CADERNOS DE MATEMÁTICA **11**, 281–330 October (2010)
ARTIGO NÚMERO SMA#339

Reduction of infinite dimensional systems to finite dimensions: Compact convergence approach

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We consider parameter dependent semilinear evolution problems for which, at the limit value of the parameter, the problem is finite dimensional. We introduce an abstract functional analytic framework that applies to many problems in the existing literature for which the study of the asymptotic dynamics can be reduced to finite dimensions via the invariant manifold theory. Some practical models are considered to show the wide applicability of the theory. October, 2010 ICMC-USP

* [1] Partially supported by CNPq 302022/2008-2, CAPES/DGU 267/2008 and FAPESP 2008/55516-3, Brazil
† [2] Partially supported by FAPESP 2009/52687-4, Brazil
‡ [3] Partially supported by FAPESP, Grant: 07/00981-0, Brazil

Publicado pelo ICMC-USP
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1. INTRODUCTION

A number of natural phenomena that appear in various branches of science, such as ecology, physics, chemistry, economics, neurobiology and many others are modeled by differential equations. Many of these models fall into a class of semilinear parabolic problems of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= DLu + f(x,u,\nabla u),\ x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} &= g(u),\ \partial \in \Omega,\ t > 0, \end{aligned} \tag{1.1}$$

where $k \in \mathbb{N}^*$, $u \in \mathbb{R}^k$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, L is a second order elliptic operator, $D = \mathrm{diag}(d_1, \cdots, d_k)$ with $d_i > 0$, $1 \le i \le k$, $d = \min\{d_i : 1 \le i \le n\}$, ν is a normal vector to $\partial\Omega$, $f : \Omega \times \mathbb{R}^k \times \mathbb{R}^{kN} \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}^k$ are suitably smooth functions.

Spatial homogenization or local spatial homogenization is a natural phenomenon in many of the problems with such models. These spatial homogenization or local spatial homogenizations are in general associated to reduction to finite dimensions in relation to the study of the asymptotic behavior of solutions.

In [19] Conway, Hoff and Smoller considered the case $g = 0$ and showed that when the diffusion coefficients are large (d large) the solutions of (1.1) are asymptotic to the solutions of

$$\frac{du}{dt} = f(u). \tag{1.2}$$

When g is a linear function Hale [21], Hale and Rocha in [23] and Carvalho and Hale [9] showed that if the 'limiting' system (an ode) has a compact attractor, for d large the parabolic system also has a compact attractor that is close to a 'limit' one (the same when $g = 0$). When $f(x,u,\nabla u) = f(u)$ and g is nonlinear Carvalho and Primo in [16] showed that there exists a system of ODEs which describes the asymptotic dynamics of the original system for large diffusion.

For $N = 1$ $f(x,u,u_x) = f(u)$ and $g = 0$, Fusco [20] and later Carvalho and Pereira in [13] and Carvalho in [5] considered the diffusion coefficient of the form $a(x,\nu)$ with a parameter $\nu > 0$ which is large except in a neighborhood of a finite number of points where it becomes small. They showed that the asymptotic dynamics of such reaction diffusion problems can be described through a system of ordinary differential equations which can be exhibited explicitly. The case when g is nonlinear is considered in [6] and the extension for higher dimensional domains of these results appear in Carvalho and Cuminato [10] (planar cell tissue).

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Morita [28] considered dumbbell type domains and $g(u) = 0$. He studied the long-time behavior of the bounded solutions and proved the existence of an invariant attracting finite-dimensional manifold (under suitable assumptions on the nonlinearity). On this manifold the system is reduced to an explicitly given system of ordinary differential equations. The dynamics of the reaction-diffusion system is then studied via this system of ODEs, for example it is shown that there exist equilibrium solutions of the original system with large spatial inhomogeneity.

The case when the domain consists of N regions D_i , $i = 1, \cdots, N$, connected by thin channels $Q_{i,j}(\epsilon)$ which approach a line segments as $\epsilon \to 0$ was considered in [29] by Morita and Jimbo. They showed that an inertial manifold exists, which is invariant and attracts solutions exponentially. Moreover the ODE, describing the dynamics on the inertial manifold, can be given in an explicit form through the analysis of the limit of the manifold as $\epsilon \to 0$.

When $d = \delta\epsilon^{-(N-1)}$, $\delta > 0$ and $\epsilon > 0$ is a small parameter, a dumbbell type domain Ω_ϵ was considered by Carvalho and Lozada-Cruz in [11, 12]. They proved the convergence of eigenvalues and eigenfunctions of the Laplace operator in Ω_ϵ as $\epsilon \to 0$, which enabled them to show that the asymptotic dynamics of the parabolic equation in this domain is equivalent to the asymptotic dynamics of a system of two diffusively coupled ordinary differential equations (here the fact that the diffusivity is large allows for less restrictions on the nonlinearity).

In all these papers a difficult part is to obtain the convergence of eigenvalues and eigenfunctions of the unbounded operator associated with the linear main part in (1.1) and to apply the theory of invariant manifolds.

In this paper we will work out an appropriate functional setting to treat a broad class of problems such as spatial homogenization, perturbations problems or air pollution models besides many of the problems in the literature cited above. This functional setting will make use concepts like convergence of sequences of elements from different spaces X_d, $d \in [1, \infty]$, compactness of families living in different spaces and first and foremost the concept of compact convergence for linear operators. Using these concepts we will develop an abstract approach that will make possible to 'reduce to finite dimensions' some infinite-dimensional evolutionary problems. For their attractors we will show that they are actually contained in an exponentially attracting finite dimensional Lipschitz manifold of dimension equal to the dimension of a 'limit' space X_∞.

We remark that application of invariant manifold technique is possible if the operators exhibit suitably nice spectral properties (existence of large gaps). Convergence of spectra with respect to the perturbation parameter is then crucial to describe the limit dynamics identifying the finite dimensional limiting system. The compact convergence approach developed in

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this paper will reduce considerably the work needed to prove the results concerning convergence of eigenvalues and eigenfunctions and existence of large gaps. In this sense, the theory developed here generalizes, unifies and simplifies the work done in the above mentioned literature.

To better explain the ideas this work is divided into two parts that we now describe briefly.

The first part is devoted to the development of the abstract functional analytic framework to treat the problem of reduction to finite dimensions of semilinear evolution problems using the invariant manifold theory. We first consider linear operators acting in Banach spaces that change with a parameter for the situation when the limiting operator (and limiting space) is finite dimensional. For this situation we introduce the notion and properties of compact convergence and prove the convergence of eigenvalues and eigenfunctions as well as the existence of a large gap. We then consider equations involving sectorial operators with compactly convergent resolvent with the limiting problem being finite dimensional and construct exponentially attracting invariant manifolds that are the graphs of Lipschitz continuous maps. Using this we study the convergence of attractors for the associated nonlinear semigroups.

The second part is devoted to applications. We consider problems originating in applied sciences such as atmospheric problems and cell tissue modeling using reaction diffusion equations with large diffusion and show that the relevant asymptotic dynamics can be described with the aid of the global attractors of nonlinear semigroups in finite dimensional spaces (originating from ODE). Using the approach developed in the first part of the paper we show that the solutions on these attractors actually lie on finite dimensional invariant manifolds and that these invariant manifolds are near a linear manifold where the limiting problem is posed. We show that the attractors of the perturbed problems are close to the attractor for the limiting problem in this linear finite dimensional manifold where the flow is described by an ODE (explicitly given using the convergence of eigenvalues, eigenfunctions and the blow up of the gap).

Acknowledgements: The authors would like to thank FAPESP, Brazil, for supporting their research. Also, the second author would like to acknowledge the hospitality of the scientist from Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo in São Carlos, Brazil and from Universidade Estadual Paulista, Campus de São Jose do Rio Preto, where this work has been carried out.

2. PART I: COMPACT CONVERGENCE AND INVARIANT MANIFOLDS

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In this section we introduce the notion and some basic properties of compact convergence of resolvent for families of closed invertible operators and discuss the continuity of the spectral properties of the operators on this family. We also consider the continuity properties of of semigroups generated by sectorial operators with compactly convergent resolvent. The latter implies suitable spectral properties needed to construct finite dimensional exponentially attracting invariant manifolds and leads to reduction to finite dimensions in some semilinear problems.

2.1. Generalities concerning operators with compactly convergent resolvent

In what follows $\{X_d, d \in [1,\infty]\}$ is a family of Banach spaces, for which we assume that X_∞ is finite dimensional, and $L(X_d)$, $d \in [1,\infty]$, denote the spaces of all bounded linear operators defined on X_d with values in X_d.

For $d \neq \infty$ spaces X_d will in general be infinite dimensional and may vary with d. Following [33], we introduce a family of connecting maps $\mathcal{E} := \{E_d\}_{d\in[1,\infty)}$, which consists of bounded linear operators $E_d : X_\infty \to X_d$, $d \in [1,\infty)$, with the property that

$$\|E_d u\|_{X_d} \stackrel{d\to\infty}{\longrightarrow} \|u\|_{X_\infty} \quad \text{for each} \quad u \in X_\infty. \tag{2.1}$$

Using the family of connecting maps we define suitable notions of convergence and compactness concerning sequences of points $u_{d_n} \in X_{d_n}$ (see [33, 2, 14]).

Definition 2.1. Given $u_\infty \in X_\infty$ and given a sequence of points $\{u_{d_n}\}$ such that $d_n \to \infty$, $u_{d_n} \in X_{d_n}$ and $\lim_{n\to\infty} \|u_{d_n} - E_{d_n} u_\infty\|_{X_{d_n}} = 0$ we say that u_{d_n} converges to u_∞ relatively to the family $\mathcal{E}$ and write this as $u_{d_n} \stackrel{\mathcal{E}}{\longrightarrow} u_\infty$ or simply $u_{d_n} \dashrightarrow u_\infty$.

We say that $\{u_{d_n}\}$ is compact relatively to the family $\mathcal{E}$ if and only if for each subsequence $\{u_{d_{n_k}}\}$ there is a subsequence $\{u_{d_{n_{k_j}}}\}$ and a certain $u_\infty \in X_\infty$ such that $u_{d_{n_{k_j}}} \dashrightarrow u_\infty$.

With the above notions concerning convergence and compactness of sequences of points we introduce suitable notions of convergence of bounded linear operators.

Definition 2.2. Given $\mathcal{B}_\infty \in L(X_\infty)$ and given a sequence of operators $\{\mathcal{B}_{d_n}\}$ such that $d_n \to \infty$ and $\mathcal{B}_{d_n} \in L(X_{d_n})$, $n \in \mathbb{N}$, we say that $\{\mathcal{B}_{d_n}\}$ converges to $\mathcal{B}_\infty \in L(X_\infty)$ relatively to the family $\mathcal{E}$, which we denote as $\mathcal{B}_{d_n} \stackrel{\mathcal{E}\mathcal{E}}{\longrightarrow} \mathcal{B}_\infty$ or simply $\mathcal{B}_{d_n} \dashrightarrow \mathcal{B}_\infty$, if and only if

$$u_{d_n} \dashrightarrow u_\infty \quad \text{implies} \quad \mathcal{B}_{d_n} u_{d_n} \dashrightarrow \mathcal{B}_\infty u_\infty.$$

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DEFINITION 2.3. Given a sequence of operators $\{\mathcal{B}_{d_n}\}$ such that $d_n \to \infty$ and $\mathcal{B}_{d_n} \in L(X_{d_n})$ are linear and compact, we say that $\{\mathcal{B}_{d_n}\}$ converges compactly to a compact linear operator $\mathcal{B}_\infty \in L(X_\infty)$, which we denote as $\mathcal{B}_{d_n} \overset{cc}{\dashrightarrow} \mathcal{B}_\infty$, if and only if the following two conditions are satisfied:

(cc1) each sequence of the form $\{\mathcal{B}_{d_n} u_{d_n}\}$, where $u_{d_n} \in X_{d_n}$ and $\|u_{d_n}\|_{X_{d_n}} = 1$ for every $n \in \mathbb{N}$, is compact relatively to the family $\mathcal{E}$ and

(cc2) $\mathcal{B}_{d_n} \dashrightarrow \mathcal{B}_\infty$.

Our further concern will be families of closed invertible operators with compactly convergent resolvent.

DEFINITION 2.4. Given a family $\{A_d\}_{d\in[1,\infty)}$ of linear operators $A_d : dom(A_d) \subset X_d \to X_d$ in the Banach spaces X_d, $d \in [1,\infty)$, we say that $\{A_d\}_{d\in[1,\infty)}$ is of the class $CC(X_d)$ if and only if $A_d^{-1} \in L(X_d)$ are compact for all $d \in [1,\infty)$ and there exists a finite dimensional Banach space X_∞, a family $\mathcal{E} = \{E_d\}_{d\in[1,\infty)}$ of connecting bounded linear maps $E_d : X_\infty \to X_d$, $d \in [1,\infty)$, satisfying (2.1) and a linear operator $A_\infty : X_\infty \to X_\infty$ such that $A^{-1} \in L(X_\infty)$ and $A_d^{-1} \overset{cc}{\dashrightarrow} A_\infty^{-1}$.

In [2] (see also [14]) families of operators of the latter type have been considered and the authors proved a number of results involving their spectral properties. The following lemma is fundamental for this.

LEMMA 2.1. *If $\{A_d\}_{d\in[1,\infty)}$ is of the class $CC(X_d)$, then for every $d \in [1,\infty]$ the spectrum of A_d consists of isolated eigenvalues only.*
Given any compact set $K \subset \rho(A_\infty)$ we have that

$$\exists_{d_K \geqslant 1}\ \forall_{d \geqslant d_K}\ K \subset \rho(A_d) \tag{2.2}$$

and

$$\exists_{M>0}\ \exists_{d_K\geqslant 1} \sup_{d\geqslant d_K} \sup_{\lambda\in K} \|(\lambda - A_d)^{-1}\|_{L(X_d)} \leq M. \tag{2.3}$$

Furthermore, whenever $d_n \to \infty$, we have

$$(\lambda I - A_{d_n})^{-1} E_{d_n} u_\infty \dashrightarrow (\lambda I - A_\infty)^{-1} u_\infty \text{ for each } u_\infty \in X_\infty,\ \lambda \in K, \tag{2.4}$$

and actually

$$(\lambda - A_d)^{-1} \overset{cc}{\dashrightarrow} (\lambda - A_\infty)^{-1} \ \text{ for every } \ \lambda \in K. \tag{2.5}$$

Proof: If (2.2) fails then there is a sequence $K \cap \sigma(A_{d_n}) \ni k_n \to k \in K$. Consequently, there is a sequence $x_{d_n} \in X_{d_n}$ for which $\|x_{d_n}\|_{X_{d_n}} = 1$ and $x_{d_n} = k_n A_{d_n}^{-1} x_{d_n}$. By assumption, taking a subsequence if necessary, $x_{d_n} \dashrightarrow x_\infty$ for a certain $x_\infty \in X_\infty$ and $A_{d_n}^{-1} x_{d_n} \dashrightarrow A_\infty^{-1} x_\infty$. Since this leads to the condition $x_\infty = k A_\infty^{-1} x_\infty$, k is an eigenvalue of A_∞ which is absurd.

If (2.3) fails, then there are sequences $K \ni k_n \to k \in K$ and $x_{d_n} \in X_{d_n}$ such that $\|x_{d_n}\|_{X_{d_n}} = 1$ and $\|(k_n I - A_{d_n})^{-1} x_{d_n}\|_{X_{d_n}} > n$. Denoting $(k_n I - A_{d_n})^{-1} x_{d_n} =: y_n$ we have $\|y_n\|_{X_{d_n}} \to \infty$, $k_n A_{d_n}^{-1} \|y_n\|_{X_{d_n}}^{-1} y_n - \|y_n\|_{X_{d_n}}^{-1} A_{d_n}^{-1} x_{d_n} = \|y_n\|_{X_{d_n}}^{-1} y_n$ and, taking subsequence if necessary, we obtain that there exists an element $y_\infty \in X_\infty$ for which $k A_\infty^{-1} y_\infty = y_\infty$. However this is again absurd as k cannot be an eigenvalue of A_∞.

