CADERNOS DE MATEMÁTICA **12**, 167–204 May (2011)
ARTIGO NÚMERO SMA#350

Autonomous dissipative semidynamical systems with impulses

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In the present paper, we study the theory of dissipative impulsive semidynamical systems. We define different types of dissipativity as point, compact, local and bounded. The center of Levinson is defined for compact dissipative impulsive semidynamical systems and its topological properties are investigated. Also, we present results giving necessary and sufficient conditions to obtain dissipativity, and we include some examples to point out that the concepts of the different kinds of dissipativity are not equivalent in general. May, 2011 ICMC-USP

Key Words: impulsive semidynamical systems, dissipativity, center of Levinson.

1. INTRODUCTION

The theory of impulsive dynamical systems play an important role to model real-world problems in science and technology. This theory has been attracting the attention of many mathematicians and the interest in the subject is still growing. In the last years, the action of impulses on dynamical systems has been intensively investigated, see for instance, [1, 6], [8, 16], [18] and the references therein.

In [7], the author develops the theory of autonomous dissipative dynamical systems. Several types of dissipativity as point, compact, local, bounded and weak one, are given. The center of Levinson of a compact dissipative dynamical system $(X, \pi, \mathbb{R})$, where X is a metric space, is defined. Cheban shows that the Levinson's center is the least compact positive invariant set that attracts all compact subsets from X. Moreover, it is proved in [7] that the center of Levinson is invariant and globally asymptotically stable, see [7, Theorem 1.6] and [7, Theorem 1.8]. Results giving necessary and sufficient conditions to

* Supported by FAPESP grant 2010/08994-7.

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obtain dissipativity are established, see for instance, Theorems 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18 and 1.19 in [7].

In this paper, we present a systematical study of autonomous dissipative impulsive semidynamical systems. We consider impulsive semidynamical systems of type $(X, \pi; M, I)$ subject to impulse action which varies in time, where X is a metric space, $(X, \pi, \mathbb{R}_+)$ is a semidynamical system, M is a non-empty closed subset of X that denotes the impulsive set and $I : M \to X$ is the impulse function.

We generalize many results of [7] for semidynamical systems with and without impulses. In some cases, we present different proofs for the results proved in [7]. Let us mention some of these results. Let $(X, \pi, \mathbb{R})$ be a dynamical system defined on a metric space X (the reader should consult [7] for the notations). Consider the following lemma.

[7, Lemma 1.6] Let $A \subset X$ and the set $\sum^+(A)$ be relatively compact. If the positive $\omega-$limit satisfies the condition $\omega(A) \subset A$, then $\omega(A) = \cap\{\pi^t(A) : t \in \mathbb{R}\}$.

Cheban uses the invariance of $\omega(A)$ to prove [7, Lemma 1.6]. However, this proof does not apply for semidynamical systems since $\omega(A)$ is only positively invariant in these systems. We generalize this result for impulsive semidynamical systems and it is stated in Lemma 3.8. When we consider the impulsive set $M = \emptyset$ in Lemma 3.8, we have the non-impulsive case and we can state the following result.

Lemma 1.1. *Let $(X, \pi, \mathbb{R}_+)$ be a semidynamical system. Let $A \subset X$ and the set $\sum^+(A)$ be relatively compact. If $\omega(A) \subset A$, then $\omega(A) = \cap\{\pi^t(A) : t \geq 0\}$.*

We also can use the proof of Lemma 1.1 to prove [7, Lemma 1.6].

In [7], the author considers a compact dissipative dynamical system and he defines the center of Levinson for this system by

$$J := \omega(K) = \cap\{\pi^t(K) : t \in \mathbb{R}\},$$

where K is a non-empty compact set that is an attractor for compacts subsets from X. It is proved that J does not depend on the choice of the set K attracting all compact subsets of the space X. Again, the invariance of limit sets is used to get the result. Our Lemma 3.10 generalizes this result for impulsive semidynamical systems and semidynamical systems ($M = \emptyset$). We can use positive invariance instead of invariance to proof that J does not depend on the choice of the set K in [7]. Indeed, we can use the proof of Lemma 3.10 by considering $M = \emptyset$.

Let $\Omega = \overline{\cup\{\omega_x : x \in X\}}$ and set $J^+(\Omega)$ by

$$J^+(\Omega) = \bigcap_{\epsilon>0} \bigcap_{t\geq 0} \overline{\bigcup\{\pi(B(\Omega, \epsilon), \tau) : \tau \geq t\}},$$

see [7] for details. Cheban proved the following corollary.

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[7, Corollary 1.4] If $(X, \pi, \mathbb{R})$ is point dissipative, $\Omega \neq \emptyset$ and it is compact, then $\Omega \subset J^+(\Omega)$.

We show that the inclusion $\Omega \subset J^+(\Omega)$ holds in general, that is, we do not need the conditions that $(X, \pi, \mathbb{R})$ is point dissipative and Ω is compact. By following the proof of Proposition 3.3 presented in subsection 3.3, we can state:

Proposition 1.1. *Let $(X, \pi, \mathbb{R})$ $((X, \pi, \mathbb{R}_+))$ be a dynamical system (semidynamical system). Then $\Omega \subset J^+(\Omega)$.*

Proposition 3.3 is a version for impulsive semidynamical systems.

Some results from this paper point out other directions to get new proofs for the results presented in [7], for instance, the proof of Theorem 3.7 gives a different proof to [7, Theorem 1.11]. Furthermore, when we consider our systems with $M = \emptyset$, this paper present a theory of autonomous dissipative semidynamical systems (without impulses).

In the next lines, we describe the organization of the paper and the main results.

In the first part of this paper, we present the basis of the theory of impulsive semidynamical systems. We divide section 2 in three parts. In Subsection 2.1, we give some basic definitions and notations about impulsive semidynamical systems. In Subsection 2.2, we discuss the continuity of a function which describes the times of meeting impulsive sets. In Subsection 2.3, we give some additional useful definitions.

The second part of the paper, namely Section 3, concerns the main results. Subsection 3.1 deals with various properties about limit sets. In Subsection 3.2, we define the concept of several types of dissipativity for the impulsive case (in the sense of [7]). We define the center of Levinson for a dissipative compact impulsive semidynamical system $(X, \pi; M, I)$ and we prove that this center is positively invariant, compact, orbitally stable and it is the attractor of the family of all compacts of X, see Theorem 3.1. In Theorem 3.4, we prove that J is globally asymptotically stable.

In Subsection 3.3, we investigate criteria to obtain compact dissipativity. In Theorem 3.6, we present a version of Ura's Theorem for impulsive semidynamical systems defined in a metric space not necessarily locally compact. The results that give necessary and sufficient conditions to obtain compact dissipativity are presented in Theorem 3.8, Theorem 3.9, Theorem 3.10 and Theorem 3.11.

Subsection 3.4 deals with local dissipativity. Since compact dissipativity does not imply in local dissipativity (see Example 3.5), we state sufficient conditions to assure this result, see Theorems 3.12, 3.13, 3.14 and 3.15.

2. PRELIMINARIES

In this section we present the basic definitions and notations of the theory of impulsive semidynamical systems. We also include some fundamental results which are necessary for understanding the basis of the theory.

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2.1. Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in X$ and $t, s \in \mathbb{R}_+$. We denote such system simply by (X, π). For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \Delta) = \cup\{F(x, t) : x \in D \text{ and } t \in \Delta\}.$$

Then a point $x \in X$ is called an *initial point*, if $F(x, t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive semidynamical system. The points of M are isolated in every trajectory of system (X, π). The set M is called the *impulsive set* and the function I is called *impulse function*. We also define

$$M^+(x) = (\pi^+(x) \cap M) \setminus \{x\}.$$

By [6], Lemma 2.1, the impulsive set M is a meager set in X.

Given an impulsive semidynamical system $(X, \pi; M, I)$ and $x \in X$ such that $M^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. This result is stated next and a proof of it can be found in [1] and in [14].

Lemma 2.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Then for every $x \in X$, there is a positive number s, $0 < s \leq +\infty$, such that $\pi(x, t) \notin M$, whenever $0 < t < s$, and $\pi(x, s) \in M$ if $M^+(x) \neq \emptyset$.*

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. By means of Lemma 2.1, it is possible to define a function $\phi : X \to (0, +\infty]$ in the following manner

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M \text{ and } \pi(x, t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

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The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an $X-$valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x,t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $M^+(x) \neq \emptyset$, it follows from Lemma 2.1 that there is a smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M$ and $\pi(x,t) \notin M$, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x,t), & 0 \le t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$ and $\phi(x) = s_0$. Let us denote x by x_0^+.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, for $s_0 \le t < +\infty$, and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t - s_0) \notin M$, for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \le t < s_0 + s_1 \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = I(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$, $n = 1, 2, 3, \dots$, and in this case the function $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set

$$\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x,t) : t \in [0, T(x))\}.$$

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies standard properties which follow straightforwardly from the definition. See the next proposition and [2] for a proof of it.

Proposition 2.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. The following properties hold:*

a)$\widetilde{\pi}(x, 0) = x$,
b)$\widetilde{\pi}(\widetilde{\pi}(x,t), s) = \widetilde{\pi}(x, t+s)$, for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [1, 6], [8, 10] and [14, 15].

2.2. Semicontinuity and continuity of ϕ

The result of this section is borrowed from [8]. It concerns the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system. Such result is applied sometimes intrinsically in the proofs of the main theorems of the next section.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

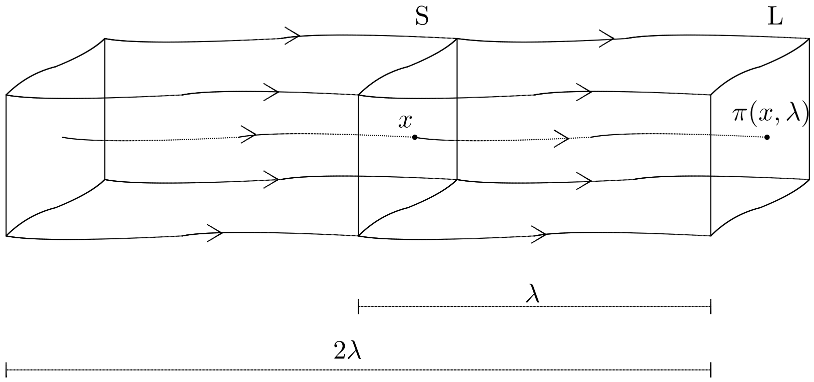

Figure 1: The λ-tube $F(L, [0, 2\lambda])$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*, see Figure 1. Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [8, Theorem 3.8].

Theorem 2.1. *Consider an impulsive semidynamical system $(X, \pi; M, I)$. Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies the condition (TC). Then ϕ is continuous at x if and only if $x \notin M$.*

Remark 2. 1. Suppose the conditions of Theorem 2.1 are true. Although the function $\widetilde{\pi}$ is not continuous, by the continuity of the impulse function $I : M \to I(M)$ and function ϕ, we can obtain the following result: Suppose $x \in X \setminus M$. Given $\varepsilon > 0$, for each

$k = 0, 1, 2, \ldots$ and $t \in [0, \phi(x_k^+)]$, there is a $\delta_k > 0$ such that $\rho(\pi(x_k^+, t), \pi(y_k^+, t)) < \varepsilon$ whenever $\rho(y_k^+, x_k^+) < \delta_k$ (ρ is a metric in X and $x_0^+ = x$). This result is applied in the proofs of the main theorems of the next section.

2.3. Additional definitions

Let us consider a metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in$ X and ratio $\delta > 0$. Given $A \subset X$, let $B(A, \delta) = \{x \in \mathrm{X} : \rho(x, A) < \delta\}$ where $\rho(x, A) = \inf\{\rho(x, y) : y \in A\}$.

Let $C(X)$ be the collection of all compact subsets from X and $B(X)$ be the collection of all bounded subsets from X.

Let A and B be bounded subsets from X. We denote by $\beta(A, B)$ the semi-deviation of A to B, that is, $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$.

Let (X, π; M, I) be an impulsive semidynamical system and $A \subset X$. We define $\widetilde{\pi}^+(A, [0, s]) = \bigcup_{x \in A} \{\widetilde{\pi}(x, t) : 0 \leq t \leq s\}$ for $s > 0$. Also, we define

$$\widetilde{\pi}^+(A) = \bigcup_{x \in A} \widetilde{\pi}^+(x) \quad \text{and} \quad \widetilde{\pi}^+(A, t) = \bigcup_{x \in A} \widetilde{\pi}(x, t)$$

for each $t \geq 0$. If $\widetilde{\pi}^+(A) \subset A$, we say that A is positively $\widetilde{\pi}-$*invariant.*

3. THE MAIN RESULTS

In this section, we present the main results from this paper. We divide this section in four subsections. In the first one, we present some basic properties of limit sets. In the second subsection, we define the concept of dissipativity for impulsive semidynamical systems and we study the center of Levinson for compact dissipative systems. In the third subsection, we study criteria to obtain compact dissipativity. Finally, in the last one, we establish necessary and sufficient conditions to obtain local dissipativity.

Throughout this section we shall consider an impulsive semidynamical system $(X, \pi; M, I)$, where (X, ρ) is a metric space. Moreover, we shall assume the following additional hypotheses:

(H1) No initial point in (X, π) belongs to the impulsive set M and each element of M satisfies the condition (STC), consequently ϕ is continuous on $X \setminus M$ (see Theorem 2.1).

(H2) $M \cap I(M) = \emptyset$.

(H3) For each $x \in X$, the motion $\widetilde{\pi}(x, t)$ is defined for every $t \geq 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$. By following [14], the impulsive systems where the motion $\widetilde{\pi}(x, t)$ is defined for all $t \geq 0$ are the most important and interesting, and, moreover, in many cases we may restrict ourselves to such systems (because of the existence of suitable isomorphisms), due to the paper [10].

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3.1. Limit sets

Definition 3.1. Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$. The limit set of A is represented by $\widetilde{L}^+(A) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \widetilde{\pi}(A, \tau)}$. If $A = \{x\}$, we set $\widetilde{L}^+(x) = \widetilde{L}^+(\{x\})$.

By the Definition 3.1, we have the following straightforward result.

