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# Non-autonomous Morse decomposition and Lyapunov functions for gradient-like processes

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We define Morse Decompositions (time dependent) for non-autonomous evolution processes (non-autonomous dynamical systems) and prove that a non-autonomous gradient-like evolution process possesses a Morse decomposition on the associated pullback attractor. We also prove the existence of an associated Lyapunov function which describes the gradient behavior of the system. Finally, we apply these abstract results to non-autonomous perturbations of autonomous gradient-like evolution processes (semigroups or autonomous dynamical systems). May, 2011 ICMC-USP

## 1. INTRODUCTION

The *Fundamental Theorem of Dynamical Systems* (see [8]) describes the flow of autonomous reversible dynamical systems (groups) in compact metric spaces as a decomposition of an ordered family of *isolated invariant sets* and connections between them which respects their ordering. In the terminology of [8], this is called a *Morse decomposition* of a compact invariant set. The extension of this result to general autonomous dynamical systems (semigroups) is done in [2] (see also [18] for the Morse-Decomposition for semigroups in compact metric spaces). When trying to understand the structure of attractors or, more generally, of invariant sets for *autonomous dynamical systems* (or *semigroups*) the Morse-Decomposition plays a fundamental role. It allows us to decompose the dynamics into a *gradient* part and the dynamics in smaller isolated invariant sets reducing the study of the structure of the attractors to the study of the structure of these isolated invariant sets.

The aim of this paper is to extend the notion of Morse-Decomposition to *non-autonomous dynamical systems* (or *non-autonomous evolution processes*) and to show the pullback attractors for these non-autonomous dynamical systems can be also decomposed as an ordered family of *isolated invariant families* and connections between them respecting their ordering. In particular, under some mild additional assumptions, we construct a non-autonomous Lyapunov function for the non-autonomous evolution process with a Morse-Decomposition. We show that this decomposition is observed for non-autonomous perturbations of *gradient autonomous dynamical systems* (or *gradient semigroups*). Some practical examples are also presented.

Our results contrast with those of [17] because, in our case, the definition of local attractors takes into account only the usual notion of attraction instead of using past and forwards notions. We also do not impose the reversibility required in [17]. Our results can be applied (under mild assumptions) to perturbations of autonomous dynamical systems (which cannot be said for the results in [17]). Furthermore, in the case of perturbations of autonomous dynamical systems and under mild assumptions, we prove some sort of continuity of the Morse-Decomposition.

To better describe the results in the paper we will need to introduce some terminology and a few definitions. Let $X$ be a metric space with metric $\mathbf{d} : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ := [0, \infty)$, and denote by $\mathcal{C}(X)$ the set of continuous maps from $X$ into $X$. Given a subset $A \subset X$, the $\epsilon$-neighborhood of $A$ is the set $\mathcal{O}_\epsilon(A) := \{z \in X : \mathbf{d}(z, a) < \epsilon \text{ for some } a \in A\}$.

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Now we introduce the notion of evolution process. An evolution process in a metric space $X$ is a two parameter family $\{T(t,s) : t \geq s\}$ in $\mathcal{C}(X)$ such that

1) $T(t,t) = I,$ for all $t \in \mathbb{R}$,
2) $T(t,\sigma)T(\sigma,s) = T(t,s),$ for $t \geq \sigma \geq s,$ and
3) $\mathcal{P}\times X \ni ((t,s),x) \mapsto T(t,s)x \in X$ is continuous, where $\mathcal{P} := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$.

An evolution process $\{T(t,s) : t \geq s\}$ is said to be autonomous if $T(t,s) = T(t-s,0)$ for all $t \geq s$, otherwise it is said to be non-autonomous. A semigroup is a family $\{S(t) : t \geq 0\} \subset \mathcal{C}(X)$ such that

1) $S(0) = I,$ for all $t \in \mathbb{R}$,
2) $S(t+s) = S(t)S(s),$ $t,s \geq 0,$ and
3) $\mathbb{R}^+ \times X \ni (t,x) \mapsto S(t)x \in X$ is continuous.

Note that $\{T(t,s) : t \geq s\}$ is an autonomous evolution process if and only if $\{T(t,0) : t \geq 0\}$ is a semigroup. A continuous function $\xi : \mathbb{R} \to X$ is called a global solution for the evolution process $\{T(t,s) : t \geq s\}$ if it satisfies

$$T(t,s)\xi(s) = \xi(t), \text{ for all } (t,s) \in \mathcal{P}.$$

Next we define Hausdorff semi-distance and Hausdorff distance. Given $A, B \subset X$, the **Hausdorff semidistance from** $A$ **to** $B$ is given by

$$\mathbf{dist}(A,B) := \sup_{a\in A}\inf_{b\in B} \mathbf{d}(a,b) = \sup_{a\in A} \mathbf{d}(a,B),$$

where $\mathbf{d}(a,B) := \inf_{b\in B} \mathbf{d}(a,b)$ is the usual distance from the point $a$ to the set $B$, and the **Hausdorff distance between** $A$ **and** $B$ is defined by

$$\mathbf{d}_{\mathrm{H}}(A,B) := \mathbf{dist}(A,B) + \mathbf{dist}(B,A).$$

Definition 1.1. We say that a family $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ of subsets of $X$ is an invariant family for the evolution process $\{T(t,s) : t \geq s\}$ if $T(t,s)\Xi(s) = \Xi(t)$ whenever $t \geq s$. An invariant family $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is said to be a pullback attractor if, for each $t \in \mathbb{R}$, $\mathscr{A}(t)$ is compact, pullback attracts bounded subsets of $X$; that is,

$$\lim_{s\to-\infty} \mathbf{dist}(T(t,s)B, \mathscr{A}(t)) = 0, \text{ for each } t \in \mathbb{R} \text{ and bounded subset } B \text{ of } X,$$

and $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ is minimal among all closed invariant families $\{\mathscr{C}(t) : t \in \mathbb{R}\}$ with the property that $\mathscr{C}(t)$ pullback attracts bounded subsets of $X$ for each $t \in \mathbb{R}$.

The notion of unstable set of an invariant family is key to the concept of attractor-repeller pair which, in turn, is key to the definition of Morse-Decomposition for non-autonomous evolution processes.

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Definition 1.2. Let $\{T(t,s) : t \geq s\}$ be an evolution process. The unstable set of an invariant family $\Xi = \{\Xi(t) : t \in \mathbb{R}\}$ is the set

$$W^{\mathrm{u}}(\Xi) := \{ (t,z) \in \mathbb{R} \times X : \text{ there is a global solution } \xi : \mathbb{R} \to X \\ \text{such that } \xi(t) = z \text{ and } \lim_{s \to -\infty} \mathbf{dist}(\xi(s), \Xi(s)) = 0\}.$$

Also, $W^{\mathrm{u}}(\Xi)(t) := \{ z \in X : (t,z) \in W^{\mathrm{u}}(\Xi)\}$.

We now define the notions of a local attractor and of an atractor-repeller pair.

Definition 1.3. Let $\{T(t,s) : t \geq s\}$ be an evolution process in a metric space $X$ with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$. We say that an invariant family $A := \{A(t) : t \in \mathbb{R}\}$, with $A(t) \subset \mathscr{A}(t)$ for all $t \in \mathbb{R}$, is a (non-autonomous) **local attractor** if $W^u(A)(t) = A(t)$ for all $t \in \mathbb{R}$.

If $A$ is a local attractor, we define its associated **repeller** $A^* := \{A^*(t) : t \in \mathbb{R}\}$ by

$$A^*(t) := \{z \in \mathscr{A}(t) : \mathbf{dist}(T(r+t,t)z, A(r+t)) \not\to 0 \text{ as } r \to \infty\}, \text{ for each } t \in \mathbb{R}. \quad (1)$$

The pair $(A, A^*)$ is called an attractor-repeller pair.

Note that this definition extends the notion of local attractor for the autonomous case. We will see that if $\{A(t) : t \in \mathbb{R}\}$ is a local attractor, it may not be chain recurrent and that $A$ and $A^*$ are invariant.

In the autonomous case a repeller is automatically closed. In the non-autonomous case that is not the case. However, if there exists $\varepsilon > 0$ such that $\mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \varnothing$, for all $t \in \mathbb{R}$, then $A^*(t)$ is closed for each real $t$.

Now, we define the notion of Morse-Decomposition for non-autonomous evolution processes.

Definition 1.4. Let $\{T(t,s) : t \geq s\}$ be an evolution process in $X$ with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ and let $A_0 = \{A_0(t) : t \in \mathbb{R}\}$, $A_1 = \{A_1(t) : t \in \mathbb{R}\}$, $\cdots$, $A_n = \{A_n(t) : t \in \mathbb{R}\}$ be $n+1$ local attractors with $\varnothing = A_0(t) \subset A_1(t) \subset \cdots \subset A_n(t) = \mathscr{A}(t)$ for each $t \in \mathbb{R}$.

Define $\Xi_j(t) := A_j(t) \cap A^*_{j-1}(t)$ for each $t \in \mathbb{R}$ and $j = 1, 2, \cdots, n$. The ordered set of invariant families $\boldsymbol{\Xi} := \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$ is called a **Morse-Decomposition** for the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$.

We prove in Section 2 that the pullback attractor of a gradient-like evolution process has a Morse-Decomposition. In Section 3 we use this Morse-Decomposition to obtain, under some mild additional assumption, a Lyapunov function for a gradient-like (non-autonomous) evolution process in the pullback attractor. In Section 4 we show that the results of Section 2 can be applied (under natural assumptions) to non-autonomous perturbations of autonomous gradient-like evolution processes. We show that the Lyapunov function can be defined and is continuous in $\mathbb{R} \times X$, and state conditions under which the Lyapunov function behaves continuously under perturbation. In Section 5 we present classes of concrete examples where the theory developed here applies.

## 2. MORSE-DECOMPOSITION OF PULLBACK ATTRACTORS FOR GENERALIZED GRADIENT-LIKE EVOLUTION PROCESSES

We first introduce the notion of gradient-like process (see [5]). To that end we first need the definition of isolated invariant set.

Definition 2.1. Let $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ be an invariant family for the evolution process $\{T(t,s) : t \geq s\}$. If there exists $\delta > 0$ such that, any global solution $\xi : \mathbb{R} \to X$ with $\xi(t) \in \mathcal{O}_\delta(\Xi(t))$ must satisfy $\xi(t) \in \Xi(t)$ for all $t \in \mathbb{R}$, then we say that $\Xi$ is an *isolated invariant family.* A set $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ is said a disjoint *set of isolated invariant families* if each $\Xi_i$, $1 \leq i \leq n$, is an isolated invariant family and there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i(t)) \cap \mathcal{O}_\delta(\Xi_j(t)) = \varnothing$, for all $t \in \mathbb{R}$ and for $1 \leq i < j \leq n$.

Let $\{T(t,s) : t \geq s\}$ be an evolution process with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ which contains a disjoint set of isolated invariant families $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$, that is, $\Xi_i(t) \subset \mathscr{A}(t)$ for each $i$ and $t$. We can now define (see [8, 13] for similar concepts in the autonomous case):

Definition 2.2. Let $\delta$ be as in Definition 2.1 and fix $\epsilon_0 \in (0, \delta)$. For $\Xi \in \mathbf{\Xi}$ and $\epsilon \in (0, \epsilon_0)$, an $\epsilon-$chain from $\Xi$ to $\Xi$ is a sequence $\ell_i$, $1 \leq i \leq k$ in $\{1, \cdots, n\}$, a sequence of real numbers $t_i, \sigma_i, \tau_i$ with $\tau_i < \sigma_i < t_i$ , and a sequence of points $z_i \in X$, $1 \leq i \leq k$ such that $z_i \in \mathcal{O}_\epsilon(\Xi_{\ell_i}(\tau_i))$, $T(\sigma_i, \tau_i)z_i \notin \mathcal{O}_{\epsilon_0}(\cup_{i=1}^n \Xi_i(\sigma_i))$ and $T(t_i, \tau_i)z_i \in \mathcal{O}_\epsilon(\Xi_{\ell_{i+1}}(t_i))$, $1 \leq i \leq k$, with $\Xi = \Xi_{\ell_{k+1}} = \Xi_{\ell_1}$. We say that $\Xi \in \mathbf{\Xi}$ is **chain recurrent** if there is an $\epsilon_0 \in (0, \delta)$ and $\epsilon-$chain from $\Xi$ to $\Xi$ for each $\epsilon \in (0, \epsilon_0)$.

*Remark 2. 1.* We note that the introduction of $\epsilon_0$ in the above definition is only needed to account for the case $k = 1$. When $k > 1$, it is automatically true that the solution must leave $\mathcal{O}_{\epsilon_0}(\cup_{i=1}^n \Xi_i(t))$, for some real $t \in \mathbb{R}$, while going from one isolated invariant family to another.

Definition 2.3. Let $X$ be a metric space and $\{T(t,s) : t \geq s\}$ be an evolution process in $X$ with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ and a set $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant families in $\{\mathscr{A}(t) : t \in \mathbb{R}\}$. We say that $\{T(t,s) : t \geq s\}$ is a **gradient-like evolution process with respect to** $\Xi$ if the following two conditions are satisfied:

(H1) Any global solution $\xi : \mathbb{R} \to X$ in $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ satisfies

$$\lim_{t\to-\infty} \mathbf{dist}(\xi(t), \Xi_i(t)) = 0 \text{ and } \lim_{t\to\infty} \mathbf{dist}(\xi(t), \Xi_j(t)) = 0,$$

for some $1 \leq i, j \leq n$.

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(H2) $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ does not contain any chain recurrent invariant family.

Closely related to the concept of chain recurrent is the notion of homoclinic structure.

DEFINITION 2.4. Let $\{T(t,s) : t \geq s\}$ be an evolution process which possesses a disjoint set $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant families. A **homoclinic-structure** associated to $\boldsymbol{\Xi}$ is a finite subset $\{\Xi_{i_1}, \cdots, \Xi_{i_p}\}$ of $\boldsymbol{\Xi}$ together with a set of global solutions $\{\xi_1, \cdots, \xi_p\}$ such that

$$\lim_{t\to-\infty} \mathbf{dist}(\xi_k(t), \Xi_{i_k}(t)) = 0 \text{ and } \lim_{t\to\infty} \mathbf{dist}(\xi_k(t), \Xi_{i_{k+1}}(t)) = 0,\ 1 \leq k \leq p$$

where $\Xi_{i_{p+1}} := \Xi_{i_1}$.

Next we prove the invariance of the repeller $A^*$ of a local attractor $A$.

PROPOSITION 2.1. *Let $\{T(t,s) : t \geq s\}$ be an evolution process, and $A = \{A(t) : t \in \mathbb{R}\}$ is a local attractor. Then, the repeller $A^*$ of $A$ is invariant.*

*Proof.* Let $s \in \mathbb{R}$, $t \geq s$ and $w \in A^*(s)$. If $T(t,s)w \notin A^*(t)$ we have that

$$\lim_{\tau\to\infty} \mathbf{dist}(T(\tau,t)T(t,s)w, A(\tau)) = 0,$$

that is,

$$\lim_{\tau\to\infty} \mathbf{dist}(T(\tau,s)w, A(\tau)) = 0,$$

which contradicts the fact that $w \in A^*(s)$. This reasoning proves that $T(t,s)A^*(s) \subset A^*(t)$.
Conversely, if $z \in A^*(t) \subset \mathscr{A}(t) = T(t,s)\mathscr{A}(s)$, let $w \in \mathscr{A}(s)$ be such that $z = T(t,s)w$. It follows that $w \in A^*(s)$ because, otherwise,

$$\begin{aligned} 0 &= \lim_{\tau\to\infty} \mathbf{dist}\,(T(\tau,s)\,w, A(\tau)) \\ &= \lim_{\tau\to\infty} \mathbf{dist}(T(\tau,t)T(t,s)w, A(\tau)) = \lim_{\tau\to\infty} \mathbf{dist}(T(\tau,t)z, A(\tau)), \end{aligned}$$

which is in contradiction with the fact that $z \in A^*(t)$. This shows that $T(t,s)A^*(s) \supset A^*(t)$. ∎

We note that, if $A_0 = \{A_0(t) : t \in \mathbb{R}\}$, $A_1 = \{A_1(t) : t \in \mathbb{R}\}, \cdots$, $A_n = \{A_n(t) : t \in \mathbb{R}\}$ are $n+1$ local attractors with $\varnothing = A_0(t) \subset A_1(t) \subset \cdots \subset A_n(t) = \mathscr{A}(t)$ for each $t \in \mathbb{R}$, we have that $\varnothing = A_n^*(t) \subset A_{n-1}^*(t) \subset \cdots \subset A_0^*(t) = \mathscr{A}(t)$, also for each $t \in \mathbb{R}$.

Next we describe the construction of a Morse-Decomposition for the pullback attractor of a generalized gradient-like evolution process relative to the disjoint set of isolated invariant families $\{\Xi_1, \cdots, \Xi_n\}$, and of the associated collection of increasing local attractors starting from the collection of isolated invariant sets. The following result plays a fundamental role on that.

Lemma 2.1. *Let* $\{T(t,s) : t \geq s\}$ *be a gradient-like evolution process with associated disjoint set of isolated invariant families* $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. *Then, there exists* $i \in \{1, \cdots, n\}$ *such that* $\Xi_i$ *is a local attractor.*

*Proof.* We first note that, each $\Xi_i$ is an invariant family which is not chain recurrent, and we need to show that there is some $i \in \{1, \cdots, n\}$ such that $W^u(\Xi_i)(t) = \Xi_i(t)$ for each $t \in \mathbb{R}$.

