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Counting singularities via Fitting ideals

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The stable singularities of differential map germs constitute the main source of studying the geometric and topological behavior of these maps. In special, one interesting problem is to find formulae which allow us to count the isolated stable singularities which appear in the discriminant of a stable deformation of a finitely determined map germ. Mond and Pellikaan in [15] showed how the Fitting ideals are related to such singularities and obtain a formula to count the number of ordinary triple points in map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, in terms of the Fitting ideals associated with the discriminant.

In this article we consider map germs from $(\mathbb{C}^{n+m}, 0)$ to $(\mathbb{C}^m, 0)$, we obtain results to count the number of isolated singularities by means of the dimension of some associated algebras to the Fitting ideals. First in the Corollary 4.1 we provide a way to compute the total sum of these singularities. In the Proposition 4.3, for $m = 3$ we show how to compute the number of ordinary triple points. In the Corollary 4.2 and with f of co-rank one, we show a way to compute the number of points formed by the intersection between a germ of a cuspidal edge and a germ of a plane.

Furthermore, we show in some examples how to calculate the number of isolated singularities using these results. May, 2011 ICMC-USP

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1. INTRODUCTION

The study of stable singularities is very important in determining conditions of triviality in families of germs of applications and there are several invariants associated to these singularities. For families of finitely determined map germs from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$, Gaffney in [6] and Gaffney-Mond in [8] showed that the constancy of the number of cusps and the number of transverse double points is sufficient to guarantee the topological triviality. In the case of germs of families of co-rank 1 from $\mathbb{C}^2$ to $\mathbb{C}^3$, Gaffney showed in [6] that the constancy of a single invariant implies the equisingularity of family. In this same case Ruas showed in [19], under some conditions, that the number of triple points, the number of cross-caps (Whitney umbrellas) and the Milnor number of the curve of double points are sufficient to ensure the Whitney equisingularity of family.

Therefore we find interesting work which show explicit formulae for counting invariants of stable singularities that can arise in the discriminant of a stable deformation of a finitely determined germ. The case of maps from the plane to the plane was studied by Rieger [18], for germs of co-rank 1 and by Fukuda and Ishikawa in [3], Gaffney and Mond in [8] for germs of any co-rank. In these work there are obtained formulae for calculating the number of cusps and the Milnor number at 0 of the transverse double points curve. In the case of map germs from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^3, 0)$, Mond presents formulae for the number of cross-caps and ordinary triple points when f is quasi-homogeneous in terms of weights and degrees of f, [14]. In the case of co-rank one map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$, Marar, Montaldi and Ruas show formulae for the number of isolated singularities, [12]. Under some conditions, Fukui, Nuño-Ballesteros and Saia calculate the number of isolated stable mono-germs in stable deformations for the case of map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$, in terms of a complex dimension of an appropriate algebra, [4].

In this work we are mainly interested in the study of isolated stable singularities of a generic deformation of finitely determined map germs $f : (\mathbb{C}^{n+m}, 0)$ to $(\mathbb{C}^m, 0)$, from the dimension of algebras associated to the Fitting ideals. In order to obtain formulae which relates these invariants we study the Fitting ideals of the discriminant set of the germ f, following the ideas of Mond and Pellikan in [15]. These ideals are a powerful tool to develop methods of counting these invariants. Mond and Pellikan applied then to study the behavior of sets of multiple points in the target of analytical applications (between) finite analytic varieties $f : \widetilde{X} \to Y$. They seek for a good analytic structure for the set of points in Y whose pre-image consists of k or more points, counting multiplicity, denoted by $M_k(f)$. When $\widetilde{X}$ is a Cohen-Macaulay analytic manifold and Y is smooth with $\dim Y = \dim \widetilde{X} + 1$, they showed that the appropriate analytical structure in this case is defined by the Fitting ideals of the $\mathcal{O}_Y$-coherent module $f_*\mathcal{O}_{\widetilde{X}}$. As an application Mond and Pellikan get a formula for the number of ordinary triple points in terms of Fitting ideals associated with the discriminant.

Here apply the results of Mond and Pellikaan to the restriction of f to the critical set of f and study the associated Fitting ideal sheaves. In the special case that $m = 3$, we obtain formulae for the calculation of each isolated stable singularities. When $m > 3$, we obtain formulae that relate the total sum of all isolated stable singularities with the dimension of algebras associated to the Fitting ideals.

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2. FINITELY DETERMINED MAP GERMS

We denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mapping from $(\mathbb{C}^n,0)$ to $(\mathbb{C}^p)$ and $\mathcal{O}_n$ denotes the ring $\mathcal{O}(n,1)$.

A map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ is *stable* in a finite set S if, by composition with families of holomorphic diffeomorphisms in source and target, every deformation is $\mathcal{A}$-trivial, where $\mathcal{A}$ denotes the usual Mather group of germs of holomorphic diffeomorphisms in the source and in the target. We call stable type, an equivalence class of stable map germs. Our interest here is in $\mathcal{A}$-finitely determined map germs $f\colon(\mathbb{C}^{n+m},0)\to(\mathbb{C}^m,0)$. We say that a germ is k-$\mathcal{A}$-*determined* if any $g\in\mathcal{O}(n+m,m)$ with the same Taylor polynomial of order k at zero than f is $\mathcal{A}$-equivalent to f. The germ f is said to be $\mathcal{A}$-*finitely determined or finitely determined* if it is k-$\mathcal{A}$-determined for some k.

The critical point set $\Sigma(f)$ of $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ is the set of points $x\in\mathbb{C}^n$ such that $J(f)(x)=0$, where $J(f)$ denotes the ideal generated by the set of $p\times p$ minors of the derivative of f. If $n<p$, then $\Sigma(f)=\mathbb{C}^n$. We denote by $\Delta(f)=f(\Sigma(f))$ the discriminant set of f.

Mather and Gaffney showed the characterization of finitely determined map germs in terms of stable germs.

Proposition 2.1. *(Mather-Gaffney) Suppose $f\in\mathcal{O}(n,p)$. Then f is finitely determined if, and only if, for each representative $\tilde f$ of f, there exist neighborhoods of the origin $U\subset\mathbb{C}^n$, $V\subset\mathbb{C}^p$ such that $\tilde f^{-1}(0)\cap U\cap\Sigma(\tilde f)=0$ and for each $y\in V$, $y\neq 0$, the germ $\tilde f_y:(\mathbb{C}^n,S_y)\to(\mathbb{C}^p,y)$ is stable, where $S_y=\tilde f^{-1}(y)\cap U\cap\Sigma(\tilde f)$ and $\Sigma(\tilde f)$ denotes the critical set of $\tilde f$.*

Definition 2.1. We say that an unfolding F is a *stabilization* of f if there is a representative $F:D\times U\to D\times\mathbb{C}^p$, where D, U are open neighborhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively such that $f_t:U\to\mathbb{C}^p$ is stable for any $t\in D\setminus\{0\}$.