Lemma 3.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$. Then*

$$\widetilde{L}^+(A) = \{y \in X : \text{there exist sequences } \{x_n\}_{n\geq 1} \subset A \text{ and } \{t_n\}_{n\geq 1} \subset \mathbb{R}_+$$

$$\text{such that } t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y\}.$$

Next, we mention an important lemma that will be very useful in the next results. The reader may consult [2] for a proof.

Lemma 3.2. *Given an impulsive semidynamical system $(X,\, \pi;\, M,\, I)$, suppose $w \in X \setminus M$ and $\{z_n\}_{n\geq 1}$ is a sequence in X which converges to the point w. Then, for any $t \geq 0$, there exists a sequence of real numbers $\{\varepsilon_n\}_{n\geq 1}$, with $\varepsilon_n \overset{n\to+\infty}{\longrightarrow} 0$, such that $\widetilde{\pi}(z_n,\, t + \varepsilon_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$.*

In Lemma 3.2, when $\widetilde{\pi}(w,\, t) \neq w_j^+ = I(w_j)$ for every $j = 1, 2, 3, \ldots$, the convergence $\widetilde{\pi}(z_n,\, t + \varepsilon_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$ does not depend on the sequence $\{\varepsilon_n\}_{n\geq 1}$, that is, $\widetilde{\pi}(z_n,\, t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$, whenever $t \neq \sum_{j=0}^{k} \phi(w_j^+)$ for every $k = 0, 1, 2, \ldots$. Recall that $w = w_0^+$. We present this fact in the next lemma whose the proof is in [5, Lemma 3.3].

Lemma 3.3. *Given an impulsive semidynamical system $(X,\, \pi;\, M,\, I)$, suppose $w \in X \setminus M$ and $\{z_n\}_{n\geq 1}$ is a sequence in X which converges to w. Then, for any $t \geq 0$ such that $t \neq \sum_{j=0}^{k} \phi(w_j^+)$, $k = 0, 1, 2, \ldots$, we have $\widetilde{\pi}(z_n,\, t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,\, t)$.*

It is clear that $\widetilde{L}^+(A)$ is closed in X by the Definition 3.1. In general, the invariance of $\widetilde{L}^+(A)$ is not true, see Example 3.1 in [3], for instance. The next lemma concerns the invariance of the limit set and its proof follows the ideas of the proof of [15, Lemma 2.6], by using Lemma 3.2 above.

Lemma 3.4. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$. If $\widetilde{L}^+(A) \cap M = \emptyset$ then $\widetilde{L}^+(A)$ is positively $\widetilde{\pi}-$invariant.*

Analogously to the continuous dynamical systems [7, Lemma 1.3], we have the next lemma to impulsive semidynamical systems.

LEMMA 3.5. *Let $A \subset X$. In the impulsive semidynamical system $(X, \pi; M, I)$ the following conditions are equivalent:*

a)for every sequences $\{x_n\}_{n\geq 1} \subset A$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, the sequence $\{\widetilde{\pi}(x_n, t_n)\}_{n\geq 1}$ is relatively compact;

b)$\widetilde{L}^+(A)$ is non-empty, compact and the following equality

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x,t), \widetilde{L}^+(A)) = 0,$$

holds;

c)there exists a non-empty compact subset $K \subset X$ such that

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x,t), K) = 0.$$

Proof. It is easy to see the implications $(b) \Rightarrow (c) \Rightarrow (a)$. The proof that (a) implies (b) follows similarly as in [7, Lemma 1.3]. ∎

Let $A \subset X$ be given. In the sequel, we give conditions to obtain the relative compactness of $\widetilde{\pi}^+(A)$. But before that, we prove an auxiliary result.

LEMMA 3.6. *Let $A \subset X$ be non-empty and relatively compact. The set $\widetilde{\pi}(A, [0, \ell])$ is relatively compact in X for each $\ell > 0$.*

Proof. Let $\{y_n\}_{n\geq 1} \subset \widetilde{\pi}(A, [0, \ell])$. Then, there are sequences $\{a_n\}_{n\geq 1} \subset A$ and $\{t_n\}_{n\geq 1} \subset [0, \ell]$ such that $y_n = \widetilde{\pi}(a_n, t_n)$. Since A is relatively compact and $\{t_n\}_{n\geq 1}$ is bounded, we may assume without loss of generality that

$$a_n \overset{n\to+\infty}{\longrightarrow} a \quad \text{and} \quad t_n \overset{n\to+\infty}{\longrightarrow} \bar{t},$$

where $a \in \overline{A}$ and $\bar{t} \in [0, \ell]$. We have two cases to consider: when $a \in M$ and when $a \notin M$.

Case 1) $a \notin M$. In this case, we need to consider when $\widetilde{\pi}(a, \bar{t}) \in M$ and when $\widetilde{\pi}(a, \bar{t}) \notin M$.

First, suppose $\widetilde{\pi}(a, \bar{t}) \notin M$. If $0 \leq \bar{t} < \phi(a)$ then $\widetilde{\pi}(a_n, t_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(a, \bar{t})$. Let us suppose $\bar{t} \geq \phi(a)$. Then there is $k \in \mathbb{N}$ such that $\bar{t} = \sum_{j=0}^{k} \phi(a_j^+) + s$ where $0 < s < \phi(a_{k+1}^+)$. Since ϕ is continuous on $X \setminus M$, we can find a sequence $\{s_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$t_n = \sum_{j=0}^{k} \phi((a_n)_j^+) + s_n,$$

$n = 1, 2, \ldots$, and $s_n \overset{n\to+\infty}{\longrightarrow} s$. By the continuity of I we have $(a_n)^+_{k+1} \overset{n\to+\infty}{\longrightarrow} a^+_{k+1}$, $k = 0, 1, 2, \ldots$, and

$$\widetilde{\pi}(a_n, t_n) = \pi((a_n)^+_{k+1}, s_n) \overset{n\to+\infty}{\longrightarrow} \pi(a^+_{k+1}, s) = \widetilde{\pi}(a, \bar{t}).$$

Now, if $\widetilde{\pi}(a, \bar{t}) \in M$, then there is $k \in \mathbb{N}$ such that $\bar{t} = \sum_{j=0}^{k} \phi(a^+_j)$. Here, we need to consider two cases: when $t_n \leq \bar{t}$ for infinitely many n and when $t_n > \bar{t}$ for infinitely many n. We are going to consider without loss of generality the cases: when $t_n \leq \bar{t}$ for each $n \in \mathbb{N}$ and when $t_n > \bar{t}$ for each $n \in \mathbb{N}$.

First, suppose $t_n \leq \bar{t}$ for each $n \in \mathbb{N}$. There exists a sequence $\{s_n\}_{n\geq 1} \subset [0, \phi(a^+_k)]$ such that

$$t_n = s_n \ \ (\text{if } k = 0) \qquad \text{or} \qquad t_n = \sum_{j=0}^{k-1} \phi((a_n)^+_j) + s_n \quad (\text{if } k = 1, 2, \ldots),$$

$n = 1, 2, \ldots$, and $s_n \overset{n\to+\infty}{\longrightarrow} \phi(a^+_k)$. Then

$$\widetilde{\pi}(a_n, t_n) = \pi((a_n)^+_k, s_n) \overset{n\to+\infty}{\longrightarrow} \pi(a^+_k, \phi(a^+_k)) = a_{k+1}.$$

On the other hand, if $t_n > \bar{t}$ for each $n \in \mathbb{N}$, there is a sequence $\{s_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$t_n = \sum_{j=0}^{k} \phi((a_n)^+_j) + s_n,$$

$n = 1, 2, \ldots$, and $s_n \overset{n\to+\infty}{\longrightarrow} 0$. Then

$$\widetilde{\pi}(a_n, t_n) = \pi((a_n)^+_{k+1}, s_n) \overset{n\to+\infty}{\longrightarrow} \pi(a^+_{k+1}, 0) = a^+_{k+1}.$$

Case 2) $a \in M$. Since M satisfies the condition STC (see hypothesis (H1)), there exists a STC-tube $F(L, [0, 2\lambda])$ through a given by a section S. Since the tube is a neighborhood of a, there is $\eta > 0$ such that

$$B(a, \eta) \subset F(L, [0, 2\lambda]).$$

Denote A_1 and A_2 by

$$A_1 = F(L, (\lambda, 2\lambda]) \cap B(a, \eta) \quad \text{and} \quad A_2 = F(L, [0, \lambda]) \cap B(a, \eta).$$

We need to study the cases when $a_n \in A_1$ for infinitely many n and when $a_n \in A_2$ for infinitely many n. Again, we are going to consider without loss of generality the cases: when $\{a_n\}_{n\geq 1} \subset A_1$ and when $\{a_n\}_{n\geq 1} \subset A_2$.

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Let us suppose that $\{a_n\}_{n\geq 1} \subset A_1$. Note that $\phi(a_n) \stackrel{n\to+\infty}{\longrightarrow} 0$ and $I(a) \notin M$ by hypothesis (H2). If $\widetilde{\pi}(I(a),\bar{t}) \notin M$ and $0 \leq \bar{t} < \phi(I(a))$ then

$$\widetilde{\pi}(a_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} \pi(I(a), \bar{t}),$$

but if $\bar{t} \geq \phi(I(a))$, then there is $k \in \mathbb{N}$ such that $\bar{t} = \displaystyle\sum_{j=0}^{k} \phi(I(a)_j^+) + s$, where $0 < s < \phi(I(a)_{k+1}^+)$. Thus, there exists a sequence $\{s_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$t_n = \sum_{j=0}^{k+1} \phi((a_n)_j^+) + s_n,$$

$n = 1, 2, \ldots,$ and $s_n \stackrel{n\to+\infty}{\longrightarrow} s$. By the continuity of I we have $(a_n)_{k+1}^+ \stackrel{n\to+\infty}{\longrightarrow} (I(a))_k^+$, $k = 0, 1, 2, \ldots$, and

$$\widetilde{\pi}(a_n, t_n) = \pi((a_n)_{k+2}^+, s_n) \stackrel{n\to+\infty}{\longrightarrow} \pi(a_{k+1}^+, s).$$

But, if $\widetilde{\pi}(I(a),\bar{t}) \in M$, then there is $k \in \mathbb{N}$ such that $\bar{t} = \displaystyle\sum_{j=0}^{k} \phi(I(a)_j^+)$. As we did before, we need to study the cases when $t_n \leq \bar{t}$ for infinitely many n and when $t_n > \bar{t}$ for infinitely many n. By supposing that $t_n \leq \bar{t}$ for all $n \in \mathbb{N}$, we can find a sequence $\{s_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$t_n = \sum_{j=0}^{k} \phi((a_n)_j^+) + s_n,$$

$n = 1, 2, \ldots,$ and $s_n \stackrel{n\to+\infty}{\longrightarrow} \phi(I(a)_k^+)$. Then

$$\widetilde{\pi}(a_n, t_n) = \pi((a_n)_{k+1}^+, s_n) \stackrel{n\to+\infty}{\longrightarrow} \pi(I(a)_k^+, \phi(I(a)_k^+)) = I(a)_{k+1}.$$

Analogously, if we assume without loss of generality that $t_n > \bar{t}$ for all $n \in \mathbb{N}$, we conclude that

$$\widetilde{\pi}(a_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} I(a)_{k+1}^+.$$

In the last case, we assume $\{a_n\}_{n\geq 1} \subset A_2$. If $\widetilde{\pi}(a,\bar{t}) \notin M$ and $0 \leq \bar{t} < \phi(a)$ then

$$\widetilde{\pi}(a_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} \pi(a, \bar{t}),$$

but if $\bar{t} \geq \phi(a)$ then there is $k \in \mathbb{N}$ such that $\bar{t} = \displaystyle\sum_{j=0}^{k} \phi(a_j^+) + s$, where $0 < s < \phi(a_{k+1}^+)$. Thus

$$\widetilde{\pi}(a_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} \pi(a_{k+1}^+, s).$$

But, if $\widetilde{\pi}(a,\bar{t}) \in M$, there is $k \in \mathbb{N}$ such that $\bar{t} = \sum_{j=0}^{k} \phi(a_j^+)$. We also have two cases to consider: If we assume without loss of generality that $t_n \leq \bar{t}$ for all $n \in \mathbb{N}$, then

$$\widetilde{\pi}(a_n, t_n) \overset{n\to+\infty}{\longrightarrow} \pi(a_k^+, \phi(a_k^+)) = a_{k+1}.$$

If $t_n > \bar{t}$ for all $n \in \mathbb{N}$, then

$$\widetilde{\pi}(a_n, t_n) \overset{n\to+\infty}{\longrightarrow} a_{k+1}^+.$$

Therefore, $\{\widetilde{\pi}(a_n, t_n)\}_{n\geq 1}$ is convergent and the lemma is complete. ▌

Proposition 3.1. *Let $A \subset X$ be non-empty and relatively compact. If $\widetilde{L}^+(A)$ is non-empty, compact and*

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x,t), \widetilde{L}^+(A)) = 0,$$

then $\widetilde{\pi}^+(A)$ is relatively compact.

Proof. By the statement, given $\epsilon > 0$ there exists $\ell = \ell(\epsilon) > 0$ such that

$$A_\epsilon = \bigcup_{t\geq \ell} \widetilde{\pi}(A,t) \subset B(\widetilde{L}^+(A), \epsilon).$$

Then

$$\lambda(\widetilde{\pi}^+(A)) = \lambda\left(\widetilde{\pi}(A,[0,\ell]) \cup A_\epsilon\right) = max\{\lambda(\widetilde{\pi}(A,[0,\ell])), \lambda(A_\epsilon)\},$$

where $\lambda(B)$ denote the measure of non-compactness of Kuratowsky of the set $B \in B(X)$, see [17] for instance. By Lemma 3.6, the set $\widetilde{\pi}(A,[0,\ell])$ is relatively compact. Thus $\lambda(\widetilde{\pi}(A,[0,\ell])) = 0$ and

$$\lambda(\widetilde{\pi}^+(A)) = \lambda(A_\epsilon) \leq 2\epsilon.$$

Hence, $\lambda(\widetilde{\pi}^+(A)) = 0$ and $\widetilde{\pi}^+(A)$ is relatively compact. ▌

3.2. Dissipative impulsive semidynamical systems and the center of Levinson

We start by defining the concept of dissipativity for an impulsive semidynamical system. The study of dissipativity for dynamical systems without impulses may be found in [7].