Arguing by contradiction, if that is not the case, for each $1 \leq i \leq n$, there is a global solution $\xi_i(t) \in \mathscr{A}(t)$ (with $\xi_i(s) \notin \Xi_i(s)$ for some $s \in \mathbb{R}$) such that $\lim_{t \to -\infty} \mathbf{dist}(\xi_i(t), \Xi_i(t)) = 0$. Since $\xi_i$ must converge to some element of $\boldsymbol{\Xi}$, this necessarily produces a homoclinic-structure and provides a contradiction with (H2). ∎

Let $\{T(t,s) : t \geq s\}$ be a gradient-like evolution process with the associated disjoint set of isolated invariant families $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. If (after possible reordering) $\Xi_1$ is a local attractor and we define $\Xi_1^*$ as in Definition 1.3, then we have that each $\Xi_i(s)$, for $i \geq 2$ and $s \in \mathbb{R}$, is contained in $\Xi_1^*(s)$ (also if $\xi : \mathbb{R} \to X$ is a global bounded solution and $\mathbf{dist}(\xi(t), \Xi_i(t)) \overset{t\to\infty}{\longrightarrow} 0$ with $i > 1$ we have that $\xi(s) \in \Xi_1^*(s)$ for all $s \in \mathbb{R}$) and then for any $z \notin \mathscr{A}(t) \backslash (\Xi_1(t) \cup \Xi_1^*(t))$ and global solution $\xi : \mathbb{R} \to X$ with $\xi(s) \in \mathscr{A}(s)$, for each real $s$, and $\xi(t) = z$ we have that

$$\lim_{s \to -\infty} \mathbf{dist}(\xi(s), \Xi_1^*(s)) = 0 \text{ and } \lim_{s \to \infty} \mathbf{dist}(\xi(s), \Xi_1(s)) = 0.$$

We can repeat the reasoning in Lemma 2.1 to conclude that there is $i \geq 2$ such that

$$W^u(\Xi_i)(t) \cap \Xi_1^*(t) = \Xi_i(t) \text{ for all } t.$$

We relabel this isolated invariant family as $\Xi_2$ and define

$$\Xi_{2,1}^*(t) := \{z \in \Xi_1^*(t) : \mathbf{dist}(T(r+t,t)z, \Xi_2(r+t)) \not\longrightarrow 0 \text{ as } r \to \infty\}. \tag{2}$$

Then, we have that, for each $t \in \mathbb{R}$ and $i = 3, \cdots, n$, $\Xi_i(t) \subset \Xi_{2,1}^*(t)$ (also if $\xi : \mathbb{R} \to X$ is a global bounded solution and $\mathbf{dist}(\xi(t), \Xi_i(t)) \overset{t\to\infty}{\longrightarrow} 0$ with $i > 2$ we have that $\xi(s) \in \Xi_1^*(s)$ for all $s \in \mathbb{R}$). As before, we can assume that $W^u(\Xi_3)(t) \cap \Xi_{2,1}^*(t) = \Xi_3(t)$ for each $t$ and define $\Xi_{3,2}^*$ in analogy to (2).

Proceeding in this way until all isolated invariant families are exhausted, we obtain a reordering of $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ in such a form that, $\Xi_1$ is a local attractor for $\{T(t,s) : t \geq s\}$, the set $\Xi_{1,0}^* := \Xi_1^*$, and

$$W^u(\Xi_i)(t) \cap \Xi_{i-1,i-2}^*(t) = \Xi_i(t) \text{ for all } t \text{ and } i = 2, \cdots, n,$$

where, for $i = 2, \cdots, n$

$$\Xi_{i,i-1}^*(t) := \{z \in \Xi_{i-1,i-2}^*(t) : \mathbf{dist}(T(r+t,t)z, \Xi_i(r+t)) \not\longrightarrow 0 \text{ as } r \to \infty\}.$$

LEMMA 2.2. *Let $\{T(t,s) : t \geq s\}$ be a gradient-like evolution process with associated (reordered) disjoint set of isolated invariant families $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. Then, any global bounded solution $\xi : \mathbb{R} \to X$ satisfies*

$$\lim_{t\to-\infty} \mathbf{dist}(\xi(t), \Xi_i(t)) = 0 \text{ and } \lim_{t\to\infty} \mathbf{dist}(\xi(t), \Xi_j(t)), \tag{3}$$

*with $i \geq j$.*

*Proof.* Indeed, if $j = 1$ in (3) there is nothing to prove. If $j = 2$ and (3) holds, we have that $\xi(t) \in \Xi_1^*(t)$ for all $t \in \mathbb{R}$ so, if $1 = i < j = 2$, we have that $\xi(t) \in W^u(\Xi_1)(t) = \Xi_1(t)$ for each $t$, which contradicts that $\Xi_1^*(t) \cap \Xi_1(t) = \varnothing$ for every $t$. It follows that $i \geq j$.

For the general case, we suppose that $j \geq 3$ and that (3) holds. Then $\xi(t) \in \Xi_{j-1,j-2}^*(t)$ for all $t$, so if $i < j$, we have that $\Xi_{j-1,j-2}^*(t) \subset \Xi_{i-1,i-2}^*(t)$ for all $t$ and $\xi(t) \in W^u(\Xi_i)(t) \cap \Xi_{i-1,i-2}^*(t) = \Xi_i(t)$ for every $t$. Thanks to the fact that the invariant families in $\mathbf{\Xi}$ are isolated, we must have $i = j$, which is a contradiction and proves the lemma. ▮

We will prove that this reordering of $\{\Xi_1, \cdots, \Xi_n\}$ (which we denote the same) is a Morse-Decomposition for $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ with a suitably chosen sequence $A_0(t) \subset A_1(t) \subset A_2(t) \subset ... \subset A_n(t)$ of local attractors:

For each $t \in \mathbb{R}$, define $A_0(t) := \varnothing$, $A_1(t) := \Xi_1(t)$ and for $j = 2, 3, \cdots, n$

$$A_j(t) := A_{j-1}(t) \cup W^{\mathrm{u}}(\Xi_j)(t) = \bigcup_{i=1}^{j} W^{\mathrm{u}}(\Xi_i)(t). \tag{4}$$

It is clear that $A_n(t) = \mathscr{A}(t)$.

THEOREM 2.1. *Let $\{T(t,s) : t \geq s\}$ be a gradient-like process with an associated disjoint set of isolated invariant families $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ reordered as explained above.*

*Assume that there exists $\delta > 0$ such that, for $j = 1, 2, \cdots, n-1$ and $t \in \mathbb{R}$, it holds that*

$$\mathcal{O}_\delta(A_j(t)) \cap \left( \bigcup_{i=j+1}^{n} \Xi_i(t) \right) = \varnothing. \tag{5}$$

*Then, for each $j = 0, 1, \cdots, n$, the invariant family $A_j$ defined in (4) is a local attractor for $\{T(t,s) : t \geq s\}$ and*

$$\Xi_j(t) = A_j(t) \cap A_{j-1}^*(t), \text{ for all } t \in \mathbb{R} \text{ and } 1 \leq j \leq n.$$

*Consequently, $\mathbf{\Xi}$ defines a Morse-Decomposition for the pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$.*

*Proof.* Clearly $\{A_j(t) : t \in \mathbb{R}\}$ is invariant and $A_j(t) \subset W^u(A_j)(t)$ for each $t \in \mathbb{R}$.

On the other hand, if $z \in W^u(A_j)(t)$, there is a global solution $\xi : \mathbb{R} \to X$ with $\xi(t) = z$ and $\lim_{s\to-\infty} \mathbf{dist}(\xi(s), A_j(s)) = 0$. Since $\{T(t,s) : t \geq s\}$ is gradient-like and from

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(5), there exists $k \in \{1, 2, \cdots, j\}$, such that $\lim_{s\to-\infty} \mathbf{dist}(\xi(s), \Xi_k(s)) = 0$. Therefore, $z \in W^u(\Xi_k)(t)$ with $k \leq j$. This implies that $z \in A_j$ and $W^u(A_j)(t) \subset A_j(t)$. This proves that $\{A_j(t) : t \in \mathbb{R}\}$ is a local attractor.

It is easy to see that $\Xi_j(t) \subset A_j(t) \cap A^*_{j-1}(t)$. For the reverse inclusion, note that, if $z \in A_j(t) \cap A^*_{j-1}(t)$, there exists a global solution $\xi : \mathbb{R} \to X$ with $\xi(t) = z$. Since $z \in A_j(t)$ we must have that $\lim_{s\to-\infty} \mathbf{dist}(\xi(s), \Xi_k(s))$ for some $k \leq j$. As $\{T(t,s) : t \geq s\}$ is gradient-like, there is $i \in \{1, 2, \cdots, n\}$ such that $\lim_{s\to\infty} \mathbf{dist}(\xi(s), \Xi_i(s))$ and, due to the fact that $z \in A^*_{j-1}(t)$, it follows that $i \geq j$. Since from Lemma 2.2 we must have that $i \leq k$, it follows that $k = i = j$, and taking again into account that $\{T(t,s) : t \geq s\}$ is gradient-like, we have that $\xi(s) \in \Xi_j(s)$ for each $s \in \mathbb{R}$ and, in particular, $z = \xi(t) \in \Xi_j(t)$. This proves that $A_j(t) \cap A^*_{j-1}(t) \subset \Xi_j(t)$ and completes the proof. ∎

The following result plays a key role in the proof of the main results in this paper concerned with the continuity of the Lyapunov function of a gradient-like evolution process. It extends Lemma 2.11 in [2] to the non-autonomous case.

Lemma 2.3. *Let $\{T(t,s) : t \geq s\}$ be an evolution process and let $A = \{A(t) : t \in \mathbb{R}\}$ be a local attractor which is not chain recurrent. Suppose that there exists $\varepsilon > 0$ with*

$$\mathscr{A}(t) \cap \mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \varnothing, \text{ for all } t \in \mathbb{R}.$$

*Then, for each $\delta \in (0, \varepsilon)$ there is $\delta' \in (0, \delta)$ such that*

$$T(t,s)(\mathscr{A}(s) \cap \mathcal{O}_{\delta'}(A(s))) \subset \mathscr{A}(t) \cap \mathcal{O}_\delta(A(t)), \text{ whenever } t \geq s.$$

*Proof.* We argue by contradiction. Assume that there is a $\delta > 0$, sequences $(s_j)_{j\in\mathbb{N}}$, $(t_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$, and $(x_j)_{j\in\mathbb{N}}$ in $X$ with $x_j \in \mathscr{A}(s_j)$ for each $j$, and such that

$$s_j \leq t_j,$$

$$\mathbf{dist}(x_j, A(s_j)) < \frac{1}{j} \text{ but } \mathbf{dist}(T(t_j, s_j)x_j, A(t_j)) \geq \delta,$$

for all $j \in \mathbb{N}$. Choose $j_0 \in \mathbb{N}$ with $\mathbf{dist}(x_j, A(s_j)) < \frac{1}{j} < \varepsilon$ for all $j \geq j_0$. By the definition of repeller, for each $j \geq j_0$, there is a $\tau_j \geq t_j$ such that

$$\mathbf{dist}(T(\tau_j, s_j)x_j, A(\tau_j)) < \frac{1}{j}.$$

This fact implies that $A$ is chain recurrent which is a contradiction and the lemma is proved. ∎

We can now show the following proposition:

Proposition 2.2. *Let $A = \{A(t) : t \in \mathbb{R}\}$ be an isolated invariant family for the evolution process $\{T(t,s) : t \geq s\}$, with pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, such that $A(t) \subset \mathscr{A}(t)$, for all $t$, and for each $\delta > 0$ there is $\delta' \in (0,\delta)$ satisfying*

$$T(t,s)(\mathscr{A}(s) \cap \mathcal{O}_{\delta'}(A(s))) \subset \mathscr{A}(t) \cap \mathcal{O}_{\delta}(A(t)), \text{ whenever } t \geq s.$$

*Then, $A = \{A(t) : t \in \mathbb{R}\}$ is a local attractor for $\{T(t,s) : t \geq s\}$ which is not chain recurrent.*

*Proof.* Since $\{A(t) : t \in \mathbb{R}\}$ is an isolated invariant family, there exists $\delta > 0$ such that if $\xi : \mathbb{R} \to X$ is a global solution with $\xi(t) \in \mathcal{O}_\delta(A(t))$ for each $t \in \mathbb{R}$, then $\xi(t) \in A(t)$ for each $t \in \mathbb{R}$. Clearly $A(t) \subset W^u(A)(t)$ for each $t \in \mathbb{R}$ and, if $\xi : \mathbb{R} \to X$ is a global solution such that $\lim_{s \to -\infty} \mathbf{dist}(\xi(s), A(s)) = 0$, choose $\delta' \in (0,\delta)$ with

$$T(t,s)(\mathscr{A}(s) \cap \mathcal{O}_{\delta'}(A(s))) \subset \mathscr{A}(t) \cap \mathcal{O}_{\delta}(A(t)), \text{ whenever } t \geq s.$$

Choosing $s_\delta \in \mathbb{R}$ such that $\mathbf{dist}(\xi(s), A(s)) < \delta'$ for all $s \leq s_\delta$, we must have $\xi(t) \in \mathcal{O}_\delta(A(t))$ for all $t \in \mathbb{R}$. Consequently, $\xi(t) \in A(t)$ for all $t \in \mathbb{R}$ and $W^u(A)(t) \subset A(t)$ for all $t \in \mathbb{R}$.

Finally, if $\{A(t) : t \in \mathbb{R}\}$ is chain recurrent, there are $\delta > 0$, sequences $(s_j)_{j\in\mathbb{N}}$, $(t_j)_{j\in\mathbb{N}}$ and $(\tau_j)_{j\in\mathbb{N}}$ of real numbers, a sequence $(x_j)_{j\in\mathbb{N}}$ in $X$, with $x_j \in \mathscr{A}(s_j)$ for each $j$, and such that,

$$s_j \leq t_j \leq \tau_j,$$

$$\mathbf{dist}(x_j, A(s_j)) < \frac{1}{j}, \quad \mathbf{dist}(T(t_j,s_j)x_j, A(t_j)) \geq \delta \quad \text{and} \quad \mathbf{dist}(T(\tau_j,s_j)x_j, A(\tau_j)) < \frac{1}{j}.$$

That is clearly in contradiction with the fact that there exists $\delta' \in (0,\delta)$ with

$$T(t,s)(\mathscr{A}(s) \cap \mathcal{O}_{\delta'}(A(s))) \subset \mathscr{A}(t) \cap \mathcal{O}_{\delta}(A(t)), \text{ whenever } t \geq s,$$

and the proof is complete. ∎

Our next result is a first consequence of Lemma 2.3.

Lemma 2.4. *Let $\{T(t,s) : t \geq s\}$ be an evolution process and $A = \{A(t) : t \in \mathbb{R}\}$ be a local attractor which is not chain recurrent. Suppose that there exists $\varepsilon > 0$ with*

$$\mathscr{A}(t) \cap \mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \varnothing, \text{ for all } t \in \mathbb{R}.$$

*If $K \subset \mathscr{A}(t)$ is compact and $K \cap A^*(t) = \varnothing$, then*

$$\lim_{\tau \to \infty} \mathbf{dist}(T(\tau,t)K, A(\tau)) = 0.$$

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*Proof.* We prove the result by contradiction. Assume that there are $\delta > 0$, sequence $\{\tau_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}$ with $\tau_j \overset{j\to\infty}{\longrightarrow} \infty$ and a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $K$ with $x_j \overset{j\to\infty}{\longrightarrow} x_0 \in K$ such that

$$\mathbf{dist}(T(\tau_j, t)x_j, A(\tau_j)) \geq \delta, \text{ for all } j. \tag{6}$$

Thanks to Lemma 2.3, we can choose $\delta' \in (0, \delta)$ such that

$$T(s, r)\mathcal{O}_{\delta'}(A(r)) \subset \mathcal{O}_\delta(A(s)) \text{ whenever } s \geq r.$$

Thus, it follows from (6) that

$$\mathbf{dist}(T(s, t)x_j, A(s)) \geq \delta' \text{ for all } s \in [t, \tau_j] \text{ and all } j. \tag{7}$$

Since $\tau_j \to \infty$, using (7), we have that

$$\mathbf{dist}(T(s, t)x_0, A(s)) \geq \delta' \text{ for all } s \geq t.$$

From the definition of $A^*(t)$ we have that $x_0 \in A^*(t)$ which is in contradiction with $K \cap A^*(t) = \varnothing$, and the proof is therefore complete. ▌

## 3. A LYAPUNOV FUNCTION FOR A GENERALIZED GRADIENT-LIKE PROCESS

We will prove in this section that gradient-like evolution processes are gradient processes. A gradient evolution process is defined as follows

Definition 3.1. We say that an evolution process $\{T(t, s) : t \geq s\}$ with a pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, and a disjoint set of isolated invariant families $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ is a **gradient evolution process with respect to $\boldsymbol{\Xi}$** if there is a function $V : \mathbb{R} \times X \to \mathbb{R}$ such that:

*i)* For each $z \in X$ and $t \in \mathbb{R}$, the real function $[0, \infty) \ni r \mapsto V(r + t, T(r + t, t)z) \in \mathbb{R}$ is non-increasing.

*ii)* Given $t \in \mathbb{R}$ and $z \in \mathscr{A}(t)$, $V(r + t, T(r + t, t)z) = V(t, z)$ for all $r \geq 0$ if and only if $z \in \bigcup_{i=1}^{n} \Xi_i(t)$ and $V(t, \Xi_i(t))$ is a unitary set for each $t \in \mathbb{R}$.

*iii)* For each $t \in \mathbb{R}$, the function $V_t : \mathscr{A}(t) \to \mathbb{R}$, given by $V_t(z) := V(t, z)$ for $z \in \mathscr{A}(t)$, is continuous.

A function $V : \mathbb{R} \times X \to \mathbb{R}$ with the properties above is called a **Lyapunov function** for the generalized gradient process $\{T(t, s) : t \geq s\}$ **with respect to $\boldsymbol{\Xi}$**.