Proving (2.4) we denote $(\lambda I - A_{d_n})^{-1} E_{d_n} u_\infty =: w_{d_n}$ to get $w_{d_n} = \lambda A_{d_n}^{-1} w_{d_n} - A_{d_n}^{-1} E_{d_n} u_\infty$. By (2.3) the sequence of norms $\{\|w_{d_n}\|_{X_{d_n}}\}$ is bounded. By assumption we may thus assume not loosing generality that $A_{d_n}^{-1} w_{d_n} \dashrightarrow z_\infty$ for a certain $z_\infty \in X_\infty$. Since $E_{d_n} u_\infty \dashrightarrow u_\infty$ we also have that $A_{d_n}^{-1} E_{d_n} u_\infty \dashrightarrow A_\infty^{-1} u_\infty$. This ensures now that $w_{d_n} \dashrightarrow w_\infty$ for a certain $w_\infty \in X_\infty$. By assumption we then have $A_{d_n}^{-1} w_{d_n} \dashrightarrow A_\infty^{-1} w_\infty$ and obtain

$$w_\infty \dashleftarrow w_{d_n} = \lambda A_{d_n}^{-1} w_{d_n} - A_{d_n}^{-1} E_{d_n} u_\infty \dashrightarrow \lambda A_\infty^{-1} w_\infty - A_\infty^{-1} u_\infty.$$

Therefore $u_\infty = (\lambda - A_\infty) w_\infty$ and we get

$$(\lambda I - A_{d_n})^{-1} E_{d_n} u_\infty = w_{d_n} \dashrightarrow w_\infty = (\lambda - A_\infty)^{-1} u_\infty.$$

To ensure (2.5) we need to show that both conditions required in the Definition 2.3 holds. As for the condition (cc1) we remark that this is a consequence of the resolvent estimates and the assumptions concerning the family $\{A_d\}_{d \in [1,\infty]}$. Finally, if $u_{d_n} \dashrightarrow u_\infty$, then the proof that $(\lambda I - A_{d_n})^{-1} u_{d_n} \dashrightarrow (\lambda - A_\infty)^{-1} u_\infty$ is fully analogous to the proof of (2.4). ▮

Remark 2. 1. Note that the convergence in (2.4) is uniform for u_∞ varying in compact subsets of X_∞ provided that the family $\{E_d\}_{d \in [1,\infty]}$ is bounded; that is $\|E_d\|_{L(X_\infty, X_d)} \le \mathcal{C}$ for all $d \in [1, \infty]$ where constant $\mathcal{C}$ does not depend on the parameter.

Lemma 2.1 is crucial to exhibit the relevant spectral properties of the considered perturbed system. Below we use the perturbation theory of linear operators [27] and give some results which follow from (2.2)-(2.5). We start from a proposition concerning projection operators and projected spaces.

PROPOSITION 2.1. *Suppose that* $\{A_d\}_{d\in[1,\infty)}$ *is of the class* $CC(X_d)$ *and let* $\lambda \in \sigma(A_\infty)$.
Then,
i) given sufficiently small $\delta > 0$*, the operators*

$$Q_d(\lambda) = \frac{1}{2\pi i}\int_{|\xi-\lambda|=\delta} (\xi I - A_d)^{-1} d\xi,\ d \in [1,\infty],$$

are projections on X_d *and*

$$Q_d(\lambda) \overset{cc}{\dashrightarrow} Q_\infty(\lambda). \tag{2.6}$$

Furthermore,
ii) all the projected spaces $W_d(\lambda) := Q_d(\lambda)X_d$ *are eventually of the same algebraic dimension; namely, there exists a certain* $d_0 \geq 1$ *such that*

$$\operatorname{rank}(Q_d(\lambda)) := \dim W_d(\lambda) = \dim W_\infty(\lambda) =: \operatorname{rank}(Q_\infty(\lambda)) \text{ for all } d \in [d_0,\infty],$$

iii) for each $u_\infty \in W_\infty(\lambda)$ *there is a sequence* $\{u_{d_n}\}$ *such that* $u_{d_n} \in W_{d_n}(\lambda)$, $d_n \to \infty$ *and* $u_{d_n} \dashrightarrow u_\infty$.
iv) given sequences $d_n \to \infty$ *and* $\{u_n\}$ *such that* $u_n \in W_{d_n}(\lambda)$ *and* $\|u_n\|_{X_{d_n}} = 1$ *for all* $n \in \mathbb{N}$*, each subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *has a subsequence* $\{u_{n_{k_l}}\}$ *such that* $u_{n_{k_l}} \dashrightarrow u_\infty$ *for a certain* $u_\infty \in W_\infty(\lambda)$.

Proof: Part i) follows from Lemma 2.1 and from [27, Theorem III.6.17].

ii) If $v_1,\ldots,v_k$ is a basis for $W_\infty(\lambda)$ one can see that, for d suitably large, the elements $Q_d(\lambda)E_dv_1,\ldots,Q_d(\lambda)E_dv_k$ are linearly independent in $W_d(\lambda)$. Otherwise we consider the relation $\sum_{j=1}^k \alpha_j^d Q_d(\lambda)E_dv_j = 0$, where $\sum_{j=1}^k |\alpha_j^d|^2 = 1$, and taking subsequences of $\{\alpha_j^d\}$ convergent to α_j (which we denote the same), we get in the limit that $\sum_{j=1}^k |\alpha_j|^2 = 1$ and $0 = \sum_{j=1}^k \alpha_j^d Q_d(\lambda)E_dv_j \dashrightarrow \sum_{j=1}^k \alpha_j Q_\infty(\lambda)v_j = \sum_{j=1}^k \alpha_j v_j$, which is absurd. Hence $\operatorname{rank}(Q_d(\lambda)) \geqslant \operatorname{rank}(Q_\infty(\lambda))$. Now, if for some sequence $d_n \to \infty$ we have $\operatorname{rank}(Q_{d_n}(\lambda)) > \operatorname{rank}(Q_\infty(\lambda))$, we infer from [27, Lemma IV.2.3] that for each $n \in \mathbb{N}$ there is a point $u_n \in W_{d_n}(\lambda)$ with $\|u_n\|_{X_{d_n}} = 1$ such that $\inf_{w\in Q_{d_n}(\lambda)E_{d_n}X_\infty} \|u_n - w\|_{X_{d_n}} = 1$. Since $Q_{d_n}(\lambda)$ are projections we have $Q_{d_n}(\lambda)\, u_n = u_n$ and hence from (2.6) we can assume without loss of generality that $u_n \dashrightarrow u_\infty$ for a certain $u_\infty \in X_\infty$. Then also $Q_{d_n}(\lambda)\, u_n = u_n \dashrightarrow Q_\infty(\lambda)u_\infty$, where $Q_\infty(\lambda)u_\infty = u_\infty \in W_\infty(\lambda)$ as $Q_\infty(\lambda)$ is a projection, and we get

$$1 = \inf_{w\in Q_{d_n}(\lambda)E_{d_n}X_\infty} \|u_n - w\|_{X_{d_n}} \leq \|u_n - Q_{d_n}(\lambda)E_{d_n}u_\infty\|_{X_{d_n}}.$$

Since $E_{d_n}u_\infty \dashrightarrow u_\infty$, then $Q_{d_n}(\lambda)E_{d_n}u_\infty \dashrightarrow Q_\infty(\lambda)u_\infty$ and consequently we infer that $\|u_n - Q_{d_n}(\lambda)E_{d_n}u_\infty\|_{X_{d_n}} \to 0$, which is absurd.

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iii) If $d_n \to \infty$ then $E_{d_n} u_\infty \dashrightarrow u_\infty$ and (2.6) implies $u_{d_n} = Q_{d_n}(\lambda) E_{d_n} u_\infty \dashrightarrow Q_\infty(\lambda) u_\infty$. Since $Q_\infty(\lambda)$ is a projection and $u_\infty \in W_\infty(\lambda)$ we also have $Q_\infty(\lambda) u_\infty = u_\infty$ which gives the result.

iv) Since Q_{d_n} are projections we have that $Q_{d_n} u_n = u_n$ for all $n \in \mathbb{N}$. By (2.6) we then infer that any subsequence $\{u_{n_k}\}$ has a subsequence $\{u_{n_{k_l}}\}$ such that $u_{n_{k_l}} \dashrightarrow u_\infty$ for a certain $u_\infty \in X_\infty$. Therefore $u_{n_{k_l}} = Q_{d_{n_{k_l}}} u_{n_{k_l}} \dashrightarrow Q_\infty u_\infty = u_\infty$, which completes the proof. ▮

We will now denote by $\mathcal{O}_\delta(z)$ an open ball in the complex plane $\mathbb{C}$ of radius $\delta > 0$ around $z \in \mathbb{C}$. The next result concerns the convergence of eiganvalues (see [2, 14]).

Corollary 2.1. *If $\{A_d\}_{d \in [1,\infty)}$ is of the class $CC(X_d)$ then*

(i)for each $\lambda \in \sigma(A_\infty)$, there exists a sequence $d_n \to \infty$ and $\lambda_n \in \sigma(A_{d_n})$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda$ as $n \to \infty$,

(ii)conditions $d_n \to \infty$, $\lambda_n \in \sigma(A_{d_n})$, $n \in \mathbb{N}$ and $\lambda_n \to \lambda$ as $n \to \infty$ imply that $\lambda \in \sigma(A_\infty)$,

Proof: (i) It suffices to note that there is a certain $d_0 \geq 1$ such that for each $\delta > 0$ and every $d \geq d_0$ we have $\sigma(A_d) \cap \mathcal{O}_\delta(\lambda) \neq \emptyset$. Indeed, if this is not the case then there are $\epsilon > 0$ and a sequence $d_n \to \infty$ such that $\int_{|\xi-\lambda|=\delta} (\xi - \lambda)^n (\xi I - A_d)^{-1} d\xi = 0$ for all $n \in \mathbb{N}$. By (2.6) we then have $\int_{|\xi-\lambda|=\delta} (\xi - \lambda)^n (\xi I - A_\infty)^{-1} d\xi = 0$ for all $n \in \mathbb{N}$, which is absurd as $(\xi I - A_\infty)^{-1}$ does not have removable singularity at $\xi = \lambda$.

(ii) Let $d_n \to \infty$, $\{\lambda_n\} \subset \sigma(A_{d_n})$, $\lambda_n \to \lambda$ and let $u_n \in X_{d_n}$ be the corresponding eigenfunctions with $\|u_n\|_{X_{d_n}} = 1$. Then $u_n = \lambda A_{d_n}^{-1} u_n$ and by assumption there is a subsequence (denoted the same) such that $u_n \dashrightarrow u_\infty$ for a certain $u_\infty \in X_\infty$. By compact convergence we obtain

$$u_\infty \dashleftarrow u_n = \lambda A_{d_n}^{-1} u_n \dashrightarrow \lambda A_\infty^{-1} u_\infty,$$

which proves that $\lambda \in \sigma(A_\infty)$. ▮

It is reasonable to describe convergence properties of spectral sets in term of the Hausdorff distance of sets.

Definition 2.5. If V is a metric space and $\operatorname{dist}(\cdot,\cdot)$ denotes the metric in V, then the Hausdorff semidistance d_H^V in V between sets $B_1, B_2 \subset V$ is defined as

$$d_H^V(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \operatorname{dist}(b_1, b_2)$$

and the Hausdorff distance dist_H^V in V between $B_1, B_2 \subset V$ is

$$\operatorname{dist}_H^V(B_1, B_2) = d_H^V(B_1, B_2) + d_H^V(B_2, B_1).$$

Given a family of sets $\{B_n : n \in \mathbb{N} \cup \{\infty\}\}$ in V we say that $\{B_n : n \in \mathbb{N} \cup \{\infty\}\}$ is upper semicontinuous at ∞ (resp. lower semicontinuous at ∞) (resp. continuous at ∞) if and only if $d_H(B_n, B_\infty) \to 0$ (resp. $d_H(B_\infty, B_n) \to 0$) (resp. $\mathrm{dist}_H(B_\infty, B_n) \to 0$).

In a similar manner we define the notion of Hausdorff distance relatively to the family $\mathcal{E}$.

Definition 2.6. For $d \in [1, \infty)$ and for sets $B_d \subset X_d$, $B_\infty \subset X_\infty$ we set

$$d_H^{\mathcal{E}}(B_d, B_\infty) := \sup_{b_d \in B_d} \inf_{b_\infty \in B_\infty} \|b_d - E_d b_\infty\|_{X_d}, \quad d_H^{\mathcal{E}}(B_\infty, B_d) := \sup_{b_\infty \in B_\infty} \inf_{b_d \in B_d} \|E_d b_\infty - b_d\|_{X_d}$$

and we define the Hausdorff distance $\mathrm{dist}_H^{\mathcal{E}}(B_d, B_\infty)$ relative to $\mathcal{E}$ as

$$\mathrm{dist}_H^{\mathcal{E}}(B_d, B_\infty) = d_H^{\mathcal{E}}(B_d, B_\infty) + d_H^{\mathcal{E}}(B_\infty, B_d).$$

If $\{B_n : n \in \mathbb{N} \cup \{\infty\}\}$ is a family of sets such that $B_n \subset X_{d_n}$ for each $n \in \mathbb{N}$ and $d_n \to \infty$, then we say that $\{B_n : n \in \mathbb{N} \cup \{\infty\}\}$ is upper semicontinuous at ∞ (resp. lower semicontinuous at ∞) (resp. continuous at ∞) relatively to $\mathcal{E}$ if and only if $d_H^{\mathcal{E}}(B_n, B_\infty) \to 0$ (resp. $d_H^{\mathcal{E}}(B_\infty, B_n) \to 0$) (resp. $\mathrm{dist}_H^{\mathcal{E}}(B_\infty, B_n) \to 0$).

The convergence of spectral sets and subsets of projected spaces can be now concluded as follows.

Corollary 2.2. *Suppose that $\{A_d\}_{d \in [1,\infty)}$ is of the class $CC(X_d)$. Let $\lambda_{\infty 1}, \lambda_{\infty 2}, \ldots, \lambda_{\infty k}$ be the all distinct eigenvalues of A_∞ and let $\delta \in (0,1)$ be such that the balls $\mathcal{O}_\delta(\lambda_{\infty j})$, $j = 1, \ldots, k$ are disjoint.*
Then,
i) for each $R > 0$ there exists a certain $d(R) > 0$ such that for any $d \geqslant d(R)$ and each $\lambda \in \sigma(A_d)$

$$\text{either } \lambda \in \cup_{j=1}^{k} \mathcal{O}_\delta(\lambda_{\infty j}) \text{ or } |\lambda| > R,$$

Furthermore,
ii) concerning the spectral sets

$$\sigma_j^d := \sigma(A_d) \cap \mathcal{O}_\delta(\lambda_{\infty j}), \quad \sigma_{k+1}^d := \sigma(A_d) \setminus \cup_{j=1}^{k} \sigma_j^d,$$
$$\sigma_j^\infty := \{\lambda_{\infty j}\}, \ 1 \leqslant j \leqslant k, \ d \in [d(R), \infty),$$

we have that

$$\operatorname{dist}^{\mathbb{C}}_H(\sigma^d_j, \sigma^\infty_j) \to 0 \text{ as } d \to \infty \text{ for each } 1 \leq j \leq k,$$

and

$$\inf\{|\lambda| : \lambda \in \sigma^d_{k+1}\} \to \infty \quad \text{as } d \to \infty, \tag{2.7}$$

iii) concerning the unit spheres $W_{dj} = \{x_d \in W_d(\lambda_j) : \|x_d\|_{X_d} = 1\}$ *in the projected spaces* $W_d(\lambda_j)$, $d \in [d(R), \infty]$, *we have that*

$$\operatorname{dist}^{\mathcal{E}}_H(W_{dj}, W_{\infty j}) \to 0 \text{ as } d \to \infty \text{ for each } 1 \leq j \leq k.$$

Proof: i) Assume that there is a certain $R > 0$ and there are sequences $d_n \overset{n\to\infty}{\longrightarrow} \infty$ and $\{\lambda_{d_n}\}$ such that $\lambda_{d_n} \in \sigma(A_{d_n})$, $|\lambda_{d_n}| \leqslant R$ and $|\lambda_{d_n} - \lambda_{\infty j}| \geqslant \delta$. Evidently there is a convergent subsequence $\{\lambda_{d_{n_k}}\}$, which via Corollary 2.1 tends to some $\lambda \in \sigma(A_\infty)$. However the latter implies that $|\lambda - \lambda_{\infty j}| \geqslant \delta$ for every $j = 1, \dots k$, which is absurd.