Definition 3.2. Let $\mathfrak{M}$ be a family of subsets from X. An impulsive semidynamical system $(X,\pi;M,I)$ is called $\mathfrak{M}-$dissipative if there exists a set $K \subset X$ such that for every $\epsilon > 0$ and $A \in \mathfrak{M}$ there exists $\ell(\epsilon, A) > 0$ such that $\widetilde{\pi}(A,t) \subset B(K,\epsilon)$ for all $t \geq \ell(\epsilon, A)$. In this case, the set K is called an attractor of the family $\mathfrak{M}$.

By following [7], the most important applications occur when K is bounded or compact and $\mathfrak{M} = \{\{x\} : x \in X\}$, or $\mathfrak{M} = C(X)$, or $\mathfrak{M} = \{B(x, \delta_x) : x \in X, \ \delta_x > 0\}$ or $\mathfrak{M} = B(X)$. From now, we are going to consider $\mathfrak{M} = \{\{x\} : x \in X\}$ and we shall drop out the symbol $\mathfrak{M}$.

In the sequel, we define some types of dissipativity for impulsive semidynamical systems.

Definition 3.3. An impulsive semidynamical system $(X, \pi; M, I)$ is called:

1. point dissipative if there exists a subset $K \subset X$ with $K \cap M = \emptyset$ such that for every $x \in X$ the limit

$$\lim_{t \to +\infty} \rho(\widetilde{\pi}(x, t), K) = 0 \tag{1}$$

holds;

2. compact dissipative if the equality (1) takes place uniformly with respect to x on the compact subsets from X;

3. locally dissipative if for any point $x \in X$ there exists $\delta_x > 0$ such that the equality (1) takes place uniformly with respect to $y \in B(x, \delta_x)$;

4. bounded dissipative if the equality (1) takes place uniformly with respect to x on every bounded subset from X.

Definition 3.4. If K is compact in the Definition 3.3, the impulsive system $(X, \pi; M, I)$ will be called $k-$dissipative.

The next result says that an impulsive semidynamical system is compact $k-$dissipative whenever it is local $k-$dissipative. The reader may see [7, Lemma 1.5] for a proof.

Lemma 3.7. *Let $(X, \pi; M, I)$ be a local $k-$dissipative system, then it is compact $k-$dissipative.*

From the Definition 3.3 and Lemma 3.7, we have the following implications

bounded dissipativity $\Rightarrow$ local dissipativity $\Rightarrow$ compact dissipativity $\Rightarrow$

$\Rightarrow$ point dissipativity.

We shall show that point dissipativity does not imply in local dissipativity in general. We will present an example to show this fact in the last subsection.

Next, we are going to define the center of Levinson of an impulsive semidynamical system. Before that, we present some auxiliaries results.

Lemma 3.8. *Let $A \subset X$ be a set such that $\widetilde{\pi}^+(A)$ is relatively compact. If $A \cap M = \emptyset$ and $\widetilde{L}^+(A) \subset A$, then*

$$\widetilde{L}^+(A) = \cap\{\widetilde{\pi}(A, t) : t \geq 0\}.$$

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Proof. By the definition of $\widetilde{L}^+(A)$ it is enough to prove that $\widetilde{L}^+(A) \subset \cap\{\widetilde{\pi}(A,t) : t \geq 0\}$. Let $y \in \widetilde{L}^+(A)$ and $t \geq 0$ arbitrary. Thus there exist sequences $\{x_n\}_{n\geq 1}$ in A and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ and

$$\widetilde{\pi}(x_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} y. \tag{2}$$

Take a positive integer n_0 such that $t_n > t$ for all $n > n_0$. Since $\widetilde{\pi}(x_n, t_n - t) \in \widetilde{\pi}^+(A)$ for $n > n_0$ and $\widetilde{\pi}^+(A)$ is relatively compact, we may assume that

$$\widetilde{\pi}(x_n, t_n - t) \stackrel{n\to+\infty}{\longrightarrow} p,$$

for some $p \in \overline{\widetilde{\pi}^+(A)}$. Note that $p \in \widetilde{L}^+(A)$ because $t_n - t \stackrel{n\to+\infty}{\longrightarrow} +\infty$.

Since $\widetilde{L}^+(A) \subset A$ and A is relatively compact, we have $\widetilde{L}^+(A)$ compact. By the condition $A \cap M = \emptyset$, it follows that $\widetilde{L}^+(A)$ is positively $\widetilde{\pi}-$invariant (see Lemma 3.4) and there is $\eta > 0$ such that $B(\widetilde{L}^+(A), \eta) \cap M = \emptyset$. Then

$$\widetilde{\pi}(a, t) = \pi(a, t) \in \widetilde{L}^+(A)$$

for all $a \in \widetilde{L}^+(A)$ and for all $t \geq 0$, that is, $\phi(a) = +\infty$ for all $a \in \widetilde{L}^+(A)$. By Lemma 3.3 we have

$$\widetilde{\pi}(\widetilde{\pi}(x_n, t_n - t), t) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(p, t),$$

that is,

$$\widetilde{\pi}(x_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(p, t). \tag{3}$$

By (2) and (3), we get $y = \widetilde{\pi}(p, t) \in \widetilde{\pi}(A, t)$. Since t is arbitrary, $y \in \widetilde{\pi}(A, t)$ for all $t \geq 0$. The proof is complete. ∎

Let $(X, \pi; M, I)$ be compact $k-$dissipative and K be a non-empty compact set such that $K \cap M = \emptyset$ and it is an attractor for all compact subsets from X. Then for every compact $A \subset X$ the equality

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x, t), K) = 0 \tag{4}$$

holds. By Lemma 3.5, we have $\widetilde{L}^+(A)$ non-empty, compact and

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x, t), \widetilde{L}^+(A)) = 0.$$

Thus by Proposition 3.1 the set $\widetilde{\pi}^+(A)$ is relatively compact. By taking $A = K$ in (4) we get $\widetilde{L}^+(K) \subset K$ and $\widetilde{\pi}^+(K)$ is relatively compact, and consequently by using Lemma 3.8 it follows that $\widetilde{L}^+(K) = \cap\{\widetilde{\pi}(K, t) : t \geq 0\}$. Let us denote J by

$$J := \widetilde{L}^+(K) = \cap\{\widetilde{\pi}(K, t) : t \geq 0\}.$$

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By using the previous information, we can state the following result.

Lemma 3.9. *Let $(X, \pi; M, I)$ be compact $k-$dissipative and K be a non-empty compact set such that $K \cap M = \emptyset$ and it is an attractor for all compact subsets from X. Then, for all $A \in C(X)$ we have:*

a)$\widetilde{L}^+(A)$ *is non-empty, positively $\widetilde{\pi}-$invariant and compact;*
b) $\lim\limits_{t\to+\infty} \sup\limits_{x\in A} \rho(\widetilde{\pi}(x,t), \widetilde{L}^+(A)) = 0;$
c)$\widetilde{L}^+(A) \subset K;$
d)$\widetilde{\pi}^+(A)$ *is relatively compact.*

In the next lemma, we show that the set J is well-defined.

Lemma 3.10. *The set J does not depend on the choice of the set K which attracts all compact subsets of X and $K \cap M = \emptyset$.*

Proof. Let $J(K) = \widetilde{L}^+(K)$ and $J(K_1) = \widetilde{L}^+(K_1)$, where K_1 is another compact set attracting all compact subsets from X such that $K_1 \cap M = \emptyset$. By using Lemma 3.9 with the set K and with the set K_1, separately, we get

$$J(K) = \widetilde{L}^+(K) \subset K, \;\; J(K_1) = \widetilde{L}^+(K_1) \subset K_1, \;\; J(K) \subset K_1 \;\text{ and }\; J(K_1) \subset K. \tag{5}$$

Now, we claim that $J(K) \subset \widetilde{\pi}(J(K), t)$ for all $t \geq 0$. In fact, let $y \in J(K)$ and $t \geq 0$. Then there exist sequences $\{x_n\}_{n\geq 1} \subset K$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and

$$\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y. \tag{6}$$

Note that there is $n_0 > 0$ such that $t_n > t$ for all $n > n_0$. Thus $\widetilde{\pi}(x_n, t_n) = \widetilde{\pi}(\widetilde{\pi}(x_n, t_n - t), t)$ for all $n > n_0$. By the compactness of $\overline{\widetilde{\pi}^+(K)}$ (see Lemma 3.9) we may assume

$$\widetilde{\pi}(x_n, t_n - t) \overset{n\to+\infty}{\longrightarrow} b,$$

where $b \in \widetilde{L}^+(K)$ because $\{x_n\}_{n\geq 1} \subset K$ and $t_n - t \overset{n\to+\infty}{\longrightarrow} +\infty$. By Lemma 3.9, the set $\widetilde{L}^+(K)$ is positively $\widetilde{\pi}-$invariant, compact and $\widetilde{L}^+(K) \cap M = \emptyset$. Then there is $\eta > 0$ such that $B(\widetilde{L}^+(K), \eta) \cap M = \emptyset$ and we can conclude that

$$\widetilde{\pi}(a, s) = \pi(a, s) \in \widetilde{L}^+(K),$$

for all $a \in \widetilde{L}^+(K)$ and for all $s \geq 0$. Thus, by Lemma 3.3 we have

$$\widetilde{\pi}(\widetilde{\pi}(x_n, t_n - t), t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(b, t),$$

that is,

$$\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(b, t). \tag{7}$$

By (6) and (7), it follows that $y = \widetilde{\pi}(b,t) \in \widetilde{\pi}(J(K), t)$ and the assertion follows. By using a similar proof we have $J(K_1) \subset \widetilde{\pi}(J(K_1), t)$ for all $t \geq 0$.

Finally, we can conclude that $J(K) = J(K_1)$. In fact, by equation (5) we have $J(K) \subset K_1$, then

$$\widetilde{\pi}(J(K), t) \subset \widetilde{\pi}(K_1, t),$$

for all $t \geq 0$. As we proved above, we have $J(K) \subset \widetilde{\pi}(J(K), t)$ for each $t \geq 0$. Therefore,

$$J(K) \subset \cap\{\widetilde{\pi}(J(K), t) : t \geq 0\} \subset \cap\{\widetilde{\pi}(K_1, t) : t \geq 0\} = J(K_1).$$

Analogously, we obtain that $J(K_1) \subset J(K)$ and the lemma is proved. ▌

Definition 3.5. The set J defined above by $J = \widetilde{L}^+(K)$ will be called the center of Levinson of the compact $k-$dissipative impulsive semidynamical system $(X, \pi; M, I)$.

Example 3.1. Let us consider the Example 3.17 from [6]. Consider the space $X = \mathbb{R}^2 \times \{0, 1\}$ and the dynamical system

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y, \end{cases}$$

on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$, independently. Now let $M_0 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$, $M_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1/4, z = 1\}$ and consider the impulsive set $M = M_0 \cup M_1$. We define the impulse function I by $I(x, y, 0) = (x, y, 1)$ for $(x, y, 0) \in M_0$ and $I(x, y, 1) = (x, y, 0)$ for $(x, y, 1) \in M_1$. Then $(X, \pi; M, I)$ is compact $k-$dissipative and $J = \{(0, 0, 0), (0, 0, 1)\}$ is its center of Levinson.

The next definitions are established from the non-impulsive case in the sense of [7].

Definition 3.6. Let $(X, \pi; M, I)$ be an impulsive semidynamical system. The stable manifold of set $A \subset X$ in the impulsive system is defined by $\widetilde{W^s}(A) = \{x \in X : \lim\limits_{t\to+\infty} \rho(\widetilde{\pi}(x,t), A) = 0\}$.

Definition 3.7. A set A in $(X, \pi; M, I)$ is said to be:

1. orbitally $\widetilde{\pi}-$stable, if given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $\rho(x, A) < \delta$ implies $\rho(\widetilde{\pi}(x, t), A) < \epsilon$ for all $t \geq 0$;
2. $\widetilde{\pi}-$attracting, if there exists $\gamma > 0$ such that $B(A, \gamma) \subset \widetilde{W^s}(A)$;
3. asymptotically $\widetilde{\pi}-$stable, if it is orbitally $\widetilde{\pi}-$stable and $\widetilde{\pi}-$attracting;

4. globally asymptotically $\widetilde{\pi}-$stable, if it is asymptotically $\widetilde{\pi}-$stable and $\widetilde{W}^s(A) = X$;
5. uniform $\widetilde{\pi}-$attracting, if there is $\gamma > 0$ such that $\lim\limits_{t\to+\infty} \sup\limits_{x\in B(A,\gamma)} \rho(\widetilde{\pi}(x,t), A) = 0$.

The next theorem concerns the compactness, positive invariance, orbital stability and attraction of the center of Levinson.

THEOREM 3.1. *Let $(X, \pi; M, I)$ be compact $k-$dissipative and J be its center of Levinson. Then:*

a)J is a compact positively $\widetilde{\pi}-$invariant set;
b)J is orbitally $\widetilde{\pi}-$stable;
c)J is the attractor of the family of all compacts of X;
d)J is the maximal compact positively $\widetilde{\pi}-$invariant set in $(X, \pi; M, I)$ such that $J \subset \widetilde{\pi}(J, t)$ for each $t \geq 0$.

Proof. Let $J = \widetilde{L}^+(K)$ where K is the non-empty compact attractor of all compact subsets from X such that $K \cap M = \emptyset$.

a) It follows by Lemma 3.9.

b) Suppose the contrary, then there are $\epsilon_0 > 0$, $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$ $(\delta_n > 0)$, $x_n \in B(J, \delta_n)$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n > 0)$ such that

$$\rho(\widetilde{\pi}(x_n, t_n), J) \geq \epsilon_0,$$

for each $n \in \mathbb{N}$. Since $x_n \in B(J, \delta_n)$, $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$ and J is compact, we can assume without loss of generality that the sequence $\{x_n\}_{n\geq1}$ is convergent. Let $\overline{x}$ its limit and consider the compact set $A = \{\overline{x}, x_1, x_2, \ldots\}$. By Lemma 3.9, $\widetilde{L}^+(A)$ is non-empty, compact and $\widetilde{\pi}^+(A)$ is relatively compact.