Before proving our main result, we need to establish the continuity of the invariant families $(A, A^*)$ of attractor–repeller pairs in the following sense:

Lemma 3.1. *Let $\{T(t, s) : t \geq s\}$ be a gradient-like evolution process in a metric space $X$ with pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ and let $(A, A^*)$ be an attractor-repeller pair for $\{T(t, s) : t \geq s\}$.*

*Then, for each* $t_0 \in \mathbb{R}$ *we have:*

$$\lim_{t\to t_0} \mathbf{d}_{\mathrm{H}}(A(t),A(t_0)) = 0 \text{ and } \lim_{t\to t_0} \mathbf{d}_{\mathrm{H}}(A^*(t),A^*(t_0)) = 0. \tag{8}$$

*Proof.* In fact, we know that $A^*$ is invariant and then, by Lemma 2.10 and the proof of Theorem 2.9, both in [6], we obtain the conclusion (8). ▌

Now, we can prove the main result in this paper.

THEOREM 3.1. *Let* $\{T(t,s) : t \geq s\}$ *be an evolution process in a metric space* $X$*, with pullback attractor* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$*, and let* $(A, A^*)$ *be an attractor-repeller pair for* $\{T(t,s) : t \geq s\}$*, and assume that* $(A, A^*)$ *is not chain recurrent. Suppose that there exists* $\varepsilon > 0$ *with*

$$\mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \varnothing, \text{ for all } t \in \mathbb{R}. \tag{9}$$

*Then, there exists a function* $k : \mathbb{R} \times X \to \mathbb{R}$ *satisfying the following four properties:*

*i) For each* $z \in X$ *and* $t \in \mathbb{R}$*, the function* $[0,\infty) \ni r \mapsto k(r+t, T(r+t,t)z) \in \mathbb{R}$ *is non-increasing.*

*ii) If* $k_t : X \to \mathbb{R}$ *is defined by* $k_t(z) := k(t,z)$*, for each* $t \in \mathbb{R}$ *and* $z \in X$*, then* $k_t^{-1}(0) = A(t)$ *and* $k_t^{-1}(1) \cap \mathscr{A}(t) = A^*(t)$*.*

*iii) Given* $t \in \mathbb{R}$ *and* $z \in \mathscr{A}(t)$*, if* $k(r+t, T(r+t,t)z) = k(t,z)$ *for all* $r \geq 0$*, then* $z \in A(t) \cup A^*(t)$*.*

*iv)* $k_t : \mathscr{A}(t) \to \mathbb{R}$ *is continuous, in* $\mathscr{A}(t)$*, for each* $t \in \mathbb{R}$*.*

*Proof.* First, with the convention that $\mathbf{dist}(z,\varnothing) = 1$, let $l : \mathbb{R} \times X \to [0,1]$ be the Uryshon function associated to the attractor-repeller pair $(A, A^*)$, that is, for each $t \in \mathbb{R}$ and $z \in X$,

$$l(t,z) := \frac{\mathbf{dist}(z,A(t))}{\mathbf{dist}(z,A(t)) + \mathbf{dist}(z,A^*(t))}.$$

We have that, $l$ is well defined, $l\,(\mathbb{R} \times X) \subset [0,1]$, is continuous in both variables $(t,z)$ and, for each $t \in \mathbb{R}$, is uniformly continuous in $X$ (that is, the family $\{l_t : X \to [0,1] : t \in \mathbb{R}\}$ is uniformly Lipschitz continuous where, $l_t : X \to \mathbb{R}$ is given by $l_t(z) := l(t,z)$, for each $t \in \mathbb{R}$ and $z \in X$). In fact, since, by (9),

$$d_0 := \inf_{t\in\mathbb{R}} \{\inf\{\mathbf{d}(x,y) : x \in A(t),\ y \in A^*(t)\}\} \geq \varepsilon > 0, \tag{10}$$

it holds that $|l(t,z) - l(t,w)| \leq \frac{2}{d_0}\mathbf{d}(z,w)$, for any $z, w \in X$ and $t \in \mathbb{R}$.

Now, from Lemma 3.1, it is not difficult to see that $l : \mathbb{R} \times X \to [0,1]$ is continuous in both variables. Moreover, it is easy to see that $l_t^{-1}(0) = A(t)$ and $l_t^{-1}(1) = A^*(t)$.

We define $k : \mathbb{R} \times X \to \mathbb{R}$ by

$$k(t,z) := \sup_{r\geq 0} l(r+t, T(r+t,t)z).$$

We now show that $k : \mathbb{R} \times X \to \mathbb{R}$ has the properties $i)$ - $iv)$ above.

Clearly $k(\mathbb{R} \times X) \subset [0,1]$. To prove that, given $t \in \mathbb{R}$ and $z \in X$, the function $[0,\infty) \ni r \mapsto k(r+t, T(r+t,t)z) \in [0,1]$ is non-increasing, observe that, if $0 \le r_1 \le r_2$, we have

$$\begin{aligned} k(r_2+t, T(r_2+t,t)z) &= \sup_{r\ge 0} l(r+r_2+t, T(r+r_2+t,t)z) \\ &= \sup_{r\ge r_2} l(r+t, T(r+t,t)z) \\ &\le \sup_{r\ge r_1} l(r+t, T(r+t,t)z) \\ &= \sup_{r\ge 0} l(r+r_1+t, T(r+r_1+t,t)z) \\ &= k(r_1+t, T(r_1+t,t)z). \end{aligned}$$

By the definition of $k$ and from the invariance of $A$ and $A^*$, it is clear that $k_t(A(t)) = \{0\}$ and $k_t(A^*(t)) = \{1\}$. Now, if $z \in X$ is such that $k(t,z) = 0$, then $l(r+t, T(r+t,t)z) = 0$ for all $r \ge 0$. In particular, $0 = l(t,z)$, and then, $z \in A(t)$, that is, $k_t^{-1}(0) \subset A(t)$ which shows that $k_t^{-1}(0) = A(t)$. On the other hand, if $z \in \mathscr{A}(t)$ is such that $k_t(z) = 1$ and $z \notin A^*(t)$, then $\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A(r+t)) = 0$. From the definition of $l$ we obtain that $\lim_{r\to\infty} l(r+t, T(r+t,t)z) = 0$. Therefore, there exists $r_0 > 0$ such that $1 = k(t,z) = \sup_{0\le r\le r_0} l(r+t, T(r+t,t)z)$. The continuity of $[0,\infty) \ni r \mapsto l(r+t, T(r+t,t)z) \in [0,1]$ implies the existence of $r' \in [0, r_0]$ such that $l(r'+t, T(r'+t,t)z) = 1$; that is, $T(r'+t,t)z \in A^*(r'+t)$ which contradicts the fact that $\lim_{r\to\infty} l(T(r+t,t)z, A(r+t)) = 0$. Thus, if $k(t,z) = 1$ for some $t \in \mathbb{R}$ and $z \in \mathscr{A}(t)$, we must have that $z \in A^*(t)$. From this, we conclude that $k_t^{-1}(1) \cap \mathscr{A}(t) \subset A^*(t)$ and, consequently, $k_t^{-1}(1) \cap \mathscr{A}(t) = A^*(t)$.

We now prove that, if $z \in \mathscr{A}(t)$ and $k(r+t, T(r+t,t)z) = k(t,z)$ for all $r \ge 0$, then $z \in A(t) \cup A^*(t)$. If $z \notin A(t) \cup A^*(t)$, $\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A(r+t)) = 0$ and, from the definition of $k$, we have that $k(t,z) = \lim_{r\to\infty} k(r+t, T(r+t,t)z) = 0$. Since $k_t^{-1}(0) = A(t)$, $z$ must belong to $A(t)$, which is a contradiction.

Next we prove the continuity of $k_t : \mathscr{A}(t) \to \mathbb{R}$. We split the proof into three cases:

*Case* 1) Continuity of $k_t : \mathscr{A}(t) \to \mathbb{R}$ in $A^*(t)$.

Since $l(t,z) \le k(t,z) \le 1$, for all $t \in \mathbb{R}$ and $z \in X$, given $z_0 \in A^*(t)$ and $z \in X$ we have

$$|k(t,z) - k(t,z_0)| = 1 - k(t,z) \le 1 - l(t,z).$$

This and the continuity of $l : \mathbb{R} \times X \to \mathbb{R}$ in $(t, z_0)$ imply the continuity of $k_t : X \to \mathbb{R}$ in $z_0$.

*Case* 2) Continuity of $k_t : \mathscr{A}(t) \to \mathbb{R}$ in $A(t)$.

From the equicontinuity of the family of functions $\{l_t : X \to \mathbb{R} : t \in \mathbb{R}\}$, given $\varepsilon > 0$, there is $\delta > 0$ such that $l_s(\mathcal{O}_\delta(A(s))) \subset [0,\varepsilon)$ for all $s \in \mathbb{R}$. Now, Lemma 2.3 implies that there exists $\delta' \in (0,\delta)$ such that $T(r+t,t)(\mathscr{A}(t) \cap \mathcal{O}_{\delta'}(A(t))) \subset \mathscr{A}(r+t) \cap \mathcal{O}_\delta(A(r+t))$, for all $r \ge 0$, from which we can conclude that $k_t(\mathscr{A}(t) \cap \mathcal{O}_{\delta'}(A(t))) \subset [0,\varepsilon]$.

*Case* 3) Continuity of $k_t : \mathscr{A}(t) \to \mathbb{R}$ in $\mathscr{A}(t)\backslash(A(t) \cup A^*(t))$.

Given $z \in \mathscr{A}(t)\backslash(A(t)\cup A^*(t))$, since $z \notin A^*(t)$, it follows that $\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A(r+t)) = 0$ and, since $z \notin A(t)$, it holds that $l(t,z) > 0$. Choose $\delta > 0$ such that $l_s(\mathcal{O}_\delta(A(s))) \subset [0, \frac{l(t,z)}{2})$ for all $s \in \mathbb{R}$ and, from Lemma 2.3, we can choose $\delta' \in (0,\delta)$ such that $T(r+t,t)(\mathscr{A}(t) \cap \mathcal{O}_{\delta'}(A(t))) \subset \mathscr{A}(r+t) \cap \mathcal{O}_\delta(A(r+t))$, for all $r \geq 0$.

From $\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A(r+t)) = 0$, there exists $\sigma > 0$ with the property that $T(r+t,t)z \in \mathcal{O}_{\delta'}(A(r+t))$, for all $r \geq \sigma$. From the continuity of $T(\sigma+t,t) : X \to X$, there is a neighborhood $U_1$ of $z$ in $X$ such that $T(\sigma+t,t)U_1 \subset \mathcal{O}_{\delta'}(A(\sigma+t))$. Then, for all $w \in U_1$ we have that $T(\sigma+t,t)w \in \mathcal{O}_{\delta'}(A(\sigma+t))$, so that $T(r+t,t)w \in \mathcal{O}_\delta(A(r+t))$ for all $r \geq \sigma$ and $w \in U_1 \cap \mathscr{A}(t)$.

Finally, from the continuity of $l_t$, let $U_2$ be a neighborhood of $z$ in $X$ such that $l(t,w) > \frac{l(t,z)}{2}$ for all $w \in U_2$ and write $U := U_1 \cap U_2 \cap \mathscr{A}(t)$, so that, for all $w \in U$, it holds that $k(t,w) = \sup_{0\leq r\leq\sigma} l(r+t, T(r+t,t)w)$, from which we obtain the continuity of $k_t$ in points of $\mathscr{A}(t)\backslash(A(t) \cup A^*(t))$. ∎

The proof of our next result is very similar to the proof of Theorem 3.1 and follows the ideas in [2]. It brings up a special case for which the continuity of the "Lyapunov functions" in both variables holds. We observe that the last theorem says that the "Lyapunov function" is continuous for each fixed $t$ and in the pullback attractor only.

PROPOSITION 3.1. *Let $\{T(t,s) : t \geq s\}$ be an evolution process in a metric space $X$, with pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$, and let $(A, A^*)$ be an attractor-repeller pair for $\{T(t,s) : t \geq s\}$ with*

$$\mathcal{O}_{\varepsilon_0}(A(t)) \cap \mathcal{O}_{\varepsilon_0}(A^*(t)) = \varnothing, \text{ for all } t \in \mathbb{R}, \tag{11}$$

*for some $\varepsilon_0 > 0$. Also assume that the following two conditions hold:*

*(a) The conclusion of the Lemma 2.3 holds considering neighborhoods of $A$ in $X$, that is, for each $\delta > 0$ there is $\delta' \in (0,\delta)$ such that*

$$T(r+t,t)(\mathcal{O}_{\delta'}(A(t))) \subset \mathcal{O}_\delta(A(r+t)), \text{ for all } r \geq 0 \text{ and } t \in \mathbb{R}.$$

*(b) For each $t \in \mathbb{R}$ and $z \in X\backslash(A(t) \cup A^*(t))$ we have*

$$\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A(r+t)) = 0 \text{ or } \lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A^*(r+t)) = 0.$$

*Then, the function $k : \mathbb{R}\times X \to \mathbb{R}$, defined in Theorem 3.1, is continuous (in both variables $(t,x)$ and for all $(t,x) \in \mathbb{R}\times X$).*

*Proof.* Let us prove the continuity of $k$ in a point $(t_0, z_0) \in \mathbb{R}\times X$. As in the proof of Theorem 3.1, we consider three cases:

*Case* 1) $z_0 \in A^*(t_0)$.

Since $l(t,z) \leq k(t,z) \leq 1$, for all $t \in \mathbb{R}$ and $z \in X$ we can write

$$|k(t,z) - k(t_0,z_0)| = 1 - k(t,z) \leq 1 - l(t,z).$$

This and the continuity (in both variables) of $l : \mathbb{R} \times X \to \mathbb{R}$ in $(t_0, z_0)$ imply the continuity of $k : \mathbb{R} \times X \to \mathbb{R}$ in $(t_0, z_0)$.

*Case* 2) $z_0 \in A(t_0)$. From

$$|l(t,z) - l(t,w)| \leq \frac{2}{\varepsilon_0}\mathbf{d}(z,w), \quad \forall z, w \in X \text{ and } t \in \mathbb{R},$$

given $\varepsilon > 0$, there is $\delta > 0$ such that $l(s, \mathcal{O}_\delta(A(s))) \subset [0, \varepsilon)$ for all $s \in \mathbb{R}$.

On the other hand, from $(a)$, there exists $\delta' \in (0, \delta)$ such that $T(r+t,t)(\mathcal{O}_{\delta'}(A(t))) \subset \mathcal{O}_\delta(A(r+t))$, for all $r \geq 0$ and $t \in \mathbb{R}$, from which we can conclude that $k(t, \mathcal{O}_{\delta'}(A(t))) \subset [0, \varepsilon]$ for each $t \in \mathbb{R}$ .

Now, by Lemma 3.1, let $\delta'' > 0$ be such that $\mathcal{O}_{\frac{\delta'}{2}}(A(t_0)) \subset \mathcal{O}_{\delta'}(A(t))$ whenever $|t - t_0| < \delta''$. Hence, for $|t - t_0| < \delta''$ and $\mathbf{d}(z, z_0) < \frac{\delta'}{2}$ we have that $k(t,z) \leq \varepsilon$. This proves the continuity of $k : \mathbb{R} \times X \to \mathbb{R}$ in $(t_0, z_0)$ with $z_0 \in A(t_0)$.

*Case* 3) $z_0 \in X \backslash (A(t_0) \cup A^*(t_0))$.

By hypothesis $(b)$ we have

$$\lim_{r\to\infty} \mathbf{dist}(T(r+t_0,t_0)z_0, A(r+t_0)) = 0 \text{ or } \lim_{r\to\infty} \mathbf{dist}(T(r+t_0,t_0)z_0, A^*(r+t_0)) = 0.$$

First, we suppose that $\lim_{r\to\infty} \mathbf{dist}(T(r+t_0,t_0)z_0, A(r+t_0)) = 0$, since $z_0 \notin A^*(t_0)$ it holds that $l(t_0,z_0) > 0$. Choose $\delta > 0$ such that $l(s, \mathcal{O}_\delta(A(s))) \subset [0, \frac{l(t_0,z_0)}{2})$ for all $s \in \mathbb{R}$ and, from hyphotesis $(a)$, we choose $\delta' \in (0,\delta)$ such that $T(r+t,t)(\mathcal{O}_{\delta'}(A(t))) \subset \mathcal{O}_\delta(A(r+t))$, for all $r \geq 0$.

From $\lim_{r\to\infty} \mathbf{dist}(T(r+t_0,t_0)z_0, A(r+t_0)) = 0$, there exists $\sigma > 0$ with the property that $T(\sigma+t_0,t_0)z_0 \in \mathcal{O}_{\frac{\delta'}{2}}(A(\sigma+t_0))$. From the continuity of the process $\{T(t,s) : t \geq s\}$ and Lemma 3.1, there is a neighborhood $U_1$ of $z_0$ in $X$ and $\delta'' > 0$ such that $T(\sigma+t,t)U_1 \subset \mathcal{O}_{\frac{\delta'}{2}}(A(\sigma+t_0)) \subset \mathcal{O}_{\delta'}(A(\sigma+t))$ whenever $|t-t_0| < \delta''$. Then, for all $z \in U_1$ and $t \in (t_0 - \delta'', t_0 + \delta'')$, we have that $T(\sigma+t,t)z \in \mathcal{O}_{\delta'}(A(\sigma+t))$, so that $T(r+t,t)z \in \mathcal{O}_\delta(A(r+t))$ for all $r \geq \sigma$, $z \in U_1$ and $t \in (t_0 - \delta'', t_0 + \delta'')$.