We call such unfolding a stable unfolding of f and denote it by F. The stable types of singularities which appear in the critical set of any finitely determined germ f are given in any stable unfolding of f. We say that f has discrete stable type if there exist a versal unfolding of f in which only a finite number of stable types occur. If the numbers (n,p) are in Mather's "nice dimensions" (see [13]) or on the boundary thereof, then every finitely determined germ $f\in\mathcal{O}(n,p)$ has discrete stable type.

3. FITTING IDEALS

In this section we recover the definition of Fitting ideals of a module M. Here we follow the notation of Mond and Pellikan in the sections 1. and 2. of [15].

Definition 3.1. Let M be a finitely presented R-module (where R is a commutative ring with unit) and let

$$R^p \xrightarrow{\lambda} R^q \xrightarrow{\alpha} M \longrightarrow 0$$

be a presentation. The k^{th} fitting ideal of M, $\mathcal{F}_k(M)$, is defined to be the ideal in R generated by all $(q-k)\times(q-k)$ minors of the matrix λ, for $q > k \geq q-p$. $\mathcal{F}_k(M) = R$, for $k \geq q$, and $\mathcal{F}_k(M) = 0$ for $k < q-p$.

By [11] pag. 59, the Fitting ideals are well defined, that is, they do not depend upon the choice of a presentation.

The concept of Fitting ideals also makes sense in the context of $\mathcal{O}_m$-modules, as we see in [11] pag. 63. Let $(X,x) \in \mathcal{O}_n$ be a multi-germ of a $(m-1)$-dimensional Cohen-Macaulay variety and $\mathcal{O}_{(X,x)}$ denotes the set of germs g in $\mathcal{O}_n$ such that $g(X) = 0$. For a fixed finite analytic map $f : (X,x) \to (\mathbb{C}^m, 0)$ we have by the Weierstrass preparation theorem that $\mathcal{O}_{(X,x)}$ is a finite $\mathcal{O}_m$-module via the function f^*. If the classes of $g_1, g_2, \ldots, g_h$ in $\frac{\mathcal{O}_{(X,x)}}{f^*\mathfrak{m}_0}$ generate it as vector space over $\mathbb{C}$, then $g_1, g_2, \ldots, g_h$ generate $\mathcal{O}_{(X,x)}$ as $\mathcal{O}_m$-module.

A presentation of $\mathcal{O}_{(X,x)}$ over $\mathcal{O}_m$ is an exact sequence

$$\mathcal{O}_m^p \xrightarrow{\lambda} \mathcal{O}_m^q \xrightarrow{\alpha} \mathcal{O}_{(X,x)} \longrightarrow 0 \tag{1}$$

of $\mathcal{O}_m$-modules. If $g_1, g_2, \ldots, g_h$ generate $\mathcal{O}_{(X,x)}$ then one may take $q = p = h$, $\alpha(e_i) = g_i$ (where e_i is the i^{th} member of usual basis). See [15] for more details.

From now on we consider finitely determined map germs $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$ and to apply the results of Mond and Pellikaan we consider the restriction of f to the singular set germ $(\Sigma(f), 0)$, $f|_{\Sigma(f)} : (\Sigma(f), 0) \to (\mathbb{C}^m, 0)$. As $\Sigma(f) = V(J(f))$ and $J(f)$ is generated by the maximal minors of the matrix of derivative of f, then $(\Sigma(f), 0)$ is a $(m-1)$-dimensional Cohen-Macaulay manifold and as $\mathbb{C}^m$ is smooth we obtain that $f_*\mathcal{O}_{(\Sigma(f),0)}$ is a coherent sheaf of $\mathcal{O}_{\mathbb{C}^m}$-module, since $f|_{\Sigma(f)} : (\Sigma(f), 0) \to (\mathbb{C}^m, 0)$ is a finite morphism.

Then we can associate the Fitting ideal sheaves $\mathcal{F}_k(f_*\mathcal{O}_{(\Sigma(f),0)})$ and [15, Proposition 1.5] tells us:

$$V(\mathcal{F}_k(f_*\mathcal{O}_{(\Sigma(f),0)})) = \left\{ y \in \mathbb{C}^m | \sum_{x \in f^{-1}(y)\cap\Sigma(f)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{(\Sigma(f),x)}}{f^*\mathfrak{m}_y} > k \right\}, \tag{2}$$

$V(\mathcal{F}_k(f_*\mathcal{O}_{(\Sigma(f),0)}))$ denotes the set of points where $f_*\mathcal{O}_{(\Sigma(f),0)}$ requires at least $k+1$ generators on $\mathcal{O}_m$, or in other words, $V(\mathcal{F}_k(f_*\mathcal{O}_{(\Sigma(f),0)}))$ is the set of points in $\mathbb{C}^m$ whose pre-image consists of $(k+1)$ or more points in $\Sigma(f)$, counting multiplicities, [15, Proposition 1.3]. For simplicity, we denote $\mathcal{F}_k(f_*\mathcal{O}_{(\Sigma(f),0)})$ by $\mathcal{F}_k(f)$ and the variety of zeros of $\mathcal{F}_k(f)$ by $V(\mathcal{F}_k(f))$.

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4. NUMBER OF SINGULARITIES $A_{\mathcal{P}}$ FOR $F : (\mathbb{C}^{N+M}, 0) \to (\mathbb{C}^M, 0)$

In this section we shall calculate the total sum of the isolated singularities of the type $A_{\mathcal{P}}$ that can appear in the discriminant of a stable deformation of finitely determined map germs $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$ in terms of the complex dimension of algebras. For the special case that $m = 3$, in addition to calculate the total sum of singularities, we also found a formula to compute the number of ordinary triple points, denoted $A_{1,1,1}$-points, in this sense we improve the formulae given by Mond and Pellikan in [15] for the calculation of these points.

We say that a stable map germ $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$ has a singularity of type A_k if f is $\mathcal{A}$-equivalent to the germ

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_{n+m}) \to (x_1, ..., x_{m-1}, x_m^{k+1} + x_1 x_m^{k-1} + ... + x_{k-1}x_m + x_{m+1}^2 \dots + x_{n+m}^2),$$

this is called a suspension of the stable singularity A_k from $(\mathbb{C}^m, 0)$ to $(\mathbb{C}^m, 0)$ which has normal form

$$(x_1, \dots, x_m) \to (x_1, ..., x_{m-1}, x_m^{k+1} + x_1 x_m^{k-1} + ... + x_{k-1}x_m).$$

We remark that not only stable singularities of type A_k, appear in the discriminant of co-rank one map germs $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$, with $m > 3$, for instance we cite the umbilic singularity, of type D_4 which can be expressed as a co-rank one map germ from $\mathbb{C}^5$ to $\mathbb{C}^4$ with normal form $(x, y, a, b, c) = (x^2 y + y^3 + ay^2 + bx + cy, a, b, c)$, see [5] for more details. In this section we are not considering map germs with such singularities, only with A_k singularities.

The notation for the A_k singularities is the usual, as introduced by Arnol'd in [1].