Part ii) follows now immediately from i) and from Corollary 2.1. Finally, part iii) is a consequence of Proposition 2.1 iii)-iv). ▌

2.2. Problems involving sectorial operators with compactly convergent resolvent

In this subsection we consider semilinear equations with sectorial operators possessing compactly convergent resolvent. For these equations we construct exponentially attracting invariant manifolds $\mathcal{S}_d$ as the graphs of the Lipschitz continuous maps Σ^*_d defined on a suitably chosen subspace Y_d of X_d.

We start from a lemma concerning linear problems

$$\begin{cases} \dot{u} = A_d u, \ t > 0, \ d \in [1, \infty), \\ u(0) = u_0 \in X_d, \end{cases} \tag{2.8}$$

where $\{A_d\}_{d\in[1,\infty)}$ is assumed to fall into a suitable class of linear operators in the Banach spaces as defined below.

Definition 2.7. Given a family $\{A_d\}_{d\in[1,\infty)}$ of linear operators $A_d : dom(A_d) \subset X_d \to X_d$ in the Banach spaces X_d, $d \in [1, \infty)$, we say that $\{A_d\}_{d\in[1,\infty)}$ is of the class $\mathcal{G}(X_d, c, \omega)$ if and only if for every $d \in [1, \infty)$ the operator $A_d : D(A_d) \subset X_d \to X_d$ generates a strongly continuous semigroup of linear operators $\{e^{A_d t} : t \geqslant 0\} \subset L(X_d)$ and

$$\|e^{A_d t}\|_{\mathcal{L}(X_d)} \leqslant c\, e^{\omega t} \quad \text{for all} \quad t \geqslant 0, \, d \in [1, \infty), \tag{2.9}$$

with constants $c \geqslant 1$ and $\omega \in \mathbb{R}$ independent of $d \in [1, \infty)$.

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We say that $\{A_d\}_{d\in[1,\infty)}$ is of the class $\mathcal{H}(X_d, M, \theta)$ if and only if for each $d \in [1,\infty)$ the resolvent set $\rho(A_d)$ of A_d contains the same sector $\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \frac{\pi}{2} + \theta\}$ and moreover

$$\|(\lambda - A_d)^{-1}\|_{L(X_d)} \leq \frac{M}{|\lambda| + 1} \quad \text{for all} \quad \lambda \in \Sigma_\theta,\ d \in [1,\infty), \tag{2.10}$$

where constants $\theta \in (0, \frac{\pi}{2})$ and $M \geq 0$ do not depend on $d \in [1,\infty)$.

Evidently $\mathcal{G}(X_d, c, \omega)$ contains $\mathcal{H}(X_d, M, \theta)$ as in the latter case operators A_d, $d \in [1,\infty)$ generate semigroups $\{e^{A_d t} : t \geq 0\}$ which are analytic and (2.10) implies (2.9) for some $\omega < 0$ and $c \geq 1$ (see [26]).

LEMMA 2.2. *Let $\{A_d\}_{d\in[1,\infty)}$ be of the class $CC(X_d) \cap \mathcal{H}(X_d, M, \theta)$. Then, for each $u_\infty \in X_\infty$, we have*

$$\|e^{A_d t} E_d u_\infty - E_d e^{A_\infty t} u_\infty\|_{X_d} \longrightarrow 0 \quad as \quad d \to \infty \tag{2.11}$$

uniformly in compact time intervals away from zero.

Proof: Since A_d is a sectorial operator then for every $d \in [1,\infty]$ we have

$$e^{A_d t} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I - A_d)^{-1} d\lambda,\ t > 0,$$

where, by assumption, Γ is a contour suitably chosen in $\cap_{d\in[1,\infty]} \rho(A_d)$ (see [26]). For each $r > 0$ we write $\Gamma = [\Gamma \cap \mathcal{O}_r(0)] \cup [\Gamma \cap (\mathbb{C} \setminus \mathcal{O}_r(0))] =: \Gamma_r^1 \cup \Gamma_r^2$ and observe that

$$\sup_{d\in[1,\infty]} \| \int_{\Gamma_r^2} e^{\lambda t} (\lambda I - A_d)^{-1} d\lambda \|_{L(X_d)} \leq -\frac{2M e^{rt\cos\phi}}{t\cos\phi},\ t > 0,$$

with a certain $\phi \in (\frac{\pi}{2}, \pi)$. With the aid of Lemma 2.1 we also obtain

$$\begin{aligned}
&\| \int_{\Gamma_r^1} e^{\lambda t} (\lambda I - A_d)^{-1} E_d u_\infty d\lambda - \int_{\Gamma_r^1} e^{\lambda t} E_d (\lambda I - A_\infty)^{-1} u_\infty d\lambda \|_{X_d} \\
&\qquad \leq r \int_{-\phi}^{\phi} \|(re^{i\theta} - A_d)^{-1} E_d u_\infty - E_d (re^{i\theta} - A_\infty)^{-1} u_\infty\|_{X_d} d\theta \stackrel{d\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which proves the result. ▮

Remark 2. 2. Note that the convergence in (2.11) is also uniform for u_∞ varying in compact sets of X_∞ provided that the family $\{E_d\}_{d\in[1,\infty)}$ of connecting maps is bounded (Remark 2.1).

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Remark 2. 3. Using that

$$(-A_d)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{A_d t} dt,$$

it follows from Lemma 2.2 that $(-A_d)^{-\alpha} \dashrightarrow (-A_\infty)^{-\alpha}$ for each $\alpha > 0$.

In further consideration we will consider perturbed semilinear problems for which we define some suitable classes of nonlinearities.

DEFINITION 2.8. We say that $\{F_d\}_{d\in[0,\infty)}$ is of the class $\mathcal{F}(X_d, C)$ if $F_d : X_d \to X_d$ are the maps in the Banach spaces X_d, $d \in [1,\infty)$ satisfying

$$\sup_{d\in[1,\infty)} \sup_{x_d\in X_d} \|F_d(x_d)\|_{X_d} \le C$$

and

$$\|F_d(x_d) - F_d(y_d)\|_{X_d} \le C\|x_d - y_d\|_{X_d} \quad \text{for all } x_d, y_d \in X_d,$$

with constant $C > 0$ being independent of $d \in [1,\infty)$.

If for all $d \in [0,\infty)$ we have $F_d = F$ and $X_d =: X$ then we say that F is of the class $\mathcal{F}(X, C)$.

For $d \in [1,\infty)$ we consider semilinear Cauchy problems

$$\begin{cases} \dot{u} = A_d u + F_d(u), \\ u(0) = u_0, \end{cases} \tag{2.12}$$

where $\{A_d\}_{d\in[1,\infty)}$ is of the class $\mathcal{H}(X_d, M, \theta)$ and $\{F_d\}_{d\in[0,\infty)}$ is of the class $\mathcal{F}(X_d, C)$.

Following [26, Definition 6.1.1] we recall that

DEFINITION 2.9. A set $\mathcal{S}_d \subset X_d$ is an invariant manifold for (2.12) if and only if for each $u_0 \in S_d$ there is a solution $u(t) : \mathbb{R} \to X_d$ of (2.12) such that $u(0) = u_0$ and $u(t) \in \mathcal{S}_d$ for all $t \in \mathbb{R}$.

We also recall that

DEFINITION 2.10. If $\{S_d(t) : t \ge 0\}$ is the semigroup in X_d, then $\mathbf{A}_d$ is a global attractor for $\{S_d(t) : t \ge 0\}$ if and only if $\mathbf{A}_d$ is compact invariant and $\text{dist}_H^{X_d}(S_d(t)B_d, A_d) \to 0$ as $t \to \infty$ whenever B_d is bounded in X_d (see [22]).

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A set $\mathcal{S}_d \subset X_d$ is exponentially attracting under $\{S_d(t) : t \geq 0\}$ if and only if there is a certain $\rho > 0$ such that $e^{\rho t}\mathrm{dist}_H^{X_d}(S(t)B_d, \mathcal{S}_d) \to 0$ as $t \to \infty$ whenever B_d is bounded in X_d.

In what follows our main concern will be to prove the following result.

THEOREM 2.1. *Suppose that* $\{F_d\}_{d\in[0,\infty)}$ *is of the class* $F(X_d, C)$ *and* $\{A_d\}_{d\in[1,\infty)}$ *is of the class* $CC(X_d) \cap \mathcal{H}(X_d, M, \theta)$.

Choose $\delta > 0$ *and* $R > 1 + \max\{|\lambda_{\infty,j}|, j = 1, \ldots, k\}$ *according to Corollary 2.2 and define the spectral sets*

$$\sigma_d^+ := \{\lambda_d \in \sigma(A_d) : |\lambda_d| < R\},\ d \geqslant d(R), \tag{2.13}$$

and the corresponding projections

$$Q_d(\sigma_d^+) : X_d \to X_d, \quad Q_d(\sigma_d^+) = \frac{1}{2\pi i}\int_{|\xi|=R} (\xi I - A_d)^{-1} d\xi,\ d \geqslant d(R). \tag{2.14}$$

Define also projected spaces

$$Y_d := Q_d(\sigma_d^+)(X_d),\ Z_d = (I - Q_d(\sigma_d^+))(X_d),\ d \geqslant d(R), \tag{2.15}$$

and projected operators

$$\Lambda_d^1 = A_d|_{Y_d},\ \Lambda_d^2 = A_d|_{Z_d},\ d \geqslant d(R).$$

Assume finally that the following condition $(\mathcal{D})$ *holds:*

$(\mathcal{D})$*the semigroups generated by* Λ_d^1, Λ_d^2 *with* $d \geq d(R)$ *satisfy the estimates*

$$\begin{aligned} \|e^{\Lambda_d^1 t} z\|_{X_d} &\leqslant M\, e^{\beta(d)t}\|z\|_{X_d}, \quad t \leqslant 0,\ z \in Y_d \\ \|e^{\Lambda_d^2 t} z\|_{X_d} &\leqslant M\, e^{-\gamma(d)t}\|z\|_{X_d}, \quad t \geqslant 0,\ z \in Z_d, \end{aligned} \tag{2.16}$$

where $|\beta(d)| \leq \beta$, $0 < \gamma(d) \overset{d\to\infty}{\to} \infty$ *and* $M > 1$ *does not depend on* $d \geq d(R)$.

Then,
i) for each d *sufficiently large there is an invariant manifold* $\mathcal{S}_d$ *for* (2.20), *which is given with the aid of a certain Lipschitz continuous map* $\Sigma_d^* : Y_d \to Z_d$ *as*

$$\mathcal{S}_d = \{u_d \in X_d : u_d = Q_d(\sigma_d^+)(u_d) + \Sigma_d^*(Q_d(\sigma_d^+)(u_d))\},$$

ii) the maps $\Sigma_d^*: Y_d \to Z_d$ satisfy the conditions

$$|||\Sigma_d^*||| := \sup_{u_d^1 \in Y_d} \|\Sigma_d^*(u_d^1)\|_{X_d} \stackrel{d\to\infty}{\longrightarrow} 0 \tag{2.17}$$

and

$$\sup_{\substack{u_d^1, \tilde{u}_d^1 \in Y_d \\ u_d^1 \neq \tilde{u}_d^1}} \frac{\|\Sigma_d^*(u_d^1) - \Sigma_d^*(\tilde{u}_d^1)\|_{X_d}}{\|u_d^1 - \tilde{u}_d^1\|_{X_d}} \leqslant L(d) \stackrel{d\to\infty}{\longrightarrow} 0, \tag{2.18}$$

iii) $\mathcal{S}_d$ is exponentially attracting.

In particular,

iv) the problem (2.12) generates in X_d a C^0 semigroup $\{S_d(t) : t \geq 0\}$ which has a global attractor $\mathbf{A}_d \subset \mathcal{S}_d$ and the flow on $\mathbf{A}_d$ is given by

$$u(t) = v(t) + \Sigma_d^*(v(t)), \ t \in \mathbb{R}$$

where

$$\dot{v} = \Lambda_d^1 v + Q_d(\sigma_d^+) F_d(v, \Sigma_d^*(v)). \tag{2.19}$$

Remark 2. 4. Note that with the above setup we have that $X_d = Y_d \oplus Z_d$, that Λ_d^2 generates a strongly continuous semigroup on Z_d and that $\Lambda_d^1 \in L(Y_d)$ for every $d \geqslant d(R)$. Also note that, if A_d are all selfadjoint bounded from above operators in Hilbert spaces X_d, $d \in [1, \infty)$, and $\{A_d\}_{d\in[1,\infty)}$ is of the class $CC(X_d)$ so that Corollary 2.2 applies, then $\mathcal{D}$ is satisfied, which is the case in many applications (see Subsections 3.1 and 3.2). To see that M in (2.16) can be chosen independently of d one can use the numerical range to estimate the resolvent of A_d or, when A_d has compact resolvent, to use Fourier series.

Theorem 2.6 will be proved in a sequence of lemmas.

Lemma 2.3. *Suppose that $\{F_d\}_{d\in[0,\infty)}$ is in $F(X_d, C)$, $\{A_d\}_{d\in[1,\infty)}$ is in $CC(X_d) \cap \mathcal{H}(X_d, M, \theta)$ and $\mathcal{D}$ holds. For $u_d(t)$ being a solution of (2.12) let*

$$u_d^1(t) = Q_d(\sigma_d^+)(u_d(t)), \quad u_d^2(t) = (I - Q_d(\sigma_d^+))(u_d(t))$$

and consider the equations

$$\dot{u}_d^1 = \Lambda_d^1 u_d^1 + H_d(u_d^1, u_d^2), \quad \dot{u}_d^2 = \Lambda_d^2 u_d^2 + G_d(u_d^1, u_d^2), \tag{2.20}$$

where

$$H_d(u_d^1, u_d^2) = Q_d(\sigma_d^+)F_d(u_d^1, u_d^2), \quad G_d(u_d^1, u_d^2) = (I - Q_d(\sigma_d^+))F_d(u_d^1, u_d^2)$$

and σ_d^+, $Q_d(\sigma_d^+)$ *are given by* (2.13)-(2.14) *with* $\delta > 0$ *and* $R > 0$ *as in Corollary 2.2.*

Then for each d *sufficiently large there is an invariant manifold* $\mathcal{S}_d$ *for* (2.20) *and*

$$\mathcal{S}_d = \{u_d \in X_d : u_d = Q_d(\sigma_d^+)(u_d) + \Sigma_d^*(Q_d(\sigma_d^+)(u_d))\}$$

where $\Sigma_d^* : Y_d \to Z_d$ *is a certain Lipschitz map and* Y_d, Z_d *are as in* (2.15).

Proof: The proof of this lemma is adapted from [3, 4] and is given here for completeness of the presentation.