Now, from the continuity of $l : \mathbb{R} \times X \to \mathbb{R}$, let $U_2$ be a neighborhood of $z_0$ in $X$ and $0 < \eta < \delta''$ such that $l(t,z) > \frac{l(t_0,z_0)}{2}$ for all $z \in U_2$ and $|t - t_0| < \eta$ and write $U := U_1 \cap U_2$, so that, for all $z \in U$ and $|t - t_0| < \eta$, it holds that $k(t,z) = \sup_{0 \leq r \leq \sigma} l(r+t, T(r+t,t)z)$, from which we obtain the continuity of $k$ in points $(t_0,z_0)$ with $z_0 \in X \backslash (A(t_0) \cup A^*(t_0))$ and $\lim_{r\to\infty} \mathbf{dist}(T(r+t_0,t_0)z_0, A(r+t_0)) = 0$.

Finally, if $\lim_{r\to\infty} \mathbf{dist}(T(r+t_0,t_0)z_0, A^*(r+t_0)) = 0$ we have that $k(t_0,z_0) = 1$ and, given $\varepsilon > 0$, we can choose $\delta > 0$ such that $l(s, \mathcal{O}_\delta(A^*(s))) \subset (1-\varepsilon, 1]$ for all real $s$. We choose $\sigma > 0$ with $T(\sigma+t_0,t_0)z_0 \in \mathcal{O}_{\frac{\delta}{2}}(A^*(\sigma+t_0))$.

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From the continuity of the process $\{T(t,s) : t \geq s\}$ and Lemma 3.1, there is a neighborhood $U$ of $z_0$ in $X$ and $\delta' > 0$ such that $T(\sigma+t,t)U \subset \mathcal{O}_{\frac{\delta}{2}}(A^*(\sigma+t_0)) \subset \mathcal{O}_\delta(A^*(\sigma+t))$ whenever $|t-t_0| < \delta'$, therefore, $k(t,z) \geq l(t,z) > 1-\varepsilon$ whenever $z \in U$ and $|t-t_0| < \delta'$ and the proof of the proposition is complete. ▮

Lemma 3.2. *Let $\{T(t,s) : t \geq s\}$ be an evolution process in a metric space $X$, with pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ satisfying that for each $z \in X$ we have*

$$\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, \mathscr{A}(r+t)) = 0,$$

*and for every $\delta > 0$ there is $\delta' \in (0,\delta)$ such that*

$$T(r+t,t)(\mathcal{O}_{\delta'}(\mathscr{A}(t))) \subset \mathcal{O}_\delta(\mathscr{A}(r+t)),\ for\ all\ r \geq 0\ and\ t \in \mathbb{R}.$$

*Then, the function $h : \mathbb{R} \times X \to \mathbb{R}$ given by*

$$h(t,z) := \sup_{r\geq 0} \mathbf{dist}(T(r+t,t)z, \mathscr{A}(r+t)),\ (t,z) \in \mathbb{R} \times X,$$

*is continuous (in both variables) and for each $(t,z) \in \mathbb{R} \times X$, the function $[0,\infty) \ni r \mapsto h(r+t, T(r+t,t)z) \in \mathbb{R}$ is non-increasing with $h(t,z) = 0$ if and only if $z \in \mathscr{A}(t)$.*

*Proof.* Indeed, the proof of the fact that the function $[0,\infty) \ni r \mapsto h(r+t, T(r+t,t)z) \in \mathbb{R}$ is non-increasing, for each $z \in X$ and $t \in \mathbb{R}$, is analogous to the proof of the same property for the function $k : \mathbb{R} \times X \to \mathbb{R}$ in Theorem 3.1. The proof of the continuity can be done in the same way as in cases 2) and 3) in the proof of the continuity of $k$ in Proposition 3.1. ▮

In the conditions of the two last results, using some ideas from [2], we can improve the conclusion of Theorem 3.1 by proving the following theorem.

Theorem 3.2. *Let $\{T(t,s) : t \geq s\}$ be an evolution process in a metric space $X$, with pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ satisfying that for each $z \in X$ we have*

$$\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, \mathscr{A}(r+t)) = 0,$$

*and for every $\delta > 0$ there is $\delta' \in (0,\delta)$ such that*

$$T(r+t,t)(\mathcal{O}_{\delta'}(\mathscr{A}(t))) \subset \mathcal{O}_\delta(\mathscr{A}(r+t)),\ for\ all\ r \geq 0\ and\ t \in \mathbb{R}.$$

*Also, let $(A, A^*)$ be an attractor-repeller pair for $\{T(t,s) : t \geq s\}$ with*

$$\mathcal{O}_\varepsilon(A(t)) \cap \mathcal{O}_\varepsilon(A^*(t)) = \varnothing,\ for\ all\ t \in \mathbb{R},$$

*for some $\varepsilon > 0$ and, furthermore, satisfying the following conditions:*

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*(a) The conclusion of Lemma 2.3 holds considering neighborhoods of $A$ in $X$, that is, for each $\delta > 0$ there is $\delta' \in (0, \delta)$ such that*

$$T(r+t,t)(\mathcal{O}_{\delta'}(A(t))) \subset \mathcal{O}_\delta(A(r+t)), \text{ for all } r \geq 0 \text{ and } t \in \mathbb{R}.$$

*(b) For each $t \in \mathbb{R}$ and $z \in X \backslash (A(t) \cup A^*(t))$ we have*

$$\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A(r+t)) = 0 \text{ or } \lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A^*(r+t)) = 0.$$

*Then, there exists a function $f : \mathbb{R} \times X \to \mathbb{R}$ satisfying the following four properties:*

*i) For each $(t, z) \in \mathbb{R} \times X$, the function $[0, \infty) \ni r \mapsto f(r+t, T(r+t,t)z) \in \mathbb{R}$ is non-increasing.*

*ii) If $t \in \mathbb{R}$ and $f_t : X \to \mathbb{R}$ is defined by $f_t(z) := f(t, z)$, for each $z \in X$, then $f_t^{-1}(0) = A(t)$ and $f_t^{-1}(1) \cap \mathscr{A}(t) = A^*(t)$.*

*iii) Given $t \in \mathbb{R}$ and $z \in X$, if $f(r+t, T(r+t,t)z) = f(t, z)$ for all $r \geq 0$, then $z \in A(t) \cup A^*(t)$.*

*iv) $f : \mathbb{R} \times X \to \mathbb{R}$ is continuous (in both variables).*

*Proof.* In fact, let $k : \mathbb{R}\times X \to \mathbb{R}$ be the function given in Theorem 3.1 and $h : \mathbb{R}\times X \to \mathbb{R}$ be the function given in Lemma 3.2. We define $f : \mathbb{R} \times X \to \mathbb{R}$ by

$$f(t, z) := k(t, z) + h(t, z), \ (t, z) \in \mathbb{R} \times X.$$

It follows, immediately from the properties of $h$ and $k$, that $f$ is continuous (in both variables) and that, for each $(t, z) \in \mathbb{R}\times X$, the function $[0, \infty) \ni r \mapsto f(r+t, T(r+t, t)z) \in \mathbb{R}$ is non-increasing.

Proof of *ii*). First, let $z \in A(t) \subset \mathscr{A}(t)$, then $h(t, z) = 0$ and, by Theorem 3.1 we have that $k(t, z) = 0$. Hence $f(t, z) = 0$ and $A(t) \subset f_t^{-1}(0)$. Conversely, if $z \in X$ is such that $f_t(z) = 0$, we have that $h(t, z) = 0$ and $k(t, z) = 0$, therefore $z \in \mathscr{A}(t)$, and by *ii*) in Theorem 3.1, $z \in A(t)$. This proves that $f_t^{-1}(0) \subset A(t)$. On the other hand, if $z \in A^*(t) \subset \mathscr{A}(t)$ we have that $h(t, z) = 0$ and, thus, $f(t, z) = k(t, z)+h(t, z) = k(t, z) = 1$ and $z \in A^*(t)$. It follows that $A^*(t) \subset f^{-1}(1) \cap \mathscr{A}(t)$. Now, let $z \in \mathscr{A}(t)$ with $f(t, z) = 1$. It follows that $h(t, z) = 0$ and $k(t, z) = 1$. Therefore $z \in A^*(t)$ and the proof of *ii*) is complete.

Proof of *iii*). Let $(t, z) \in \mathbb{R} \times X$ with $f(r+t, T(r+t, t)z) = f(t, z)$ for all $r \geq 0$, if $z \in \mathscr{A}(t)$ we have $h(r+t, T(r+t, t)z) = 0$ for all $r \geq 0$ and $k(r+t, T(r+t, t)z) = k(t, z)$ for all $r \geq 0$ and, from *iii*) in Theorem 3.1, the conclusion follows. On the other hand, if $z \in X\backslash\mathscr{A}(t)$ then $\lim_{r\to\infty} \mathbf{dist}(T(r+t, t)z, A^*(r+t)) = 0$. Indeed, if that was not the case, by assumption $\lim_{r\to\infty} \mathbf{dist}(T(r+t, t)z, A(r+t)) = 0$, and it is easy to see that

$$\begin{aligned} f(t, z) &= \lim_{r\to\infty} f(r+t, T(r+t, t)z) \\ &= \lim_{r\to\infty} k(r+t, T(r+t, t)z) + \lim_{r\to\infty} h(r+t, T(r+t, t)z) = 0. \end{aligned} \tag{12}$$

Thus, $k(t,z) = 0$ and $z \in A(t)$ which is in contradiction with the fact that $z \in X \backslash \mathscr{A}(t)$. It follows that $\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A^*(r+t)) = 0$. Now, using the same reasoning as in (12), we obtain

$$\begin{aligned} f(t,z) &= \lim_{r\to\infty} f(r+t, T(r+t,t)z) \\ &= \lim_{r\to\infty} k(r+t, T(r+t,t)z) + \lim_{r\to\infty} h(r+t, T(r+t,t)z) = 1 + 0 = 1, \end{aligned}$$

but this cannot be true since the fact that $k(t,z) \geq k(r+t, T(r+t,t)z)$ for all $r \geq 0$, implies $1 \geq k(t,z) \geq \lim_{r\to\infty} k(r+t, T(r+t,t)z) = 1$, that is, $k(t,z) = 1$, therefore $z \in A^*(t)$, contradicting $z \in X \backslash \mathscr{A}(t)$. This completes the proof. ∎

*Remark 3. 1.* We note that the function $h$ plays a fundamental role in the proof of *iii*).

Finally, analogously to Theorem 3.4 in [2], joining all of the results in this section, the following result holds.

THEOREM 3.3. *Let $\{T(t,s) : t \geq s\}$ be a gradient-like evolution process with respect to the isolated invariant families $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$, reordered in the way described in Theorem 2.1, and with pullback attractor $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ .*

*Let $A_j(t) = \bigcup_{i=1}^{j} W^u(\Xi_i)(t)$, for $1 \leq j \leq n$ and $t \in \mathbb{R}$.*

*Assume that the following conditions hold:*

*$(S1)$ There exists $\delta > 0$ such that $\mathcal{O}_\varepsilon(A_j(t)) \cap \mathcal{O}_\varepsilon(A_j^*(t)) = \varnothing$, for all $1 \leq j \leq n$ and $t \in \mathbb{R}$,*

*$(S2)$ Given $\delta > 0$, there is $\delta' \in (0,\delta)$ such that*

$$T(r+t,t)(\mathscr{A}(t) \cap \mathcal{O}_{\delta'}(A_j(t))) \subset \mathscr{A}(t+r) \cap \mathcal{O}_\delta(A_j(r+t)),$$

*for all $r \geq 0$, $1 \leq j \leq n$ and $t \in \mathbb{R}$.*

*Then, there exists a Lyapunov function $V : \mathbb{R} \times X \to \mathbb{R}$ with properties $i$)-$iii$) of Definition 3.1, and such that $V(t, \Xi_k(t)) = \{k-1\}$, for all $t \in \mathbb{R}$ and $k = 1, \cdots, n$.*

*Moreover, if $(S2)$ holds for neighborhoods of the $A_j{}'s$ in $X$ and given $(t,z) \in \mathbb{R} \times X$ and $1 \leq j \leq n$ $\lim_{r\to\infty} \mathbf{dist}(T(r+t,t)z, A_j(r+t) \cup A_j^*(r+t)) = 0$, then the Lyapunov function $V : \mathbb{R} \times X \to \mathbb{R}$ can be chosen continuous in both variables.*

## 4. NON–AUTONOMOUS PERTURBATIONS OF AUTONOMOUS EVOLUTION PROCESSES

Now, we will exhibit a class of examples to which our previous abstract theory can be applied. This class consists of the non-autonomous perturbations of gradient-like semigroups.

To begin with, we recall Theorem 3.9 in [5]:

THEOREM 4.1. *Let $X$ be a metric space and, for each $\eta \in [0,1]$, let $\{T_\eta(t,s) : t \geq s\}$ be an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$. Assume that the following conditions hold:*

$(a)$ $\overline{\bigcup_{\eta\in[0,1]} \bigcup_{t\in\mathbb{R}} \mathscr{A}_\eta(t)}$ *is compact.*

$(b)\{T_0(t,s) : t \geq s\}$ *is an autonomous evolution process and* $\{T_0(t,0) = S(t) : t \geq 0\}$ *is a gradient-like semigroup relatively to the set of equilibria* $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$.

$(c)$*For each* $\eta \in (0,1]$, $\{T_\eta(t,s) : t \geq s\}$ *possesses n isolated global solutions* $\xi^*_{i,\eta} : \mathbb{R} \to X$ $i = 1, 2, \cdots, n$, $\eta \in (0,1]$, *and* $\lim_{\eta\to 0^+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$.

$(d)$*For each compact set* $K \subset \mathbb{R}^+ \times X$,

$$\lim_{\eta\to 0^+} \sup_{s\in\mathbb{R}} \sup_{(t,x)\in K} \mathbf{d}(T_\eta(t+s,s)x, T_0(t+s,s)x) = 0.$$

$(e)$*There exist* $\mu > 0$ *and* $\eta_1 \in (0,1]$ *such that, if* $\xi_\eta : \mathbb{R} \to X$ *is a bounded solution of* $\{T_\eta(t,s) : t \geq s\}$ *with* $\eta \leq \eta_1$ *so that there exists* $t_0 \in \mathbb{R}$ *and some* $i \in \{1, \cdots, n\}$ *with* $\sup_{t\leq t_0} \mathbf{dist}(\xi_\eta(t), \xi^*_{i,\eta}(\mathbb{R})) < \mu$ *(resp.* $\sup_{t\geq t_0} \mathbf{dist}(\xi_\eta(t), \xi^*_{i,\eta}(\mathbb{R})) < \mu$*), then* $\lim_{t\to-\infty} \mathbf{d}(\xi_\eta(t), \xi^*_{i,\eta}(t)) = 0$ *(resp.* $\lim_{t\to\infty} \mathbf{d}(\xi_\eta(t), \xi^*_{i,\eta}(t)) = 0$*).*

*Then, there exists* $\eta_0 \in (0, \eta_1]$ *such that, for each* $\eta \in (0, \eta_0]$, $\{T_\eta(t,s) : t \geq s\}$ *is a non-autonomous gradient-like evolution process with respect to the disjoint set of isolated invariant families* $\mathcal{E}_\eta = \{\xi^*_{1,\eta}, \cdots, \xi^*_{n,\eta}\}$.

The following lemma will be crucial to establish our main results in this section, (see Lemma 3.1 in [7] for the proof).

LEMMA 4.1. *Let $X$ be a metric space and $\{T_\eta(t,s) : t \geq s\}_{\eta\in[0,1]}$ be a family of evolution processes on $X$ with pullback attractors $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta\in[0,1]}$. Assume that conditions $(a)$, $(b)$ and $(d)$ of Theorem 4.1 are satisfied.*

*Let* $\{\eta_k\}_{k\in\mathbb{N}}$ *be a sequence in* $(0,\infty)$ *with* $\eta_k \overset{k\to\infty}{\longrightarrow} 0$, $\{t_k\}_{k\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}$ *and* $\{\xi_k\}_{k\in\mathbb{N}}$ *be a sequence in* $C(\mathbb{R}, X)$ *such that, for each* $k \in \mathbb{N}$, $\xi_k : \mathbb{R} \to X$ *is a bounded global solution for* $\{T_{\eta_k}(t,s) : t \geq s\}$.

*Then, there is a subsequence of* $\{\xi_k\}_{k\in\mathbb{N}}$, *which we denote the same, and a global solution* $\xi : \mathbb{R} \to \mathscr{A}_0$ *for* $\{T_0(t,0) = S(t) : t \geq 0\}$, *such that*

$$\lim_{k\to\infty} \sup_{|t|\leq R} \mathbf{d}(\xi_k(t+t_k), \xi(t)) = 0, \text{ for each } R > 0.$$

Let $X$ be a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ which satisfies conditions $(a)$-$(e)$ in Theorem 4.1. We suppose that the set of equilibria, $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$, of $\{T_0(t,0) = S(t) : t \geq 0\}$, is reordered so that it is a Morse-Decomposition of the global attractor $\mathscr{A}_0$, as in Theorem 2.1 (see also Theorem 2.17 in [2] for the autonomous framework).