For a fixed partition $\mathcal{P} = (k_1, k_2, \dots, k_s)$ of m, we call $A_{\mathcal{P}}$ the singularity in the discriminant formed by the multiple points given as the normal crossing of the singularities A_{k_ℓ} with $\ell = 1, \dots, s$.

We are counting here the isolated singularities in the target and we denote them according to the number of pre images which they have, for example, if $\mathcal{P} = (m)$ we have only one pre-image or a mono germ of type A_m, if $\mathcal{P} = (1, \dots, 1)$ (1 counting m-times) the singularity $A_{(1,\dots,1)}$ is an ordinary m-multiple point in the target.

As we can see in [15], the first and the second Fitting ideals ($\mathcal{F}_1(f)$ and $\mathcal{F}_2(f)$) are Cohen-Macaulay, and for $\mathcal{F}_2(f)$ we need to consider that $(\Sigma(f), 0)$ is Gorenstein and the codimension of $V(\mathcal{F}_2(f))$ should be greater or equal than 3.

Hence to count isolated singularities using Fitting ideals for map germs from $(\mathbb{C}^{n+m}, 0)$ to $(\mathbb{C}^m, 0)$ it is necessary that the ideal $\mathcal{F}_{m-1}(f)$ is Cohen-Macaulay.

According to the Theorem 5.2. of [15] the ideals $\mathcal{F}_k(f)$ for $1 \leq k \leq (m-1)$ are determinantals if: $\mathcal{O}_{(\Sigma(f),0)}$ is cyclic over $\mathcal{O}_m$, i.e. $\frac{\mathcal{O}_{(\Sigma(f),0)}}{f^*\mathfrak{m}_m} \cong \frac{\mathbb{C}[t]}{t^h}$, $\operatorname{codim} M_{k+1}(f) = k + 1$ and $\operatorname{codim} M_{k+2}(f) = k + 2$.

Moreover for $\mathcal{F}_{m-1}(f)$ to be a determinantal ideal we need to suppose that there exists a deformation $F : (\Sigma(f), 0) \to (\mathbb{C}^{m+1}, 0)$ such that these hypothesis for the codimension of $\mathcal{F}_{m-1}(F)$ and $\mathcal{F}_m(F)$ hold.

We remark that if f has an $A_{\mathcal{P}}$ singularity in the origin, then a straightforward calculation shows us that $\mathcal{O}_{(\Sigma(f),0)}$ is cyclic over $\mathcal{O}_m$. Then we have the following:

Let $F : (\mathbb{C} \times \mathbb{C}^{n+m}, 0) \to (\mathbb{C} \times \mathbb{C}^m, 0)$ with $F(t, x) = (t, f_t(x))$ an stable unfolding to 1-parameter of f, f_t is a deformation of f.

Proposition 4.1. *Let $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$ be a finitely determined map germ with only singularities of type A_k ($k \leq m$) in its discriminant, then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(f)} = \sum_{|\mathcal{P}|=m} C_{\mathcal{P}} \sharp A_{\mathcal{P}}.$$

where $C_{\mathcal{P}}$ denotes the local contribution of the singularity $A_{\mathcal{P}}$ in the ideal $\mathcal{F}_{m-1}(f)$.

Proof: From the Theorem 5.2 of [15], we have that $\mathcal{F}_{m-1}(f)$ is a determinantal ideal, hence it is Cohen-Macaulay and $M_m(F) \to \mathbb{C}$ is a flat deformation on the basi $\mathbb{C}$, then as $\Sigma(f) \times \mathbb{C}, 0)$ is Cohen-Macaulay too, we have the commutative diagram below:

$$\begin{array}{ccc} V(\mathcal{F}_{m-1}(f_* \mathcal{O}_{\Sigma(f)})) & \longrightarrow & V(\mathcal{F}_{m-1}(F_* \mathcal{O}_{(\Sigma(f) \times \mathbb{C})})) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

and by the conservation law of the multiplicity, we have that:

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(f_* \mathcal{O}_{\Sigma(f)})} = \sum_{z \in \mathbb{C}^m} \dim_{\mathbb{C}} \frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(f_{t_*} \mathcal{O}_{\Sigma(f_t)})},$$

where z goes through all the singularities $A_{\mathcal{P}}$ in the discriminant of versal unfolding f_t. Since the ideal $\mathcal{F}_{m-1}(f_{t_*} \mathcal{O}_{\Sigma(f_t)})$ is Cohen-Macaulay we obtain the equality

$$\sum_{z \in \mathbb{C}^m} \dim_{\mathbb{C}} \frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(f_{t_*} \mathcal{O}_{\Sigma(f_t)})} = \sum_{|\mathcal{P}|=m} C_{\mathcal{P}} \sharp A_{\mathcal{P}}.$$

□

To improve this result we show in the next Lemmas how to compute the contribution of the singularity $A_{\mathcal{P}}$ in the ideal $\mathcal{F}_{m-1}(f)$. In fact, we show that $C_{\mathcal{P}} = 1$ for all partition $\mathcal{P}$.

Lemma 4.1. *Consider the 0-stable monogerm singularity A_k from $(\mathbb{C}^k, 0)$ to $(\mathbb{C}^k, 0)$ which has normal form $(x_1, \ldots, x_k) \mapsto (x_1, ..., x_{k-1}, x_k^{k+1} + x_1 x_k^{k-1} + ... + x_{k-1} x_k)$. Then, the local contribution C_k in the $k-1$-th Fitting ideal $\mathcal{F}_{k-1}(A_k)$ of A_k is 1.*

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Proof: The jacobian ideal of A_k is:

$$J = (k+1)x_k^k + (k-1)x_1x_k^{k-2} + \cdots + 2x_{k-2}x_k + x_{k-1}.$$

Restricting A_k to $\Sigma(A_k) := V(J)$, we have the following map:

$$A_{k|_{\Sigma(A_k)}} : \Sigma(A_k) \mapsto (x_1, \dots, x_{k-1}, x_k^{k+1} + x_1x_k^{k-1} + \cdots + x_{k-2}x_k^2 + x_{k-1}x_k)$$

and the generators set of $\mathcal{O}_{(\Sigma(A_k)),0)}$ over $\mathcal{O}_k$ (target) via A_k^* is $Ger =< g_1, \dots, g_{k-1} >=< 1, x_k, \dots, x_k^{k-1} >$. Let X_i be the new variable in the target of $A_{k|_{\Sigma(A_k)}}$, where X_i is the i-th entries of $A_{k|_{\Sigma(A_k)}}$, i.e.,

$$X_1 := x_1, X_2 := x_2, \dots, X_{k-1} := x_{k-1}, X_k := x_k^{k+1} + x_1x_k^{k-1} + \cdots + x_{k-2}x_k^2 + x_{k-1}x_k. \quad (3)$$

We now obtain the first relation between the new variables and the set of generators $\langle 1, x_k, \dots, x_k^{k-1}\rangle$ module the ideal J.