Given any $L, D > 0$ we define the complete metric space

$$\mathfrak{X}_d = \{\Sigma_d : Y_d \to Z_d;\ \|\Sigma_d\| \leqslant D,\ \|\Sigma_d(u_d^1) - \Sigma_d(\tilde{u}_d^1)\|_{X_d} \leqslant L\|u_d^1 - \tilde{u}_d^1\|_{X_d}\}$$

with distance between $\Sigma_d, \tilde{\Sigma}_d \in \mathfrak{X}_d$ given by

$$\|\Sigma_d - \tilde{\Sigma}_d\| := \sup_{z \in Y_d} \|\Sigma_d(z) - \tilde{\Sigma}_d(z)\|_{X_d}.$$

By assumption there is a certain $\rho > 0$ such that for all $u_d^1, \tilde{u}_d^1 \in Y_d$ and $u_d^2, \tilde{u}_d^2 \in Z_d$ we have

$$\begin{aligned}
&\|H_d(u_d^1, u_d^2)\|_{X_d} \leqslant \varrho, \quad \|G_d(u_d^1, u_d^2)\|_{X_d} \leqslant \varrho, \\
&\|H_d(u_d^1, u_d^2) - H_d(\tilde{u}_d^1, \tilde{u}_d^2)\|_{X_d} \leqslant \varrho(\|u_d^1 - \tilde{u}_d^1\|_{X_d} + \|u_d^2 - \tilde{u}_d^2\|_{X_d}), \\
&\|G_d(u_d^1, u_d^2) - G_d(\tilde{u}_d^1, \tilde{u}_d^2)\|_{X_d} \leqslant \varrho(\|u_d^1 - \tilde{u}_d^1\|_{X_d} + \|u_d^2 - \tilde{u}_d^2\|_{X_d}).
\end{aligned} \tag{2.21}$$

Since $(\mathcal{D})$ holds there is a certain $d_0 = d_0(D, L) \geq 1$ such that for all $d \geq d_0$ we have

$$\frac{\varrho M}{\gamma(d)} \leqslant D, \quad \frac{\varrho(1+L)M^2}{[\gamma(d) + \beta(d) - \varrho(1+L)M]} \leqslant L, \tag{2.22}$$

and

$$\begin{aligned}
I_\Sigma(d) &:= \frac{\varrho M}{\gamma(d)} + \frac{\varrho^2 M^2 (1+L)}{(\varrho(1+L)M - \beta(d))(\gamma(d) + \beta(d) - \varrho(1+L)\ M)} < 1 \\
I_\zeta(d) &:= \frac{\varrho(1+L)M^2 + M}{\gamma(d) + \beta(d) - \varrho(1+L)M} \leq L.
\end{aligned} \tag{2.23}$$

In what follows we are going to find $\Sigma_d^* \in \mathfrak{X}_d$ such that, if $\tau \in \mathbb{R}$ and $(\zeta, \Sigma_d^*(\zeta)) \in Y_d \times Z_d$, then the solution of (2.20) starting at $u_d^1(\tau) = \zeta$ and $u_d^2(\tau) = \Sigma_d^*(\zeta)$ is such that $u_d^2(t) = \Sigma_d^*(u_d^1(t))$ for all $t \geqslant 0$ in which case (2.20) reads

$$\dot{u}_d^1 = \Lambda_d^1 u_d^1 + H_d(u_d^1, \Sigma_d^*(u_d^1)), \quad \dot{u}_d^2 = \Lambda_d^2 u_d^2 + G_d(u_d^1, \Sigma_d^*(u_d^1)). \tag{2.24}$$

Since

$$u_d^2(t) = e^{\Lambda_d^2(t-t_0)} u_d^2(t_0) + \int_{t_0}^{t} e^{\Lambda_d^2(t-s)} G_d(u_d^1(s), \Sigma_d^*(u_d^1(s)))ds,$$

then, letting $t_0 \to -\infty$, we have

$$u_d^2(t) = \Sigma_d^*(u_d^1(t)) = \int_{-\infty}^{t} e^{\Lambda_d^2(t-s)} G_d(u_d^1(s), \Sigma_d^*(u_d^1(s)))ds,$$

and

$$\Sigma_d^*(\zeta) = \Sigma_d^*(u_d^1(\tau)) = u_d^2(\tau) = \int_{-\infty}^{\tau} e^{\Lambda_d^2(t-s)} G_d(u_d^1(s), \Sigma_d^*(u_d^1(s)))ds.$$

The existence of such Σ^* will follow from the Banach fixed point theorem.

For $\tau \in \mathbb{R}$ and arbitrary $\zeta \in Y_d$, $\Sigma_d \in \mathfrak{X}_d$ we denote by $u_d^1(t) = \psi(t, \tau, \zeta, \Sigma_d)$ the solution of

$$\dot{u}_d^1 = \Lambda_d^1 u_d^1 + H_d(u_d^1, \Sigma_d(u_d^1)),\ t < \tau, \quad u_d^1(\tau) = \zeta, \tag{2.25}$$

and define $\Phi_d : \mathfrak{X}_d \to \mathfrak{X}_d$ by

$$\Phi_d(\Sigma_d)(\zeta) = \int_{-\infty}^{\tau} e^{\Lambda_d^2(\tau-s)} G_d(u_d^1(s), \Sigma_d(u_d^1(s)))ds. \tag{2.26}$$

Note that by (2.16) we then have

$$\|\Phi_d(\Sigma_d)(\zeta)\|_{X_d} \leqslant \varrho M \int_{-\infty}^{\tau} e^{-\gamma(d)(\tau-s)} ds = \frac{\varrho M}{\gamma(d)}, \tag{2.27}$$

and hence $\|\Phi_d(\Sigma_d)(\zeta)\|_{X_d} \leqslant D$ via (2.22).

For $\Sigma_d, \tilde{\Sigma}_d \in \mathfrak{X}_d$, $\zeta, \tilde{\zeta} \in Y_d$ denote $u_d^1 = \psi(t, \tau, \zeta, \Sigma_d)$, $\tilde{u}_d^1 = \psi(t, \tau, \tilde{\zeta}, \tilde{\Sigma}_d)$. Then

$$\begin{aligned} u_d^1 - \tilde{u}_d^1 &= e^{\Lambda_d^1(t-\tau)}(\zeta - \tilde{\zeta}) \\ &\quad + \int_{\tau}^{t} e^{\Lambda_d^1(t-\tau)} [H_d(u_d^1(s), \Sigma_d(u_d^1(s))) - H_d(\tilde{u}_d^1(s), \tilde{\Sigma}_d(\tilde{u}_d^1(s)))]ds \end{aligned}$$

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and with (2.16), (2.21) we obtain

$$\begin{aligned}&\|u_d^1-\tilde u_d^1\|_{X_d}\leqslant Me^{\beta(d)(t-\tau)}\|\zeta-\tilde\zeta\|_{X_d}\\&+M\int_t^\tau e^{\beta(d)(t-s)}\|H_d(u_d^1(s),\Sigma_d(u_d^1(s)))-H_d(\tilde u_d^1(s),\tilde\Sigma_d(\tilde u_d^1(s)))\|_{X_d}\,ds\\&\leqslant M\,e^{\beta(d)(t-\tau)}\|\zeta-\tilde\zeta\|_{X_d}\\&+\varrho(1+L)M\int_t^\tau e^{\beta(d)(t-s)}\|u_d^1-\tilde u_d^1\|_{X_d}\,ds+\varrho M|||\Sigma_d-\tilde\Sigma_d|||\int_t^\tau e^{\beta(d)(t-s)}ds.\end{aligned}$$

Since for $\phi(t)=e^{-\beta(d)(t-\tau)}\|u_d^1-\tilde u_d^1\|_{X_d}$ we have

$$\begin{aligned}\phi(t)\leqslant M\|\zeta-\tilde\zeta\|_{X_d}&+\varrho M|||\Sigma_d-\tilde\Sigma_d|||\int_t^\tau e^{\beta(d)(\tau-s)}ds\\&+\varrho(1+L)M\int_t^\tau\phi(s)ds,\end{aligned}$$

then

$$\begin{aligned}\|u_d^1(t)-\tilde u_d^1(t)\|_{X_d}\leqslant{}&\frac{M\varrho(1+L)+1}{\rho(1+L)}\|\zeta-\tilde\zeta\|_{X_d}e^{[\varrho(1+L)M-\beta(d)](\tau-t)}\\&+\frac{\varrho M}{\varrho(1+L)M-\beta(d)}|||\Sigma_d-\tilde\Sigma_d|||e^{[\varrho(1+L)M-\beta(d)](\tau-t)}.\end{aligned}\tag{2.28}$$

Now, with the same argument above, it follows that

$$\begin{aligned}&\|\Phi_d(\Sigma)(\zeta)-\Phi(\tilde\Sigma)(\tilde\zeta)\|_{X_d}\\&\qquad\leqslant\varrho M\int_{-\infty}^\tau e^{-\gamma(d)(\tau-s)}\big[(1+L)\|u_d^1-\tilde u_d^1\|_{X_d}+|||\Sigma_d-\tilde\Sigma_d|||\big]\,ds\end{aligned}$$

and using (2.28) we get

$$\|\Phi_d(\Sigma_d)(\zeta)-\Phi(\tilde\Sigma_d)(\tilde\zeta)\|_{X_d}\leqslant I_\Sigma(d)|||\Sigma_d-\tilde\Sigma_d|||+I_\zeta(d)\|\zeta-\tilde\zeta\|_{X_d},\tag{2.29}$$

with $I_\Sigma(d)$ and $I_\zeta(d)$ as in (2.23). Therefore, there is a unique fixed point $\Phi(\Sigma_d^*)=\Sigma_d^*\in\mathfrak{X}_d$.

Now let $(x_0,y_0)\in\mathcal{S}_d$, $y_0=\Sigma_d^*(x_0)$ and let $x_d^*(t)$ be the solution of

$$\frac{dx}{dt}=\Lambda_d^1x+H_d(x,\Sigma_d^*(x)),\quad x(0)=x_0.$$

This defines a curve $(x_d^*(t),\Sigma_d^*(x_d^*(t))\in\mathcal{S}_d$, $t\in\mathbb{R}$. But there is a solution of

$$\dot y=\Lambda_d^2y+G_d(x_d^*(t),\Sigma_d^*(x_d^*(t))),$$

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given by the integral formula

$$y_d^*(t) = \int_{-\infty}^{t} e^{\Lambda_d^2(t-s)} G_d(x_d^*(s), \Sigma_d^*(x_d^*(s)) ds = \Sigma_d^*(x_d^*),$$

which stays bounded as $t \to -\infty$. Therefore, $(x_d^*(t), \Sigma_d^*(x_d^*))$, $t \in \mathbb{R}$, is a solution of (2.20) through (x_0, y_0) and thus $\mathcal{S}_d$ satisfies the requirements of the Definition 2.9. ▌

LEMMA 2.4. *Under the assumptions and notation of Lemma 2.3 both* (2.17) *and* (2.18) *hold.*

Proof: Note that (2.27) implies

$$\sup_{\zeta \in Y_d} \|\Sigma_d^*(\zeta)\|_{X_d} = \sup_{\zeta \in Y_d} \|\Phi_d(\Sigma_d^*)(\zeta)\|_{X_d} \leqslant \varrho M \int_{-\infty}^{\tau} e^{-\gamma(d)(\tau - s)} ds = \frac{\varrho M}{\gamma(d)}$$

where $\gamma(d) \to 0$ as $d \to \infty$ by assumption $(\mathcal{D})$. Using (2.29) we also have

$$\|\Sigma_d^*(\zeta) - \Sigma_d^*(\tilde{\zeta})\|_{X_d} = \|\Phi_d(\Sigma_d^*)(\zeta) - \Phi(\tilde{\Sigma}_d^*)(\tilde{\zeta})\|_{X_d} \leqslant I_\zeta(d)\|\zeta - \tilde{\zeta}\|_{X_d}$$

with $I_\zeta(d)$ as in (2.23); i.e. $I_\zeta(d)$ converging to zero as $d \to \infty$. ▌

We now show an exponential attracting property of the invariant manifold constructed in Lemma 2.3 above.

LEMMA 2.5. *Under the assumptions and notation of Lemma 2.3 for any* $\bar{\gamma} > 0$ *there is a certain* $d(\bar{\gamma}) \geq 1$ *such that for all* $d \geq d(\bar{\gamma})$ *and for any solution* $(u_d^1(t), u_d^2(t))$, $t \geq t_0$, *of* (2.20) *we have*

$$\|u_d^2(t) - \Sigma_d^*(u_d^1(t))\|_{X_d} \leqslant M e^{-\bar{\gamma}(t-t_0)} \|u_d^2(t_0) - \Sigma_d^*(u_d^1(t_0))\|_{X_d}, \ t \geqslant t_0, \tag{2.30}$$

where M *is a constant appearing in the condition* $(\mathcal{D})$.

Proof: Let $\xi_d(t) = u_d^2(t) - \Sigma_d^*(u_d^1(t))$ and $y^1(s,t)$, $s < t$, be the solution to

$$\dot{y}^1 = \Lambda_d^1 y^1 + H_d(y^1, \Sigma_d^*(y^1)), \ s < t, \quad y^1(t,t) = u_d^1(t).$$

Then,

$$\begin{aligned} &\|y^1(s,t) - u_d^1(s)\|_{X_d} \\ &\leqslant \varrho M \int_s^t e^{\beta(d)(s-\theta)} \left[(1+L)\|y^1(\theta,t) - u_d^1(\theta)\|_{X_d} + \|\xi_d(\theta)\|_{X_d}\right] d\theta. \end{aligned}$$

and $z(s) := e^{-\beta(d)s}\|y^1(s,t) - u_d^1(s)\|_{X_d}$ satisfies the relation

$$z(s) \leqslant \varrho M \int_s^t e^{-\beta(d)\theta} \|\xi_d(\theta)\|_{X_d} d\theta + \varrho M(1+L) \int_s^t , z(\theta)\ d\theta,$$

which implies via Gronwall's inequality that

$$\|y^1(s,t) - u_d^1(s)\|_{X_d} \leqslant \varrho M \int_s^t e^{-[\beta(d)-\varrho(1+L)M](\theta-s)} \; \|\xi(\theta)\|_{X_d}, \; s \leqslant t. \tag{2.31}$$

Similarly, for $s \leqslant t_0 \leqslant t$ we also obtain with the aid of (2.31) that

$$\begin{aligned}
&\|y^1(s,t) - y^1(s,t_0)\|_{X_d} \leqslant \|e^{\Lambda_d^1(s-t_0)} \left[y^1(t_0,t) - u_d^1(t_0)\right] \|_{X_d} + \\
&\|\int_{t_0}^s e^{\Lambda_d^1(s-\theta)}] \, [H_d(y^1(\theta,t), \Sigma_d^*(y^1(\theta,t))) - H_d(y^1(\theta,t_0), \Sigma_d^*(y^1(\theta,t_0))) \;] d\theta \|_{X_d} \\
&\leqslant \varrho M^2 e^{\beta(d)(s-t_0)} \int_{t_0}^t e^{-[\beta(d)-\varrho(1+L)M](\theta-t_0)} \|\xi_d(\theta)\|_{X_d} d\theta + \\
&\varrho(1+L)M \int_s^{t_0} e^{\beta(d)(s-\theta)} \|y^1(\theta,t) - y^1(\theta,t_0)\|_{X_d} d\theta
\end{aligned}$$

and using again Gronwall's inequality we have

$$\|y^1(s,t) - y^1(s,t_0)\|_{X_d} \leqslant \varrho M^2 \int_{t_0}^t e^{-[\beta(d)-\varrho(1+L)M](\theta-s)} \|\xi_d(\theta)\|_{X_d} d\theta. \tag{2.32}$$

In what follows we estimate $\xi_d(t)$. Note that

$$\begin{aligned}
&\xi_d(t) - e^{\Lambda_d^2(t-t_0)} \xi_d(t_0) \\
&= \int_{t_0}^t e^{\Lambda_d^2(t-s)} [G_d(u_d^1(s), u_d^2(s)) - G_d(y^1(s,t), \Sigma_d^*(y^1(s,t)))] ds \\
&- \int_{-\infty}^{t_0} e^{\Lambda_d^2(t-s)} [G_d(y^1(s,t), \Sigma_d^*(y^1(s,t))) - G_d(y^1(s,t_0), \Sigma_d^*(y^1(s,t_0))] ds.
\end{aligned}$$

Thus, using (2.31) and (2.32), we obtain

$$\begin{aligned}
&\|\xi_d(t) - e^{\Lambda_d^2(t-t_0)}\xi_d(t_0)\|_{X_d} \\
&\leqslant \varrho M \int_{t_0}^{t} e^{-\gamma(d)(t-s)} \|\xi_d(s)\|_{X_d} ds \\
&+ \varrho^2 L M^2 \int_{t_0}^{t} e^{-\gamma(d)(t-s)} \int_{s}^{t} e^{-[\beta(d)-\varrho(1+L)M](\theta-s)} \|\xi_d(\theta)\|_{X_d} d\theta ds \\
&+ \varrho^2 (1+L) M^3 \int_{-\infty}^{t_0} e^{-\gamma(d)(t-s)} \int_{t_0}^{t} e^{-[\beta(d)-\varrho(1+L)M](\theta-s)} \|\xi_d(\theta)\|_{X_d} d\theta ds,
\end{aligned}$$

so that we get

$$\begin{aligned}
&\|\xi_d(t) - e^{\Lambda_d^2(t-t_0)}\xi_d(t_0)\|_{X_d} \leqslant \varrho M \int_{t_0}^{t} e^{-\gamma(d)(t-s)} \|\xi_d(s)\|_{X_d} ds \\
&+ \varrho^2 (1+L) M^3 \int_{t_0}^{t} \int_{-\infty}^{\theta} e^{-\gamma(d)(t-s)} ds \, e^{-[\beta(d)-\varrho(1+L)M](\theta-s)} \|\xi_d(\theta)\|_{X_d} \; d\theta.
\end{aligned}$$