For $\eta \in [0,1]$, reordering the $\mathcal{E}_\eta = \{\xi^*_{1,\eta}, \cdots, \xi^*_{n,\eta}\}$ so that $\sup_{1\le i\le n}\sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) \overset{\eta\to 0^+}{\longrightarrow} 0$, let $A_{0,\eta}(t) := \varnothing$, $A_{1,\eta}(t) := \{\xi^*_{1,\eta}(t)\}$ and

$$A_{j,\eta}(t) := A_{j-1,\eta}(t) \cup W^u(\xi^*_{j,\eta})(t) \; j = 2, \cdots, n, \tag{13}$$

for each $t \in \mathbb{R}$. Also, for each $\eta \in [0,1]$, $1 \le j \le n$ and $t \in \mathbb{R}$, let

$$A^*_{j,\eta}(t) := \{z \in \mathscr{A}_\eta(t) : \mathbf{dist}(T_\eta(r+t,t)z, A_{j,\eta}(r+t)) \not\to 0 \text{ as } r \to \infty\}. \tag{14}$$

If we assume these constructions, we can prove the following results:

Lemma 4.2. *Consider $X$ a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \ge s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ which satisfies conditions (a)-(e) in Theorem 4.1. Assume that $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t,0) = S(t) : t \ge 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta\to 0^+}\sup_{1\le i\le n}\sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$.*

*Then, the families of $A_{j,\eta}$ and $A^*_{j,\eta}$ defined by (13) and (14) behave upper semicontinuously as $\eta \to 0^+$, that is,*

$$\lim_{\eta\to 0^+}\sup_{t\in\mathbb{R}} \mathbf{dist}(A_{j,\eta}(t), A_{j,0}) = 0 \tag{15}$$

*and*

$$\lim_{\eta\to 0^+}\sup_{t\in\mathbb{R}} \mathbf{dist}(A^*_{j,\eta}(t), A^*_{j,0}) = 0 \tag{16}$$

*for each $j = 1, 2, \cdots, n$.*

*Proof.* Indeed, thanks to the fact that $A_{j,0}$ and $A^*_{j,0}$ are disjoint compact sets (see [2]) for all $j$, we can pick $\varepsilon > 0$ such that, for all $j = 1, 2, \cdots, n$

$$\mathcal{O}_\varepsilon(A_{j,0}) \cap \mathcal{O}_\varepsilon\left(A^*_{j,0}\right) = \varnothing. \tag{17}$$

First, we will prove that $\lim_{\eta\to 0^+}\max_{1\le j\le n}\sup_{t\in\mathbb{R}} \mathbf{dist}(A_{j,\eta}(t), A_{j,0}) = 0$. If this is not true, there exists $\delta \in (0,\varepsilon)$, $j = 1, 2, \cdots, n$, a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in $(0,1]$ with $\eta_k \overset{k\to\infty}{\longrightarrow} 0^+$, a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ and a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $X$ such that, for each $k \in \mathbb{N}$, $x_k \in A_{j,\eta_k}(\tau_k)$ and

$$\mathbf{dist}(x_k, A_{j,0}) \ge \delta. \tag{18}$$

Now, by the definition of $A_{j,\eta}(t)$, we can assume that there is $i \le j$, fixed, such that for each natural $k$, there is a global solution $\xi_k : \mathbb{R} \to X$ for $\{T_{\eta_k}(t,s) : t \ge s\}$ with $\xi_k(\tau_k) = x_k$ and

$$\lim_{t\to -\infty} \mathbf{d}(\xi_k(t), \xi^*_{i,\eta_k}(t)) = 0. \tag{19}$$

On the other hand, by hypothesis $(c)$, it follows the existence of $\eta_\delta > 0$ such that, for all $\eta \le \eta_\delta$,

$$\sup_{t\in\mathbb{R}} \mathbf{dist}(\xi^*_{i,\eta}(t), A_{j,0}) < \frac{\delta}{2}.$$

Consequently, by (18) and (19), for each $k$ there exists $t_k \in \mathbb{R}$ such that

$$\mathbf{dist}(\xi_k(t), A_{j,0}) < \delta \text{ for all } t < t_k, \tag{20}$$

and

$$\mathbf{dist}(\xi_k(t_k), A_{j,0}) = \delta. \tag{21}$$

Thus, we define $\widetilde{\xi}_k : \mathbb{R} \to X$ by $\widetilde{\xi}_k(t) := \xi_k(t + t_k)$, $t \in \mathbb{R}$, and, by Lemma 4.1, we can assume that there is a global solution $\xi : \mathbb{R} \to X$ for the semigroup $\{T_0(t, 0) = S(t) : t \ge 0\}$ such that, for each $R > 0$

$$\lim_{k\to\infty} \sup_{|t|\le R} \mathbf{d}(\widetilde{\xi}_k(t), \xi(t)) = 0.$$

By (20), we have that $\mathbf{dist}(\xi(t), A_{j,0}) \le \delta$ for all $t \le 0$ so, by (17), we must have $\xi(0) \in A_{j,0}$, but (21) means that $\mathbf{dist}(\xi(0), A_{j,0}) = \delta$. This contradiction proves (15).

Now, we prove that $\lim_{\eta\to 0^+} \max_{1\le j\le n} \sup_{t\in\mathbb{R}} \mathbf{dist}(A^*_{j,\eta}(t), A^*_{j,0}) = 0$. If it does not hold, there exist $\delta \in (0, \varepsilon)$, $j = 1, 2, \cdots, n$, a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in $(0, 1]$ with $\eta_k \overset{k\to\infty}{\longrightarrow} 0$, a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$, and a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $X$ such that, for each $k \in \mathbb{N}$, $x_k \in A^*_{j,\eta_k}(\tau_k)$ and

$$\mathbf{dist}(x_k, A^*_{j,0}) \ge \delta. \tag{22}$$

Now, by the definition of $A^*_{j,\eta}(t)$ (and Theorem 4.1), we can assume that there is $i \ge j+1$, fixed, such that for each $k \in \mathbb{N}$, there is a global solution $\xi_k : \mathbb{R} \to X$ for $\{T_{\eta_k}(t, s) : t \ge s\}$ with $\xi_k(\tau_k) = x_k$ and

$$\lim_{t\to\infty} \mathbf{d}(\xi_k(t), \xi^*_{i,\eta_k}(t)) = 0. \tag{23}$$

On the other hand, hyphotesis $(c)$ implies the existence of $\eta^*_\delta > 0$ such that, for all $\eta \le \eta^*_\delta$,

$$\sup_{t\in\mathbb{R}} \mathbf{dist}(\xi^*_{i,\eta}(t), A^*_{j,0}) < \frac{\delta}{2},$$

whence, by (22) and (23), for each $k \in \mathbb{N}$, there exists $t_k \in \mathbb{R}$ such that

$$\mathbf{dist}(\xi_k(t), A^*_{j,0}) < \delta \text{ for all } t > t_k, \tag{24}$$

and

$$\mathbf{dist}(\xi_k(t_k), A^*_{j,0}) = \delta. \tag{25}$$

Thus, we define $\widetilde{\xi}_k : \mathbb{R} \to X$ by $\widetilde{\xi}_k(t) := \xi_k(t + t_k)$, $t \in \mathbb{R}$, and, by Lemma 4.1, we can assume that there is a global solution $\xi : \mathbb{R} \to X$ for the semigroup $\{T_0(t, 0) = S(t) : t \geq 0\}$ such that, for each $R > 0$,

$$\lim_{k\to\infty} \sup_{|t|\leq R} \mathbf{d}(\widetilde{\xi}_k(t), \xi(t)) = 0.$$

By (24) we have that $\mathbf{dist}(\xi(t), A^*_{j,0}) \leq \delta$ for all $t > 0$ and, as a consequence of (17), we must have $\xi(0) \in A^*_{j,0}$. But (25) implies that, and $\mathbf{dist}(\xi(0), A^*_{j,0}) = \delta$. This contradiction proves (16) and completes the proof of the lemma. ▮

Lemma 4.3. *Let $X$ be a metric space and, for each $\eta \in [0, 1]$, $\{T_\eta(t, s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ which satisfies conditions (a)-(e) in Theorem 4.1. Assume that $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t, 0) = S(t) : t \geq 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta\to 0^+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$.*

*If the families of local unstable sets $\{W^u_{\eta,\rho}(\xi^*_j)(t)\}_{\eta\in[0,1]}$ behave lower semicontinuously, then the families of $A_{j,\eta}$ behave lower semicontinuously as $\eta \to 0^+$, that is,*

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \mathbf{dist}(A_{j,0}, A_{j,\eta}(t)) = 0.$$

*If the families of local stable sets $\{W^s_{\eta,\rho}(\xi^*_j)(t) \cap \mathscr{A}_\eta(t)\}_{\eta\in[0,1]}$ behave lower semicontinuously and $T_\eta(t, s) : A(s) \to A(t)$ is injective for each $t \geq s$, $\eta \in [0, 1]$, then*

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \mathbf{dist}(A^*_{j,0}, A_{j,\eta}(t)^*) = 0.$$

*Proof.* It follows analogously to the proof of Theorem 2.13 in [6]. ▮

Next we exhibit an example (pictorial) for which we have lower semicontinuity of repellers but for which we do not have structural stability. The purpose of this example is to show that the systems which satisfy the hypothesis imposed in the last part of Lemma 4.3 are a larger class than that of the structurally stable systems.

Below, the figure labelled 0i.a corresponds to the perturbed attractor-repeller pair $(A_{i,\eta}, A^*_{i,\eta})$ with $A_{i,\eta}$ pictured in black and $A^*_{i,\eta}$ pictured in red, $0 \leq i \leq 5$. The figure labelled 0i.b corresponds to the limiting attractor-repeller $(A_i, A^*_i)$ pair with $A_i$ pictured in black and $A^*_i$ pictured in red, $1 \leq i \leq 5$. Of course $A_{5,\eta}$ $(A^*_{0,\eta})$ and $A_5$ $(A^*_0)$ correspond to the global attractor whereas $A^*_{5,\eta}$ $(A_{0,\eta})$ and $A^*_5$ $(A_0)$ correspond to the empty set.

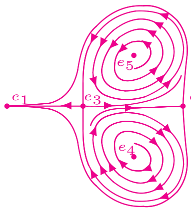

*Figure 00.a*

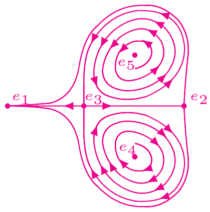

*Figure 00.b*

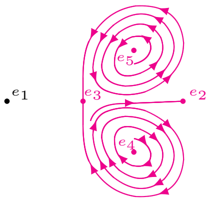

*Figure 01.a*

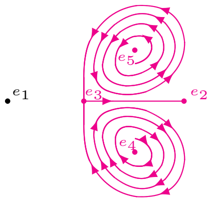

*Figure 01.b*

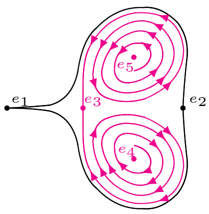

*Figure 02.a*

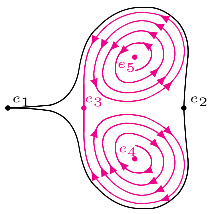

*Figure 02.b*

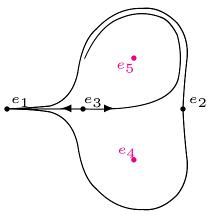

*Figure 03.a*

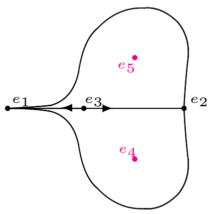

*Figure 03.b*

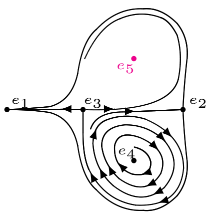

*Figure 04.a*

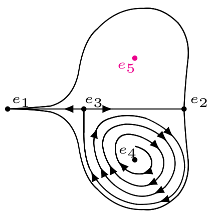

*Figure 04.b*

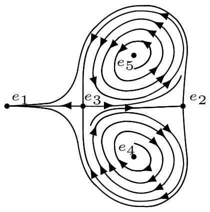


*Figure 05.a*

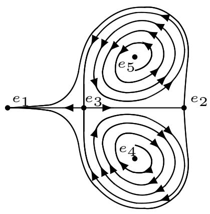


*Figure 05.b*

PROPOSITION 4.1. *Consider $X$ a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ which satisfies conditions (a)-(e) in Theorem 4.1. Assume that $\mathcal{E}_0 = \{y_{1,0}^*, \cdots, y_{n,0}^*\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t,0) = S(t) : t \geq 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta \to 0^+} \sup_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \mathbf{d}(\xi_{i,\eta}^*(t), y_{i,0}^*) = 0$.*

*Then, there exists $\varepsilon > 0$ and $\eta_0 > 0$ such that*

$$\mathcal{O}_\varepsilon(A_{j,\eta}(t)) \cap \mathcal{O}_\varepsilon\left(\bigcup_{i=j+1}^{n} \{\xi_{i,\eta}^*(t)\}\right) = \varnothing$$

*for each $j = 1, 2, \cdots, n-1$, $t \in \mathbb{R}$ and $\eta \in [0, \eta_0]$.*

*Proof.* Indeed, by Theorem 2.17 in [2], we can take $\varepsilon > 0$ such that, for each $j = 1, 2, \cdots, n-1$ and $t \in \mathbb{R}$

$$\mathcal{O}_{2\varepsilon}(A_{j,0}) \cap \mathcal{O}_{2\varepsilon}\left(\bigcup_{i=j+1}^{n} \{y_{i,0}^*\}\right) = \varnothing. \tag{26}$$

We note that the proposition will be accomplished if we show the following stronger conditions:

For each $\delta \in (0, \varepsilon]$ there is $\eta_\delta > 0$ such that

$$A_{j,\eta}(t) \subset \mathcal{O}_\delta(A_{j,0}) \tag{27}$$

and

$$\bigcup_{i=j+1}^{n} \{\xi_{i,\eta}^*(t)\} \subset \mathcal{O}_\delta\left(\bigcup_{i=j+1}^{n} \{y_{i,0}^*\}\right) \tag{28}$$

for all $t \in \mathbb{R}$, $j = 1, 2, \cdots, n-1$ and $\eta \in [0, \eta_\delta]$.

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Clearly (28) follows from hyphotesis ($c$), and (27) is a direct consequence from (15) in Lemma 4.2. This completes the proof of the proposition. ∎

Using Proposition 4.1 we can show that the sets $A_{j,\eta}$, defined above, are all local attractors for all suitably $\eta$. This is proved in the following proposition.

PROPOSITION 4.2. *Let $X$ be a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ which satisfies conditions ($a$)-($e$) from Theorem 4.1. Assume that $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t,0) = S(t) : t \geq 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta\to 0+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$.*

*Then, there exists $\eta_0 > 0$ such that the invariant family $A_{j,\eta} := \{A_{j,\eta}(t) : t \in \mathbb{R}\}$ is a local attractor for the evolution process $\{T_\eta(t,s) : t \geq s\}$, for each $j = 1, 2, \cdots, n$ and $\eta \in (0, \eta_0]$.*

*Proof.* Indeed, by Lemma 4.2 and the proof of Proposition 4.1, we can choose $\varepsilon > 0$, $\delta \in (0, \varepsilon]$ and $\eta_0 > 0$ such that

$$A_{j,\eta}(t) \subset \mathcal{O}_\delta(A_{j,0})$$

and

$$\bigcup_{i=j+1}^{n} \{\xi^*_{i,\eta}(t)\} \subset \mathcal{O}_\delta \left( \bigcup_{i=j+1}^{n} \{y^*_{i,0}\} \right)$$

for each $j = 1, 2, \cdots, n-1$, $t \in \mathbb{R}$ and $\eta \in [0, \eta_0]$, where $\varepsilon > 0$ satisfies

$$\mathcal{O}_{2\varepsilon}(A_{j,0}) \cap \mathcal{O}_{2\varepsilon} \left( \bigcup_{i=j+1}^{n} \{y^*_{i,0}\} \right) = \varnothing.$$

Thus, if $\Gamma_{j,\eta}$ is the trace of the family $A_{j,\eta}$, it follows that $\Gamma_{j,\eta} \subset \mathcal{O}_\varepsilon(A_{j,0})$ for $j = 1, 2, \cdots, n-1$ and $\eta \in [0, \eta_0]$, and therefore

$$\mathcal{O}_\varepsilon(\Gamma_{j,\eta}) \cap \mathcal{O}_\varepsilon \left( \bigcup_{i=j+1}^{n} \{\xi^*_{i,\eta}(t)\} \right) = \varnothing \tag{29}$$

for each $j = 1, 2, \cdots, n-1$, $t \in \mathbb{R}$ and $\eta \in [0, \eta_0]$.

Now, for a fixed $\eta \in [0, \eta_0]$, let $\xi_\eta : \mathbb{R} \to X$ be a global solution for $\{T_\eta(t,s) : t \geq s\}$ with $\xi_\eta(t) \in \mathcal{O}_\varepsilon(\Gamma_{j,\eta})$ for all $t \in \mathbb{R}$ and, recalling that $\{T_\eta(t,s) : t \geq s\}$ is a gradient-like evolution process with respect to $\mathcal{E}_\eta = \{\xi^*_{1,\eta}, \cdots, \xi^*_{n,\eta}\}$ for $\eta \in [0, \eta_0]$ (by Theorem 4.1), let $i \in \{1, 2, \cdots, n\}$ such that

$$\lim_{t\to-\infty} \mathbf{d}(\xi_\eta(t), \xi^*_{i,\eta}(t)) = 0.$$

By (29), we must have $i \leq j$, so $\xi_\eta(t) \in W^u(\xi^*_{i,\eta})(t) \subset A_{j,\eta}(t)$ for all $t \in \mathbb{R}$, which tells us that the family $A_{j,\eta} = \{A_{j,\eta}(t) : t \in \mathbb{R}\}$ is invariant and isolated for each $j = 1, 2, \cdots, n$ and $\eta \in [0, \eta_0]$.