By equation (3), and the generators set, we have that:

$$X_k.g_1 - x_k^{k+1} - X_1.g_k - X_2.g_{k-1} - \cdots - X_{k-2}.g_3 - X_{k-1}.g_2 = 0. \quad (4)$$

Multiplying J by x_k, using the new variables and the generators set, we obtain:

$$(k+1)x_k^{k+1} + (k-1)X_1.g_k + (k-2)X_2.g_{k-1} + (k-3)X_3.g_{k-2} + \cdots + 2X_{k-2}.g_3 + X_{k-1}.g_2 = 0. \quad (5)$$

Multiplying (4) by $(k+1)$ and summand whith (5) we obtain the following equality:

$$(k+1)X_k.g_1 - 2X_1.g_k - 3X_2.g_{k-1} - 4X_3.g_{k-2} - \cdots - (k-1)X_{k-2}.g_3 - kX_{k-1}.g_2 = 0.$$

Thus, we have the first relation of a presentation of $\mathcal{O}_{(\Sigma(A_k)),0)}$ over $\mathcal{O}_k$, and the matrix λ has the form:

$$\lambda = \begin{bmatrix} (k+1)X_k & -kX_{k-1} & -(k-1)X_{k-2} & \cdots & -4X_3 & -3X_2 & -2X_1 \\ & & & \cdots & & & \\ & & & * & & & \end{bmatrix}_{k\times k}$$

It follows that the elements of the first line of the λ generate the maximal ideal $\langle X_1, X_2, \dots, X_k\rangle$ in $\mathcal{O}_k$. By definition, we have that $\mathcal{F}_{k-1}(A_k) = \langle X_1, X_2, \dots, X_k\rangle$. Therefore, the contribution of points A_k of the A_k singularities of $(\mathbb{C}^k, 0)$ to $(\mathbb{C}^k, 0)$ in the ideal $\mathcal{F}_{k-1}(A_k)$ is 1. ($*$ denotes the others relations, but it is not necessary to obtain them). $\square$

LEMMA 4.2. *Consider the* 0*-stable multi-germ singularity* $A_{(k_1,k_2,\dots,k_s)}$ *from* $(\mathbb{C}^m,0)$ *to* $(\mathbb{C}^m,0)$*. Then, the local contribution* $C_{(k_1,k_2,\dots,k_s)}$ *in the* $k-1$*-th Fitting ideal* $\mathcal{F}_{m-1}(A_{(k_1,k_2,\dots,k_s)})$ *of* $A_{(k_1,k_2,\dots,k_s)}$ *is* 1.

Proof: For the 0-stable multi-germ singularity $A_{(k_1,k_2,\dots,k_s)}$ from $(\mathbb{C}^m,0)$ to $(\mathbb{C}^m,0)$ (transversal intersection of the A_{k_ℓ} singularities , $\ell=1,\dots s$ and $k_1+\cdots+k_s=m$), we can consider $\overset{m}{\mathcal{O}_m}\cong\overset{k_1}{\mathcal{O}_m}\oplus\cdots\oplus\overset{k_s}{\mathcal{O}_m}$. Thus, the matrix λ for a presentation of $A_{(k_1,k_2,\dots,k_s)}$, has the following form:

$$\lambda=\begin{bmatrix}\lambda_{k_1} & 0 & 0 & \cdots & 0\\ 0 & \lambda_{k_2} & 0 & \cdots & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & \lambda_{k_s}\end{bmatrix}_{m\times m}$$

where λ is a matrix with blocks in its diagonal, where each λ_{k_ℓ} is a $(k_\ell\times k_\ell)$ matrix corresponding to the $\overset{k_\ell}{\mathcal{O}_m}$ ring, $\ell=1,\dots,s$. According to Lemma 4.1, we have

$$\lambda_{k_\ell}=\begin{bmatrix}(k_\ell+1)X^\ell_{k_\ell} & -k_\ell X^\ell_{k_\ell-1} & -(k_\ell-1)X^\ell_{k_\ell-2} & \cdots & -4X^\ell_3 & -3X^\ell_2 & -2X^\ell_1\\ & & & \cdots & & & \\ & & & * & & & \end{bmatrix}_{k_\ell\times k_\ell}.$$

The variables $X^\ell_{k_\ell-i}$, $i=0,\dots,k_\ell-1$, are in the ring $\overset{k_\ell}{\mathcal{O}_m}$. By Lemma 4.1, the elements of the first line of each λ_{k_ℓ} generate the maximal ideal in $\overset{k_\ell}{\mathcal{O}_m}$, $\ell=1,\dots,s$. Hence, by definition of Fitting ideal of a matrix and considering the matrix λ for a presentation of the 0-stable multi-germ singularity $A_{(k_1,k_2,\dots,k_s)}$ from $(\mathbb{C}^m,0)$ to $(\mathbb{C}^m,0)$, we conclude that: $\mathcal{F}_{m-1}(A_{(k_1,k_2,\dots,k_s)})=\langle X^1_1,\dots,X^1_{k_1},\dots,X^s_1,\dots,X^s_{k_s}\rangle\subset\mathcal{O}_m$.

As $\mathcal{F}_{m-1}(A_{(k_1,k_2,\dots,k_s)})$ is a maximal ideal in $\mathcal{O}_m$ then, $\dim_{\mathbb{C}}\frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(A_{(k_1,k_2,\dots,k_s)})}=1$, as required. ∎

Remark 4. 1. If a germ from $(\mathbb{C}^{n+m},0)$ to $(\mathbb{C}^m,0)$,

$$f(x_1,\dots,x_{n+m})\to(x_1,\dots,x_{m-1},h(x_1,\dots,x_m)+x^2_{m+1}+\dots+x^2_{n+m}),$$

is a suspension of a germ from $(\mathbb{C}^m,0)$ to $(\mathbb{C}^m,0)$, then a presentation for $\mathcal{O}_{(\Sigma(f),0)}$ and $\mathcal{O}_{(\Sigma(g),0)}$ is the same, since these quotient rings are isomorphic,i.e.,

$$\mathcal{O}_{(\Sigma(g),0)}\cong\frac{\mathcal{O}_m}{J(g)}\cong\frac{\mathcal{O}_{m+n}}{\langle J(g),x_{m+1},\dots,x_{m+n}\rangle}\cong\mathcal{O}_{(\Sigma(f),0)},$$

and, clearly the generators of $\mathcal{O}_{(\Sigma(f),0)}$ and $\mathcal{O}_{(\Sigma(g),0)}$ as $\mathcal{O}_m$-modules are the same.

COROLLARY 4.1. *Let $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$ be a finitely determined map germ with only singularities $A_{\mathcal{P}}$ in its discriminant, then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(f)} = \sum_{|\mathcal{P}|=m} \sharp A_{\mathcal{P}}.$$

Proof: By Lemmas 4.1 and 4.2, $C_{\mathcal{P}} = 1$ for all partition $\mathcal{P}$, with $|\mathcal{P}| = m$. Therefore, by Proposition 4.1 we have the desired. □

Analyzing the case $m = 3$.

As the Fitting ideals does not distinguish ordinary k-multiple points from the other points whose multiplicity is k, then the ideal $\mathcal{F}_1(f)$ defines the union of the ordinary double points curve with the cuspidal edge curve of f in the target.