Consequently, letting $b(d) := \varrho M + \dfrac{\varrho^2(1+L)M^3}{\gamma(d)+\beta(d)-\varrho(1+L)M}$, we infer that

$$\|\xi_d(t) - e^{\Lambda_d^2(t-t_0)}\xi_d(t_0)\|_{X_d} \leqslant b(d) \int_{t_0}^{t} e^{-\gamma(d)(t-\theta)} \|\xi_d(\theta)\|_{X_d} d\theta.$$

Thus

$$\|\xi_d(t)\|_{X_d} \leqslant M e^{-\gamma(d)(t-t_0)} \|\xi_d(t_0)\|_{X_d} + b(d) \int_{t_0}^{t} e^{-\gamma(d)(t-\theta)} \|\xi_d(\theta)\|_{X_d} d\theta$$

and we conclude that

$$\|\xi_d(t)\|_{X_d} \leqslant M \|\xi_d(t_0)\|_{X_d} e^{-(\gamma(d)-b(d))(t-t_0)}.$$

Since $\gamma(d) \to \infty$ and $b(d) \to \varrho M$ as $d \to \infty$ the proof is complete. ▌

LEMMA 2.6. *Under the assumptions of Lemma 2.3 the problem* (2.12) *generates in* X_d*,* $d \geq 1$*, a* C^0 *semigroup* $\{S_d(t) : t \geq 0\}$ *of global solutions which has a global attractor* $\mathbf{A}_d$*. For all* d *sufficiently large the attractor* $\mathbf{A}_d$ *is contained in the invariant manifold* $\mathcal{S}_d$ *constructed in Lemma 2.3 and* $\mathcal{S}_d$ *is exponentially attracting,*

Proof: By assumption all solutions of (2.12) exist globally in time and are given by the variation of constants formula

$$u(t) = e^{A_d t} u_0 + \int_0^t e^{A_d(t-s)} F_d(u(s)) ds, \ t \geq 0, \tag{2.33}$$

where the linear semigroup $\{e^{A_d t} : t \geq 0\}$ is asymptotically decaying. Consequently, there is a bounded set in X_d absorbing bounded sets and, since the resolvent of A_d is compact, we infer that the semigroup $S_d(t) : X_d \to X_d$, $t \geq 0$, of global solutions of (2.12) has a global attractor $\mathbf{A}_d$.

¿From Lemma 2.5, we conclude that $\mathbf{A}_d \subset \mathcal{S}_d$. Indeed, if $S_d(t)u_{d0} = u_d^1(t) + u_d^2(t)$, where $t \in \mathbb{R}$, denotes the solution passing at 'time' zero through a given point $u_{d0} = u_{d0}^1 + u_{d0}^2 \in \mathbf{A}_d$, then (2.30) reads

$$\|u_d^2(t) - \Sigma_d^*(u_d^1(t))\|_{X_d} \leqslant Me^{-\bar{\gamma}(t-t_0)} \|u_d^2(t_0) - \Sigma_d^*(u_d^1(t_0))\|_{X_d}, \ t \geqslant t_0. \tag{2.34}$$

Since $\gamma_d(u_{d0}) = \{S_d(t)u_{d0} : t \in \mathbb{R}\} \subset \mathbf{A}_d$, then $\gamma_d(u_{d0})$ is bounded and letting $t_0 \to -\infty$ we obtain that $S_d(t)u_{d0} = u_d^1(t) + \Sigma_d^*(u_d^1(t)) \in \mathcal{S}_d$ for every $t \in \mathbb{R}$, which proves the inclusion.

Finally, if $B_d \subset X_d$ is bounded and $u_{d0} = u_{d0}^1 + u_{d0}^2 \in B_d$, then using again (2.30) we conclude that $S_d(t)u_{d0} = u_d^1(t) + u_d^2(t)$ satisfies the relation

$$\begin{aligned} \sup_{u_d \in B_d} \inf_{v \in \mathcal{S}_d} \|S_d(t)u_{d0} - v\|_{X_d} &\leqslant \sup_{u_{d0} \in B_d} \|u_d^2(t) - \Sigma_d^*(u_d^1(t))\|_{X_d} \\ &\leqslant Me^{-\bar{\gamma} t} \sup_{u_{d0} \in B_d} \|u_{d0}^2 - \Sigma_d^*(u_{d0}^1)\|_{X_d}. \end{aligned}$$

Thus

$$\text{dist}_H^{X_d}(S_d(t)B_d, \mathcal{S}_d) \leq c(B_d)e^{-\bar{\gamma} t} \text{ whenever } B_d \text{ is bounded in } X_d. \tag{2.35}$$

and thus the proof is complete. ▌

Remark 2. 5. Note that due to Lemma 2.5 the exponent $\bar{\gamma}$ in (2.35) can be chosen as large as we wish.

We remark that the above consideration can be generalized to include the case when the nonlinear term F_d in (2.12) is defined on the fractional power space X_d^α, $\alpha \in (0, 1)$.

DEFINITION 2.11. Given one-sided fractional power scales $\{X_d^\alpha, \ \alpha \geq 0\}$ and $\alpha \in (0, 1)$ we say that $\{F_d\}_{d \in [1,\infty)}$ is of the class $\mathcal{F}(X_d^\alpha, C)$ if and only if $F_d : X_d^\alpha \to X_d$ is differentiable and satisfies

$$\sup_{d\in[1,\infty)} \sup_{x_d\in X_d^\alpha} \|F_d(x_d)\|_{X_d} \le C$$

and

$$\|F_d(x_d) - F_d(y_d)\|_{X_d} \le C\|x_d - y_d\|_{X_d^\alpha} \quad \text{for all } x_d, y_d \in X_d^\alpha,$$

where constant $C > 0$ is independent of $d \in [1,\infty)$. Of course, the second condition above implies that $\sup_{d\in[1,\infty)} \sup_{u\in X^\alpha} \|F_d'(u)\|_{L(X_d^\alpha, X_d)} \le C$.

In a similar way as in Lemma 2.3 and [26, Theorem 6.1.7] the following result can be proved.

PROPOSITION 2.2. *Suppose that* $\{F_d\}_{d\in[1,\infty)}$ *is of the class* $F(X_d^\alpha, C)$, $\{A_d\}_{d\in[1,\infty)}$ *is of the class* $CC(X_d) \cap \mathcal{H}(X_d, M, \theta)$ *and the condition* $\mathcal{D}$ *holds. For* $u_d(t)$ *being a solution of* (2.12) *let*

$$u_d^1(t) = Q_d(\sigma_d^+)(u_d(t)), \quad u_d^2(t) = (I - Q_d(\sigma_d^+))(u_d(t))$$

and consider the equations

$$\dot u_d^1 = \Lambda_d^1 u_d^1 + H_d(u_d^1, u_d^2), \quad \dot u_d^2 = \Lambda_d^2 u_d^2 + G_d(u_d^1, u_d^2), \tag{2.36}$$

where

$$H_d(u_d^1, u_d^2) = Q_d(\sigma_d^+) F(u_d^1, u_d^2), \quad G_d(u_d^1, u_d^2) = (I - Q_d(\sigma_d^+)) F(u_d^1, u_d^2)$$

and σ_d^+, $Q_d(\sigma_d^+)$ *are given by* (2.13)-(2.14) *with* $\delta > 0$ *and* $R > 0$ *as in Corollary 2.2.*

Then for all d sufficiently large there is an exponentially attracting invariant manifold

$$\mathcal{S}_d = \{u_d \in X_d : u_d = Q_d(\sigma_d^+)(u_d) + \Sigma_d^*(Q_d(\sigma_d^+)(u_d))\}$$

for (2.20). *The map* $\Sigma_d^* : Y_d \to Z_d^\alpha$ *is differentiable,*

$$\|\Sigma_d^*\| := \sup_{u_d^1\in Y_d} \|\Sigma_d^*(u_d^1)\|_{X_d^\alpha} \overset{d\to\infty}{\longrightarrow} 0,$$

$$\|\Sigma_d^*(u_d^1) - \Sigma_d^*(\tilde u_d^1)\|_{X_d^\alpha} \leqslant L(d)\|u_d^1 - \tilde u_d^1\|_{X_d}, \quad L(d) \overset{d\to\infty}{\longrightarrow} 0$$

and from this

$$\sup_{u_d\in Y_d} \|\Sigma_d^{*\prime}(u_d)\|_{L(Y_d, Z_d^\alpha)} \le L(d) \overset{d\to\infty}{\to} 0.$$

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3. PART II: APPLICATIONS

In this section we will apply abstract results developed in Section 2 to sample problems originating in applications. These will be atmospherics' and cells' modeling reaction-diffusion equations.

3.1. A cell tissue reaction-diffusion problem

In this section we consider a perturbed one dimensional scalar parabolic equation with Neumann boundary condition and with the diffusivity being large except in neighborhoods of some chosen points where it becomes small (see [5, 6, 10, 13, 20]). This situation can be found in modeling of a linearly coupled cell tissue for which the diffusivity is small in the membranes and large elsewhere.

Consider the problem

$$\begin{cases} u_t = (a^d u_x)_x + f(u), & x \in (0,1),\ t > 0 \\ u_x(0) = u_x(1) = 0, & t > 0, \end{cases} \tag{3.1}$$

where we assume that

(a_1) $a^d \in C^2([0,1],\mathbb{R})$ and $a^d(x) > 0$ for each $x \in [0,1]$,

(a_2) if $0 = x_0 < x_1 < \ldots < x_n = 1$ is a partition of $[0,1]$, $e_1,\ldots,e_n$, $l_0,\ldots l_n$, $a_0^d,\ldots,a_n^d$ are positive constants and ${l'}_0^d,\ldots,{l'}_n^d$, ${a'}_0^d,\ldots,{a'}_n^d$ are functions depending on $d \geq 1$ and approaching (respectively) $l_0,\ldots l_n$, $a_0,\ldots,a_n$ from above as $d \to +\infty$, then

$$a^d(x) := \begin{cases} \geqslant de_i, & \text{for } x_{i-1} + d^{-1}l'_{i-1} \leqslant x \leqslant x_i - d^{-1}l'_i,\ i = 1,\ldots,n, \\ \geqslant d^{-1}a_i, & \text{for } x_i - d^{-1}l'_i \leqslant x \leqslant x_i + d^{-1}l'_i,\ i = 0,\ldots,n, \\ \leqslant d^{-1}a'_i, & \text{for } x_i - d^{-1}l_i \leqslant x \leqslant x_i + d^{-1}l_i,\ i = 0,\ldots,n, \end{cases} \tag{3.2}$$

where we set $x_0 - d^{-1}l'_0 = x_0 - d^{-1}l_0 = 0, \quad x_n + d^{-1}l'_n = x_n + d^{-1}l_n = 1,$

and

(a_3) $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \theta < 0. \tag{3.3}$$

We remark that the coefficient a^d in the interval $[0,1]$ is thus large except neighborhoods of some chosen points where it becomes small.

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In the analysis of (3.1) out primary concern will be the eigenvalue problem of the Sturmian type

$$\begin{cases} -(a^d u_x)_x + u = \lambda u, & x \in (0,1),\ t > 0 \\ u_x(0) = u_x(1) = 0, & t > 0. \end{cases} \tag{3.4}$$

In fact we will show that compact convergence approach of Section 2 leads to the following result.

THEOREM 3.1. *Suppose that the conditions $(a_1) - (a_2)$ are satisfied. Let $\{\lambda_k^d\}_{k=1}^{\infty}$ be the set of ordered eigenvalues for* (3.4) *(counting their algebraic multiplicity) and let $\{\phi_k^d\}_{k=1}^{\infty}$ be the corresponding set of orthonormal eigenfunctions. Also let $\{\lambda_k^{\infty}\}_{k=1}^{n}$ be the ordered set of eigenvalues (counting their algebraic multiplicity) for*

$$B_\infty \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \lambda \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}, \tag{3.5}$$

where

$$B_\infty := \mathrm{diag}(L_0^{-1}, \ldots, L_{n-1}^{-1}) \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & \cdots & & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \cdots & & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & \alpha_4 & \cdots & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{n-2} & \beta_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}, \tag{3.6}$$

$$\begin{aligned} &L_{k-1} = x_k - x_{k-1},\ k = 1, \ldots, n, \quad \alpha_1 = \frac{a_1}{2l_1} + L_0, \\ &\alpha_k = \frac{a_{k-1}}{2l_{k-1}} + \frac{a_k}{2l_k} + L_{k-1},\ k = 2, \ldots, n-1, \quad \alpha_n = \frac{a_{n-1}}{2l_{n-1}} + L_{n-1}, \\ &\beta_k = -\frac{a_k}{2l_k}, \quad k = 1, 2, \ldots, n-1, \end{aligned} \tag{3.7}$$

and let $\{\phi_k^{\infty}\}_{k=1}^{n}$ be the corresponding set of orthonormal eigenfunctions.
Then
i) $\lambda_k^d \to \lambda_k^{\infty}$ as $d \to \infty$ for each $k = 1, 2, \cdots, n$,
ii) $\lambda_k^d \to \infty$ as $d \to \infty$ for all $k > n$,
iii) $\|\phi_k^d - E\phi_k^{\infty}\|_{L^2(0,1)} \to 0$ as $d \to \infty$ for every $k = 1, 2, \cdots, n$, where

$$E\mathbf{c} = \begin{cases} c_1, & x \in (x_0, x_1), \\ \vdots & \vdots \quad \vdots \\ c_n, & x \in (x_{n-1}, x_n), \end{cases} \qquad \textit{for} \quad \mathbf{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n. \tag{3.8}$$

Theorem 3.1 will follow from Corollary 2.2 and Lemma 3.1 below, in which we establish a suitable compact convergence property.

Lemma 3.1. *For each* $d \in [1, \infty)$ *let* $X_d = L^2(0,1)$ *and* $B_d : D(B_d) \subset L^2(0,1) \to L^2(0,1)$ *be given by*

$$B_d u = -(au_x)_x + u \quad \text{for} \quad u \in D(B_d) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\}. \tag{3.9}$$

Then $\{B_d\}_{d\in[1,\infty)}$ *is of the class* $CC(L^2(0,1))$. *In fact,*

$$B_d \overset{cc}{\dashrightarrow} B_\infty,$$

where the operator $B_\infty : X_\infty \to X_\infty$ *is given by the matrix in* (3.6) *on the Hilbert space* $X_\infty = \mathbb{R}^n$ *with the scalar product*

$$\langle x_\infty, y_\infty \rangle_{X_\infty} = \sum_{k=1}^{n} L_k x_\infty^k y_\infty^k,$$

$$x_\infty = (x_\infty^1, \dots, x_\infty^n), \quad y_\infty = (y_\infty^1, \dots, y_\infty^n) \in X_\infty,$$

and the connecting map $E : X_\infty \to L^2(0,1)$ *is defined by* (3.8) *(thus the connecting maps do not depend on* $d \geq 1$*).*

Proof: Consider $\tilde{B}_\infty : \mathbb{R}^n \to \mathbb{R}^n$ given by the matrix

$$\tilde{B}_\infty := \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & \cdots & & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \cdots & & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & \alpha_4 & \cdots & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{n-2} & \beta_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}. \tag{3.10}$$

Consider also symmetric coercive bilinear forms $\tilde{b}_d : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$, $d \geq 1$, and $\tilde{b}_\infty : X_\infty \times X_\infty \to \mathbb{R}$,

$$\begin{aligned} b_d(\varphi, \psi) &= \int_0^1 (a_d \varphi' \psi' + \varphi \psi) dx, \\ \tilde{b}_\infty(\mathbf{c}, \hat{\mathbf{c}}) &= \sum_{j=1}^{n-1} \frac{a_j}{2l_j} (c_j - c_{j-1})(\hat{c}_j - \hat{c}_{j-1}) + \sum_{j=0}^{n-1} L_j c_j \hat{c}_j, \end{aligned} \tag{3.11}$$

where $\mathbf{c} = (c_0, c_1, \cdots, c_{n-1})$, $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \dots, \hat{c}_{n-1})$, $L_j = x_{j+1} - x_j$ and $j = 0, 1, \cdots, n-1$.