On the other hand, let $\xi_\eta : \mathbb{R} \to X$ be a global solution for $\{T_\eta(t,s) : t \geq s\}$ with $\eta \in [0, \eta_0]$ and $\lim_{t\to-\infty} \mathbf{dist}(\xi_\eta(t), A_{j,\eta}(t)) = 0$. If $t_0$ is such that $\mathbf{dist}(\xi_\eta(t), A_{j,\eta}(t)) < \varepsilon$ for $t \leq t_0$, as $\mathcal{O}_\varepsilon(A_{j,\eta}(t)) \cap \mathcal{O}_\varepsilon\left(\bigcup_{i=j+1}^{n} \{\xi^*_{i,\eta}(t)\}\right) = \varnothing$ for all $t \in \mathbb{R}$, using the above reasoning, we must have $\xi_\eta(t) \in A_{j,\eta}(t)$ for all $t \in \mathbb{R}$, which shows that $W^u(A_{j,\eta})(t) \subset A_{j,\eta}(t)$ for each $t \in \mathbb{R}$. Now, the inclusion $A_{j,\eta}(t) \subset W^u(A_{j,\eta})(t)$, for each $t \in \mathbb{R}$, holds thanks to the invariance of the family $A_{j,\eta}$, showing that $W^u(A_{j,\eta})(t) = A_{j,\eta}(t)$ for all $t \in \mathbb{R}$, and completing the proof of the proposition. ∎

The separation property between the local attractor and its repeller is also satisfied for the class of the non-autonomous perturbation of a gradient-like semigroup, this is what we will show in the next proposition.

PROPOSITION 4.3. *Let $X$ be a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$, which satisfies conditions (a)-(e) in Theorem 4.1. Assume that $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t,0) : t \geq 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta\to 0^+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$. Consider the families of local attractors and corresponding repellers defined in* (13) *and in* (14).

*Then, there exist $\varepsilon > 0$ and $\eta_0 > 0$ such that, for all $t \in \mathbb{R}$, $j = 1, 2, \cdots, n$ and $\eta \in [0, \eta_0]$, we have*

$$\mathcal{O}_\varepsilon(A_{j,\eta}(t)) \cap \mathcal{O}_\varepsilon\left(A^*_{j,\eta}(t)\right) = \varnothing.$$

*Proof.* Indeed, thanks to the fact that $A_{j,0}$ and $A^*_{j,0}$ are disjoint compact sets (see [2]) for all $j$, we can choose $\varepsilon > 0$ such that, for all $j = 1, 2, \cdots, n$

$$\mathcal{O}_{2\varepsilon}(A_{j,0}) \cap \mathcal{O}_{2\varepsilon}\left(A^*_{j,0}\right) = \varnothing. \tag{30}$$

By Lemma 15, for each $\delta \in (0, \varepsilon]$ , there is $\eta_\delta > 0$ such that

$$A_{j,\eta}(t) \subset \mathcal{O}_\delta(A_{j,0})$$

and

$$A^*_{j,\eta}(t) \subset \mathcal{O}_\delta(A^*_{j,0}), \tag{31}$$

for every $\eta \in [0, \eta^*_\delta]$, real $t \in \mathbb{R}$ and $j = 1, 2, \cdots, n$ and the result easily follows. ∎

THEOREM 4.2. *Let $X$ be a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ a nonlinear evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$, which satisfies* *conditions (a)-(e) of Theorem 4.1. Assume that* $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$ *is reordered in such a way that it is a Morse-Decomposition for* $\{T_0(t,0) : t \geq 0\}$ *and that* $\mathcal{E}_\eta$ *is reordered in such a way that* $\lim_{\eta\to 0^+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$. *Consider also the families of local attractors* $A_{j,\eta} = \{A_{j,\eta}(t) : t \in \mathbb{R}\}$ *defined in* (13), *and their associated repellers* $A^*_{j,\eta} = \{A^*_{j,\eta}(t) : t \in \mathbb{R}\}$ *given by* (14).

*Then, there exists* $\eta_0 > 0$ *such that for all* $\eta \in (0, \eta_0]$, $t \in \mathbb{R}$ *and* $j = 1, 2, \cdots, n$

$$A_{j,\eta}(t) \cap A^*_{j-1,\eta}(t) = \{\xi^*_{j,\eta}(t)\},$$

*that is, the set* $\mathcal{E}_\eta = \{\xi^*_{1,\eta}, \cdots, \xi^*_{n,\eta}\}$ *determines a Morse decomposition of the pullback attractor* $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ *for* $\{T_\eta(t,s) : t \geq s\}$ *for each* $\eta \in (0, \eta_0]$. *In particular, such Morse-Decomposition behaves continuously.*

*Proof.* Indeed, it is clear that $\xi^*_{j,\eta}(t) \in W^u(\xi^*_{j,\eta})(t) \subset A_{j,\eta}(t)$ for each $1 \leq j \leq n$, $t \in \mathbb{R}$ and $\eta \in [0, 1]$. Now, using Proposition 4.1, let $\varepsilon > 0$ and $\eta_0 > 0$ be such that, for all $t \in \mathbb{R}$, $2 \leq j \leq n$ and $\eta \in [0, \eta_0]$, we have

$$\mathcal{O}_{2\varepsilon}(A_{j-1,\eta}(t)) \cap \mathcal{O}_{2\varepsilon}\left(\bigcup_{i=j}^{n} \{\xi^*_{i,\eta}(t)\}\right) = \varnothing. \tag{32}$$

From (32) $\mathbf{dist}(T_\eta(r+t,t)\xi^*_{j,\eta}(t), A_{j-1,\eta}(t+r)) = \mathbf{dist}(\xi^*_{j,\eta}(t+r), A_{j-1,\eta}(t+r)) \geq \varepsilon$ for all $r \geq 0$. Hence $\xi^*_{j,\eta}(t) \in A^*_{j-1,\eta}(t)$ for all $t \in \mathbb{R}$. Therefore, $\{\xi^*_{j,\eta}(t)\} \subset A_{j,\eta}(t) \cap A^*_{j-1,\eta}(t)$ for every $\eta \in (0, \eta_0]$, $t \in \mathbb{R}$ and $1 \leq j \leq n$.

Conversely, we note that, by Lemma 4.2, given $\delta \in (0, \varepsilon]$ there exists $\eta_\delta > 0$ such that for all $\eta \in (0, \eta_\delta]$, $t \in \mathbb{R}$ and $1 \leq j \leq n$ we have that

$$A_{j,\eta}(t) \cap A^*_{j-1,\eta}(t) \subset \mathcal{O}_{\frac{\delta}{2}}(A_{j,0}) \cap \mathcal{O}_{\frac{\delta}{2}}\left(A^*_{j-1,0}\right).$$

Now, if the statement

$A_{j,\eta}(t) \cap A^*_{j-1,\eta}(t) \subset \{\xi^*_{j,\eta}(t)\}$ for all $t \in \mathbb{R}$, $1 \leq j \leq n$ and for all $\eta > 0$ suitably small

does not hold, there are $1 \leq j \leq n$, a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in $(0, 1]$ with $\eta_k \to 0^+$, a sequence $\{t_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ and a sequence $\{z_k\}_{k\in\mathbb{N}}$ in $X$ with $z_k \in A_{j,\eta_k}(t_k) \cap A^*_{j-1,\eta_k}(t_k)$ but $z_k \neq \xi^*_{j,\eta_k}(t_k)$ for all $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there exists a global solution $\xi_k : \mathbb{R} \to X$ for $\{T_{\eta_k}(t,s) : t \geq s\}$ with $\xi_k(t_k) = z_k$ and, we can assume that, there are $i \leq j$ and $l \geq j$, fixed, with

$$\lim_{t\to-\infty} \mathbf{d}(\xi_k(t), \xi^*_{i,\eta_k}(t)) = 0 \text{ and } \lim_{t\to\infty} \mathbf{d}(\xi_k(t), \xi^*_{l,\eta_k}(t)) = 0.$$

Moreover, $\xi_k(t) \in \mathcal{O}_{\frac{\delta}{2}}(A_{j,0}) \cap \mathcal{O}_{\frac{\delta}{2}}\left(A^*_{j-1,0}\right)$ for each $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Therefore, if we choose, for suitably large $k$, $\tau_k$ and $\sigma_k$ with

$$\mathbf{d}(\xi_k(t), y^*_{i,0}) < \frac{\delta}{2} \text{ for } t \leq \tau_k \text{ and } \mathbf{d}(\xi_k(t), y^*_{l,0}) < \frac{\delta}{2} \text{ for } t \geq \sigma_k,$$

we must have $y_{i,0}^*$, $y_{l,0}^* \in \mathcal{O}_\delta(A_{j,0}) \cap \mathcal{O}_\delta\left(A_{j-1,0}^*\right)$. Then, by (32), $i = l = j$, that is, $y_{i,0}^* = y_{l,0}^* = y_{j,0}^*$, thus $\xi_{i,\eta_k}^* = \xi_{l,\eta_k}^* = \xi_{j,\eta_k}^*$, which means that the solution $\xi_k : \mathbb{R} \to X$ is a homoclinic solution (because $z_k \neq \xi_{j,\eta_k}^*(t_k)$ for every $k$), contradicting the fact that $\{T_{\eta_k}(t,s) : t \geq s\}$ is gradient-like respect to $\mathcal{E}_{\eta_k} = \{\xi_{1,\eta_k}^*, \cdots, \xi_{n,\eta_k}^*\}$, completing the proof. ▮

Definition 4.1. We say that the family $\{T_\eta(t,s) : t \geq s\}_{\eta\in[0,1]}$ of evolution processes in a metric space $X$ is **collectively asymptotically compact**, if $\{T_{\eta_k}(t_k+\tau_k, \tau_k)x_k\}_{k\in\mathbb{N}}$ is relatively compact whenever $\{\eta_k\}_{k\in\mathbb{N}}$ is a sequence in $(0,1]$, $\{x_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $X$, $\{t_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ and $\{\tau_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ with with $\eta_k \stackrel{k\to\infty}{\longrightarrow} 0$ and $t_k \stackrel{k\to\infty}{\longrightarrow} \infty$.

Lemma 4.4. *Let $\{T_\eta(t,s) : t \geq s\}_{\eta\in[0,1]}$ be a family of evolution processes in a metric space $X$ with a corresponding family of pullback attractors $\{\mathscr{A}_\eta\}_{\eta\in[0,1]} := \{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta\in[0,1]}$, which satisfies conditions $(a)$ and $(d)$ in Theorem 4.1. Assume that the family $\{T_\eta(t,s) : t \geq s\}_{\eta\in[0,1]}$ is collectively asymptotically compact and that $T_0(t,s) = S(t-s)$ for all $t \geq s$, where $\{S(t) : t \geq 0\}$ is a semigroup.*

*Let $\{A_\eta\}_{\eta\in[0,1]} := \{A_\eta(t) : t \in \mathbb{R}\}_{\eta\in[0,1]}$ be such that $A_\eta(t) \subset \mathscr{A}_\eta(t)$ for every $t \in \mathbb{R}$ with $A_0$ a local attractor for the semigroup $\{S(t) : t \geq 0\}$ (that is, $\omega(\mathcal{O}_\varepsilon(A_0)) = A_0$ for some $\varepsilon > 0$).*

*If $\{A_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$, that is,*

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \mathbf{d}_H(A_\eta(t), A_0) = 0,$$

*then, given $\delta \in (0,\varepsilon)$, there exist $\delta' \in (0,\delta)$ and $\eta_\delta > 0$ such that, for all $\eta \in [0,\eta_\delta]$, it holds*

$$T_\eta(t,s)(\mathcal{O}_{\delta'}(A_\eta(s))) \subset \mathcal{O}_\delta(A_\eta(t)) \text{ whenever } t \geq s.$$

*Proof.* If not, there are $\delta > 0$ and sequences $\{z_j\}_{j\in\mathbb{N}}$ in $X$, $\{\eta_j\}_{j\in\mathbb{N}}$ in $(0,1]$, $\{t_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^+$ and $\{\tau_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}$ such that $\eta_j \stackrel{j\to\infty}{\longrightarrow} 0^+$, $t_j \stackrel{j\to\infty}{\longrightarrow} \infty$, $\mathbf{dist}(z_j, A_{\eta_j}(\tau_j)) < \frac{1}{j}$ for all $j$,

$$\mathbf{dist}(T_{\eta_j}(t+\tau_j,\tau_j)z_j, A_{\eta_j}(t+\tau_j)) < \delta \text{ for all } t \in [0,t_j) \text{ and all } j \in \mathbb{N}$$

and

$$\mathbf{dist}(T_{\eta_j}(t_j+\tau_j,\tau_j)z_j, A_{\eta_j}(t_j+\tau_j)) = \delta \text{ for all } j \in \mathbb{N}.$$

If, for each $j$, we now define $\xi_j : [-t_j,\infty) \to X$ by $\xi_j(t) := T_{\eta_j}(t+t_j+\tau_j,\tau_j)z_j$, from the collective asymptotic compactness and the uniform convergence in compact sets (hypothesis $(d)$), it is not difficult to see that there exists a bounded global solution $\xi_0 : \mathbb{R} \to X$ for $\{T_0(t,0) = S(t) : t \geq 0\}$ and a subsequence of $\{\xi_j\}_{j\in\mathbb{N}}$, denoted the same, such that for all $t$, $\xi_0(t) = \lim_{j\to\infty} \xi_j(t)$.

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On the other hand, given $t < 0$, for all $j$ big enough it holds

$$\mathbf{dist}(\xi_j(t), A_0) \leq \mathbf{dist}(\xi_j(t), A_{\eta_j}(t+\tau_j)) + \mathbf{dist}(A_{\eta_j}(t+\tau_j), A_0),$$

from where, by the upper-semicontinuity of $\{A_\eta\}_{\eta\in[0,1]}$, we obtain that for all $t < 0$

$$\mathbf{dist}(\xi_0(t), A_0) \leq \delta,$$

and from $\delta = \mathbf{dist}(\xi_j(0), A_{\eta_j}(t_j+\tau_j)) \leq \mathbf{dist}(\xi_j(0), A_0) + \mathbf{dist}(A_0, A_{\eta_j}(t_j+\tau_j))$, by the lower-semicontinuity of $\{A_\eta\}_{\eta\in[0,1]}$, it follows that $\mathbf{dist}(\xi_0(0), A_0) = \delta$.

But, as $\delta < \varepsilon$, then $A_0$ attracts $K = \{\xi_0(t) : t \leq 0\}$, which contradicts the fact that $\mathbf{dist}(\xi_0(0), A_0) = \delta$. ▌

PROPOSITION 4.4. *Let $X$ be a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$, which satisfies conditions (a)-(e) in Theorem 4.1. Assume that $\mathcal{E}_0 = \{y^*_{1,0}, \cdots, y^*_{n,0}\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t,0) : t \geq 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta\to 0^+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi^*_{i,\eta}(t), y^*_{i,0}) = 0$. Consider the families of local attractors and corresponding repellers defined in* (13) *and in* (14) *and assume that the families of local unstable sets $\{W^u_{\eta,\rho}(\xi^*_j)(t)\}_{\eta\in[0,1]}$ behave lower semicontinuously.*

*Then, there exists $\eta_0 > 0$ such that for every $\eta \in [0,\eta_0]$ and every $j = 1, 2, \cdots, n$ the local atractor $A_{j,\eta} = \{A_{j,\eta}(t) : t \in \mathbb{R}\}$ satisfies the conclusion of Lemma 2.3 for a neighborhood of $A_{j,\eta}$ in $X$ (see hypothesis* (a) *in Proposition 3.1).*

*Proof.* Indeed, we fix $\delta_1 \in (0,\varepsilon)$, $\eta_1 > 0$ and $\delta'_1 \in (0,\delta_1)$ such that the conclusion of Lemma 4.4 holds, that is, for every $\eta \in [0,\eta_1]$ and each $j = 1, 2, \cdots, n$, we have

$$T_\eta(t,s)(\mathcal{O}_{\delta'_1}(A_{j,\eta}(s))) \subset \mathcal{O}_{\delta_1}(A_{j,\eta}(t)) \text{ whenever } t \geq s. \tag{33}$$

On the other hand, from Lemma 4.3, there are $0 < \delta'_3 < \delta'_2 < \delta'_1$ and $\eta_2 \in (0,\eta_1]$ such that $\mathcal{O}_{\delta'_3}(A_{j,\eta}(s)) \subset \mathcal{O}_{\delta'_2}(A_{j,0}) \subset \mathcal{O}_{\delta'_1}(A_{j,\eta}(s))$ for each $\eta \in [0,\eta_2]$, $j = 1, 2, \cdots, n$ and $s \in \mathbb{R}$. Hence, by (33), $\omega_\eta(\mathcal{O}_{\delta'_2}(A_{j,0}), s)$[1] $\subset \mathcal{O}_{\delta_1}(A_{j,\eta}(s))$ for all real $s$, $\eta \in [0,\eta_2]$ and $j = 1, 2, \cdots, n$.

It is clear that $\omega_\eta(\mathcal{O}_{\delta'_2}(A_{j,0}), s)$ is contained in $A_{j,\eta}(s)$ for all $s \in \mathbb{R}$, otherwise, there would be a global solution $\xi : \mathbb{R} \to X$ with $\xi(t) \in \mathcal{O}_{\delta_1}(A_{j,\eta}(t))$ for all $t \in \mathbb{R}$ and such that $\xi(t_0) \notin A_{j,\eta}(t_0)$ for some $t_0 \in \mathbb{R}$, but that contradicts the result of Proposition 4.3.

Now $\omega_\eta(\mathcal{O}_{\delta'_2}(A_{j,0}), t)$ attracts pullback $\mathcal{O}_{\delta'_2}(A_{j,0})$ and, given $\delta > 0$ and $t \in \mathbb{R}$, there exists $\tau_t > 0$ such that

$$T_\eta(t, t-\tau)\mathcal{O}_{\delta'_3}(A_{j,\eta}(t-\tau)) \subset \mathcal{O}_\delta(A_{j,\eta}(t))$$

[1] $\omega_\eta(\mathcal{O}_{\delta'_2}(A_{j,0}); s)$ indicates the $\omega$-limit set of $\mathcal{O}_{\delta'_2}(A_{j,0})$ in pullback sense and at time $s$ respect to the process $\{T_\eta(t,s) : t \geq s\}$. We have that the set $\omega_\eta(\mathcal{O}_{\delta'_2}(A_{j,0}); s)$ attracts the set $\mathcal{O}_{\delta'_2}(A_{j,0})$ in pullback sense and at time $s$ (see [4]).

for all $\tau \geq \tau_t$.