By the compactness of K, there is $\epsilon > 0$ such that $B(K, \epsilon) \cap M = \emptyset$ and by the compact dissipativity of $(X, \pi; M, I)$, there is $\ell = \ell(\epsilon) > 0$ such that

$$\overline{\widetilde{\pi}(A, t)} \subset B(K, \epsilon) \quad \text{for all} \quad t \geq \ell.$$

Let $B = \overline{\widetilde{\pi}^+(\widetilde{\pi}(A, \ell))}$. Then $B \cap M = \emptyset$ and B is compact because $\widetilde{\pi}^+(A)$ is relatively compact. Let $K' = K \cup B$, then K' is an attractor for the family of all compacts from X such that $K' \cap M = \emptyset$. By Lemma 3.10, $\widetilde{L}^+(K') = \widetilde{L}^+(K) = J$. In particular, $\widetilde{L}^+(B) \subset \widetilde{L}^+(K') = J$. By compactness of B the sequence $\{\widetilde{\pi}(x_n, t_n)\}_{n\geq1}$ can be considered convergent. Let $p = \lim\limits_{n\to+\infty} \widetilde{\pi}(x_n, t_n)$. Then,

$$p = \lim_{n\to+\infty} \widetilde{\pi}(\widetilde{\pi}(x_n, \ell), t_n - \ell)$$

and $p \in \widetilde{L}^+(B) \subset J$. On the other hand, the inequality $\rho(\widetilde{\pi}(x_n, t_n), J) \geq \epsilon_0$, $n \in \mathbb{N}$, implies that $p \notin J$ and it is a contradiction.

c) Let $A \in C(X)$. By Lemma 3.9 we have $\widetilde{\pi}^+(A)$ relatively compact, $\widetilde{L}^+(A)$ compact and $\widetilde{L}^+(A) \cap M = \emptyset$. By the compactness of $\widetilde{L}^+(A)$ there is $\epsilon > 0$ such that $B(\widetilde{L}^+(A), \epsilon) \cap M = \emptyset$. By item b) of Lemma 3.9, there is $\ell = \ell(\epsilon) > 0$ such that

$$\overline{\widetilde{\pi}(A,t)} \subset B(\widetilde{L}^+(A), \epsilon) \tag{8}$$

for all $t \geq \ell$. Let $B = \overline{\widetilde{\pi}^+(\widetilde{\pi}(A, \ell))}$. Then $B \cap M = \emptyset$ and $B \subset \overline{\widetilde{\pi}^+(A)}$ is compact.
Define $K' = K \cup B$, then K' is an attractor for the family of all compacts from X such that $K' \cap M = \emptyset$. Then $\widetilde{L}^+(K') = \widetilde{L}^+(K) = J$ (see Lemma 3.10) and

$$\widetilde{L}^+(B) \subset \widetilde{L}^+(K') = J.$$

Since $\widetilde{\pi}(A, \ell) \subset B$, we have $\widetilde{L}^+(A) \subset \widetilde{L}^+(B)$. In fact, let $z \in \widetilde{L}^+(A)$, then there are sequences $\{a_n\}_{n\geq 1} \subset A$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\widetilde{\pi}(a_n, t_n) \overset{n\to+\infty}{\longrightarrow} z,$$

that is,

$$\widetilde{\pi}(\widetilde{\pi}(a_n, \ell), t_n - \ell) \overset{n\to+\infty}{\longrightarrow} z.$$

Since $\widetilde{\pi}(a_n, \ell) \in \widetilde{\pi}(A, \ell) \subset B$, $n = 1, 2, \ldots$, and $t_n - \ell \overset{n\to+\infty}{\longrightarrow} +\infty$, we have $z \in \widetilde{L}^+(B)$. Therefore,

$$\widetilde{L}^+(A) \subset J$$

and hence by using (8) we have

$$\beta(\widetilde{\pi}(A,t), J) \leq \beta(\widetilde{\pi}(A,t), \widetilde{L}^+(A)) < \epsilon,$$

for all $t \geq \ell$, that is, $\lim\limits_{t\to+\infty} \beta(\widetilde{\pi}(A,t), J) = 0$.

d) Let J_1 be a compact positively $\widetilde{\pi}-$invariant set such that $J_1 \subset \widetilde{\pi}(J_1, t)$ for all $t \geq 0$. By the positive $\widetilde{\pi}-$invariance of J_1 we have $\widetilde{\pi}(J_1, t) \subset J_1$ for all $t \geq 0$. Then $J_1 = \widetilde{\pi}(J_1, t)$ for all $t \geq 0$. The proof that $J_1 \subset J$ is the same proof presented in item (4) of [7, Theorem 1.6].

∎

Let $\{K_\lambda : \lambda \in \Lambda\}$ be the family of all non-empty compact positively $\widetilde{\pi}-$invariant sets that attract all compact subsets from X. Also, we assume that $K_\lambda \cap M = \emptyset$ for all $\lambda \in \Lambda$.

Theorem 3.2. *Let $(X, \pi; M, I)$ be a compact $k-$dissipative semidynamical system with impulses and J be its center of Levinson. Then*

$$J = \cap\{K_\lambda : \lambda \in \Lambda\},$$

that is, J is the least compact positively $\widetilde{\pi}-$invariant set attracting all compacts from X.

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Proof. Note that $J = \widetilde{L}^+(K_\lambda) \subset K_\lambda$, for all $\lambda \in \Lambda$. On the other hand, since J attracts all compacts from X and it is non-empty, positively $\widetilde{\pi}-$invariant with $J \cap M = \emptyset$, then $J \in \{K_\lambda : \lambda \in \Lambda\}$. Hence, the lemma is proved. ▌

THEOREM 3.3. *Let A be a non-empty compact positively $\widetilde{\pi}-$invariant asymptotically $\widetilde{\pi}-$stable set. Suppose $I(\widetilde{W}^s(A)) \subset \widetilde{W}^s(A)$. Then the following statements hold:*

a)the domain of attraction $\widetilde{W}^s(A)$ of the set A is open in X;
b)the equality

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(K,t),A) = 0,$$

takes place for every compact K from $\widetilde{W}^s(A)$.

Proof.

a) Since A is $\widetilde{\pi}-$attracting, there is $\gamma > 0$ such that $B(A,\gamma) \subset \widetilde{W}^s(A)$. It is enough to prove that for each point $x \in \widetilde{W}^s(A) \setminus B(A,\gamma)$, there is $\delta_x > 0$ such that $B(x,\delta_x) \subset \widetilde{W}^s(A)$. Let $x \in \widetilde{W}^s(A) \setminus B(A,\gamma)$ and $0 < \epsilon < \gamma$. By the orbital stability of A, there is $\delta = \delta(\epsilon) > 0$ such that $\widetilde{\pi}(B(A,\delta),[0,+\infty)) \subset B(A,\epsilon)$. By the other side, there is $t_1 = t_1(x,\epsilon) > 0$ with $t_1 \neq \phi(x_k^+)$, $k = 0,1,2,\ldots$, such that

$$\widetilde{\pi}(x,t_1) \in B(A,\delta).$$

Then, there is $\nu > 0$ such that $B(\widetilde{\pi}(x,t_1),\nu) \subset B(A,\delta)$. We have two cases to consider: when $x \in M$ and otherwise.

First, suppose $x \notin M$. By the continuity of π and I, there is $\nu_1 > 0$ such that

$$\widetilde{\pi}(B(x,\nu_1),t_1) \subset B(\widetilde{\pi}(x,t_1),\nu) \subset B(A,\delta).$$

By the orbital stability, $\widetilde{\pi}(B(x,\nu_1),t) \subset B(A,\epsilon)$ for all $t \geq t_1(x,\epsilon)$. Hence, $B(x,\nu_1) \subset \widetilde{W}^s(A)$.

Second, suppose $x \in M$. Since M satisfies the condition STC (by hypothesis (H1)), there exists a STC-tube $F(L,[0,2\lambda])$ through x given by a section S. Since the tube is a neighborhood of x, there is $\eta > 0$ such that

$$B(x,\eta) \subset F(L,[0,2\lambda]).$$

Denote H_1 and H_2 by

$$H_1 = F(L,(\lambda,2\lambda]) \cap B(x,\eta) \quad \text{and} \quad H_2 = F(L,[0,\lambda]) \cap B(x,\eta).$$

By the continuity of π and I, we can obtain $\eta_1 > 0$, $\eta_1 < \eta$, such that

$$\widetilde{\pi}(B(x,\eta_1) \cap H_2, t_1) \subset B(\widetilde{\pi}(x,t_1),\nu) \subset B(A,\delta).$$

On the other hand, note that $I(x) \in \widetilde{W}^s(A) \setminus M$ because $I(\widetilde{W}^s(A)) \subset \widetilde{W}^s(A)$ and we have hypothesis (H2). Thus, there is $t_2 = t_2(x,\epsilon) > 0$ with $t_2 \neq \phi(I(x)_k^+)$, $k = 0, 1, 2, \ldots$, such that

$$\widetilde{\pi}(I(x), t_2) \in B(A,\delta).$$

Then, there is $\nu_2 > 0$ such that $B(\widetilde{\pi}(I(x),t_2),\nu_2) \subset B(A,\delta)$. As we did before, there is $\eta_2 > 0$, $\eta_2 < \eta$, such that

$$\widetilde{\pi}(B(x,\eta_2) \cap H_1, t_2) \subset B(\widetilde{\pi}(I(x),t_2),\nu_2) \subset B(A,\delta).$$

By taking $\eta_3 < \min\{\eta_1,\eta_2\}$, it follows by the orbital stability that

$$\widetilde{\pi}(B(x,\eta_3), t) \subset B(A,\epsilon)$$

for all $t \geq \max\{t_1(x,\epsilon), t_2(x,\epsilon)\}$. Thus, we can conclude that $B(x,\eta_3) \subset \widetilde{W}^s(A)$. Therefore, $\widetilde{W}^s(A)$ is open in X.

b) Since A is $\widetilde{\pi}-$attracting, there is $\gamma > 0$ such that $B(A,\gamma) \subset \widetilde{W}^s(A)$. Let $\epsilon > 0$, $\epsilon < \gamma$, and K be a compact from $\widetilde{W}^s(A)$. For the number $\epsilon > 0$ we choose $\delta = \delta(\epsilon) > 0$ taking in account the stability of A. Since A attracts points from $W^s(A)$, given $x \in K$ there is $t(x,\epsilon) > 0$ such that

$$\widetilde{\pi}(x,t) \in B(A,\delta),$$

for all $t \geq t(x,\epsilon)$. By using the same ideas from item a), we can find $\rho = \rho(x,\epsilon) > 0$ and $T(x,\epsilon)$ such that

$$\widetilde{\pi}(B(x,\rho),t) \subset B(A,\epsilon),$$

for all $t \geq T(x,\epsilon)$. The result follows by the compactness of K, the reader may consult [7, Lemma 1.8] for a proof.

∎

If A is globally asymptotically $\widetilde{\pi}-$stable in Theorem 3.3, we can drop out the condition $I(\widetilde{W}^s(A)) \subset \widetilde{W}^s(A)$ as show the next result.

Corollary 3.1. *Let A be a non-empty compact positively $\widetilde{\pi}-$invariant globally asymptotically $\widetilde{\pi}-$stable set. Then $\widetilde{W}^s(A)$ is open in X and the limit* $\lim\limits_{t\to+\infty} \beta(\widetilde{\pi}(K,t),A) = 0$ *takes place for every compact K from $\widetilde{W}^s(A)$.*

Denote by $\{H_\lambda : \lambda \in \Lambda\}$ the family of all non-empty compact positively $\widetilde{\pi}-$invariant and globally asymptotically $\widetilde{\pi}-$stable sets from X.

THEOREM 3.4. *Let $(X, \pi; M, I)$ be compact $k-$dissipative and J be its center of Levinson. Then $J = \cap\{H_\lambda : \lambda \in \Lambda\}$.*

For a proof of Theorem 3.4, use Theorem 3.2, Corollary 3.1 and the proof of [7, Theorem 1.8].

THEOREM 3.5. *Let $(X, \pi; M, I)$ be compact $k-$dissipative and K be a non-empty compact positive $\widetilde{\pi}-$invariant set from X. Suppose $K \subset \widetilde{\pi}(K, t)$ for all $t \geq 0$ and $K \cap M = \emptyset$. Then the following statements are equivalents:*

a)K is the center of Levinson of $(X, \pi; M, I)$;
b)K is globally asymptotically $\widetilde{\pi}-$stable;
c)K is maximal compact positively $\widetilde{\pi}-$invariant set in X such that $K \subset \widetilde{\pi}(K, t)$ for all $t \geq 0$.

Proof. By using Theorem 3.1 and Theorem 3.4 we get the result. The reader may see [7, Theorem 1.9]. ▌

3.3. Criteria of compact dissipativity

Let $\Omega = \overline{\cup\{\widetilde{L}^+(x) : x \in X\}}$. Let $(X, \pi; M, I)$ be a compact $k-$dissipative impulsive semidynamical system and J be its Levinson's center. By Theorem 3.1, we have $\widetilde{L}^+(x) \subset J$ for all $x \in X$, and therefore $\Omega \subset J$.

Given $A \subset X$, we define $\widetilde{D}^+(A)$ and $\widetilde{J}^+(A)$ by

$$\widetilde{D}^+(A) = \bigcap_{\epsilon>0} \overline{\bigcup\{\widetilde{\pi}(B(A, \epsilon), t) : t \geq 0\}}$$

and

$$\widetilde{J}^+(A) = \bigcap_{\epsilon>0} \bigcap_{t\geq 0} \overline{\bigcup\{\widetilde{\pi}(B(A, \epsilon), \tau) : \tau \geq t\}}.$$

We also define $\widetilde{D}^+(x) = \widetilde{D}^+(\{x\})$ and $\widetilde{J}^+(x) = \widetilde{J}^+(\{x\})$ for each $x \in X$. From the definition of these sets, we have the following straightforward result.