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Thus the solution u of $B_d u = g \in L^2(0,1)$ is characterized as the minimizer of the functional $H^1(0,1) \ni (\varphi,\varphi) \mapsto \frac{1}{2} b_d(\varphi,\varphi) - \langle g,\varphi\rangle_{L^2(0,1)} \in \mathbb{R}$; i.e.

$$\begin{aligned}\mu_d := \frac{1}{2} b_d(u,u) - \int_0^1 gu &= \min_{\varphi \in H^1(0,1)} \Big\{ \frac{1}{2}\ b_d(\varphi,\varphi) - \int_0^1 g\varphi \Big\} \\ &= \min_{\varphi \in H^1(0,1)} \Big\{ \frac{1}{2}\ \int_0^1 (a(x)|\varphi'|^2 + \varphi^2) - \int_0^1 g\varphi \Big\}.\end{aligned}$$

For $\mathbf{m} = (m_0, m_1, \cdots, m_{n-1}) \in X_\infty$, the solution $\mathbf{r}$ of $B_\infty \mathbf{r} = \mathbf{m}$ is the solution of

$$\tilde{B}_\infty \mathbf{r} = D_L \mathbf{m}, \quad \text{where } D_L := \operatorname{diag}(L_0, \ldots, L_{n-1}), \tag{3.12}$$

and thus it is characterized as the minimizer of the functional $\mathbb{R}^n \ni \mathbf{s} \to \frac{1}{2}\tilde{b}_\infty(\mathbf{s},\mathbf{s}) - \langle \mathbf{m},\mathbf{s}\rangle_{X_\infty} \in \mathbb{R}$; namely,

$$\begin{aligned}\mu_\infty := {} & \frac{1}{2}\tilde{b}_\infty(\mathbf{r},\mathbf{r}) - \langle \mathbf{m},\mathbf{r}\rangle_{X_\infty} \\ = {} & \min_{\mathbf{s}\in\mathbb{R}^n} \Big\{ \sum_{j=1}^{n-1} \frac{a_j}{4l_j}(s_j - s_{j-1})^2 + \sum_{j=0}^{n-1} \frac{L_j}{2} s_j^2 - \sum_{j=0}^{n-1} L_j m_j s_j \Big\}.\end{aligned}$$

We now denote by $\mathcal{M} : L^2(0,1) \to X_\infty$ the operator given by

$$\mathcal{M}g = \Big(\frac{\int_{x_0}^{x_1} g}{L_0}, \frac{\int_{x_1}^{x_2} g}{L_1}, \cdots, \frac{\int_{x_{n-1}}^{x_n} g}{L_{n-1}} \Big),\ g \in L^2(0,1),$$

and we will show that

$$\text{if } \ \mathbf{m} := \mathcal{M}g \ \text{ then } \ \lim_{d\to+\infty} \mu_d = \mu_\infty. \tag{3.13}$$

Given $k_0, \ldots, k_{n-1} \in \mathbb{R}$ define $\psi_d \in H^1(0,1)$, $d \geq 1$, by

$$\psi_d(x) = \begin{cases} k_0,\ x \in [0, x_1 - \frac{l_1}{d}], \\ k_{j-1} + d\frac{k_j - k_{j-1}}{2l_j}(x - x_j + \frac{l_j}{d}),\ x \in [x_j - \frac{l_j}{d}, x_j + \frac{l_j}{d}],\ 1 \leqslant j \leqslant n-1, \\ k_j,\ x \in [x_j + \frac{l_j}{d}, x_{j+1} - \frac{l_{j+1}}{d}],\ j = 1,2,\ldots,n-1, \end{cases}$$

we obtain

$$\begin{aligned}\mu_d &\leqslant \frac{1}{2}\int_0^1 a^d|\psi_d'|^2 + \frac{1}{2}\int_0^1 \psi_d^2 - \int_0^1 g\psi_d \\ &\leqslant \frac{1}{2}\sum_{j=1}^{n-1}\int_{x_j-\frac{l_j}{d}}^{x_j+\frac{l_j}{d}} \Big(\frac{a_j'}{d}\Big|d\frac{k_j-k_{j-1}}{2l_j}\Big|^2 + |\psi_d|^2\Big) \\ &\quad + \frac{k_0^2}{2}(x_1-\frac{l_1}{d}) + \frac{k_j^2}{2}\sum_{j=1}^{n-1}[(x_{j+1}-\frac{l_{j+1}}{d})-(x_j+\frac{l_j}{d})] \\ &\quad - \Big[\sum_{j=1}^{n-1}\int_{x_j-\frac{1}{d}l_j}^{x_j+\frac{1}{d}l_j} g\psi_d + k_0\int_0^{x_1-\frac{1}{d}l_1} g + \sum_{j=1}^{n-1}k_j\int_{x_j+\frac{1}{d}l_j}^{x_{j+1}-\frac{1}{d}l_{j+1}} g\Big].\end{aligned}$$

Since

$$\int_{x_j-\frac{1}{d}l_j}^{x_j+\frac{1}{d}l_j}|\psi_d|^2 \to 0, \qquad \int_{x_j-\frac{1}{d}l_j}^{x_j+\frac{1}{d}l_j} g\psi_d \to 0 \ \text{ as } \ d\to\infty,\ j=1,2,\dots,n-1.$$

then recalling that $L_j = x_{j+1}-x_j$ and $m_j = L_j^{-1}\int_{x_j}^{x_{j+1}} g$, $j=0,1,\dots,n-1$, we get

$$\limsup_{d\to+\infty}\mu_d \leqslant \frac{1}{2}\sum_{j=1}^{n-1}\frac{a_j}{2l_j}(k_j-k_{j-1})^2 + \sum_{j=0}^{n-1}\frac{L_j}{2}k_j^2 - \sum_{j=0}^{n-1}L_jm_jk_j. \tag{3.14}$$

This ensures that $\limsup_{d\to\infty}\mu_d \leq \mu_\infty$ and we will show below that $\liminf_{d\to\infty}\mu_d \geq \mu_\infty$, for which we first write

$$\begin{aligned}\mu_d &= \frac{1}{2}\int_0^1[a^du'^2+u^2] - \int_0^1 gu \\ &\geqslant \frac{1}{2}\sum_{j=1}^{n-1}\int_{x_j-\frac{l_j'}{d}}^{x_j+\frac{l_j'}{d}} a^du'^2 + \frac{1}{2}\int_0^{x_1-\frac{l_1'}{d}} u^2 + \frac{1}{2}\sum_{j=1}^{n-1}\int_{x_j+\frac{l_j'}{d}}^{x_{j+1}-\frac{l_{j+1}'}{d}} u^2 \\ &\quad - \Big[\int_0^{x_1-\frac{l_1'}{d}} gu + \sum_{j=1}^{n-1}\int_{x_j+\frac{l_j'}{d}}^{x_{j+1}-\frac{l_{j+1}'}{d}} gu + \sum_{j=1}^{n-1}\int_{x_j-\frac{l_j'}{d}}^{x_j+\frac{l_j'}{d}} gu\Big].\end{aligned} \tag{3.15}$$

Note that for $j=1,\dots,n-1$ we have

$$\begin{aligned}\int_{x_j-\frac{l_j'}{d}}^{x_j+\frac{l_j'}{d}} a^du'^2 \geq \int_{x_j-\frac{l_j'}{d}}^{x_j+\frac{l_j'}{d}}\frac{a_j}{d}u'^2 \geq \frac{a_j}{2l_j'}\Big(\int_{x_j-\frac{l_j'}{d}}^{x_j+\frac{l_j'}{d}} u'\Big)^2 \\ = \frac{a_j}{2l_j'}\big(u(x_j+\frac{l_j'}{d}) - u(x_j-\frac{l_j'}{d})\big)^2.\end{aligned}$$

Next, since in each of the intervals $[x_j + \frac{l'_j}{d}, x_{j+1} - \frac{l'_{j+1}}{d}]$ the diffusion coefficient a^d becomes larger than $e_j d$ and we know that $\int_0^1 a^d u'^2$ remains bounded as $d \to \infty$, then $de_j \int_{x_j + \frac{l'_j}{d}}^{x_{j+1} - \frac{l'_{j+1}}{d}} u'^2$ is bounded for all d sufficiently large and hence

$$\|u'\|_{L^2\left(x_j + \frac{l'_j}{d}, x_{j+1} - \frac{l'_{j+1}}{d}\right)} \stackrel{d \to \infty}{\longrightarrow} 0. \tag{3.16}$$

Furthermore boundedness of the $L^2(0,1)$-norm of u as $d \to \infty$ implies boundedness of the average $\int_0^1 u$ as $d \to \infty$.

Consequently, using Poincaré inequality, from any sequence $d_n \to \infty$ we can choose a subsequence (denoted the same) such that $u = u^{d_n}$ converges almost everywhere on (x_j, x_{j+1}) to a certain constant, which we denote by r_j, $j = 0, 1, \ldots, n-1$. Actually, choosing a certain $x \in (x_j, x_{j+1})$ such that $|u(x) - r_j| \to 0$ as $d_n \to \infty$, we infer that

$$\begin{aligned} |u(x_j + d_n^{-1} l'_j) - r_j| &\leq |u(x) - r_j| + |u(x) - u(x_j + d_n^{-1} l'_j)| \\ &= |u(x) - r_j| + |\int_{x_j + \frac{l'_j}{d_n}}^{x} u'| \\ &\leq |u(x) - r_j| + \int_{x_j + \frac{l'_j}{d_n}}^{x_{j+1} - \frac{l'_{j+1}}{d_n}} |u'|^2 \stackrel{d_n \to \infty}{\longrightarrow} 0. \end{aligned} \tag{3.17}$$

Similarly, choosing a certain $x \in (x_{j-1}, x_j)$ such that $|u(x) - r_{j-1}| \to 0$ as $d_n \to \infty$ we obtain

$$\begin{aligned} |u(x_j - d_n^{-1} l'_j) - r_{j-1}| &\leq |u(x) - r_{j-1}| + |u(x_j - d_n^{-1} l'_j) - u(x)| \\ &= |u(x) - r_{j-1}| + |\int_{x}^{x_j - \frac{l'_j}{d_n}} u'| \\ &\leq |u(x) - r_{j-1}| + \int_{x_{j-1} - \frac{l'_{j-1}}{d_n}}^{x_j - \frac{l'_j}{d_n}} |u'|^2 \stackrel{d_n \to \infty}{\longrightarrow} 0. \end{aligned} \tag{3.18}$$

We also remark that

$$\int_{x_j - \frac{l'_j}{d_n}}^{x_j + \frac{l'_j}{d_n}} gu \to 0 \quad \text{as} \quad d_n \to \infty. \tag{3.19}$$

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Combining now (3.15)-(3.19) and choosing a subsequence of $\{d_n\}$ if necessary (denoted the same) we infer that

$$\liminf \mu_{d_n} \geqslant \frac{1}{2}\Big[\sum_{j=1}^{n-1} \frac{a_j}{2l_j}(r_j - r_{j-1})^2 + \sum_{j=0}^{n-1} L_j r_j^2\Big] - \sum_{j=0}^{n-1} L_j m_j r_j \geq \mu_\infty.$$

Thus (3.13) is proved and we infer that $\mathbf{r}$ obtained above is a unique minimizer of the functional $\mathbb{R}^n \ni \mathbf{s} \to \frac{1}{2}\tilde{b}_\infty(\mathbf{s},\mathbf{s}) - \langle \mathbf{m}, \mathbf{s}\rangle_{X_\infty} \in \mathbb{R}$ so that $\mathbf{r} = B_\infty^{-1}\mathbf{m}$ where $\mathbf{m}$ is as in (3.13). In particular, recalling from (3.8) that $E\mathbf{r}$ is a 'step function' such that $E\mathbf{r}(x) = r_j$ for $x \in (x_{j-1}, x_j)$ we conclude from what was said above that

$$\begin{aligned}\big\|B_d^{-1}g - EB_\infty^{-1}\mathcal{M}g\big\|_{L^2(0,1)} &= \|u - E\mathbf{r}\|^2_{L^2(0,1)} \\ &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |u(x) - r_j|^2 dx \overset{d\to\infty}{\to} 0.\end{aligned} \tag{3.20}$$

Next we suppose that $h_d \in L^2(0,1)$, $d \geq 1$, $h_\infty \in X_\infty$ and $h_d \dashrightarrow h_\infty$; that is

$$\|h_d - Eh_\infty\|_{L^2(0,1)} \overset{d\to\infty}{\longrightarrow} 0.$$

Since $\|B_d^{-1}\|_{L(L^2(0,1))} \leqslant C$, where C is a constant which does not depend on d, then

$$\|B_d^{-1}(h_d - Eh_\infty)\|_{L^2(0,1)} \leqslant C\|h_d - Eh_\infty\|_{L^2(0,1)} \overset{d\to+\infty}{\longrightarrow} 0. \tag{3.21}$$

On the other hand letting $Eh_\infty =: g$ we obtain from (3.20):

$$\begin{aligned}\|B_d^{-1}Eh_\infty - EB_\infty^{-1}\mathcal{M}(Eh_\infty)\|_{L^2(0,1)} &= \|B_d^{-1}g - EB_\infty^{-1}\mathcal{M}g\|_{L^2(0,1)} \\ &\overset{d\to+\infty}{\longrightarrow} 0.\end{aligned} \tag{3.22}$$

Thus from the relations (3.21)-(3.22) and from the fact that $\mathcal{M}Eh_\infty = h_\infty$ we infer that $\|B_d^{-1}h_d - EB_\infty^{-1}h_\infty\|_{L^2(\Omega)} \to 0$ as $d \to \infty$; that is

$$B_d^{-1}h_d \dashrightarrow B_\infty^{-1}h_\infty.$$

Condition (cc1) required in the Definition 2.3 can be now obtained with the similar argument as above because $\|g_n\|_{L^2(0,1)} = 1$ implies for $u_n = B_{d_n}^{-1}g_n$ with $d_n \to \infty$ that

$$\int_0^1 a^{d_n} {u_n'}^2 + \int_0^1 u_n^2 = b_{d_n}(u_n, u_n) = \|g_n\|^2_{L^2(0,1)} = 1;$$

consequently, the average $\int_0^1 u_n$ is bounded and (3.16) holds with $d = d_n$. ▮

Having proved Theorem 3.1 we now turn our attention to the nonlinear problem (3.1).

LEMMA 3.2. *Suppose that the conditions $(a_1) - (a_3)$ are satisfied.*

Then, there is a C^0 semigroup $\{S_d(t)\}$ of global solutions corresponding to (3.1) *in $H^1(0,1)$ which possesses a global attractor $\mathbf{A}_d$. Furthermore, the union $\cup_{d\in[1,\infty)}\mathbf{A}_d$ is bounded in $L^\infty(0,1)$.*

Proof: Since $H^1(0,1) \hookrightarrow L^\infty(0,1)$, then (3.1) is locally well posed in $H^1(0,1)$. Note also that the linear semigroup $\{e^{B_d t}\}$ is monotone and the solutions of the ODE

$$\dot{U} = f(U), \quad U(0) = U_0 \in \mathbb{R}, \tag{3.23}$$

are the solutions of (3.1) through $u(0) \equiv U(0)$. Using comparison, if $t \mapsto u(t, u_0)$ is a solution of (3.1) with initial condition u_0, $\|u_0\|_{L^\infty(0,1)} \le U_0$, we obtain that

$$|u(t, u_0)(x)| \le U(t, U_0)$$

as long as this solution exists. Since, by the dissipativeness assumption, (3.23) has a global attractor, there is also a global attractor $\mathbf{A}_d$ for (3.1). Finally, if $\phi_d \in \mathbf{A}_d$, then $r_{\min} \le \phi_d(x) \le r_{\max}$, where $r_{\min}, r_{\max}$ are the extremal equilibria for (3.23) (see [17]). ▮

We consider next in $L^2(0,1)$ subspaces

$$Y_d = \mathrm{span}[\phi_1^d, \phi_2^d, \cdots, \phi_n^d], \tag{3.24}$$

where $\phi_1^d, \phi_2^d, \cdots, \phi_n^d$ are the first n eigenfunctions of B_d, and denote by $Y_d^\perp$ the orthogonal complement of Y_d in $L^2(0,1)$. Then $\chi \in L^2(0,1)$ can be decomposed as

$$\chi = v^d + w^d = v_1^d\phi_1^d + \ldots + v_n^d\phi_n^d + w^d, \ d \ge 1,$$

where

$$v_j^d = \langle \chi, \phi_j^d \rangle_{L^2(0,1)} \quad \text{for} \quad j = 1, 2, \ldots, n,$$

and

$$w^d = \chi - v_1^d\phi_1^d - \ldots - v_n^d\phi_n^d.$$

Combining the results already proved in this subsection with Theorem 2.1 we obtain the following conclusion.