Therefore according to the equation (2) we have that the discriminant of f is:

$$V(\mathcal{F}_0(f)) = \Delta(f) = f(\Sigma(f)).$$

Analogously, $V(\mathcal{F}_1(f)) = A_{(1,1)} \cup A_2$ and $V(\mathcal{F}_2(f)) = A_{(1,2)} \cup A_{(1,1,1)} \cup A_3$.

Note that $V(\mathcal{F}_2(f)) \subset V(\mathcal{F}_1(f)) \subset V(\mathcal{F}_0(f)) \subset \mathbb{C}^3$.

Mond and Pellikaan showed in the Theorem 4.3 in [15] that if $f : (X, x) \to (\mathbb{C}^m, 0)$ is a finite mapping where (X, x) is a $(m-1)$-dimensional Gorenstein space, f of degree 1 onto its image, and that codimension of $\mathcal{F}_2(f) \geq 3$ implies that $\mathcal{F}_2(f)$ is an symmetric determinantal ideal, hence $\mathcal{F}_2(f)$ is Cohen-Macaulay. In particular for $m = 3$, for any finite map germ f from X to $\mathbb{C}^3$ with just ordinary triple points ($A_{(1,1,1)}$-points) as isolated singularities in a stable perturbation, the number of these points (denoted T by Mond and Pellikaan and by $\sharp A_{(1,1,1)}$ in the general context) is given by the complex dimension of the algebra $\frac{\mathcal{O}_3}{\mathcal{F}_2(f)}$, or

$$T = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)}.$$

Here we do analogous study for the case of finitely determined map germs f from $\mathbb{C}^{n+3}$ to $\mathbb{C}^3$. In this case, we also have points of type $A_{(1,2)}$ and A_3 appearing as isolated stable singularities and we show that the dimension $\dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)}$ provides us the total sum of all these 0-stable singularities. To show this result we apply the results of Mond and Pellikaan to the set $(X, x) = (\Sigma(f), 0)$, which is 2-dimensional.

PROPOSITION 4.2. *Let $f : (\mathbb{C}^{n+3}, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ with $(\Sigma(f), 0)$ Gorenstein and codimension of $V(\mathcal{F}_2(f))$ equals to 3, then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} = \sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3.$$

Proof: As $\Sigma(f)$ is Gorenstein, by Theorem 4.1 or 4.3 of [15], $\mathcal{F}_2(f)$ is a symmetric determinantal ideal, hence is Cohen-Macaulay. As there are no stable singularities with co-rank greater than one in map germs in these dimesions, it follows that independent of co-rank of f, will appear only singularities of type A_k, $(k = 1, 2, 3)$ in a stable perturbation of the germ. Thus, the result follows by Corollary 4.1. □

Remark 4. 2. If f has only ordinary triple points in its stable perturbation, then the proposition above is the affirmation of the Theorem 4.3 in [15] applied to the set $(X, x) = (\Sigma(f), 0)$.

Remark 4. 3. If $f : (\mathbb{C}^{n+3}, 0) \to (\mathbb{C}^3, 0)$ is a finitely determined map germ of co-rank one, then $\Sigma(f)$ is a complete intersection with isolated singularity, hence $\Sigma(f)$ is Gorenstein. If $n = 0$ and f is of any co-rank, $\Sigma(f)$ is a hypersurface, hence Gorenstein.

The disadvantage of the Proposition 4.2 above is that it gives us only the sum of the isolated singularities appearing in the discriminant of a stable perturbation of the germ f.

However, the next proposition shows to us a way to calculate the number of ordinary triple points appearing in a stable perturbation of f, for this we just need to know the defining equation of the ideal of the ordinary double point curve of f in the target.

We denote by $I(A_2)$ the defining ideal of the cuspidal edge and by $I(A_{1,1})$ the defining ideal of the ordinary double points curve of f in the target.

Proposition 4.3. *With the same hypothesis of the Proposition 4.2 we have,*

$$\sharp A_{(1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))}.$$

The notation $(I : J)$ means the quotient ideal of an ideal I by other ideal J in a ring A or

$$(I : J) := \{a \in A \ ; \ aJ \subset I\}.$$

Proof: By [15, Propositon 4.1] we have that $\mathcal{F}_2(f) = (\mathcal{F}_1(f)^2 : \mathcal{F}_0(f))$. As the ideal $\mathcal{F}_1(f)$ defines the set of the points in the target with multiplicity 2, then $\mathcal{F}_1(f) = I(A_{1,1}) \cap I(A_2)$. We remark that to have points of the type A_3 and $A_{(1,2)}$ in the discriminant, it is necessary the to have the curve A_2 in it. On the other side, if f has ordinary triple points in a deformation, it follows that $(I(A_{1,1})^2 : \mathcal{F}_0(f))$ is the ideal that defines the set of ordinary triple points, since these points do not belong to the curve A_2. □

As a consequence of these results we obtain:

Corollary 4.2. *With the same hypothesis of Proposition 4.2 we have,*

$$\sharp A_{(1,2)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} - \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))} - \sharp A_3.$$

Now, if we consider germs $f : (\mathbb{C}^{n+3}, 0) \to (\mathbb{C}^3, 0)$ of co-rank 1, we see in [4, Corollary 4.4] that we can compute the number of points A_3 as

$$\sharp A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{J_{n+1,1,1}},$$

where $J_{n+1,1,1}$ denotes the iterated Jacobian ideal associated to this set.

Therefore, with the same hypothesis of Proposition 4.2 and furthermore f being of co-rank 1, we have all these numbers expressed by dimensions since we have the number of points of the type $A_{(1,2)}$, given by a sum of dimensions, or:

$$\sharp A_{(1,2)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} - \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))} - \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+3}}{J_{n+1,1,1}}.$$

To finish this section we conclude that for map germs f from $(\mathbb{C}^{n+3}, 0)$ to $(\mathbb{C}^3, 0)$:

1. If f is of co-rank 1, it is possible to count each one of the isolated singularities.

2. If f is not of co-rank 1, we know how to calculate the total sum of three singularities and the number of ordinary triple points and consequently the sum $\sharp A_{(1,2)} + \sharp A_3$.

5. EXAMPLES

We now apply the results shown here to some examples. The first example below is for a finitely determined map germ from $\mathbb{C}^4$ to $\mathbb{C}^3$ of co-rank 1. The second example is for a map germ from $\mathbb{C}^3$ to $\mathbb{C}^3$ of co-rank 1 and in the third example we show how to compute such numbers for the co-rank 2 case and finally an exemple from $\mathbb{C}^4$ to $\mathbb{C}^4$.

Example 5.1. Let $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ be defined by: $f(x, y, z, w) = (x, y, w^4 + z^2 w + yz + xw)$.

First we need to find the matrix of presentation for the $\mathcal{O}_{\Sigma(f)}$-module. The jacobian ideal of f is given by: $J(f) = \langle y + 2zw,\ 4w^3 + z^2 + x\rangle$, $(\Sigma(f), 0)$ is a germ of a 2-dimensional Cohen-Macaulay variety and also is Gorenstein, since it is a smooth complete intersection.