LEMMA 3.11. *The following statements hold:*

a)$y \in \widetilde{J}^+(A)$ if and only if there exist sequences $\{x_n\}_{n\geq 1} \subset X$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $\rho(x_n, A) \overset{n\to+\infty}{\longrightarrow} 0$, $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y$;
b)$y \in \widetilde{D}^+(A)$ if and only if there exist sequences $\{x_n\}_{n\geq 1} \subset X$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $\rho(x_n, A) \overset{n\to+\infty}{\longrightarrow} 0$ and $\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y$;
c)the sets $\widetilde{D}^+(A)$ and $\widetilde{J}^+(A)$ are closed;
d)If $\widetilde{J}^+(A) \cap M = \emptyset$ then $\widetilde{J}^+(A)$ is positively $\widetilde{\pi}-$invariant;

e)If $\widetilde{D}^+(A) \cap M = \emptyset$ then $\widetilde{D}^+(A)$ is positively $\widetilde{\pi}-$invariant.

In [15], Kaul considers an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, where $\Omega \subset X$ is an open set in a metric space X and the continuous impulse function I is defined on the boundary $\partial\Omega$ of Ω in X and takes values in Ω. He proves that $\widetilde{D}^+(x) = \overline{\widetilde{\pi}^+(x)} \cup \widetilde{J}^+(x)$ for each $x \in \Omega$ (see [15, Lemma 2.11]). We also have this result as show the next lemma.

Lemma 3.12. *Given $x \in X$, we have $\widetilde{D}^+(x) = \overline{\widetilde{\pi}^+(x)} \cup \widetilde{J}^+(x)$.*

Proof. It is enough to prove that $\widetilde{D}^+(x) \subset \overline{\widetilde{\pi}^+(x)} \cup \widetilde{J}^+(x)$. Let $y \in \widetilde{D}^+(x)$, then there are sequences $\{w_n\}_{n\geq 1} \subset X$, $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $w_n \overset{n\to+\infty}{\longrightarrow} x$ and

$$\widetilde{\pi}(w_n, t_n) \overset{n\to+\infty}{\longrightarrow} y.$$

If $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ then $y \in \widetilde{J}^+(x)$. In contrary, $\{t_n\}_{n\geq 1}$ admits a convergent subsequence, we say

$$t_{n_\ell} \overset{\ell\to+\infty}{\longrightarrow} \bar{t},$$

where $\bar{t} \geq 0$. Then $\widetilde{\pi}(w_{n_\ell}, t_{n_\ell}) \overset{\ell\to+\infty}{\longrightarrow} y$.

By following the proof of Lemma 3.6, we conclude that the sequence $\{\widetilde{\pi}(w_{n_\ell}, t_{n_\ell})\}_{\ell\geq 1}$ converges to a point in $\overline{\widetilde{\pi}^+(x)}$.

Therefore, $y \in \overline{\widetilde{\pi}^+(x)} \cup \widetilde{J}^+(x)$ and the result is proved. ∎

For any subset $A \subset X$ we have $\cup\{\widetilde{D}^+(a) : a \in A\} \subset \widetilde{D}^+(A)$ and $\cup\{\widetilde{J}^+(a) : a \in A\} \subset \widetilde{J}^+(A)$. The equality does not hold in general as show the next example.

Example 3.2. Consider the impulsive differential system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x_1} = -x_1 \\ \dot{x_2} = x_2 \\ I : M \to N, \end{cases}$$

where $M = \{(3, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}$, $N = \{(2, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}$ and the impulse function assigns to every point $x \in M$ a point $y \in N$ which is on the ray joining x to the origin in $\mathbb{R}^2$. Let $A = \{(x_1, 0) \in \mathbb{R}^2 : 2 < x_1 < 4\}$. For any $a \in A$ we have

$$\widetilde{D}^+(a) = \widetilde{\pi}^+(a) \cup \{(0, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}.$$

Then,

$$\cup\{\widetilde{D}^+(a) : a \in A\} = \{(x_1, 0) \in \mathbb{R}^2 : 0 < x_1 < 4\} \cup \{(0, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}.$$

On the other hand,

$$\widetilde{D}^+(A) = \{(x_1, 0) \in \mathbb{R}^2 : 0 < x_1 \leq 4\} \cup \{(0, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}.$$

Note that $(4, 0) \in \widetilde{D}^+(A)$ because given $\{(x_n, 0)\}_{n\geq 1} \subset \mathbb{R}^2$ and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $x_n \overset{n\to+\infty}{\longrightarrow} 4$ and $t_n \overset{n\to+\infty}{\longrightarrow} 0$, we have

$$\widetilde{\pi}((x_n, 0), t_n) \overset{n\to+\infty}{\longrightarrow} (4, 0).$$

Thus, $\cup\{\widetilde{D}^+(a) : a \in A\} \subsetneq \widetilde{D}^+(A)$.

However, if A is compact we have the equality as show the next proposition.

Proposition 3.2. *If the set $A \subset X$ is compact, then $\widetilde{D}^+(A) = \cup\{\widetilde{D}^+(a) : a \in A\}$ and $\widetilde{J}^+(A) = \cup\{\widetilde{J}^+(a) : a \in A\}$.*

Proof. Let $x \in \widetilde{D}^+(A)$ $(\widetilde{J}^+(A))$. Then there exist sequences $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ $(t_n \overset{n\to+\infty}{\longrightarrow} +\infty)$, $\{x_n\}_{n\geq 1} \subset X$ such that $\rho(x_n, A) \overset{n\to+\infty}{\longrightarrow} 0$ and $\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} x$. Since A is compact, we may assume

$$x_n \overset{n\to+\infty}{\longrightarrow} y \in A.$$

Then $x \in \widetilde{D}^+(y) \subset \cup\{\widetilde{D}^+(a) : a \in A\}$ $(x \in \widetilde{J}^+(y) \subset \cup\{\widetilde{J}^+(a) : a \in A\})$. ▮

Next, we prove that $\Omega \subset \widetilde{J}^+(\Omega)$. But before that, we present an auxiliary result.

Lemma 3.13. *Let $x \notin M$ and $y \in \widetilde{L}^+(x)$, then $\widetilde{J}^+(x) \subset \widetilde{J}^+(y)$.*

Proof. Let $y \in \widetilde{L}^+(x)$ and take $z \in \widetilde{J}^+(x)$, then there are sequences $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$, $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ and $\{x_n\}_{n\geq 1} \subset X$ where $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, $\tau_n \overset{n\to+\infty}{\longrightarrow} +\infty$, $x_n \overset{n\to+\infty}{\longrightarrow} x$, $\widetilde{\pi}(x, t_n) \overset{n\to+\infty}{\longrightarrow} y$ and $\widetilde{\pi}(x_n, \tau_n) \overset{n\to+\infty}{\longrightarrow} z$. We may assume without loss of generality that $\{\tau_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ are increasing sequences and $\tau_n - t_n \geq n$ for all $n \in \mathbb{N}$.

For each fixed t_k, $k = 1, 2, \ldots$, it follows by Lemma 3.2 that there is a sequence $\{\varepsilon_n^k\}_{n\geq 1} \subset \mathbb{R}$, $\varepsilon_n^k \overset{n\to+\infty}{\longrightarrow} 0$ such that

$$\widetilde{\pi}(x_n, t_k + \varepsilon_n^k) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(x, t_k).$$

Thus for each natural $k > 0$ there exists $n_k \geq k$ such that

$$\rho(\widetilde{\pi}(x_{n_k}, t_k + \varepsilon_{n_k}^k), \widetilde{\pi}(x, t_k)) \leq \frac{1}{k}.$$

Note that

$$\rho(y, \widetilde{\pi}(x_{n_k}, t_k + \varepsilon_{n_k}^k)) \leq \rho(y, \widetilde{\pi}(x, t_k)) + \rho(\widetilde{\pi}(x, t_k), \widetilde{\pi}(x_{n_k}, t_k + \varepsilon_{n_k}^k)) \leq \rho(y, \widetilde{\pi}(x, t_k)) + \frac{1}{k}.$$

Then $\widetilde{\pi}(x_{n_k}, t_k + \varepsilon_{n_k}^k) \overset{k\to+\infty}{\longrightarrow} y$ because $\widetilde{\pi}(x, t_k) \overset{k\to+\infty}{\longrightarrow} y$. Since

$$\widetilde{\pi}(x_{n_k}, \tau_{n_k}) = \widetilde{\pi}(\widetilde{\pi}(x_{n_k}, t_k + \varepsilon_{n_k}^k), \tau_{n_k} - t_k - \varepsilon_{n_k}^k) \overset{k\to+\infty}{\longrightarrow} z,$$

$\widetilde{\pi}(x_{n_k}, t_k+\varepsilon_{n_k}^k) \overset{k\to+\infty}{\longrightarrow} y$ and $\tau_{n_k} - t_k - \varepsilon_{n_k}^k \overset{k\to+\infty}{\longrightarrow} +\infty$ (because $\tau_{n_k} - t_k - \varepsilon_{n_k}^k \geq \tau_k - t_k - \varepsilon_{n_k}^k$), we have $z \in \widetilde{J}^+(y)$. ∎

In [7, Corollary 1.4], the author shows that if the dynamical system (X, π) is point $k-$dissipative, $\Omega \neq \emptyset$ and Ω is compact, then $\Omega \subset J^+(\Omega)$. In Proposition 3.3 below, we show that we do not need any hypothesis to get this result. Consequently, we improve the result presented in [7, Corollary 1.4].

Proposition 3.3. *The inclusion* $\Omega \subset \widetilde{J}^+(\Omega)$ *holds.*

Proof. Let us prove that $\cup\{\widetilde{L}^+(x) : x \in X\} \subset \widetilde{J}^+(\Omega)$. Indeed, let $y \in \cup\{\widetilde{L}^+(x) : x \in X\}$, then $y \in \widetilde{L}^+(x)$ for some $x \in X$.

First suppose $x \notin M$. Since $y \in \widetilde{L}^+(x)$ it follows by Lemma 3.13 that $\widetilde{J}^+(x) \subset \widetilde{J}^+(y)$. Since $y \in \Omega$ then $\widetilde{J}^+(y) \subset \widetilde{J}^+(\Omega)$. Thus

$$y \in \widetilde{L}^+(x) \subset \widetilde{J}^+(x) \subset \widetilde{J}^+(y) \subset \widetilde{J}^+(\Omega).$$

Now suppose $x \in M$. Let $\lambda > 0$ such that $\widetilde{\pi}(x, \lambda) = \pi(x, \lambda) := z \notin M$. We note that $\widetilde{L}^+(x) \subset \widetilde{L}^+(z)$ because if $w \in \widetilde{L}^+(x)$, there is a sequence $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n > 0)$ such that $\widetilde{\pi}(x, t_n) \overset{n\to+\infty}{\longrightarrow} w$. Then $\widetilde{\pi}(z, t_n - \lambda) \overset{n\to+\infty}{\longrightarrow} w$ and $w \in \widetilde{L}^+(z)$. Consequently,

$$y \in \widetilde{L}^+(x) \subset \widetilde{L}^+(z) \subset \widetilde{J}^+(z) \subset \widetilde{J}^+(y) \subset \widetilde{J}^+(\Omega),$$

where the inclusion $\widetilde{J}^+(z) \subset \widetilde{J}^+(y)$ follows by Lemma 3.13, since $z \notin M$ and $y \in \widetilde{L}^+(z)$. Since $\widetilde{J}^+(\Omega)$ is closed, the result follows. ∎

Proposition 3.4. *If the impulsive semidynamical system* $(X, \pi; M, I)$ *is point* $k-$*dissipative and* $\widetilde{D}^+(\Omega)$ $(\widetilde{J}^+(\Omega))$ *is compact such that* $\widetilde{D}^+(\Omega) \cap M = \emptyset$ $(\widetilde{J}^+(\Omega) \cap M = \emptyset)$, *then* $\widetilde{D}^+(\Omega) = \widetilde{D}^+(\widetilde{D}^+(\Omega))$ $(\widetilde{J}^+(\Omega) = \widetilde{J}^+(\widetilde{J}^+(\Omega)))$.

Proof. Since $\Omega \subset \widetilde{D}^+(\Omega)$ we have $\widetilde{D}^+(\Omega) \subset \widetilde{D}^+(\widetilde{D}^+(\Omega))$. Let us show the other set inclusion. By the compactness of $\widetilde{D}^+(\Omega)$, given $y \in \widetilde{D}^+(\widetilde{D}^+(\Omega))$ there is $x \in \widetilde{D}^+(\Omega)$ such that $y \in \widetilde{D}^+(x)$, see Proposition 3.2.

By Lemma 3.11 we have $\widetilde{D}^+(\Omega)$ positively $\widetilde{\pi}-$invariant. Since $\widetilde{D}^+(\Omega)$ is compact, positively $\widetilde{\pi}-$invariant and $x \in \widetilde{D}^+(\Omega)$ it follows that $\overline{\widetilde{\pi}^+(x)} \subset \widetilde{D}^+(\Omega)$, $\widetilde{L}^+(x) \neq \emptyset$ and $\widetilde{L}^+(x) \subset \widetilde{D}^+(\Omega)$. Let $z \in \widetilde{L}^+(x)$, then by Lemma 3.13 it follows

$$\widetilde{J}^+(x) \subset \widetilde{J}^+(z).$$

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Now, by the compactness of Ω we have $\widetilde{J}^+(\Omega) = \cup\{\widetilde{J}^+(w) : w \in \Omega\}$. Therefore, $\widetilde{J}^+(z) \subset \widetilde{J}^+(\Omega)$ and

$$y \in \widetilde{D}^+(x) = \overline{\widetilde{\pi}^+(x)} \cup \widetilde{J}^+(x) \subset \widetilde{D}^+(\Omega) \cup \widetilde{J}^+(z) \subset \widetilde{D}^+(\Omega) \cup \widetilde{J}^+(\Omega) \subset \widetilde{D}^+(\Omega).$$

We can prove that $\widetilde{J}^+(\Omega) = \widetilde{J}^+(\widetilde{J}^+(\Omega))$ by using the same ideas. We note that by Proposition 3.3 we have $\Omega \subset \widetilde{J}^+(\Omega)$ and then $\widetilde{J}^+(\Omega) \subset \widetilde{J}^+(\widetilde{J}^+(\Omega))$. ∎

When the metric space X is locally compact, a compact set A is orbitally $\widetilde{\pi}-$stable if and only if $\widetilde{D}^+(A) = A$. This result is an impulsive version of the Theorem of Ura and a proof of it may be found in [9]. Theorem 3.6 deals with this result for compact $k-$dissipative systems where X is not necessarily locally compact.