COROLLARY 3.1. *Let $\tilde{f}(s)$ be a bounded, globally Lipschitz function such that $\tilde{f}(s) = f(s) + s$ for $x \in [-r_{min}, r_{max}]$, $\tilde{f}(s) = r_{\max}$ for $s > r_{max}$ and*

$f(s) = r_{\min}$ for $s < r_{\min}$, where $r_{\min}, r_{\max}$ are the extremal equilibria for (3.23). Then,

i) the Nemitskĭi operator $\tilde{F}$ associated with $\tilde{f}$ is of the class $\mathcal{F}(L^2(0,1), C)$, whereas $\{-B_d\}_{d\in[1,\infty)}$ is of the class $CC(L^2(0,1)) \cap \mathcal{H}(L^2(0,1), M, \theta)$,

ii) there is a global attractor $\tilde{\mathbf{A}}_d$ for the problem

$$\begin{cases} \dot{u} = -B_d u + \tilde{f}(u), \ t > 0 \\ u(0) \in L^2(0,1), \end{cases} \tag{3.25}$$

and $\tilde{\mathbf{A}}_d = \mathbf{A}_d$; in particular for any solution $u_d(t)$ of (3.1) 'lying' on the global attractor $\mathbf{A}_d$ we have that

$$u^d(t)(x) \in [r_{\min}, r_{\max}], \ x \in [0,1], \ t \in \mathbb{R},$$

iii) the problem (3.25) can be viewed as

$$\begin{aligned} \dot{v}^d &= \Lambda_d v^d + H(v^d, w^d), \\ \dot{w}^d &= -B_d w^d + G(v^d, w^d), \end{aligned} \tag{3.26}$$

where

$$v^d = (v_1^d, \ldots, v_n^d), \quad \Lambda_d = \operatorname{diag}(-\lambda_1^d, \ldots, -\lambda_n^d), \quad \tilde{H}^d = (\tilde{H}_1^d, \ldots, \tilde{H}_n^d), \tag{3.27}$$

$$\begin{aligned} \tilde{H}_j^d(v^d, w^d) &= \langle \tilde{f}(v_1^d \phi_1^d + \ldots + v_n^d \phi_n^d + w^d), \phi_j^d \rangle_{L^2(0,1)}, \ j = 1, 2, \cdots, n, \\ \tilde{G}^d(v^d, w^d) &= \tilde{f}(v_1^d \phi_1^d + \ldots + v_n^d \phi_n^d + w^d) - \sum_{j=1}^n \tilde{H}_j(v^d, w^d)\phi_j^d, \ j = 1, 2, \cdots, n, \end{aligned} \tag{3.28}$$

and there exist Lipschitz maps $\Sigma_d^* : Y_d \to Y_d^{\perp}$, $d \geq d_0$, such that

$$\|\Sigma_d^*(y_d)\|_{L^2(0,1)} \to 0 \quad \text{as} \quad d \to \infty \quad \text{uniformly for} \quad y_d \in Y_d \tag{3.29}$$

and

$$\mathbf{A}_d = \tilde{\mathbf{A}}_d \subset \mathcal{S}_d = \{\psi \in L^2(0,1) : \chi = y_d + \Sigma_d^*(y_d), \ y_d \in Y_d\},$$

where $\mathcal{S}_d$ is an exponentially attracting invariant manifold for (3.25) and Y_d is as in (3.24).

Using the decomposition $L^2(0,1) = Y_d \oplus Y_d^{\perp}$ and Corollary 3.1 we obtain the following two lemmas.

LEMMA 3.3. *If $d \geq d_0$ and $u^d(t)$ is a solution of* (3.1) *'lying' on $\mathbf{A}_d$, then*
i) $u^d(t) = v_1^d(t)\phi_1^d + \ldots + v_n^d(t)\phi_n^d + w^d(t)$, where $v^d(t)$, $w^d(t)$ satisfy

$$\begin{aligned} \dot{v}_j^d(t) &= -\lambda_j^d v_j(t) + \langle \tilde{f}(u^d(t)), \phi_j^d \rangle_{X^d},\ j = 1, \ldots, n \\ \dot{w}^d(t) &= -B_d w^d(t) + \tilde{f}(u^d(t)) - \sum_{j=1}^{n} \langle \tilde{f}(u^d(t)), \phi_j^d \rangle_{X_d} \phi_j^d, \end{aligned} \tag{3.30}$$

ii) for each $j = 1, \ldots, n$ functions $v_j^d(t) : \mathbb{R} \to \mathbb{R}$, $d \in [1, \infty)$, are equicontinuous and equibounded,
iii) $w^d(t)$ has the property that

$$\|w^d(t)\|_{L^2(0,1)} \overset{d \to \infty}{\to} 0 \quad \textit{uniformly for} \quad t \in \mathbb{R}. \tag{3.31}$$

Proof: Part i) is straightforward. From Theorem 3.1 we know that $\{\lambda_j^d\}$ and $\{\|\phi_j^d\|_{L^2(0,1)}\}$ are bounded uniformly with respect to $d \geq 1$ for every $j = 1, \ldots, n$. From Corollary 3.1 we also know that $\|u^d\|_{L^\infty(0,1)} \leq r$. Hence both $v_j^d(t) = \langle u_j^d(t), \phi_j^d \rangle_{L^2(0,1)}$ and $\dot{v}_j^d(t)$ are bounded uniformly with respect to the parameter $d \geq 1$, which proves ii). Finally, from Corollary 3.1 iii) we have

$$w^d(t) = \Sigma_d^*(v_1^d(t)\phi_1^d + \ldots + v_n^d(t)\phi_n^d),$$

so that part iii) is a consequence of (3.29). ▮

LEMMA 3.4. *If $d_k \to \infty$ and u^{d_k} is a solution of* (3.1) *'lying' on $\mathbf{A}_{d_k}$, then*
i) there is a subsequence, denoted the same, and functions $v_j^\infty : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, n$, such that uniformly for t in bounded time intervals

$$v_j^{d_k}(t) \overset{d_k \to \infty}{\to} v_j^\infty(t),$$

$$\|u^{d_k}(t) - E(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty)\|_{L^2(0,1)} \overset{d_k \to \infty}{\to} 0 \tag{3.32}$$

and, by Lipschitz continuity of $\tilde{f}$,

$$\|\tilde{f}(u^{d_k}(t)) - \tilde{f}(E(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty))\|_{L^2(0,1)} \overset{d_k \to \infty}{\to} 0.$$

Also
ii) defining

$$P_j(s_1, \ldots, s_n) = s_j \quad \textit{for} \quad (s_1, \ldots s_n) \in \mathbb{R}^n,\ j = 1, \ldots, n, \tag{3.33}$$

and

$$\tilde{\mathbf{f}}_P := (\tilde{f} \circ P_1, \dots, \tilde{f} \circ P_n) : \mathbb{R}^n \to \mathbb{R}^n, \tag{3.34}$$

we have

$$\tilde{f}(E(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty)) = E(\tilde{\mathbf{f}}_P(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty)),$$

$$\tilde{f}(u^{d_k}(t)) \dashrightarrow \tilde{\mathbf{f}}_P(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty),$$

and

$$|\langle \tilde{f}(u^{d_k}(t)), \phi_j^{d_k}\rangle_{L^2(0,1)} - \langle E\tilde{\mathbf{f}}_P(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty), E\phi_j^\infty\rangle_{L^2(0,1)}| \overset{d_k \to \infty}{\to} 0.$$

Furthermore,
iii) for each $j = 1, 2, \dots n$ *the limit function* v_j^∞ *solves*

$$\dot{v}_j^\infty(t) = -\lambda_j^\infty v_j^\infty(t) + \langle \tilde{\mathbf{f}}_P(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty), \phi_j^\infty\rangle_{X_\infty}. \tag{3.35}$$

Proof: Part i) follows from Lemma 3.3. Part ii) is a consequence of i). For the proof of iii) we first remark that

$$\langle E\phi_\infty, E\psi_\infty\rangle_{L^2(0,1)} = \langle \phi_\infty, \psi_\infty\rangle_{X_\infty} \text{ for } d \ge 1, \ \phi_\infty, \psi_\infty \in X_\infty.$$

Then writing the first equations in (3.30) in the integral form

$$v_j^{d_k}(t) = v_j^{d_k}(0) - \lambda_j^{d_k} \int_0^t v_j(s)ds + \int_0^t \langle \tilde{f}(u^{d_k}(s)), \phi_j^{d_k}\rangle_{L^2(0,1)} ds$$

and passing to the limit we obtain

$$\begin{aligned} v_j^\infty(t) &= v_j^\infty(0) \\ &\quad - \lambda_j^\infty \int_0^t v_j^\infty(s)ds + \int_0^t \langle \tilde{\mathbf{f}}_P(v_1^\infty(s)\phi_1^\infty + \ldots + v_n^\infty(s)\phi_n^\infty), \phi_j^\infty\rangle_{X^\infty} ds, \end{aligned}$$

for every $j = 1, \dots, n$. Thus each $v_j^\infty(t)$ is differentiable and we get the result. ▌

Remark 3. 1. Recall from Corollary 3.1 that for $u^{d_k}(t)$ 'lying' on the attractor $\mathbf{A}_{d_k}$ we have that $u^{d_k}(t)(x) \in [r_{\min}, r_{\max}]$ for all $t \in \mathbb{R}$ and $x \in [0,1]$. Also note that (3.32) implies that for each fixed $t \in \mathbb{R}$ a certain subsequence exists, denoted the same, such that

$$u^{d_k}(t)(x) \to E(v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty) \text{ for a.e. } x \in (0,1).$$

Letting $v^\infty(t) = v_1^\infty(t)\phi_1^\infty + \ldots + v_n^\infty(t)\phi_n^\infty$ this ensures that $Ev^\infty(t) \in [r_{\min}, r_{\max}]$ and hence

$$v^\infty(t) \in [r_{\min}, r_{\max}]^n \quad \text{for every} \quad t \in \mathbb{R}.$$

We now obtain the following conclusion.

COROLLARY 3.2. *If $d_k \to \infty$ and u^{d_k} is a solution of (3.1) 'lying' on $\mathbf{A}_{d_k}$, for a certain subsequence which we denote the same, we have that*
i) $\langle u^{d_k}(t), \phi_j^k\rangle_{L^2(0,1)} \stackrel{d_k\to\infty}{\to} \langle Ev^\infty, E\phi^\infty\rangle_{L^2(0,1)} = \langle v^\infty(t), \phi_j^\infty\rangle_{X_\infty}$, $j = 1, \ldots, n$,
ii) $v^\infty : \mathbb{R} \to \mathbb{R}^n$ is a solution of the system of ordinary differential equations in $\mathbb{R}^n$

$$\dot{v}^\infty = -B_\infty v^\infty + \mathbf{f}_P(v^\infty) + v^\infty \tag{3.36}$$

with $\mathbf{f}_P = (f \circ P_1, \ldots, f \circ P_n) : \mathbb{R}^n \to \mathbb{R}^n$,
iii) $v^\infty(t) \in \mathbf{A}_\infty$ for every $t \in \mathbb{R}$ where $\mathbf{A}_\infty$ is the global attractor for (3.36).

Proof: Parts i) follows from Lemma 3.4 and Remark 3.1. Concerning part ii) note that, if $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ denotes the standard scalar product in $\mathbb{R}^n$ and D_L, $\tilde{B}_\infty$ are as in (3.12), then

$$\langle w_\infty, z_\infty\rangle_{X_\infty} = \langle w_\infty, D_L z_\infty\rangle_{\mathbb{R}^n} = \langle D_L w_\infty, z_\infty\rangle_{\mathbb{R}^n} \text{ for } w_\infty, z_\infty \in X_\infty. \tag{3.37}$$

Then from (3.35) and Remark 3.1 we infer that

$$\begin{aligned}
\langle \dot{v}^\infty, \phi_j^\infty\rangle_{X^\infty} - \langle \mathbf{f}_P(v^\infty(s)) + v^\infty(s), \phi_j^\infty\rangle_{X_\infty} &= \langle v^\infty, -B_\infty\phi_j^\infty\rangle_{X_\infty} \\
&= \langle v^\infty, -D_L B_\infty \phi_j^\infty\rangle_{\mathbb{R}^n} = \langle v^\infty, -\tilde{B}_\infty\phi_j^\infty\rangle_{\mathbb{R}^n} = \langle -\tilde{B}_\infty v^\infty, \phi_j^\infty\rangle_{\mathbb{R}^n} \\
&= \langle -D_L^{-1}\tilde{B}_\infty v^\infty, D_L\phi_j^\infty\rangle_{\mathbb{R}^n} = \langle -B_\infty v^\infty, D_L\phi_j^\infty\rangle_{\mathbb{R}^n} \\
&= \langle -B_\infty v^\infty, \phi_j^\infty\rangle_{X^\infty}
\end{aligned}$$

for each $j = 0, \ldots, n$, which gives the result.

For the proof of iii) we recall that for each $j = 1, \ldots, n$ condition (3.3) reads $sf(s) \le \theta s^2 + C_\theta$ (recall that $\theta < 0$) whenever $|s| \ge s_0$ and constants $s_0 > 0$, $C_\theta > 0$ are suitably chosen. Consequently, using (3.37) and bilinear form $\tilde{b}_\infty$ defined in (3.11) we observe that, as long as the solution u of (3.39) exists,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{X_\infty}^2 &= \langle -B_\infty u, u\rangle_{X_\infty} + \langle \mathbf{f}_P(u) + u, u\rangle_{X_\infty} \\
&= \langle -D_L B_\infty u, u\rangle_{\mathbb{R}^n} + \langle \mathbf{f}_P(u) + u, u\rangle_{X_\infty} \\
&= -\tilde{b}_\infty(u,u) + \sum_{j=1}^n L_{j+1} u_j f(u_j) + \|u\|_{X_\infty}^2 \le \theta\|u\|_{X_\infty}^2 + C_\theta.
\end{aligned}$$

The existence of the global attractor is now straightforward and $\{v^\infty(t) : t \in \mathbb{R}\}$ is contained in the attractor as this latter set is bounded and invariant. ▌

Summarizing the consideration of this subsection we have proved that the following theorem holds.

