } (H^m(X;R)) \geq \text{rank } (E_{m+1}^{0,m}) \overset{(6)}{\geq} \beta_{m+1}(BG;R),$$

which contradicts the hypothesis.

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Thus, the homomorphism $h^* : H^{m+1}(BG; R) \to H^{m+1}(X/G; R)$ is nontrivial. ▌

Proof (Theorem 1.1). Suppose that $f : X \to Y$ is a G-equivariant map. Since Y is a Hausdorff paracompact space, one can take a classifying map $g : Y/G \to BG$ for the principal G-bundle $Y \to Y/G$. Then the map $h = g \circ \bar{f} : X/G \to BG$ can be taken as a classifying map for the principal G-bundle $X \to X/G$, where $\bar{f} : X/G \to Y/G$ is the map induced by f between the orbit spaces. Since by hypothesis $H^{m+1}(Y/G; R) = 0$ one has that $g^* : H^{m+1}(BG; R) \to H^{m+1}(Y/G; R)$ is trivial and consequently $h^* : H^{m+1}(BG; R) \to H^{m+1}(X/G; R)$ is the zero homomorphism, which contradicts Lemma 3.2. ▌

4. SOME CONSIDERATIONS ON THEOREM 1.1

Clapp and Puppe proved in [3] the following theorem:

Theorem 4.1. *Let G be a p-torus or a torus (that is, $G = \mathbb{Z}_p^n$ or $G = S^1 \times \ldots \times S^1$). Let X and Y be G-spaces with fixed-points-free actions; moreover, in the case of a torus action assume additionally that Y has finitely many orbit types. Suppose that $\tilde{H}^i(X) = 0$ for $i < N$, Y is compact or paracompact and finite-dimensional, and $H^i(Y) = 0$ for $i \geq N$; here cohomology is considered with coefficients in $\mathbb{Z}_p$ in the case of a p-torus action and in $\mathbb{Q}$ in the case of a torus action. Then there exists no G-equivariant map of X into Y.*

Remark 4. 1. Let $G = S^1 \times S^1$, $X = S^5 \times S^5$ and $Y = S^3 \times S^3$, which admit free action of G, as in the Example 1.1. One has that $\tilde{H}^i(X; \mathbb{Q}) = 0$, for $i < N = 5$ and $H^6(Y; \mathbb{Q}) \neq 0$. Then, in this example, it does not valid $H^i(Y; \mathbb{Q}) = 0$ for $i \geq 5$. Therefore, we can not say that there is no G-equivariant map from X to Y, which does not happen if we use the Corollary 1.1. This shows that, in some situations, the Theorem 1.1 is more efficient than the Theorem 4.1, although the results are not comparable due to their hypothesis.

More generally, in the special case where $G = \mathbb{Z}_p^n$, X and Y are free G-spaces, the Theorem 1.1 extends Theorem 4.1 since the condition $\tilde{H}^i(X; \mathbb{Z}_p) = 0$ for $i < N$ implies $i(X; R) \geq N + 1 > N$ and $\beta_{N-1}(X; \mathbb{Z}_p) = 0 < \beta_N(BG; \mathbb{Z}_p)$, the condition Y compact or paracompact and finite-dimensional and $H^i(Y; \mathbb{Z}_p) = 0$ for $i \geq N$ implies $H^N(Y/G; \mathbb{Z}_p) = 0$ ([6, Proposition A.11]). Thus, taking $m = N - 1$ in the Theorem 1.1, we conclude that there is no G-equivariant map.

A similar result to Theorems 1.1 and 4.1 is the following theorem:

Theorem 4.2. *Let R be a PID, G a compact Lie group, X a free G-space Hausdorff paracompact and Y a G-space Hausdorff compact or paracompact finite-dimensional which admits a free G-action. Suppose that $i(X; R) \geq m + 1$, $i(Y; R) < \infty$ and $H^i(Y; R) = 0$, for $i \geq m$. Then, there is no G-equivariant map $f : X \to Y$.*

Proof. Suppose that there exists G-equivariant map $f : X \to Y$. Then, by Proposition 2.1(i), $i(X;R) \leq i(Y;R)$. Since $i(Y;R) < \infty$ and $H^i(Y;R) = 0$ for $i \geq m$, it follows from Proposition 2.1(iii) that $i(Y;R) \leq m$. Therefore, $i(X;R) \leq i(Y;R) \leq m$, which is a contradiction. ∎

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