THEOREM 3.6. *Let $(X, \pi; M, I)$ be compact $k-$dissipative. The compact set $A \subset X$ is orbitally $\widetilde{\pi}-$stable if and only if $\widetilde{D}^+(A) = A$.*

Proof. Let us prove the necessary condition. It is enough to prove that $\widetilde{D}^+(A) \subset A$. Let $z \in \widetilde{D}^+(A)$. Since A is compact it follows by Proposition 3.2 that there is $x \in A$ such that $z \in \widetilde{D}^+(x)$. Then there are sequences $\{x_n\}_{n\geq 1} \subset X$, $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ with $x_n \overset{n\to+\infty}{\longrightarrow} x$ and

$$\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} z.$$

Since A is orbitally $\widetilde{\pi}-$stable, given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \epsilon).$$

Then, for n sufficiently large, we have $\widetilde{\pi}(x_n, t_n) \in B(A, \epsilon)$ and then

$$z \in \overline{B(A, \epsilon)}.$$

Since $\epsilon > 0$ is arbitrary, we have $z \in \bigcap_{\epsilon>0} \overline{B(A, \epsilon)} = \overline{A} = A$. Therefore, $\widetilde{D}^+(A) = A$.

Let us prove the sufficient condition. Suppose A is not orbitally $\widetilde{\pi}-$stable. Then there are $\epsilon_0 > 0$, $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$ $(\delta_n > 0)$, $x_n \in B(A, \delta_n)$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n > 0)$ such that

$$\rho(\widetilde{\pi}(x_n, t_n), A) \geq \epsilon_0, \tag{9}$$

$n \in \mathbb{N}$. We may assume that $x_n \overset{n\to+\infty}{\longrightarrow} \overline{x} \in A$. By Lemma 3.9 the set $\widetilde{\pi}^+(B)$ is relatively compact, where $B = \{\overline{x}, x_1, x_2, \ldots\}$. Consequently, we may assume $\{\widetilde{\pi}(x_n, t_n)\}_{n\geq 1}$ convergent with limit y. Since $x_n \overset{n\to+\infty}{\longrightarrow} \overline{x} \in A$, we have $y \in \widetilde{D}^+(A) = A$. But by (9) we have $y \notin A$ and it is a contradiction. ∎

PROPOSITION 3.5. *If $(X, \pi; M, I)$ is compact $k-$dissipative, then $\widetilde{D}^+(\Omega) = \widetilde{D}^+(\widetilde{D}^+(\Omega))$ $(\widetilde{J}^+(\Omega) = \widetilde{J}^+(\widetilde{J}^+(\Omega)))$.*

Proof. Let J be the Levinson's center of $(X, \pi; M, I)$. By Theorems 3.1 and 3.6 we have

$$\widetilde{D}^+(J) = J.$$

Note that $\widetilde{D}^+(\Omega) \subset \widetilde{D}^+(J) = J$ because $\Omega \subset J$. Since J is compact and $\widetilde{D}^+(\Omega)$ is closed we have $\widetilde{D}^+(\Omega)$ compact. On the other hand, since $\widetilde{D}^+(\Omega) \subset J$ and $J \cap M = \emptyset$ it follows $\widetilde{D}^+(\Omega) \cap M = \emptyset$. The result follows by Proposition 3.4.

To prove the other equality $\widetilde{J}^+(\Omega) = \widetilde{J}^+(\widetilde{J}^+(\Omega))$ it is enough to note that $\widetilde{J}^+(\Omega) \subset \widetilde{D}^+(\Omega) \subset J$. ▌

If the impulsive semidynamical system is compact $k-$dissipative, we can show that its center of Levinson is the set $\widetilde{J}^+(\Omega)$. See Theorem 3.7.

THEOREM 3.7. *If the impulsive semidynamical system $(X, \pi; M, I)$ is compact $k-$dissipative, then $J = \widetilde{J}^+(\Omega)$.*

Proof. Since $\Omega \subset J$ we have $\widetilde{J}^+(\Omega) \subset \widetilde{J}^+(J) \subset \widetilde{D}^+(J) = J$. Let us show that $J \subset \widetilde{J}^+(\Omega)$. In fact, note that $\widetilde{J}^+(\Omega)$ is non-empty ($\Omega \subset \widetilde{J}^+(\Omega)$ by Proposition 3.3), compact (because $\widetilde{J}^+(\Omega)$ is closed and J is compact) and positively $\widetilde{\pi}-$invariant (because $\widetilde{J}^+(\Omega) \subset J$ and $J \cap M = \emptyset$).

Now, we are going to show that $\widetilde{J}^+(\Omega)$ is globally asymptotically $\widetilde{\pi}-$stable. First, let us prove that it is orbitally $\widetilde{\pi}-$stable. Since $\widetilde{J}^+(\Omega) \subset \widetilde{D}^+(\widetilde{J}^+(\Omega))$ we need just to prove the another set inclusion and use Theorem 3.6. Let $z \in \widetilde{D}^+(\widetilde{J}^+(\Omega))$ then $z \in \widetilde{D}^+(y)$ for some $y \in \widetilde{J}^+(\Omega)$. By Lemma 3.12, we have $z \in \overline{\widetilde{\pi}^+(y)}$ or $z \in \widetilde{J}^+(y)$. If $z \in \overline{\widetilde{\pi}^+(y)}$, since $\widetilde{J}^+(\Omega)$ is closed and positively $\widetilde{\pi}-$invariant, it follows that $\overline{\widetilde{\pi}^+(y)} \subset \widetilde{J}^+(\Omega)$, consequently $z \in \widetilde{J}^+(\Omega)$. But if $z \in \widetilde{J}^+(y)$, then by Proposition 3.5 we have $z \in \widetilde{J}^+(\widetilde{J}^+(\Omega)) = \widetilde{J}^+(\Omega)$. Thus $z \in \widetilde{J}^+(\Omega)$ and therefore $\widetilde{D}^+(\widetilde{J}^+(\Omega)) = \widetilde{J}^+(\Omega)$. Hence, $\widetilde{J}^+(\Omega)$ is orbitally $\widetilde{\pi}-$stable by Theorem 3.6.

Now, let us show that $\widetilde{W}^s(\widetilde{J}^+(\Omega)) = X$. Suppose there is $x \in X$ such that $x \notin \widetilde{W}^s(\widetilde{J}^+(\Omega))$. Then there is $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$ there is $t_n > n$ with

$$\rho(\widetilde{\pi}(x, t_n), \widetilde{J}^+(\Omega)) \geq \epsilon_0. \tag{10}$$

Since $\widetilde{\pi}^+(x)$ is relatively compact (see Lemma 3.9), we can assume

$$\widetilde{\pi}(x, t_n) \overset{n \to +\infty}{\longrightarrow} p \in \widetilde{L}^+(x).$$

Then, $p \in \widetilde{L}^+(x) \subset \Omega \subset \widetilde{J}^+(\Omega)$. On the other hand, by (10) we have $p \notin \widetilde{J}^+(\Omega)$ which is a contradiction.

Consequently, $\widetilde{J}^+(\Omega)$ is non-empty, compact, positively $\widetilde{\pi}-$invariant and globally asymptotically $\widetilde{\pi}-$stable. Then by Theorem 3.4 we have $J \subset \widetilde{J}^+(\Omega)$ and the theorem is proved. ▌

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Since $\widetilde{J}^+(\Omega) \subset \widetilde{D}^+(\Omega) \subset J$, we have the following result.

COROLLARY 3.2. *If $(X, \pi; M, I)$ is compact $k-$dissipative, then $J = \widetilde{D}^+(\Omega)$.*

COROLLARY 3.3. *If $(X, \pi; M, I)$ is compact $k-$dissipative, then $J = \Omega$ if and only if Ω is orbitally $\widetilde{\pi}-$stable.*

We have the following result for compact $k-$dissipative impulsive systems.

LEMMA 3.14. *Let $(X, \pi; M, I)$ be compact $k-$dissipative. Given $\varepsilon > 0$ and $x \in X$, there exist $\gamma(x, \varepsilon) > 0$ and $\ell(x, \varepsilon) > 0$ such that*

$$\widetilde{\pi}(B(x, \gamma(x, \varepsilon)), t) \subset B(J, \varepsilon)$$

for all $t \geq \ell(x, \varepsilon)$, where J is the Levinson's center of $(X, \pi; M, I)$.

Proof. Let $\varepsilon > 0$ and $x \in X$ be given. There is $\delta = \delta(\varepsilon) > 0$ such that

$$\widetilde{\pi}(B(J, \delta), [0, +\infty)) \subset B(J, \varepsilon), \tag{11}$$

because J is orbitally $\widetilde{\pi}-$stable. Since $\widetilde{L}^+(x) \subset J$, there is $t_1 = t_1(x, \varepsilon) > 0$, $t_1 \neq \phi(x_k^+)$, $k = 0, 1, 2, \ldots$, such that

$$\widetilde{\pi}(x, t_1) \in B(J, \delta).$$

By the openness of $B(J, \delta)$ there exists $\nu > 0$ such that $B(\widetilde{\pi}(x, t_1), \nu) \subset B(J, \delta)$. We have two cases to consider: when $x \in M$ and when $x \notin M$.

First, suppose $x \notin M$. It follows by the continuity of π and I that there is $\gamma = \gamma(x, \varepsilon) > 0$ such that

$$\widetilde{\pi}(B(x, \gamma), t_1) \subset B(\widetilde{\pi}(x, t_1), \nu) \subset B(J, \delta).$$

By (11), we have $\widetilde{\pi}(B(x, \gamma), t) \subset B(J, \varepsilon)$ for all $t \geq t_1(x, \varepsilon)$. The result is proved in this case.

By the other side, if $x \in M$, there exists a STC-tube $F(L, [0, 2\lambda])$ through x given by a section S, because M satisfies the condition STC (by hypothesis (H1)). Since the tube is a neighborhood of x, there is $\eta > 0$ such that

$$B(x, \eta) \subset F(L, [0, 2\lambda]).$$

Denote H_1 and H_2 by

$$H_1 = F(L, (\lambda, 2\lambda]) \cap B(x, \eta) \quad \text{and} \quad H_2 = F(L, [0, \lambda]) \cap B(x, \eta).$$

Then, there exists $\eta_1 > 0$, $\eta_1 < \eta$, such that $\widetilde{\pi}(B(x, \eta_1) \cap H_2, t_1) \subset B(\widetilde{\pi}(x, t_1), \nu) \subset B(J, \delta)$. Again, by (11) we have $\widetilde{\pi}(B(x, \eta_1) \cap H_2, t) \subset B(J, \varepsilon)$ for all $t \geq t_1(x, \varepsilon)$.

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Since $I(x) \notin M$ because $M \cap I(M) = \emptyset$ (see hypothesis (H2)) and $\widetilde{L}^+(I(x)) \subset J$, there is $t_2 = t_2(x, \varepsilon) > 0$, $t_2 \neq \phi(I(x)_k^+)$, $k = 0, 1, 2, \ldots$, such that $\widetilde{\pi}(I(x), t_2) \in B(J, \delta)$. Since $B(J, \delta)$ is open there is $\nu_1 > 0$ such that $B(\widetilde{\pi}(I(x), t_2), \nu_1) \subset B(J, \delta)$, and by the continuity of π and I there is $0 < \eta_2 < \eta$ such that

$$\widetilde{\pi}(B(x, \eta_2) \cap H_1, t_2) \subset B(\widetilde{\pi}(I(x), t_2), \nu_1) \subset B(J, \delta).$$

Hence, by (11) we conclude that $\widetilde{\pi}(B(x, \eta_2) \cap H_1, t) \subset B(J, \varepsilon)$ for all $t \geq t_2(x, \varepsilon)$.

By taking $\gamma = \gamma(x, \varepsilon) < \min\{\eta_1, \eta_2\}$, we have

$$\widetilde{\pi}(B(x, \gamma), t) \subset B(J, \varepsilon),$$

for all $t \geq \max\{t_1(x, \varepsilon), t_2(x, \varepsilon)\}$. Therefore, the proof is complete. ∎

The next four results deal with necessary and sufficient conditions to obtain compact dissipativity.

Theorem 3.8. *For the impulsive semidynamical system $(X, \pi; M, I)$ to be compact $k-$dissipative, it is necessary and sufficient that there exists a non-empty compact set $K \subset X$, $K \cap M = \emptyset$, satisfying the condition: for every $\varepsilon > 0$ and $x \in X$, there exist $\gamma(x, \varepsilon) > 0$ and $\ell(x, \varepsilon) > 0$ such that*

$$\widetilde{\pi}(B(x, \gamma(x, \varepsilon)), t) \subset B(K, \varepsilon)$$

for all $t \geq \ell(x, \varepsilon)$.

Proof. The necessary condition follows by Lemma 3.14 and the sufficient condition follows by the proof of [7, Theorem 1.12]. ∎

Theorem 3.9. *Let $(X, \pi; M, I)$ be point $k-$dissipative. For the impulsive semidynamical system $(X, \pi; M, I)$ to be compact $k-$dissipative, it is necessary and sufficient that there exists a non-empty compact set A possessing the following properties:*

a)$A \cap M = \emptyset$;
b)$\Omega \subset A$;
c)A is orbital $\widetilde{\pi}-$stable.

In this case $J \subset A$ where J is the center of Levinson of $(X, \pi; M, I)$.

Proof. Take $A = J$ to prove the necessary condition. Let us prove the sufficient condition. By Theorem 3.8 we need to prove that for every $\varepsilon > 0$ and $x \in X$, there exist $\gamma(x, \varepsilon) > 0$ and $\ell(x, \varepsilon) > 0$ such that

$$\widetilde{\pi}(B(x, \gamma(x, \varepsilon)), t) \subset B(A, \varepsilon)$$

for all $t \geq \ell(x,\varepsilon)$. But its proof is the same proof of Lemma 3.14 by replacing J by A, because A is orbitally $\widetilde{\pi}-$stable and $\widetilde{L}^+(x) \subset \Omega \subset A$.