From the continuity of the evolution process, choose $\delta' \in (0, \min\{\delta, \delta_3'\})$ such that

$$T_\eta(t, t-\tau)\mathcal{O}_{\delta'}(A_{j,\eta}(t-\tau)) \subset \mathcal{O}_\delta(A_{j,\eta}(t))$$

for all $\tau \in [0, \tau_t]$.

This completes the proof of the proposition. ▮

We now conclude that the class of non-autonomous perturbation of a gradient-like semigroup satisfies all the hypothesis that we have used to develop the abstract framework in the previous section. We can therefore state the following:

THEOREM 4.3. *Let $X$ be a metric space and, for each $\eta \in [0,1]$, $\{T_\eta(t,s) : t \geq s\}$ an evolution process on $X$ with a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$, which satisfies conditions (a)-(e) in Theorem 4.1. Assume that $\mathcal{E}_0 = \{y_{1,0}^*, \cdots, y_{n,0}^*\}$ is reordered in such a way that it is a Morse-Decomposition for $\{T_0(t,0) : t \geq 0\}$ and that $\mathcal{E}_\eta$ is reordered in such a way that $\lim_{\eta\to 0^+} \sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \mathbf{d}(\xi_{i,\eta}^*(t), y_{i,0}^*) = 0$. Then, there exists $\eta_0 > 0$ such that for all $\eta \in [0, \eta_0]$ the n-upla $(\xi_{1,\eta}^*, \cdots, \xi_{n,\eta}^*)$, determines a Morse-Decomposition for the pullback attractor of $\{T_\eta(t,s) : t \geq s\}$. Furthermore, $\{T_\eta(t,s) : t \geq s\}$ is a gradient evolution process with respect to the set $\mathcal{E}_\eta = \{\xi_{1,\eta}^*, \cdots, \xi_{n,\eta}^*\}$, of isolated global solutions, in the sense of Definition 3.1, with Lyapunov function $V_\eta : \mathbb{R} \times X \to \mathbb{R}$ continuous in both variables and satisfying $V_\eta(t, \xi_{i,\eta}^*(t)) = i - 1$, for all $t \in \mathbb{R}$ and $i = 1, \cdots, n$.*

### 4.1. Convergence of the Lyapunov functions

Now, following [3], we are going to study the problem of continuity of the Lyapunov functions $V_\eta$ as $\eta$ tends to zero.

THEOREM 4.4.

*Let $\{T_\eta(t,s) : t \geq s\}_{\eta\in[0,1]}$ be a family of evolution processes in a metric space $X$ with a corresponding family of pullback attractors $\{\mathscr{A}_\eta\}_{\eta\in[0,1]} := \{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}_{\eta\in[0,1]}$, which satisfies conditions (a) and (d) in Theorem 4.1. Assume that $T_0(t,s) = S(t-s)$ for all $t \geq s$, where $\{S(t) : t \geq 0\}$ is a semigroup.*

*For each $\eta \in [0,1]$, let $\{A_\eta\} := \{A_\eta(t) : t \in \mathbb{R}\}$ be a local attractor for $\{T_\eta(t,s) : t \geq s\}$ with associated repeller $\{A_\eta^*\} := \{A_\eta^*(t) : t \in \mathbb{R}\}$ and such that $A_0(t) = A_0$ where $A_0$ is a local attractor for $\{T_0(t,0) = S(t) : t \geq 0\}$.*

*Suppose that $\{A_\eta\}_{\eta\in[0,1]}$, $\{A_\eta^*\}_{\eta\in[0,1]}$, and $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ are continuous at $\eta = 0$, that is,*

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \mathbf{d}_H(A_\eta(t), A_0) = 0$$

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \mathbf{d}_H(A_\eta^*(t), A_0^*) = 0$$

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \mathbf{d}_H(\mathscr{A}_\eta(t), \mathscr{A}_0) = 0.$$

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*Finally, for each* $\eta \in [0,1]$*, let* $f_\eta : \mathbb{R} \times X \to \mathbb{R}$ *be the Lyapunov function associated to the attractor repeller pair* $(A_\eta, A_\eta^*)$ *defined by*

$$f_\eta(t,z) := k_\eta(t,z) + h_\eta(t,z),\ (t,z) \in \mathbb{R} \times X,$$

*where*

$$h_\eta(t,z) := \sup_{r\geq 0} \mathbf{dist}(T_\eta(r+t,t)z, \mathscr{A}_\eta(t+r)),\ (t,z) \in \mathbb{R} \times X,$$

$$k_\eta(t,z) := \sup_{r\geq 0} \frac{\mathbf{dist}(z, A_\eta(t))}{\mathbf{dist}(z, A_\eta(t)) + \mathbf{dist}(z, A_\eta^*(t))},\ (t,z) \in \mathbb{R} \times X.$$

*Then, for each compact subset* $K$ *of* $X$ *we have*

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \sup_{z\in K} |f_\eta(t,z) - f_0(z)| = 0.$$

*Proof.*
We split the proof into three steps:
*Step 1:* We have the following convergence

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \sup_{z\in X} |l_\eta(t,z) - l_0(z)| = 0.$$

Note that, for each $\eta \in [0,1]$, $z \in X$ and $t \in \mathbb{R}$

$$|\mathbf{dist}(z, A_\eta(t)) - \mathbf{dist}(z, A_0)| \leq \mathbf{d}_H(A_\eta(t), A_0) \tag{34}$$

and

$$\left|\mathbf{dist}(z, A_\eta^*(t)) - \mathbf{dist}(z, A_0^*)\right| \leq \mathbf{d}_H(A_\eta^*(t), A_0^*). \tag{35}$$

Now, given $\eta \in [0,1]$, $t \in \mathbb{R}$ and $z \in X$, we have

$$\begin{aligned} &l_\eta(z,t) - l_0(z) \\ &= \frac{[\mathbf{dist}(z,A_\eta(t)) - \mathbf{dist}(z,A_0)]\mathbf{dist}(z,A_0^*) + \mathbf{dist}(z,A_0)[\mathbf{dist}(z,A_0^*) - \mathbf{dist}(z,A_\eta^*(t))]}{[\mathbf{dist}(z, A_\eta(t)) + \mathbf{dist}(z, A_\eta^*(t))][\mathbf{dist}(z, A_0) + \mathbf{dist}(z, A_0^*)]}. \end{aligned}$$

Since $\mathbf{d}(A_0, A_0^*) > \mu$ for some $\mu > 0$, using that the families of local attractors and their corresponding repellers are continuous, there exists $\tilde{\eta} \in (0,1]$ such that $\mathbf{d}(A_\eta(t), A_\eta^*(t)) \geq \mu$, for all $\eta \in [0, \tilde{\eta}]$. From (34) and (35)

$$\begin{aligned} |l_\eta(t,z) - l_0(z)| &\leq \frac{1}{\mathbf{dist}(z, A_\eta(t)) + \mathbf{dist}(z, A_\eta^*(t))}[\mathbf{d}_H(A_\eta(t), A_0) + \mathbf{d}_{\mathrm{H}}(A_\eta^*(t), A_0^*)] \\ &\leq \frac{1}{\mu}[\mathbf{d}_H(A_\eta(t), A_0) + \mathbf{d}_H(A_\eta^*(t), A_0^*)], \end{aligned}$$

for each $z \in X$ and $\eta \in [0, \tilde{\eta}]$. Hence

$$|l_\eta(t,z) - l_0(z)| \le \frac{1}{\mu}[\mathbf{d}_H(A_\eta(t), A_0) + \mathbf{d}_H(A^*_\eta(t), A^*_0)],$$

for all $t \in \mathbb{R}$, $z \in X$ and $\eta \in [0, \tilde{\eta}]$. From the continuity assumptions (on the local attractors and corresponding repellers) the uniform convergence (in $\mathbb{R} \times X$) of $l_\eta$ to $l_0$ follows.

*Step 2:* For every compact $K \subset X$ we have

$$\lim_{\eta \to 0^+} \sup_{t \in \mathbb{R}} \sup_{z \in K} |k_\eta(t,z) - k_0(z)| = 0.$$

Indeed, given $z \in X$ we know, by Lemma 2.13-(*iii*) from [2], that $S(t)z \to A_0 \cup A^*_0$, then, consider the following three cases:

*Case 1:* $S(t)z \overset{t\to\infty}{\to} A_0$ with $l_0(z) > 0$.

Choose $0 < \theta < l_0(z)$. Since $l_0 : X \to \mathbb{R}$ is continuous, there is a $\sigma_1 > 0$ such that $l_0(\mathcal{O}_{\sigma_1}(z)) \subset (\frac{\theta}{2}, 1]$. From Step 1, there is a $\eta_0 \in (0, 1]$ such that $l_\eta(t, \mathcal{O}_{\sigma_1}(z)) \subset (\theta, 1]$ for all $\eta \in [0, \eta_0]$ and $t \in \mathbb{R}$.

Using again the continuity of $l_0 : X \to \mathbb{R}$, given $0 < \alpha < \frac{\theta}{2}$, there is a $\delta > 0$ such that $l_0(\mathcal{O}_\delta(A_0)) \subset [0, \alpha)$. Now, from Lemma 4.4, there is a $\delta' \in (0, \frac{\delta}{2})$ and $\eta_1 \in (0, \eta_0]$ such that

$$T_\eta(t,s)(\mathcal{O}_{\delta'}(A_\eta(s))) \subset \mathcal{O}_{\frac{\delta}{2}}(A_\eta(t)) \text{ whenever } t \ge s, \ \forall \eta \in [0, \eta_1]. \tag{36}$$

From the lower semicontinuity of $\{A_\eta\}_{\eta \in [0,1]}$ at $\eta = 0$, there is $\eta_2 \in (0, \eta_1]$ such that

$$A_0 \subset \mathcal{O}_{\frac{\delta'}{2}}(A_\eta(t)) \text{ for all } t \in \mathbb{R} \ \forall \eta \in [0, \eta_2]. \tag{37}$$

From the fact that $T_0(t)z \overset{t\to\infty}{\to} A_0$, choose $t_0 > 0$ such that $T_0(t_0)z \in \mathcal{O}_{\frac{\delta'}{4}}(A_0)$, and from the continuity of $T_0(t_0) : X \to X$ choose $\sigma_2 \in (0, \sigma_1]$ such that $T_0(t_0)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_{\frac{\delta'}{4}}(A_0)$. From hypothesis $(d)$, we can find $\sigma_3 \in (0, \sigma_2]$ and $\eta_3 \in (0, \eta_2]$ such that for all $\eta \in [0, \eta_3]$ we have $T_\eta(t_0 + s, s)(\mathcal{O}_{\sigma_3}(z)) \subset \mathcal{O}_{\frac{\delta'}{2}}(A_0)$ for all $s \in \mathbb{R}$. From this and (37), we obtain that $T_\eta(t_0 + s, s)(\mathcal{O}_{\sigma_3}(z)) \subset \mathcal{O}_{\delta'}(A_\eta(t))$ for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$ when $\eta \in [0, \eta_3]$, and from (36), in particular, we conclude that

$$T_\eta(t+s, s)((\mathcal{O}_{\sigma_3}(z)) \subset \mathcal{O}_{\frac{\delta}{2}}(A_\eta(t+s)) \text{ for all } t \ge t_0, \ \eta \in [0, \eta_3] \text{ and } s \in \mathbb{R}. \tag{38}$$

On the other hand, observe that, from the uniform convergence of $l_\eta \overset{\eta\to 0^+}{\to} l_0$ in $\mathbb{R} \times X$, we obtain $\eta_4 \in (0, \eta_3]$ so that, for each $\eta \in [0, \eta_4]$, it holds $l_\eta(t, \mathcal{O}_\delta(A_0)) \subset [0, 2\alpha)$ for all $t \in \mathbb{R}$. From the upper semicontinuity of $\{A_\eta\}_{\eta \in [0,1]}$ at $\eta = 0$, there exists $\eta_5 \in (0, \eta_4]$ such that, if $\eta \in [0, \eta_5]$, then $A_\eta(t) \subset \mathcal{O}_{\frac{\delta}{2}}(A_0)$ for all $t \in \mathbb{R}$. Therefore, $\mathcal{O}_{\frac{\delta}{2}}(A_\eta(t)) \subset \mathcal{O}_\delta(A_0)$ for all $\eta \in [0, \eta_5]$ and $t \in \mathbb{R}$. So $l_\eta(t, \mathcal{O}_{\frac{\delta}{2}}(A_\eta(\tau))) \subset [0, 2\alpha)$ for all $\eta \in [0, \eta_5]$ and $t, \tau$ in $\mathbb{R}$. From (38), we have that

$$\sup_{t \ge t_0} l_\eta(t+s, T_\eta(t+s, s)w) \le 2\alpha < \theta < l_\eta(s, w) \le k_\eta(s, w),$$

for each $\eta \in [0, \eta_5]$, $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\sigma_3}(z) \subset \mathcal{O}_{\sigma_1}(z)$. Consequently $k_\eta(s, w) = \sup_{0 \le t \le t_0} l_\eta(t + s, T_\eta(t+s, s)w)$ for all $\eta \in [0, \eta_5]$, $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\sigma_3}(z)$.

Finally, given $\varepsilon > 0$, from Step 1, there exists $\eta_6 \in (0, \eta_5]$ such that

$$|l_\eta(t, x) - l_0(x)| < \frac{\varepsilon}{2} \text{ for all } \eta \in [0, \eta_6],\ x \in X \text{ and } t \in \mathbb{R}.$$

From the uniform continuity of the function $l_0 : X \to \mathbb{R}$, consider $\beta > 0$ such that if $x, x' \in X$ satisfy $\mathbf{d}(x, x') < \beta$ then $|l_0(x) - l_0(x')| < \frac{\varepsilon}{2}$ so that, by the convergence in $(d)$, we can choose $\eta_7 \in (0, \eta_6]$ and $\sigma_4 \in (0, \sigma_3]$ such that

$$\sup_{s \in \mathbb{R}} \sup_{w \in \mathcal{O}_{\sigma_4}(z)} \sup_{0 \le t \le t_0} \mathbf{d}(T_\eta(t+s, s)w, S(t)w) < \beta \text{ for all } \eta \in [0, \eta_7].$$

Thus

$$|l_\eta(t+s, T_\eta(t+s, s)w) - l_0(S(t)w)| \le$$

$$|l_\eta(t+s, T_\eta(t+s, s)w) - l_0(T_\eta(t+s, s)w)| + |l_0(T_\eta(t+s, s)w) - l_0(T_0(t)w)| < \varepsilon,$$

for all $w \in \mathcal{O}_{\sigma_4}(z)$, $t \in [0, t_0]$, $s \in \mathbb{R}$ and $\eta \in [0, \eta_7]$. This implies that

$$\sup_{s \in \mathbb{R}} \sup_{w \in \mathcal{O}_{\sigma_4}(z)} |k_\eta(s, w) - k_0(w)| \le \varepsilon \text{ for all } \eta \in [0, \eta_7], \tag{39}$$

where $\sigma_4 > 0$ and $\eta_7$ only depends on $z \in X$ and $\varepsilon > 0$.

*Case 2:* $l_0(z) = 0$.

Under these conditions, note that $z \in A_0$ and, consequently, $k_0(z) = 0$.

Given $\varepsilon > 0$, by the continuity of $l_0 : X \to \mathbb{R}$ , take $\delta > 0$ such that $l_0(\mathcal{O}_\delta(A_0)) \subset [0, \frac{\varepsilon}{4})$.

Now, the uniform convergence of $(l_\eta)_{\eta \in [0,1]}$ to $l_0$ in $\mathbb{R} \times X$ implies the existence of $\eta_0 \in (0, 1]$ such that

$$l_\eta(t, \mathcal{O}_\delta(A_0)) \subset [0, \frac{\varepsilon}{2}) \text{ for each } \eta \in [0, \eta_0] \text{ and each } t \in \mathbb{R}. \tag{40}$$

By the upper semicontinuity of $\{A_\eta\}_{\eta \in [0,1]}$ at $\eta = 0$, we have the existence of $\eta_1 \in (0, \eta_0]$ such that for all $\eta \in [0, \eta_1]$ and all $t \in \mathbb{R}$ we have $A_\eta(t) \subset \mathcal{O}_{\frac{\delta}{2}}(A_0)$, from which $\mathcal{O}_{\frac{\delta}{2}}(A_\eta(t)) \subset \mathcal{O}_\delta(A_0)$ if $\eta \in [0, \eta_1]$ and $t \in \mathbb{R}$. And from (40) we conclude that for all $\eta \in [0, \eta_1]$

$$l_\eta(t, \mathcal{O}_{\frac{\delta}{2}}(A_\eta(s))) \subset [0, \frac{\varepsilon}{2}) \text{ for all } s, t \in \mathbb{R}. \tag{41}$$

Choose $\eta_2 \in (0, \eta_1]$ and $\delta' \in (0, \frac{\delta}{2})$, by Lemma 4.4, such that

$$T_\eta(t, s)(\mathcal{O}_{\delta'}(A_\eta(s))) \subset \mathcal{O}_{\frac{\delta}{2}}(A_\eta(t)) \text{ whenever } t \ge s \text{ and for all } \eta \in [0, \eta_2]. \tag{42}$$

Finally, from the lower semicontinuity of $\{A_\eta\}_{\eta \in [0,1]}$ at $\eta = 0$, there is a $\eta_3 \in (0, \eta_2]$ such that

$$A_0 \subset \mathcal{O}_{\frac{\delta'}{2}}(A_\eta(t)) \text{ for all } \eta \in [0, \eta_3] \text{ and } t \in \mathbb{R}. \tag{43}$$

Thus, from (43) and (42), for $\eta \in [0,\eta_3]$, for $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\frac{\delta'}{2}}(A_0) \subset \mathcal{O}_{\delta'}(A_\eta(s))$, we have that $T_\eta(t+s,s)w \in \mathcal{O}_{\frac{\delta}{2}}(A_\eta(t+s))$ for all $t \geq 0$. From (41) it follows that

$$k_\eta(s,w) = \sup_{t\geq 0} l_\eta(t+s, T_\eta(t+s,s)w) \leq \frac{\varepsilon}{2},$$

for all $\eta \in [0,\eta_3]$, $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\frac{\delta'}{2}}(A_0)$. In particular,

$$\sup_{s\in\mathbb{R}} \sup_{w\in\mathcal{O}_{\frac{\delta'}{2}}(A_0)} |k_\eta(s,w) - k_0(w)| \leq \varepsilon \text{ for all } \eta \in [0,\eta_3], \tag{44}$$

where $\delta' > 0$ and $\eta_3$ only depend on $\varepsilon > 0$ and $A_0$.