A presentation of $\mathcal{O}_{(\Sigma(f),0)}$ on $\mathcal{O}_3$ is an exact sequence

$$\mathcal{O}_3^5 \xrightarrow{\lambda} \mathcal{O}_3^5 \xrightarrow{\alpha} \mathcal{O}_{(\Sigma(f),0)} \to 0$$

of $\mathcal{O}_3$-modules.

A base for the quotient $\dfrac{\mathcal{O}_{(\Sigma(f),0)}}{f^*\mathfrak{m}_0}$ is: $\{1, z, z^2, w, w^2\}$, therefore the matrix λ for the exact sequence is given by:

$$\lambda = \begin{bmatrix} -Z & \frac{5}{8}Y & 0 & \frac{3}{4}X & 0 \\ -\frac{3}{8}XY & -Z & \frac{5}{8}Y & 0 & 0 \\ 0 & -XY & -Z & 0 & \frac{5}{4}Y^2 \\ -\frac{5}{16}Y^2 & 0 & 0 & -Z & \frac{3}{4}X \\ -\frac{3}{16}X^2 & 0 & -\frac{3}{16}X & -\frac{5}{16}Y^2 & -Z \end{bmatrix}_{5\times 5}$$

We now calculate the k^{th} Fitting ideal of the matrix λ, for $3 > k \geq 0$, obtaining: $\mathcal{F}_k(f) = I_k(\mathcal{O}_{(\Sigma(f),0)})$.
Therefore, we have the following minors of ideals of the matrix λ:

$$\mathcal{F}_0(f) = \langle 3125Y^8 - 6912X^5Y^2 - 36000X^2Y^4Z - 6912X^4Z^2 - 102400XY^2Z^3 - 65536Z^5\rangle$$

$$\begin{aligned} \mathcal{F}_1(f) = & \langle 27X^4Z + 75XY^2Z^2 + 160Z^4, \ \ 375XY^4 - 432X^3Z + 1600Y^2Z^2, \\ & 90X^3Y^2 + 125Y^4Z + 144X^2Z^2, \ \ 27X^4Y + 75XY^3Z + 160YZ^3, \\ & 625Y^6 - 2640X^2Y^2Z - 3072XZ^3\rangle \end{aligned}$$

$$\mathcal{F}_2(f) = \langle Z^3, \ YZ^2, \ XZ^2, \ Y^2Z, \ XYZ, \ X^2Z, \ XY^2, \ X^2Y, \ X^3, \ Y^4\rangle$$

Once obtained the ideals $\mathcal{F}_1(f)$ and $\mathcal{F}_2(f)$ we calculate: $\#A_3$, $\#A_{(1,2)}$ and $\#A_{(1,1,1)}$.

From the primary decomposition of ideal $\mathcal{F}_1(f)$, we obtain:
$\mathcal{F}_1(f) = I(A_{(1,1)}) \cap I(A_2)$, where,

$$\begin{aligned} I(A_2) = & \ \langle 15XY^2 + 16Z^2, \ \ 9X^3 - 25Y^2Z, \ \ 125Y^4 + 48X^2Z\rangle \\ I(A_{(1,1)}) = & \ \langle 5XY^2 + 32Z^2, \ \ 125Y^4 - 432X^2Z, \ \ 27X^3Y + 50Y^3Z\rangle. \end{aligned}$$

By Proposition 4.2, we obtain the sum of the isolated singularities:

$$\sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3 = 11 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)}.$$

As the quotient $(I(A_{1,1})^2 : \mathcal{F}_0(f)) = \langle X, Y, Z\rangle$ is the maximal ideal in $\mathcal{O}_3$, it follows from the Proposition 4.3 that,

$$\sharp A_{(1,1,1)} = 1.$$

We compute now the points A_3 using the equation $\sharp A_3 = \dim_{\mathbb{C}} \dfrac{\mathcal{O}_4}{J_{2,1,1}} = 5$ and from the Corollary 4.2 we have that, $\sharp A_{(1,2)} = 5$.

Example 5.2. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be defined by $f(x, y, z) = (x, y, z^8 + xz^2 + yz)$.

Calculating a presentation for the $\mathcal{O}_{\Sigma(f)}$-module results in the presentation matrix λ:

$$\lambda = \begin{bmatrix} -8Z & 7Y & 6X & 0 & 0 & 0 & 0 \\ 0 & -8Z & 7Y & 6X & 0 & 0 & 0 \\ 0 & 0 & -8Z & 7Y & 6X & 0 & 0 \\ 0 & 0 & 0 & -8Z & 7Y & 6X & 0 \\ 0 & 0 & 0 & 0 & -8Z & 7Y & 6X \\ -3XY & -6X^2 & 0 & 0 & 0 & -32Z & 28Y \\ 16XZ+7Y^2 & 6XY & 0 & 0 & 0 & 0 & 64Z \end{bmatrix}_{7\times 7}.$$

Calculating the k^{th} Fitting ideals of the matrix λ, for $0 \leq k \leq 3$, we obtain:

$$\begin{aligned} \mathcal{F}_0(f) = \; & \langle 16777216Z^7 + 823543Y^8 + 7529536XY^6Z + 22127616X^2Y^4Z^2 + \\ & 21676032X^3Y^2Z^3 + 3538944X^4Z^4 + 46656X^7Y^2 + 186624X^8Z. \rangle \\ \mathcal{F}_1(f) \; = \; & \langle 117649Y^6 + 576240XY^4Z + 677376X^2Y^2Z^2 + 110592X^3Z^3 + 11664X^7, \\ & 16807Y^5Z + 65856XY^3Z^2 + 48384X^2YZ^3 - 729X^6Y, \\ & 38416Y^4Z^2 + 112896XY^2Z^3 + 36864X^2Z^4 + 1701X^5Y^2 + 3888X^6Z, \\ & 87808Y^3Z^3 + 172032XYZ^4 - 3969X^4Y^3 - 12960X^5YZ, \\ & 200704Y^2Z^4 + 196608XZ^5 + 9261X^3Y^4 + 39312X^4Y^2Z + 20736X^5Z^2, \\ & 458752YZ^5 - 21609X^2Y^5 - 112896X^3Y^3Z - 117504X^4YZ^2, \\ & 1048576Z^6 + 50421XY^6 + 312816X^2Y^4Z + 483840X^3Y^2Z^2 + 110592X^4Z^3 \rangle \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_2(f) = \; & \langle X^5, X^4Y, X^3Y^2, X^2Y^3, XY^4, Y^5, X^4Z, X^3YZ, X^2Y^2Z, XY^3Z, Y^4Z, X^3Z^2, \\ & X^2YZ^2, XY^2Z^2, Y^3Z^2, X^2Z^3, XYZ^3, Y^2Z^3, XZ^4, YZ^4, Z^5 \rangle \end{aligned}$$