By using Corollary 3.2, it follows from $\Omega \subset A$ that $J = \widetilde{D}^+(\Omega) \subset \widetilde{D}^+(A) = A$. ∎

As consequence of Corollary 3.2 and Theorem 3.9, we have the following straightforward result.

THEOREM 3.10. *Let $(X,\pi;M,I)$ be point $k-$dissipative. For the impulsive semidynamical system $(X,\pi;M,I)$ to be compact $k-$dissipative, it is necessary and sufficient that the set $\widetilde{D}^+(\Omega)$ $(\widetilde{J}^+(\Omega))$ be compact, orbitally $\widetilde{\pi}-$stable and $\widetilde{D}^+(\Omega) \cap M = \emptyset$ $(\widetilde{J}^+(\Omega) \cap M = \emptyset)$. In this case $J = \widetilde{D}^+(\Omega)$ $(J = \widetilde{J}^+(\Omega))$ where J is the center of Levinson of $(X,\pi;M,I)$.*

THEOREM 3.11. *Let $(X,\pi;M,I)$ be point $k-$dissipative. For the impulsive semidynamical system $(X,\pi;M,I)$ to be compact $k-$dissipative, it is necessary and sufficient that $\widetilde{D}^+(\Omega) \cap M = \emptyset$ and $\widetilde{\pi}^+(A)$ be relatively compact for any compact $A \subset X$.*

Proof. The proof of the necessary condition follows by Lemma 3.9 and Theorem 3.10. Let us prove the sufficient condition. At first, let us show that $\widetilde{D}^+(\Omega)$ is non-empty and compact. Indeed, since $(X,\pi;M,I)$ is point $k-$dissipative it follows that $\Omega \neq \emptyset$. Thus $\widetilde{D}^+(\Omega)$ is non-empty. Now, let $\{y_n\}_{n\geq 1} \subset \widetilde{D}^+(\Omega)$ and $\epsilon_n \overset{n\to+\infty}{\longrightarrow} 0$ $(\epsilon_n > 0,\ n = 1,2,\ldots)$, then there exist $x_n \in \Omega$, $\overline{x}_n \in B(x_n,\epsilon_n)$ and $t_n > 0$ such that $y_n \in \widetilde{D}^+(x_n)$ and

$$\rho(y_n, \widetilde{\pi}(\overline{x}_n, t_n)) < \epsilon_n, \tag{12}$$

$n = 1,2,\ldots$. Since $\{x_n\}_{n\geq 1}$ is relatively compact because Ω is compact (since the system is point $k-$dissipative), we have $\{\overline{x}_n\}_{n\geq 1}$ relatively compact. By the statement, it follows that $\{\widetilde{\pi}(\overline{x}_n,t_n)\}_{n\geq 1}$ is relatively compact. By (12) we have $\{y_n\}_{n\geq 1}$ relatively compact. Hence, $\widetilde{D}^+(\Omega)$ is compact.

We claim that $\widetilde{D}^+(\Omega)$ is orbitally $\widetilde{\pi}-$stable. Suppose the contrary, then there are $\epsilon_0 > 0$, $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$ $(\delta_n > 0)$, $x_n \in B(\widetilde{D}^+(\Omega), \delta_n)$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n > 0)$ such that

$$\rho(\widetilde{\pi}(x_n,t_n), \widetilde{D}^+(\Omega)) \geq \epsilon_0, \tag{13}$$

$n \in \mathbb{N}$. We can assume $x_n \overset{n\to+\infty}{\longrightarrow} \overline{x} \in \widetilde{D}^+(\Omega)$ since $\widetilde{D}^+(\Omega)$ is compact. By the statement, the set $\widetilde{\pi}^+(B)$ is relatively compact, where $B = \{\overline{x}, x_1, x_2, \ldots\}$. Consequently, we may assume $\{\widetilde{\pi}(x_n,t_n)\}_{n\geq 1}$ convergent with limit y. By (13) we get

$$y \notin \widetilde{D}^+(\Omega)). \tag{14}$$

On the other hand, since $x_n \overset{n\to+\infty}{\longrightarrow} \overline{x} \in \widetilde{D}^+(\Omega)$ and $\widetilde{D}^+(\widetilde{D}^+(\Omega)) = \widetilde{D}^+(\Omega)$ (see Proposition 3.4), we have $y \in \widetilde{D}^+(\widetilde{D}^+(\Omega)) = \widetilde{D}^+(\Omega)$ which contradicts (14). Thus $\widetilde{D}^+(\Omega)$ is orbitally $\widetilde{\pi}-$stable. Hence, by Theorem 3.10 we have the result. ∎

Example 3.3. Consider the impulsive differential system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = 0 \\ I : M \to N, \end{cases}$$

where $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 9\}$, $N = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ and the impulse function $I : M \to N$ assigns to every point $x \in M$ a point $y \in N$ which is on the ray joining x to the origin in $\mathbb{R}^2$. In [3, Example 3.1], we showed that $\widetilde{L}^+((1,2)) = [1,3] \times \{0\}$. Then $\Omega \cap M \neq \emptyset$ because $\widetilde{L}^+((1,2)) \cap M \neq \emptyset$. Hence, the impulsive semidynamical systems is not point $k-$dissipative.

Example 3.4. Given $c > 0$, we define $\varphi_c : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_c(t) = \begin{cases} 0, & t \leq -1 - \dfrac{1}{c} \\ \exp\left(\dfrac{1}{t^2 - (1 + c^{-1})^2}\right), & -1 - \dfrac{1}{c} < t < 1 + \dfrac{1}{c} \\ 0, & t \geq 1 + \dfrac{1}{c}. \end{cases}$$

Also, define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \begin{cases} 0, & t \leq -1 \\ \exp\left(\dfrac{1}{t^2 - 1}\right), & -1 < t < 1 \\ 0, & t \geq 1. \end{cases}$$

Set $X = \{\psi, \varphi_c : c > 0\} \subset C(\mathbb{R}, \mathbb{R})$ and define $\pi : X \times \mathbb{R}_+ \to X$ by

$$\pi(\varphi_c, t) = \varphi_{c+t} \quad \text{if} \quad c > 0,$$

and

$$\pi(\psi, t) = \psi,$$

for all $t \geq 0$. Then (X, π) is a semidynamical system. By defining $M = \{\varphi_{c_1}, ..., \varphi_{c_k}\}$ with $0 < c_1 < \ldots < c_k$ let $I : M \to X$ be the impulse function given by $I(\varphi_{c_j}) = \varphi_{c_j + \alpha}$, $j = 1, 2, \ldots, k$, where $\alpha > 0$ is such that $I(M) \cap M = \emptyset$. Hence, we have the impulsive semidynamical system $(X, \pi; M, I)$ associated to (X, π).

Since $\lim\limits_{c \to +\infty} \varphi_c(t) = \psi(t)$ for all $t \in \mathbb{R}$, and $I(\varphi_{c_j}) = \varphi_{c_j + \alpha}$, $j = 1, 2, \ldots, k$, we have the following properties:

1) $\widetilde{L}^+(\phi) = \{\psi\}$ for all $\phi \in X$;
2) $(X, \pi; M, I)$ is point $k-$dissipative (take $K = \{\psi\}$);

3) $\Omega = \{\psi\}$ and $\widetilde{D}^+(\Omega) = \Omega$ is orbitally $\widetilde{\pi}-$stable.

Therefore, by Theorem 3.10, $(X, \pi; M, I)$ is compact $k-$dissipative.

3.4. Local dissipative impulsive semidynamical systems

This subsection deals with local $k-$dissipative impulsive semidynamical systems. The first result gives conditions for a compact $k-$dissipative impulsive system be local $k-$dissipative.

THEOREM 3.12. *For the compact $k-$dissipative impulsive semidynamical system $(X, \pi; M, I)$ to be locally $k-$dissipative, it is necessary and sufficient that for every point $x \in X$ there exists $\delta = \delta(x) > 0$ such that*

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta), t), J) = 0,$$

where J is the center of Levinson of $(X, \pi; M, I)$.

Proof. It is enough to prove the necessary condition. By the local dissipativity, there is a non-empty compact set $K \subset X$, $K \cap M = \emptyset$, such that for every point $x \in X$ there exists $\delta_x > 0$ such that

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta_x), t), K) = 0. \tag{15}$$

By Lemma 3.5, $\omega_x := \widetilde{L}^+(B(x, \delta_x))$ is non-empty, compact and

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta_x), t), \omega_x) = 0. \tag{16}$$

If we prove that $\omega_x = \widetilde{L}^+(\omega_x)$ then the result follows. Indeed, first let us show that ω_x is positively $\widetilde{\pi}-$invariant. Since K is compact and $K \cap M = \emptyset$, there is $\epsilon > 0$ such that $B(K, \epsilon) \cap M = \emptyset$. Thus by using (15) we have

$$\omega_x \subset B(K, \epsilon) \qquad \text{and} \qquad B(K, \epsilon) \cap M = \emptyset. \tag{17}$$

Then from Lemma 3.4 we have ω_x positively $\widetilde{\pi}-$invariant, that is, $\widetilde{\pi}(\omega_x, t) \subset \omega_x$ for all $t \geq 0$. Now, we claim that $\omega_x \subset \widetilde{\pi}(\omega_x, t)$ for all $t \geq 0$. In fact, let $z \in \omega_x$ and $t \geq 0$. Then there exist sequences $\{w_n\}_{n\geq1} \subset B(x, \delta_x)$ and $\{t_n\}_{n\geq1} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and

$$\widetilde{\pi}(w_n, t_n) \overset{n\to+\infty}{\longrightarrow} z. \tag{18}$$

Note that there is $n_0 > 0$ such that $t_n > t$ for all $n > n_0$. Thus $\widetilde{\pi}(w_n, t_n) = \widetilde{\pi}(\widetilde{\pi}(w_n, t_n - t), t)$ for all $n > n_0$. By Lemma 3.5 we may assume

$$\widetilde{\pi}(w_n, t_n - t) \overset{n\to+\infty}{\longrightarrow} b,$$

where $b \in \widetilde{L}^+(B(x,\delta_x)) = \omega_x$ because $\{w_n\}_{n\geq 1} \subset B(x,\delta_x)$ and $t_n - t \stackrel{n\to+\infty}{\longrightarrow} +\infty$. Since ω_x is positively $\widetilde{\pi}-$invariant, it follows by equation (17) that

$$\widetilde{\pi}(a,s) = \pi(a,s) \in \omega_x,$$

for all $a \in \omega_x$ and for all $s \geq 0$. Thus, by Lemma 3.3 we have

$$\widetilde{\pi}(\widetilde{\pi}(w_n, t_n - t), t) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(b,t),$$

that is,

$$\widetilde{\pi}(w_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(b,t). \tag{19}$$

By (18) and (19), it follows that $z = \widetilde{\pi}(b,t) \in \widetilde{\pi}(\omega_x, t)$ and the assertion follows. Therefore, $\omega_x = \widetilde{\pi}(\omega_x, t)$ for all $t \geq 0$, consequently $\widetilde{L}^+(\omega_x) = \omega_x$. Since J is an attractor of compacts sets, we have

$$\omega_x = \widetilde{L}^+(\omega_x) \subset J.$$

From the last inclusion and (16) it follows

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x,\delta_x),t), J) = 0.$$

The result is proved. ∎

Next, we present an auxiliary result whose proof is in [7, Lemma 1.9].

Lemma 3.15. *Let $K \subset X$ be a non-empty compact set, $x_i \in X$ and $\delta_i > 0$, $i = 1,\dots,m$. If $K \subset \cup\{B(x_i,\delta_i) : i = 1,\dots,m\}$, then there is $\gamma > 0$ such that*

$$B(K,\gamma) \subset \cup\{B(x_i,\delta_i) : i = 1,\dots,m\}.$$

Theorem 3.13. *Let $(X,\pi;M,I)$ be compact $k-$dissipative. Then $(X,\pi;M,I)$ is locally $k-$dissipative if and only if its Levinson's center J is uniformly $\widetilde{\pi}-$attracting.*

Proof. First, let us prove the necessary condition. Let $(X,\pi;M,I)$ be locally $k-$dissipative and let J its center of Levinson. Given $x \in J$, by Theorem 3.12, there exists $\delta_x > 0$ such that

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x,\delta_x),t), J) = 0.$$

Since $\{B(x,\delta_x) : x \in J\}$ is an open covering of J, from its compactness we can extract a finite sub-covering $\{B(x_i,\delta_{x_i}) : i = 1,...,m\}$. According to Lemma 3.15, there is $\gamma > 0$ such that

$$B(J,\gamma) \subset \bigcup_{i=1}^{m} B(x_i,\delta_{x_i}).$$

Then,

$$\lim_{t\to+\infty}\beta(\widetilde{\pi}(B(J,\gamma),t),J)=0$$

holds and J is uniformly $\widetilde{\pi}-$attracting.

Now, we prove the sufficient condition. Suppose that the center of Levinson J is uniformly $\widetilde{\pi}-$attracting set, that is, there is $\gamma>0$ such that

$$\lim_{t\to+\infty}\beta(\widetilde{\pi}(B(J,\gamma),t),J)=0. \tag{20}$$

Given $x\in X$, there is $\ell_1=\ell_1(x)>0$, $\ell_1\neq\phi(x_k^+)$, $k=0,1,2,\ldots$, such that $\rho(\widetilde{\pi}(x,t),J)<\gamma$, for all $t\geq\ell_1$. Choose $\nu>0$ such that $B(\widetilde{\pi}(x,\ell_1),\nu)\subset B(J,\gamma)$. By equation (20), given $\epsilon>0$ there is $\ell_2=\ell_2(\epsilon)>0$ such that

$$\rho(\widetilde{\pi}(y,t),J)<\epsilon, \tag{21}$$

for all $t\geq\ell_2$ and $y\in B(J,\gamma)$. We have two cases to consider: when $x\in M$ and otherwise.