*Case 3:* $T_0(t)z \overset{t\to\infty}{\to} A_0^*$.

In this case $k_0(z) = 1$. By the continuity of $l_0 : X \to \mathbb{R}$, given $\varepsilon > 0$, let $\delta > 0$ such that

$$l_0(\mathcal{O}_\delta(A_0^*)) \subset (1 - \frac{\varepsilon}{4}, 1]$$

and, by the uniform convergence $l_\eta \overset{\eta\to 0^+}{\to} l_0$ in $\mathbb{R}\times X$, take $\eta_0 \in (0,1]$ such that

$$l_\eta(t, \mathcal{O}_\delta(A_0^*)) \subset (1 - \frac{\varepsilon}{2}, 1] \text{ for all } \eta \in [0,\eta_0] \text{ and } t \in \mathbb{R}. \tag{45}$$

On the other hand, consider $t_0 > 0$ such that $T_0(t_0)z \in \mathcal{O}_{\frac{\delta}{2}}(A_0^*)$ and, from the continuity of $T_0(t_0) : X \to X$, take $\sigma_1 > 0$ such that $T_0(t_0)(\mathcal{O}_{\sigma_1}(z)) \subset \mathcal{O}_{\frac{\delta}{2}}(A_0^*)$. By hypothesis $(d)$, let $\eta_1 \in (0,\eta_0]$ and $\sigma_2 \in (0,\sigma_1]$ such that $T_\eta(t_0+s,s)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_\delta(A_0^*)$ for all $\eta \in [0,\eta_1]$ and all $s \in \mathbb{R}$.

Finally, from (45) we deduce that $l_\eta(t_0+s, T_\eta(t_0+s,s)(\mathcal{O}_{\sigma_2}(z))) \subset (1-\frac{\varepsilon}{2},1]$ for all $\eta \in [0,\eta_1]$ and all $s \in \mathbb{R}$. Thus, $1 - \frac{\varepsilon}{2} < l_\eta(t_0+s, T_\eta(t_0+s,s)w) \leq k_\eta(s,w) \leq 1$ for all $w \in \mathcal{O}_{\sigma_2}(z)$, $s \in \mathbb{R}$ and $\eta \in [0,\eta_1]$. This implies that $|k_\eta(s,w) - k_0(w)| \leq \varepsilon$ for $\eta \in [0,\eta_1]$, $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\sigma_2}(z)$ and

$$\sup_{s\in\mathbb{R}} \sup_{w\in\mathcal{O}_{\sigma_2}(z)} |k_\eta(s,w) - k_0(w)| \leq \varepsilon \text{ for } \eta \in [0,\eta_1], \tag{46}$$

where $\sigma_2 > 0$ and $\eta_1$ only depends on $A_0^*$ and $\varepsilon > 0$.

Now, joining the cases 1, 2 and 3, we have that:

Given a compact subset $K \subset X$ and $\varepsilon > 0$, by (39), (44) and (46), there exist an open subset $U = U(\varepsilon, K) \subset X$ with $K \subset U$, and an index $\eta' = \eta'(\varepsilon, K) > 0$ such that

$$\sup_{s\in\mathbb{R}} \sup_{w\in U} |k_\eta(s,w) - k_0(w)| \leq \varepsilon \text{ for all } \eta \in [0,\eta'],$$

from which we can conclude that $\lim_{\eta\to 0^+} \sup_{s\in\mathbb{R}} \sup_{w\in K} |k_\eta(s,w) - k_0(w)| = 0$.

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*Step 3*: for every compact $K \subset X$ we have

$$\lim_{\eta \to 0^+} \sup_{t \in \mathbb{R}} \sup_{z \in K} |h_\eta(t,z) - h_0(z)| = 0.$$

Indeed, given $z \in X$ consider now two cases:
*Case 1:* $\mathbf{dist}(z, \mathscr{A}_0) > 0$.

Given $\alpha > 0$ with $0 < \alpha < \mathbf{dist}(z, \mathscr{A}_0)$, let, by Lemma 4.4, $\alpha' \in (0, \alpha)$ and $\eta_0 \in (0,1]$ such that

$$T_\eta(t,s)(\mathcal{O}_{\alpha'}(\mathscr{A}_\eta(s))) \subset \mathcal{O}_\alpha(\mathscr{A}_\eta(t)) \text{ whenever } t \geq s, \text{ for all } \eta \in [0, \eta_0]. \tag{47}$$

Choose $t_0 > 0$ such that $T_0(t_0)z \in \mathcal{O}_{\frac{\alpha'}{4}}(\mathscr{A}_0)$ and, by the continuity of $T_0(t_0) : X \to X$, let $\sigma_1 > 0$ such that $T_0(t_0)(\mathcal{O}_{\sigma_1}(z)) \subset \mathcal{O}_{\frac{\alpha'}{4}}(\mathscr{A}_0)$.

Now, by hypothesis of convergence $(d)$, let $\eta_1 \in (0, \eta_0]$ and $\sigma_2 \in (0, \sigma_1]$ such that $T_\eta(t_0 + s, s)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_{\frac{\alpha'}{2}}(\mathscr{A}_0)$ for each $\eta \in [0, \eta_1]$ and each $s \in \mathbb{R}$, and, by the lower semicontinuity of $\{\mathscr{A}_\eta\}_{\eta \in [0,1]}$ at $\eta = 0$, choose $\eta_2 \in (0, \eta_1]$ such that $\mathscr{A}_0 \subset \mathcal{O}_{\frac{\alpha'}{2}}(\mathscr{A}_\eta(t))$ for all $\eta \in [0, \eta_2]$ and $t \in \mathbb{R}$. Hence, $\mathcal{O}_{\frac{\alpha'}{2}}(\mathscr{A}_0) \subset \mathcal{O}_{\alpha'}(\mathscr{A}_\eta(t))$ for $\eta \in [0, \eta_2]$ and $t \in \mathbb{R}$. In particular, $T_\eta(t_0 + s, s)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_{\alpha'}(\mathscr{A}_\eta(t_0 + s))$ for all $\eta \in [0, \eta_2]$ and $s \in \mathbb{R}$. From (47) we obtain that $T_\eta(t+s, s)(\mathcal{O}_{\sigma_2}(z)) \subset \mathcal{O}_\alpha(\mathscr{A}_\eta(t+s))$ for all $\eta \in [0, \eta_2]$, all $s \in \mathbb{R}$ and $t \geq t_0$. Consequently,

$$\sup_{t \geq t_0} \mathbf{dist}(T_\eta(t+s, s)w, \mathscr{A}_\eta(t+s)) \leq \alpha \text{ for all } \eta \in [0, \eta_2], s \in \mathbb{R}, \ w \in \mathcal{O}_{\sigma_2}(z). \tag{48}$$

On the other hand, for all $w \in X$, all $t \in \mathbb{R}$ and all $\eta \in [0,1]$ we have

$$|\mathbf{dist}(w, \mathscr{A}_\eta(t)) - \mathbf{dist}(w, \mathscr{A}_0)| \leq \mathbf{d}_H(\mathscr{A}_\eta(t), \mathscr{A}_0). \tag{49}$$

Then, we can choose $\eta_3 \in (0, \eta_2]$ and $\sigma_3 \in (0, \sigma_2]$ such that $\mathbf{dist}(w, \mathscr{A}_\eta(t)) > \alpha$ for all $\eta \in [0, \eta_3]$, $t \in \mathbb{R}$ and $w \in \mathcal{O}_{\sigma_3}(z)$. This and (48) implies that

$$\sup_{t \geq t_0} \mathbf{dist}(T_\eta(t+s, s)w, \mathscr{A}_\eta(t+s)) \leq \alpha < \mathbf{dist}(w, \mathscr{A}_\eta(s)),$$

for all $\eta \in [0, \eta_3]$, $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\sigma_3}(z)$. So, $h_\eta(s,w) = \sup_{0 \leq t \leq t_0} \mathbf{dist}(T_\eta(t+s, s)w, \mathscr{A}_\eta(t+s))$ for $\eta \in [0, \eta_3]$, $s \in \mathbb{R}$ and $w \in \mathcal{O}_{\sigma_3}(z)$.

Note that, for all $w \in X$, $\eta \in [0,1]$, $s \in \mathbb{R}$ and $t \geq 0$ we have that

$$\begin{aligned} |\mathbf{dist}(T_\eta(t+s, s)w, \mathscr{A}_\eta(t+s)) - \mathbf{dist}(S(t)w, \mathscr{A}_0)| &\leq \mathbf{d}_H(\mathscr{A}_\eta(t+s), \mathscr{A}_0) \\ &\quad + \mathbf{d}(T_\eta(t+s, s)w, S(t)w), \end{aligned}$$

so that, for all $\eta \in [0, \eta_3]$

$$\begin{aligned} \sup_{s \in \mathbb{R}} \sup_{w \in \mathcal{O}_{\sigma_3}(z)} |h_\eta(s,w) - h_0(w)| &\leq \sup_{s \in \mathbb{R}} \mathbf{d}_H(\mathscr{A}_\eta(s), \mathscr{A}_0) \\ &\quad + \sup_{s \in \mathbb{R}} \sup_{w \in \mathcal{O}_{\sigma_3}(z)} \sup_{0 \leq t \leq t_0} \mathbf{d}(T_\eta(t+s, s)w, S(t)w), \end{aligned}$$

and so, it is easy to see that, given $\varepsilon > 0$ there exist $\sigma \in (0, \sigma_3]$ and $\eta_4 \in (0, \eta_3]$ such that

$$\sup_{s\in\mathbb{R}} \sup_{w\in\mathcal{O}_\sigma(z)} |h_\eta(s,w) - h_0(w)| \leq \varepsilon \text{ for all } \eta \in [0, \eta_4].$$

*Case 2:* $\mathbf{dist}(z, \mathscr{A}_0) = 0$; that is, $z \in \mathscr{A}_0$.

From Lemma 4.4, given $\varepsilon > 0$, there are $\varepsilon' \in (0, \frac{\varepsilon}{2})$ and $\eta_0 \in (0,1]$ such that

$$T_\eta(t,s)(\mathcal{O}_{\varepsilon'}(\mathscr{A}_\eta(s))) \subset \mathcal{O}_{\frac{\varepsilon}{2}}(\mathscr{A}_\eta(t)) \text{ for each } t \geq s \text{ and } \eta \in [0, \eta_0]. \tag{50}$$

Also, since $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$, we can choose $\eta_1 \in (0, \eta_0]$ such that $\mathscr{A}_0 \subset \mathcal{O}_{\frac{\varepsilon'}{2}}(\mathscr{A}_\eta(s))$ whenever $\eta \in [0, \eta_1]$ and $s \in \mathbb{R}$. It follows that $\mathcal{O}_{\frac{\varepsilon'}{2}}(\mathscr{A}_0) \subset \mathcal{O}_{\varepsilon'}(\mathscr{A}_\eta(s))$ for all $\eta \in [0, \eta_1]$ and $s \in \mathbb{R}$. From this and (50) we have that

$$T_\eta(t,s)(\mathcal{O}_{\frac{\varepsilon'}{2}}(\mathscr{A}_0)) \subset \mathcal{O}_{\frac{\varepsilon}{2}}(\mathscr{A}_\eta(t)) \text{ if } \eta \in [0, \eta_1] \text{ and } t \geq s.$$

Consequently, $h_\eta(s,w) = \sup_{t\geq 0} \mathbf{dist}(T_\eta(t+s,s)w, \mathscr{A}_\eta(t+s)) \leq \frac{\varepsilon}{2}$ for all $\eta \in [0, \eta_1]$ and $w \in \mathcal{O}_{\frac{\varepsilon'}{2}}(\mathscr{A}_0)$, so that

$$\sup_{s\in\mathbb{R}} \sup_{w\in\mathcal{O}_{\frac{\varepsilon'}{2}}(\mathscr{A}_0)} |h_\eta(s,w) - h_0(w)| \leq \varepsilon \text{ for all } \eta \in [0, \eta_1].$$

In these conditions, given $\varepsilon > 0$ and $z \in X$, there are $\sigma = \sigma(\varepsilon, z) > 0$ and $\eta' = \eta'(\varepsilon, z) > 0$ such that

$$\sup_{s\in\mathbb{R}} \sup_{w\in\mathcal{O}_\sigma(z)} |h_\eta(s,w) - h_0(w)| \leq \varepsilon \text{ for all } \eta \in [0, \eta'].$$

We now conclude the convergence $\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \sup_{z\in K} |h_\eta(t,z) - h_0(z)| = 0$ by an argument similar to that of Step 2. This completes the proof. ∎

## 5. APPLICATIONS

In this section we consider some applications of the abstract results in the previous sections, and conclude that some non-autonomous evolutions processes possess a Morse-Decomposition and, therefore, a Lyapunov function.

Example 5.1. Consider the the initial boundary value problem

$$\begin{cases} u_t = u_{xx} + \lambda f_\eta(t,u), \ x \in (0,\pi), \ t > \tau \\ u_x(0,t) = u_x(\pi,t) = 0, \ t > \tau \\ u(\cdot,\tau) = \phi_0 \in H_0^1(0,\pi) \end{cases} \tag{51}$$

Where $\lambda \in [0, \infty)$, $\eta \in [0,1]$, $f_\eta \in C^2(\mathbb{R}^2, \mathbb{R})$ with $f_\eta$ satisfying:

1) There exists $M > 0$ such that $f_\eta : \mathbb{R}^2 \to \mathbb{R}$ is such that $f_\eta(t,u)u < 0$ for all $(t,u) \in \mathbb{R}^2$ with $|u| > M$. Suppose also that $f_0(t,u) = f_0(u)$ for all $t \in \mathbb{R}$.

2) For each $r > 0$

$$\lim_{\eta \to 0+} \sup_{t\in\mathbb{R}} \sup_{|u|\leq r} \{|f_\eta(t,u) - f_0(u)| + |(f_\eta)_u(t,u) - f_0'(u)| + |(f_\eta)_{uu}(t,u) - f_0''(u)|\} = 0.$$

It is not difficult to see that, under these assumptions, problem (51) possesses a pullback attractor $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$.

If (51) with $\eta = 0$ has a finite number of equilibria, all of them hyperbolic. The hypotheses of Theorem 4.3 are clearly satisfied and there exists $\eta_0 > 0$ such that $\{\mathscr{A}_\eta(t) : t \in \mathbb{R}\}$ has a Morse-Decomposition, and a continuous (uniformly in $\mathbb{R} \times K$ for each compact subset $K$ of $H_0^1(0,\pi)$) Lyapunov function $V_\eta : \mathbb{R} \times H_0^1(0,\pi) \to \mathbb{R}$ (in the sense of Definition 3.1).

Example 5.2. *Consider a general cascade system of the form*

$$\begin{cases} \dot{u}_1 &= f_1(u_1) \\ \dot{u}_2 &= f_2(u_1,u_2) \\ \vdots & \vdots \qquad \vdots \\ \dot{u}_n &= f_n(u_1,\cdots,u_n), \end{cases} \tag{52}$$

*where $f_j : \mathbb{R}^j \to \mathbb{R}$, $1 \leq j \leq n$, are $C^1$ functions. Assume that there is $\xi > 0$ such that $f_j(u_1,\cdots,u_j)u_j < 0$ for all $(u_1,\cdots,u_j) \in \mathbb{R}^j$ such that $|u_j| \geq \xi$ and for all $1 \leq j \leq n$. Assume that all equilibria of* (52) *are hyperbolic. It follows from the results in [2] that the semigroup associated to* (52) *is gradient-like (therefore gradient). Now we consider a small non-autonomous perturbation of the above problem, that is,*

$$\begin{cases} \dot{u}_1 &= f_1(u_1) + g_1(t,u_1,\cdots,u_n) \\ \dot{u}_2 &= f_2(u_1,u_2) + g_2(t,u_1,\cdots,u_n) \\ \vdots & \vdots \qquad \vdots \\ \dot{u}_n &= f_n(u_1,\cdots,u_n) + g_n(t,u_1,\cdots,u_n). \end{cases} \tag{53}$$

*If $g_j(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a $C^1(\mathbb{R}^n,\mathbb{R})$ function with the property that*

$$\sup_{t\in\mathbb{R}} \sup_{(u_1,\cdots,u_n)\in\mathbb{R}^n} \left\{ |g_j(t,u_1,\cdots,u_n)| + \sum_{i=1}^n \left| \frac{\partial g_j}{\partial u_i}(t,u_1,\cdots,u_n) \right| \right\}$$

*is suitably small, it follows from the results in [5] that the evolution process $\{T_g(t,s) : t \geq s\}$ associated to* (53) *is gradient-like and, as a consequence of the results in Section 4 that the pullback attractor of $\{T_g(t,s) : t \geq s\}$ has a Morse-Decomposition and that there is a Lyapunov function for it.*

One can easily see that, cascade systems of semilinear parabolic problems can also be considered.

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