From the primary decomposition of the ideal $\mathcal{F}_1(f)$, we obtain the defining ideals of the curves $I(A_{(1,1)})$ and $I(A_2)$:

$$I(A_2) = \left\langle 49Y^2 + 192XZ, 27X^4 - 1792Z^3 \right\rangle,$$

$$\begin{aligned} I(A_{(1,1)}) \; = \; & \langle 945X^4Y^2 + 864X^5Z + 12544Y^2Z^3 + 8192XZ^4, -729X^5Y + 10976Y^3Z^2 + \\ & 16128XYZ^3, 729X^6 + 12005Y^4Z + 28224XY^2Z^2 + 6912X^2Z^3, 2401Y^5 + \\ & 7840XY^3Z + 4608X^2YZ^2, 9261X^2Y^4 + 24192X^3Y^2Z + 6912X^4Z^2 + \\ & 65536Z^5, 1323X^3Y^3 + 2376X^4YZ - 14336YZ^4 \rangle \end{aligned}$$

Since

$$(I(A_{1,1})^2 : \mathcal{F}_0(f)) = \left\langle Z^3, YZ^2, XZ^2, Y^2Z, XYZ, X^2Z, Y^3, XY^2, X^2Y, X^3 \right\rangle.$$

Then we obtain

$$\sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} = 35,$$

$$\sharp A_{(1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))} = 10,$$

$$\sharp A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{J_{1,1,1}} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle z^5, x - 140z^6, y + 120z^7 \rangle} = 5,$$

$$\sharp A_{(1,2)} = 35 - 10 - 5 = 20.$$

We remark here that this germ is of co-rank one and weight-homogeneous with weights $\omega_1 = 6$, $\omega_2 = 7$, $\omega_0 = 1$ and degree $d = 8$, then we can check these numbers applying the formulae given by Marar, Montaldi and Ruas in [12] to count the isolated singularities that appear in the discriminant of the a weighted-homogeneous finitely determined germ of co-rank one $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, in terms of weights and degrees of f.

Example 5.3. Now we compute the quantity of isolated singularities of the co-rank 2 map germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ defined by $f(x, y, z) = (x, yz, z^2 + y^2 + xz)$.

Nuño-Ballesteros and Saia in [17] showed that this germ has two A_3-points in a generic deformation of f, but in this example we have $\dim_{\mathbb{C}} \dfrac{\mathcal{O}_3}{J_{1,1,1}} = 4$, hence this dimension does not gives the number of points A_3. The main problem here is that the iterated jacobian ideal $J_{1,1,1}(f)$ is not Cohen-Macaulay, therefore the set $\Sigma^{1,1,1}(f)$ is not Cohen-Macaulay and this is a necessary condition to get the equality of the dimension and the number of points A_3.

However this does not causes any problem to use the Fitting ideals to calculate all isolated stable singularities, as we show below.

First we calculate a presentation for the $\mathcal{O}_{\Sigma(f)}$-module.

The jacobian ideal of f is generated by $J(x, y, z) = 2z^2 - 2y^2 + xz$, hence $\Sigma(f) = V(J)$ is a surface, hence a complete intersection, with isolated singularity and therefore $(\Sigma(f), 0)$ is a Cohen-Macaulay variety. Moreover we also have that this set is Gorenstein, therefore we can apply the results of Mond and Pelikaan.

The restriction of f to $\Sigma(f)$, $f = f|_{\Sigma(f)} : (\Sigma(f), 0) \to (x, yz, z^2 + y^2 + xz)$ is a finite analytic map and $\mathcal{O}_{(\Sigma(f),0)}$ is a finite $\mathcal{O}_3$-module via f^*.

Here, we find $\{1, y, z\}$ as the generators of $\mathcal{O}_{(\Sigma(f),0)}$ over $\mathcal{O}_3$ (target) via f^*.

Now we write the relations to obtain the matrix λ of the presentation for $\mathcal{O}_{(\Sigma(f),0)}$.

Let $X := x$, $Y := yz$, $Z := z^2 + y^2 + xz$ and consider the generators $\{1, y, z\}$. These 3 relations sought are:

$$\begin{array}{rcrcrcl} -\frac{3}{4}XY.1 & + & \frac{1}{2}Z.y & - & Y.z & = & 0 \\ \frac{1}{8}XZ.1 & + & Y.y & + & (-\frac{1}{2}Z - \frac{3}{16}X^2).z & = & 0 \\ (-Y^2 + \frac{1}{4}Z^2).1 & - & \frac{3}{4}XY.y & - & \frac{1}{8}XZ.z & = & 0 \end{array}$$

Therefore the matrix λ of the relations is given by:

$$\lambda = \begin{bmatrix} -\frac{3}{4}XY & \frac{1}{2}Z & -Y \\ \frac{1}{8}XZ & Y & -\frac{1}{2}Z - \frac{3}{16}X^2 \\ -Y^2 + \frac{1}{4}Z^2 & -\frac{3}{4}XY & -\frac{1}{8}XZ \end{bmatrix}_{3\times 3}$$

and we obtain the following Fitting ideals of λ:

$$\begin{aligned} \mathcal{F}_0(f) = &\ \langle 256Y^4 - 128Y^2Z^2 + 16Z^4 - 144X^2Y^2Z + 4X^2Z^3 - 27X^4Y^2 \rangle, \\ \mathcal{F}_1(f) = &\ \langle 32Y^2 - 8Z^2 - 3X^2Z,\ \ 32XYZ + 9X^3Y,\ \ 32XZ^2 + 9X^3Z \rangle, \\ \mathcal{F}_2(f) = &\ \langle X^2,\ \ Y,\ \ Z \rangle. \end{aligned}$$

From the primary decomposition, we obtain: $\mathcal{F}_1(f) = I(A_{(1,1)}) \cap I(A_2)$ where,

$$I(A_{(1,1)}) = \ \langle 128Y^2 - 3X^2Z,\ \ 32YZ + 9X^2Y,\ \ 32Z^2 + 9X^2Z \rangle \text{ and}$$

$$I(A_2) = \ \langle X,\ \ 4Y^2 - Z^2 \rangle.$$

The number of points in $V(\mathcal{F}_2(f))$ is

$$2 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} = \sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3.$$

As this germ has 2 points A_3, see [17, Example 4.5], and the total sum of points with multiplicity 3 is 2, it follows that this germ has not points of the type $A_{(1,2)}$ and $A_{(1,1,1)}$.

Observe also that on the other hand, the quotient $(I(A_{1,1})^2 : \mathcal{F}_0(f)) = \mathcal{O}_3$, therefore by Proposition 4.3, we obtain the result.

Example 5.4. We fix $n = 0$ and $m = 4$ to show an example satisfying the hypothesis of the Corollary 4.1 and all 0-stable $A_{\mathcal{P}}$ singularities with $|\mathcal{P}| = 4$, which are $\mathcal{P} = (1,1,1,1)$, $\mathcal{P} = (1,1,2)$, $\mathcal{P} = (1,3)$, $\mathcal{P} = (2,2)$ and $\mathcal{P} = (4)$, appear in the discriminant of a stabilization of the germ.