At first, let us consider the case when $x\notin M$. Then, we can find $\delta_x>0$ such that

$$\widetilde{\pi}(B(x,\delta_x),\ell_1)\subset B(\widetilde{\pi}(x,\ell_1),\nu)\subset B(J,\gamma).$$

By virtue of (21),

$$\widetilde{\pi}(B(x,\delta_x),t+\ell_1)\subset B(J,\epsilon),$$

for all $t\geq\ell_2$. By taking $\ell(x,\epsilon)=\ell_1+\ell_2$, we have

$$\widetilde{\pi}(B(x,\delta_x),t)\subset B(J,\epsilon),$$

for all $t\geq\ell(x,\epsilon)$. In this case, the result follows by Theorem 3.12.

Now, suppose $x\in M$. Since M satisfies the condition STC (see hypothesis (H1)), there is a STC-tube $F(L,[0,2\lambda])$ through x given by a section S. By the properties of a tube, there is $\eta>0$ such that $B(x,\eta)\subset F(L,[0,2\lambda])$. Denote H_1 and H_2 by

$$H_1=F(L,(\lambda,2\lambda])\cap B(x,\eta)\quad\text{and}\quad H_2=F(L,[0,\lambda])\cap B(x,\eta).$$

By the continuity of π and I, there is $\eta_1>0$, $\eta_1<\eta$, such that $\widetilde{\pi}(z,\ell_1)\in B(\widetilde{\pi}(x,\ell_1),\nu)\subset B(J,\gamma)$ for all $z\in B(x,\eta_1)\cap H_2$. Thus, $\widetilde{\pi}(B(x,\eta_1)\cap H_2,t)\subset B(J,\epsilon)$ for all $t\geq\ell_1+\ell_2$. Also, there is $\ell_3=\ell_3(x)>0$, $\ell_3\neq\phi(I(x)_k^+)$, $k=0,1,2,\ldots$, such that $\rho(\widetilde{\pi}(I(x),t),J)<\gamma$, for all $t\geq\ell_3$. Set $\nu_2>0$ such that $B(\widetilde{\pi}(I(x),\ell_3),\nu_2)\subset B(J,\gamma)$. Thus, there is $\eta_2>0$, $\eta_2<\eta$, such that $\widetilde{\pi}(z,t)\in B(J,\epsilon)$ for all $z\in B(x,\eta_2)\cap H_1$ and $t\geq\ell_2+\ell_3$. Taking $\eta_3<\min\{\eta_1,\eta_2\}$, we have

$$\widetilde{\pi}(B(x,\eta_3),t)\subset B(J,\epsilon)$$

for all $t\geq\max\{\ell_1+\ell_2,\ell_2+\ell_3\}$. In this case, the result also follows by Theorem 3.12. ∎

Next, we present a theorem which give conditions for a point $k-$dissipative system be local $k-$dissipative. Before that, we prove a auxiliary result.

Lemma 3.16. *Let $A \subset X$ be a non-empty compact set in $(X, \pi; M, I)$. If A is uniformly $\widetilde{\pi}-$attracting then it is orbitally $\widetilde{\pi}-$stable.*

Proof. Suppose the contrary. Then, there are $\epsilon_0 > 0$, $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$ $(\delta_n > 0)$, $x_n \in B(A, \delta_n)$ and $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n > 0)$ such that

$$\rho(\widetilde{\pi}(x_n, t_n), A) \geq \epsilon_0, \tag{22}$$

$n = 1, 2, 3, \ldots$. Since A is uniformly $\widetilde{\pi}-$attracting, there is $\gamma > 0$ such that

$$\lim_{t\to+\infty} \sup_{x\in B(A,\gamma)} \rho(\widetilde{\pi}(x, t), A) = 0.$$

For the number $\epsilon_0 > 0$, there exists a positive number $\ell = \ell(\epsilon_0) > 0$ such that

$$\rho(\widetilde{\pi}(x, t), A) < \frac{\epsilon_0}{2} \tag{23}$$

for all $x \in B(A, \gamma)$ and $t \geq \ell$.

On the other hand, since $x_n \in B(A, \delta_n)$ and $\delta_n \overset{n\to+\infty}{\longrightarrow} 0$, the sequence $\{x_n\}_{n\geq 1}$ can be considered convergent. Let $x_0 = \lim\limits_{n\to+\infty} x_n$. Then $x_0 \in A$ and $t_n \geq \ell$ for sufficiently large n. By (23) we have

$$\rho(\widetilde{\pi}(x_n, t_n), A) < \frac{\epsilon_0}{2},$$

for a sufficiently large n, which contradicts (22). ∎

Theorem 3.14. *For a point $k-$dissipative impulsive semidynamical system $(X, \pi; M, I)$ to be local $k-$dissipative, it is necessary and sufficient that the following three conditions hold:*

a) $\widetilde{D}^+(\Omega) \cap M = \emptyset$;

b) $\widetilde{D}^+(\Omega)$ *is compact;*

c) $\widetilde{D}^+(\Omega)$ *is uniformly* $\widetilde{\pi}-$*attracting.*

Proof. Suppose conditions a), b) and c) hold. By Lemma 3.16, $\widetilde{D}^+(\Omega)$ is orbitally $\widetilde{\pi}-$stable. By Theorem 3.10, the system $(X, \pi; M, I)$ is compact $k-$dissipative and $\widetilde{D}^+(\Omega) = J$. By Theorem 3.13 the result follows.

Now, suppose the impulsive system is local $k-$dissipative. By Lemma 3.7, $(X, \pi; M, I)$ is compact $k-$dissipative. Then, by Theorem 3.10, $J = \widetilde{D}^+(\Omega)$ is compact, orbitally $\widetilde{\pi}-$stable and $\widetilde{D}^+(\Omega) \cap M = \emptyset$. By Theorem 3.13, $\widetilde{D}^+(\Omega)$ is uniformly $\widetilde{\pi}-$attracting. ∎

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Definition 3.8. The impulsive semidynamical system $(X, \pi; M, I)$ is called local asymptotically $\widetilde{\pi}-$condensing, if for every point $x \in X$ there are $\delta_x > 0$ and a non-empty compact $K_x \subset X$ with $K_x \cap M = \emptyset$ such that

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta_x), t), K_x) = 0.$$

Theorem 3.15. *Let $(X, \pi; M, I)$ be point $k-$dissipative. For $(X, \pi; M, I)$ be local $k-$dissipative it is necessary and sufficient that $(X, \pi; M, I)$ be locally asymptotically $\widetilde{\pi}-$condensing and $\widetilde{D}^+(\Omega) \cap M = \emptyset$.*

Proof. The second part of the necessary condition, that is, $\widetilde{D}^+(\Omega) \cap M = \emptyset$ follows by Theorem 3.14.

Let us prove the sufficient condition. First we prove that $(X, \pi; M, I)$ is compact $k-$dissipative. In fact, let $A \in C(X)$. Given $x \in A$, there are $\delta_x > 0$ and a non-empty compact set K_x with $K_x \cap M = \emptyset$ such that

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta_x), t), K_x) = 0.$$

By the compactness of A, the open covering $\{B(x, \delta_x) : x \in A\}$ admits a finite sub-covering $\{B(x_i, \delta_{x_i}) : i = 1, ..., m\}$. Let $K = K_{x_1} \cup ... \cup K_{x_m}$. Then K is compact and

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x, t), K) = 0.$$

By Lemma 3.5, $\widetilde{L}^+(A)$ is non-empty, compact and

$$\lim_{t\to+\infty} \sup_{x\in A} \rho(\widetilde{\pi}(x, t), \widetilde{L}^+(A)) = 0.$$

By Proposition 3.1, $\widetilde{\pi}^+(A)$ is relatively compact. Hence, by Theorem 3.11 the system $(X, \pi; M, I)$ is compact $k-$dissipative.

Now, given $x \in X$ there are $\delta_x > 0$ and $K_x \in C(X)$, $K_x \cap M = \emptyset$, such that $\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta_x), t), K_x) = 0$. By Lemma 3.5, $\widetilde{L}^+(B(x, \delta_x))$ is non-empty, compact and

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x, \delta_x), t), \widetilde{L}^+(B(x, \delta_x))) = 0. \tag{24}$$

Let $\omega_x = \widetilde{L}^+(B(x, \delta_x))$. Since $\omega_x \subset K_x$ and $K_x \cap M = \emptyset$, it follows by Lemma 3.4 that ω_x is positively $\widetilde{\pi}-$invariant. By the proof of Theorem 3.12, we also have that $\omega_x \subset \widetilde{\pi}(\omega_x, t)$ for all $t \geq 0$. Then $\omega_x = \widetilde{L}^+(\omega_x)$ and since J is the attractor of compact sets, we have

$$\omega_x = \widetilde{L}^+(\omega_x) \subset J.$$

Hence, by using equation (24) we have

$$\lim_{t\to+\infty} \beta(\widetilde{\pi}(B(x,\delta_x),t),J) = 0$$

and by Theorem 3.12 the result is proved. ∎

Now, we present an example of an impulsive semidynamical system which is compact $k-$dissipative but it is not local $k-$dissipative. We consider the Example 1.8 presented in [7] with impulsive action.

EXAMPLE 3.5. Consider a linear differential equation $\dot{x} = Ax$ in the Hilbert space $H = L_2[0,1]$, with the continuous operator $A : L_2[0,1] \to L_2[0,1]$ defined by

$$(A\varphi)(\tau) = -\tau\varphi(\tau),$$

for all $\tau \in [0,1]$ and $\varphi \in L_2[0,1]$. If $U(t)$ represents the Cauchy operator, we have $(U(t)\varphi)(\tau) = e^{-\tau t}\varphi(\tau)$, for all $t \in \mathbb{R}$ and $\varphi \in L_2[0,1]$. Thus, the dynamical system generated by $\dot{x} = Ax$ is given by $(L_2[0,1],\pi)$ where

$$\pi(\varphi,t) = U(t)\varphi,$$

for all $\varphi \in L_2[0,1]$ and $t \in \mathbb{R}$.

Consider the closed set $M = \left\{\psi \in L_2[0,1] : \int_0^1 |\psi(s)|^2 ds = 1\right\}$ and let the impulse function $I : M \to L_2[0,1]$ satisfying

$$\|I(\psi)\|_{L_2} \leq \alpha\|\psi\|_{L_2},$$

for all $\psi \in M$, where $0 < \alpha < 1$. Note that $I(M) \cap M = \emptyset$. Thus, we have the associate impulsive system $(L_2[0,1],\pi;M,I)$.

Since $\|\pi(\varphi,t)\|_{L_2} \stackrel{t\to+\infty}{\longrightarrow} 0$ for each $\varphi \in L_2[0,1]$ (see [7, Example 1.8]) and $\|I(\psi)\|_{L_2} < \|\psi\|_{L_2}$ for $\psi \in M$, we obtain

$$\|\widetilde{\pi}(\varphi,t)\|_{L_2} \stackrel{t\to+\infty}{\longrightarrow} 0,$$

for all $\varphi \in L_2[0,1]$. Hence, $(L_2[0,1],\pi;M,I)$ is point $k-$dissipative and $\widetilde{L}^+(\varphi) = \{0\}$ for every $\varphi \in L_2[0,1]$. Consequently, $\Omega = \{0\}$.

Now, since $\|\pi(\varphi,t)\|_{L_2} \leq \|\varphi\|_{L_2}$ for $\varphi \in L_2[0,1]$ and $t \geq 0$ (see [7, Example 1.8]), and $\|I(\psi)\|_{L_2} \leq \|\psi\|_{L_2}$ for all $\psi \in M$, we have

$$\|\widetilde{\pi}(\varphi,t)\|_{L_2} \leq \|\varphi\|_{L_2} \tag{25}$$

for all $\varphi \in L_2[0,1]$ and for all $t \geq 0$. Let us show that $\widetilde{D}^+(\Omega) = \{0\}$. Indeed, given $\varphi \in \widetilde{D}^+(\Omega)$ there are sequences $\{\psi_n\}_{n\geq1} \subset L_2[0,1]$, $\{t_n\}_{n\geq1} \subset \mathbb{R}_+$ such that $\|\psi_n - 0\|_{L_2} \stackrel{n\to+\infty}{\longrightarrow} 0$ and

$$\|\widetilde{\pi}(\psi_n,t_n) - \varphi\|_{L_2} \stackrel{n\to+\infty}{\longrightarrow} 0.$$

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Since $\|\widetilde{\pi}(\psi_n, t_n)\|_{L_2} \leq \|\psi_n\|_{L_2}$ and $\|\psi_n\|_{L_2} \stackrel{n\to+\infty}{\longrightarrow} 0$ we have $\varphi = 0$. Then $\widetilde{D}^+(\Omega) = \{0\}$. Therefore, $\widetilde{D}^+(\Omega)$ is compact, orbitally $\widetilde{\pi}-$stable (by (25)) and $\widetilde{D}^+(\Omega) \cap M = \emptyset$. By Theorem 3.10 the impulsive system is compact $k-$dissipative and its center of Levinson is $J = \{0\}$.

Let us prove that $(L_2[0,1], \pi; M, I)$ is not local $k-$dissipative. Suppose the contrary, then by Theorem 3.13 the center of Levinson $J = \{0\}$ is uniformly $\widetilde{\pi}-$attracting, that is, there is $\gamma > 0$ such that

$$\lim_{t\to+\infty} \sup_{\|\varphi\|_{L_2}\leq\gamma} \|\widetilde{\pi}(\varphi, t)\|_{L_2} = 0. \tag{26}$$

Define $\psi_n \in L_2[0,1]$, $n = 1, 2, \ldots$, by

$$\psi_n(t) = \gamma\sqrt{n}\chi_{[0,\frac{1}{n}]}(t),$$

for $t \in [0,1]$, where $\chi_{[0,\frac{1}{n}]}(t)$ is the characteristic function of the set $[0, \frac{1}{n}]$. Note that $\|\psi_n\|_{L_2} = \gamma$ for all $n = 1, 2, \ldots$. Moreover, by considering $t_n = \dfrac{n}{2}$, $n \in \mathbb{N}$, we have

$$\lim_{n\to+\infty} \|\pi(\psi_n, t_n)\|^2_{L_2} = \lim_{n\to+\infty} \int_0^{\frac{1}{n}} ne^{-2t_n s} ds = 1 - \frac{1}{e} \neq 0,$$

which contradicts (26). Then the system is not local $k-$dissipative.

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