Let $f : (\mathbb{C}^4, 0) \longmapsto (\mathbb{C}^4, 0)$ be given by:

$$f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4^8 + x_4^3x_1 + x_4^2x_2 + x_4x_3).$$

f is a co-rank one map germ and $\frac{\mathcal{O}_{(\Sigma(f),0)}}{f^*\mathfrak{m}_4} \cong \frac{\mathbb{C}[x_4]}{\langle x_4^7 \rangle}$, or $\mathcal{O}_{(\Sigma(f),0)}$ is cyclic over $\mathcal{O}_4$.

Calculating the presentation for $\mathcal{O}_{(\Sigma(f),0)}$ we obtain the following 7×7-matrix:

$$\lambda = \begin{bmatrix} -W & \frac{7}{8}Z & \frac{3}{4}Y & \frac{5}{8}X & 0 & 0 & 0 \\ 0 & -W & \frac{7}{8}Z & \frac{3}{4}Y & \frac{5}{8}X & 0 & 0 \\ 0 & 0 & -W & \frac{7}{8}Z & \frac{3}{4}Y & \frac{5}{8}X & 0 \\ 0 & 0 & 0 & -W & \frac{7}{8}Z & \frac{3}{4}Y & \frac{5}{8}X \\ -\frac{5}{64}XZ & -\frac{5}{32}XY & -\frac{15}{64}X^2 & 0 & -W & \frac{7}{8}Z & \frac{3}{4}Y \\ -\frac{3}{32}YZ & -\frac{3}{16}Y^2 - \frac{5}{64}XZ & -\frac{7}{16}XY & -\frac{15}{64}X^2 & 0 & -W & \frac{7}{8}Z \\ -\frac{7}{64}Z^3 & -\frac{5}{16}YZ & -\frac{3}{16}Y^2 - \frac{13}{32}XZ & -\frac{7}{16}XY & -\frac{15}{64}X^2 & 0 & -W \end{bmatrix}_{7\times 7}$$

and the third Fitting ideal of the matrix λ is:

$$\begin{aligned}\mathcal{F}_3(f) = \ & \langle X^4, X^3Y, X^2Y^2, 36XY^3, 54Y^4, X^3Z, 21X^2YZ, 252XY^2Z, 252Y^3Z, 49X^2Z^2,\\ & 147XYZ^2, 1764Y^2Z^2, 49XZ^3, 1029YZ^3, 2401Z^4, X^3W, X^2YW, 36XY^2W,\\ & 18Y^3W, X^2ZW, XYZW, 36Y^2ZW, 49XZ^2W, 21YZ^2W, 49Z^3W, X^2W^2,\\ & XYW^2, 36Y^2W^2, XZW^2, 21YZW^2, Z^2W^2, XW^3, YW^3, ZW^3, W^4\rangle\end{aligned}$$

and

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_4}{\mathcal{F}_3(f)} = 35.$$

On the other side we have that f is an weighted homogeneous germ, hence we can apply the results of [19] to count each one of the isolated singularities associated to this germ. Then we have:

$$\sharp A_4 = 4, \ \ \sharp A_{(1,3)} = 12, \ \ \sharp A_{(2,2)} = 6, \ \ \sharp A_{(1,1,2)} = 12, \ \ \sharp A_{(1,1,1,1)} = 1.$$

We remark that the sum of these singularities is 35, or

$$\sharp A_{(1,1,1,1)} + \sharp A_{(1,1,2)} + \sharp A_{(1,3)} + \sharp A_{(2,2)} + \sharp A_4 = 35 = \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{\mathcal{F}_3(f)},$$

as desired.

Remark 5. 1. In the special case of co-rank one map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ of type $(x, y, g(x, y, z))$, we see in [21, 7.1] calculations involving the number of swallowtails and of ordinary triple points $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$. There the computation for the number of swallowtails is done using the formula $\sharp A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{<g_z, g_{zz}, g_{zzz}>}$. However, the number of triple points is computed only for the very special cases that the critical set $\Sigma(f)$ is smooth, and in this case it is applied the following Theorem, due to D. Mond:

THEOREM 5.1. *[16] A generic deformation of a germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ with rank 1 at 0, has $\frac{1}{6} \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{I_3(f)}$ triple points.*

In the general case of a map germ with any co-rank and the critical set is not smooth, the computations done in [21, 7.1] are not possible. They are possible to be done as in [21, 7.1] because the critical set $\Sigma(f)$ is smooth and the restriction $F = f_{|\Sigma(f)}$ can be considered as a map germ $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ parametrizing the discriminant, which is generally not $\mathcal{A}$-finitely determined, due to the presence of cuspidal edges.

Here we can compute such numbers for any map germ $f : (\mathbb{C}^{n+3}, 0) \to (\mathbb{C}^3, 0)$ with no restrictions for the co-rank of the map and the critical set being non smooth. For instance, the critical set of the map $f(x, y, z) = (x, yz, z^9 + y^2 + xz)$ is non smooth, but we can compute all invariants using our results. Here we have a total of 219 triple points, being 98 of these ordinary triple points and the others of type A_3 and $A_{1,2}$.

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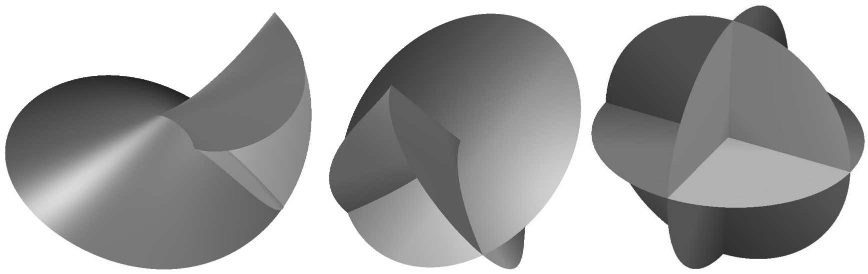
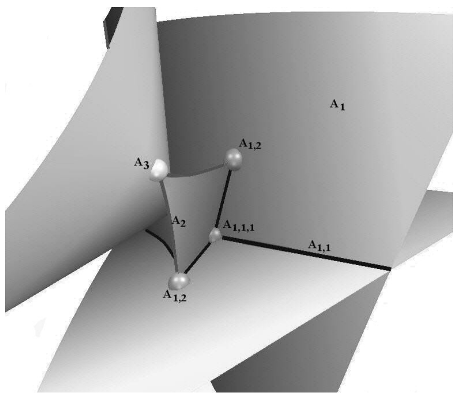
6. PICTURES



Geometric models: A_3, $A_{(1,2)}$ and $A_{(1,1,1)}$ respectively.

The figure below shows the real part of the discriminant of a 1-parameter deformation of the germ $f(x, y, z) = (x, y, z^8 + xz^2 + yz)$, where for determined real values of t, it is possible to visualize all stable types.



Real part of the discriminant of: $f_t(x, y, z) = (x, y, z^8 + xz + yz^2 + tz^4)$.

All pictures shown in this article were done with the software Surfex, [7] and several calculations where done with the help of the softwares Maple, [20] and Singular, [9].

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