THEOREM 3.2. *Suppose that the conditions* $(a_1) - (a_3)$ *are satisfied. Then,*
i) there are the global attractors $\mathbf{A}_d$ *for* (3.1) *in* $H^1(0,1)$,
ii) there exist functions $\Sigma_d^* : Y_d \to Y_d^\perp$, $d > d_0$, *such that*

$$\mathbf{A}_d \subset \{\chi = y_d + \Sigma_d^*(y_d),\ y_d \in Y_d\},$$

and

$$\|\Sigma_d^*(y_d)\|_{L^2(0,1)} \to 0 \text{ as } d \to \infty \text{ uniformly for } y_d \in Y_d,$$

where $Y_d = span[\phi_1^d, \phi_2^d, \cdots, \phi_n^d]$ *and* $\phi_1^d, \phi_2^d, \cdots, \phi_n^d$ *are the eigenfunctions corresponding to the first eigenvalues of* B_d *in* $L^2(0,1)$.
iii) the flow on $\mathbf{A}_d$ *is given by*

$$u^d(t,x) = v^d(t) + \Sigma_d^*(v^d(t))(x),\ t \in \mathbb{R},\ x \in (0,1),$$

where

$$\dot{v}^d(t) = \Lambda_d v^d(t) + H(v^d(t), \Sigma_d^*(v^d(t))), \tag{3.38}$$

Λ_d *is as in* (3.27), $H^d = (H_1^d, \cdots, H_n^d)$ *and*

$$H_j^d(v^d(t), \Sigma_d^*(v^d(t))) = \langle f(v^d(t), \Sigma_d^*(v^d(t))), \phi_j^d\rangle_{L^2(0,1)},\ j = 1, 2, \cdots, n,$$

iv) the family of attractors $\{\mathbf{A}_d\}_{d \geqslant d_0}$ *is upper semicontinuous at infinity relatively to E; that is,*

$$\sup_{\psi_d \in \mathbf{A}_d} \inf_{\psi_\infty \in \mathbf{A}_\infty} \|\psi_d - E\psi_\infty\|_{L^2(0,1)} \to 0 \quad as \quad d \to \infty,$$

where $\mathbf{A}_\infty$ *is the attractor for the semigroup defined in* $\mathbb{R}^n$ *by the system of ordinary differential equations of the form*

$$\begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} = -B_\infty \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} f(u_1)+u_1 \\ \vdots \\ f(u_n)+u_n \end{bmatrix},\ t \in \mathbb{R}, \tag{3.39}$$

Remark 3. 2. Recall that the flow of (3.39) is structurally stable in $\mathbf{A}_\infty$ if for any suitably small C^1 perturbation of the vector field the resulting equation has a global attractor $\mathbf{A}$ and the flow in $\mathbf{A}$ is topologically equivalent to the flow in $\mathbf{A}_\infty$. Note that if the flow in $\mathbf{A}_\infty$ is structurally stable, applying Proposition 2.2 instead of Theorem 2.1, then the flow in $\mathbf{A}_d$ is topologically equivalent to the flow in $\mathbf{A}_\infty$.

3.2. Spatial homogeneity in atmospherics' type problem

We will consider a bounded domain which decomposes into finitely many regions in which diffusivity is very large and the mass transfer is appropriately balanced at the boundaries.

This occurs in the air pollution modeling (see [31]), where the domain can be viewed as an interstitial air, including subregions; the droplets.

We thus assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ and that Ω_i, $1 \leq i \leq n$, are the subdomains of Ω with smooth boundaries Γ_i satisfying $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for all $1 \leq i < j \leq n$. We also let $\Omega_0 := \Omega \backslash \cup_{i=1}^n \overline{\Omega}_i$ and denote by $\vec{\nu}$ the outward normal vector to Γ and by $\vec{\nu}_i$ the inward normal vector to Γ_i $(i = 1, 2, \ldots, n)$. For simplicity of the notation we assume from now on that $|\Omega_i| = 1$ for every $i = 1, 2, \cdots, n$, which will not lower in any essential way generality of the consideration.

Our primary concern will be the eigenvalue problem of the form

$$\begin{cases} -d_i \Delta u_i + u_i = \lambda u_i \ \text{ in } \ \Omega_i, \quad i = 0, 1, \cdots, n, \\ d_0 \dfrac{\partial u_0}{\partial \vec{\nu}} = 0 \quad \text{on } \ \Gamma, \\ d_0 \dfrac{\partial u_0}{\partial \vec{\nu}_i} = d_i \dfrac{\partial u_i}{\partial(-\vec{\nu}_i)} = (u_i - u_0) \ \text{ on } \ \Gamma_i, \quad i = 1, \cdots, n. \end{cases} \tag{3.40}$$

Since the diffusivities become very large one expect to exhibit in the analysis that the solutions will become homogeneous in each of the subregions.

Letting for $d_0, \ldots, d_n \in [1, \infty)$

$$d := \min\{d_0, d_1, \ldots, d_n\}$$

we will show that the compact convergence approach of Section 2 lead to the following result.

THEOREM 3.3. *Denote by* $\{\lambda_k^d\}_{k=1}^\infty$ *the ordered set of eigenvalues for* (3.40) *(counting its algebraic multiplicity) and let* $\{\phi_k^d\}_{k=1}^\infty$ *be the corresponding set of orthonormal eigenfunctions. Also denote by* $\{\lambda_k^\infty\}_{k=1}^\infty$ *the*

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ordered set of eigenvalues for

$$\begin{bmatrix} 1+|\Omega_0|^{-1}\sum_{i=1}^{n}|\Gamma_i| & -|\Omega_0|^{-1}|\Gamma_1| & -|\Omega_0|^{-1}|\Gamma_2| & \cdots & -|\Omega_0|^{-1}|\Gamma_n| \\ -|\Gamma_1| & 1+|\Gamma_1| & 0 & \cdots & 0 \\ -|\Gamma_2| & 0 & 1+|\Gamma_2| & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -|\Gamma_n| & 0 & 0 & \cdots & 1+|\Gamma_n| \end{bmatrix} \begin{bmatrix} s_0 \\ \vdots \\ s_n \end{bmatrix} = \lambda \begin{bmatrix} s_0 \\ \vdots \\ s_n \end{bmatrix}, \tag{3.41}$$

and let $\{\phi_k^\infty\}_{k=1}^\infty$ *be the corresponding set of orthonormal eigenfunctions.*

Then we have

i) $\lambda_k^d \to \lambda_k^\infty$ *as* $d \to \infty$, $k = 1, 2, \cdots, n+1$.

ii) $\lambda_k^d \to \infty$ *for all* $k > n+1$.

iii) $\|\phi_k^d - \phi_k^\infty\|_{L^2(\Omega_0)\times\cdots\times L^2(\Omega_n)} \to 0$ *as* $d \to \infty$ *for every* $k = 1, 2, \cdots, n+1$.

The proof of Theorem 3.3 will follow from the abstract results of Corollary 2.2 and from Lemma 3.5 below.

Throughout the rest of this subsection we consider the Hilbert space

$$\mathbf{L}^2 := L^2(\Omega_0) \times \cdots \times L^2(\Omega_n),\ d \in [1, \infty),$$

with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=0}^{n} \int_{\Omega_i} u_i v_i, \quad \mathbf{u} = (u_0, \cdots, u_n), \mathbf{v} = (v_0, \cdots, v_n) \in \mathbf{L}^2,$$

and let

$$X_d = \mathbf{L}^2,\ d \in [1, \infty).$$

We also define

$$\mathbf{H}^1 = H^1(\Omega_0) \times \ldots \times H^1(\Omega_n)$$

and consider bilinear forms $a_d : \mathbf{H}^1 \times \mathbf{H}^1 \to \mathbb{R}$, $d \geq 1$, given by the formula

$$a_d(\mathbf{w}, \mathbf{z}) = \sum_{i=0}^{n} \int_{\Omega_i} (d_i \nabla w_i \nabla z_i + w_i z_i) + \sum_{i=1}^{n} \int_{\Gamma_i} (w_i - w_0)(z_i - z_0),\ \mathbf{w}, \mathbf{z} \in \mathbf{H}^1. \tag{3.42}$$

We remark that

$$a_d(u, u) \geq \|u\|^2_{\mathbf{H}^1},\ u \in \mathbf{H}^1,\ d \geq 1, \tag{3.43}$$

and that the estimate known from from trace theorem,

$$\|\psi\|_{L^2(\Gamma_i)} \leq c\|\psi\|_{H^1(\Omega)},\ \psi \in H^1(\Omega_j),\ i = 0, \ldots, n,$$

implies the relation

$$|a_d(u,v)| \leq c\|u\|_{\mathbf{H}^1}\|v\|_{\mathbf{H}^1},\ u,v \in \mathbf{H}^1.$$

The form $a_d : \mathbf{H}^1 \times \mathbf{H}^1 \to \mathbb{R}$ given by (3.42) is thus symmetric, continuous and coercive. and Riesz theorem together with Lax-Milgram lemma lead to the following conclusion.

Proposition 3.1. *For all $d \geq 1$ the form a_d defines a linear operator A_d which is an isomorphism from $\mathbf{H}^1$ to its dual $(\mathbf{H}^1)'$. A_d considered with the domain*

$$X_d^1 := \{u \in \mathbf{H}^1 : A_d u \in X_d\}$$

is then a linear selfadjoint positive definite operator in X_d which is an isomorphism from X_d^1 (considered with the graph norm) onto X_d.

In addition, since the embedding $\mathbf{H}^1 \hookrightarrow X_d$ is compact, we have that

Corollary 3.3. *A_d^{-1}, $d \geq 1$, is a compact operator.*

Remark 3. 3. The problem (3.40) can be viewed in the abstract form

$$A_d u = \lambda u, \tag{3.44}$$

where A_d is given by Corollary 3.1. Also, one can observe that

$$A_d = \mathrm{diag}(-d_0\Delta, \ldots, -d_n\Delta) + Id \tag{3.45}$$

on the domain

$$\begin{aligned} D(A_d) = \big\{ &\mathbf{v} \in X_d : v_i \in H^2(\Omega_i), \frac{\partial v_0}{\partial \vec{\nu}}|_\Gamma = 0, \\ &\frac{\partial v_0}{\partial \vec{\nu}_i}|_{\Gamma_i} = d_i \frac{\partial v_i}{\partial(-\vec{\nu}_i)}|_{\Gamma_i} = (v_i - v_0)|_{\Gamma_i},\ 1 \leqslant i \leqslant n,\ \big\}. \end{aligned}$$

In what follows we verify the compact convergence property that is needed to apply Corollary 2.2 of Section 2.

Lemma 3.5. *With the above notation we have that $\{A_d\}_{d\in[1,\infty)}$ is of the class $CC(\mathbf{L}^2)$. In fact*

$$A_d \overset{cc}{\dashrightarrow} A_\infty,$$

where the operator $A_\infty =: X_\infty \to X_\infty$ is given by

$$A_\infty := \begin{bmatrix} 1+|\Omega_0|^{-1}\sum_{i=1}^n|\Gamma_i| & -|\Omega_0|^{-1}|\Gamma_1| & -|\Omega_0|^{-1}|\Gamma_2| & \cdots & -|\Omega_0|^{-1}|\Gamma_n| \\ -|\Gamma_1| & 1+|\Gamma_1| & 0 & \cdots & 0 \\ -|\Gamma_2| & 0 & 1+|\Gamma_2| & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -|\Gamma_n| & 0 & 0 & \cdots & 1+|\Gamma_n| \end{bmatrix} \tag{3.46}$$

on a Hilbert space $X_\infty = \mathbb{R}^{n+1}$ equipped with the inner product

$$\langle \mathbf{r}, \mathbf{s}\rangle_\infty = |\Omega_0| r_0 s_0 + \sum_{i=1}^n r_i s_i, \quad \mathbf{r} = (r_0 \ \cdots \ r_n),\ \mathbf{s} = (s_0 \ \cdots \ s_n) \in X_\infty,$$

and the connecting map $E : X_\infty \to \mathbf{L}^2$ is given by

$$E(h_0, \ldots, h_n) = (h_0, \ldots, h_n) \in \mathbf{L}^2,\ (h_0, \ldots, h_n) \in X_\infty,$$

(thus the connecting maps are independent of $d \in [1, \infty)$).

Proof: Let $a_d : \mathbf{H}^1 \times \mathbf{H}^1 \to \mathbb{R}$ be given by (3.42). Let us also introduce the operator $\tilde{A}_\infty : X_\infty \to X_\infty$,

$$\tilde{A}_\infty := \operatorname{diag}(|\Omega_0|, 1, \ldots, 1) A_\infty, \tag{3.47}$$

and a symmetric coercive bilinear form $\tilde{a}_\infty : X_\infty \times X_\infty \to \mathbb{R}$ defined by

$$\tilde{a}_\infty(\mathbf{s}, \mathbf{t}) = |\Omega_0| s_0 t_0 + \sum_{i=1}^n s_i t_i + \sum_{i=1}^n |\Gamma_i| (s_i - s_0)(t_i - t_0).$$

The solution $\mathbf{u} \in \mathbf{H}^1$ of the equation $A_d \mathbf{u} = \mathbf{g}$ with $\mathbf{g} \in X_d$ is then characterized via Lax Milgram Theorem as

$$l_d := \frac{1}{2} a_d(\mathbf{u}, \mathbf{u}) - \langle \mathbf{g}, \mathbf{u}\rangle = \min_{\mathbf{v} \in \mathbf{H}^1} \Big\{ \frac{1}{2}\ a_d(\mathbf{v}, \mathbf{v}) - \langle \mathbf{g}, \mathbf{v}\rangle \Big\}.$$

If

$$D = \operatorname{diag}(|\Omega_0|, 1, \ldots, 1) \tag{3.48}$$

then the solution of $A_\infty \mathbf{r} = \mathbf{m}$ is the solution of $\tilde{A}_\infty \mathbf{r} = D\mathbf{m}$ and it can be characterized via relation

$$\begin{aligned} l_\infty &:= \frac{1}{2} \tilde{a}_\infty(\mathbf{r}, \mathbf{r}) - \langle \mathbf{m}, \mathbf{r}\rangle_\infty = \min_{\mathbf{s} \in X_\infty} \{ \frac{1}{2} \tilde{a}_\infty(\mathbf{s}, \mathbf{s}) - \langle \mathbf{m}, \mathbf{s}\rangle_\infty \} \\ &= \min_{\mathbf{s} \in X_\infty} \Big\{ \frac{1}{2} \big[(|\Omega_0| + \sum_{i=1}^n |\Gamma_i|) s_0^2 + \sum_{i=1}^n (1 + |\Gamma_i|)\, s_i^2 \big] - \sum_{i=1}^n |\Gamma_i|\, s_0 s_i - \langle \mathbf{m}, \mathbf{s}\rangle_\infty \Big\}. \end{aligned}$$

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We will make use of an auxiliary map $P : \mathbf{L}^2 \to X_\infty$,

$$P\mathbf{g} = P(g_0, \dots, g_n) = (\bar{g}_0, \dots, \bar{g}_n), \tag{3.49}$$

where

$$\bar{g}_i := |\Omega_i|^{-1} \int_{\Omega_i} g_i \quad \text{for} \quad i = 0, 1, \dots, n. \tag{3.50}$$

We will now prove that

$$\text{if} \quad \mathbf{m} = (\bar{g}_0, \dots, \bar{g}_n) \quad \text{then} \quad \lim_{d\to\infty} l_d = l_\infty. \tag{3.51}$$

Actually, since $\mathbf{r} = (r_0, r_1, \cdots, r_n)$ can be viewed as an element of $\mathbf{H}^1$, it is easy to see that

$$l_d \leqslant l_\infty.$$

This implies the relation $\limsup_{d\to\infty} l_d \leqslant l_\infty$. Below we show that $\liminf\limits_{d\to\infty} l_d \geqslant l_\infty$.

Note that for any $\mathbf{s} \in X_\infty$ we have

$$\begin{aligned} l_d :&= \frac{1}{2}\Big\{\sum_{i=0}^{n} \int_{\Omega_i} \big(d_i|\nabla u_i|^2 + u_i^2\big) + \sum_{i=1}^{n} \int_{\Gamma_i} (u_0 - u_i)^2 \Big\} - \sum_{i=0}^{n} \int_{\Omega_i} g_i u_i \\ &\geqslant \frac{1}{2}\Big\{\sum_{i=0}^{n} \int_{\Omega_i} (u_i^2 - s_i^2) + \sum_{i=1}^{n} \int_{\Gamma_i} [(u_0 - u_i)^2 - (s_0 - s_i)^2]\Big\} \\ &- \sum_{i=0}^{n} \int_{\Omega_i} g_i[u_i - s_i] + l_\infty. \end{aligned}$$

Since l_d is bounded as $d \to \infty$ then $d_i \int_{\Omega_i} |\nabla u_i|^2$ and $\int_{\Omega_i} |u_i|^2$ are also bounded as $d \to \infty$. In particular

$$\bar{u}_i \text{ is bounded as } d \to \infty \text{ and } \int_{\Omega_i} |\nabla u_i|^2 \overset{d\to\infty}{\to} 0. \tag{3.52}$$

Writing now explicitly the dependence of u on d we thus infer via Poincaré inequality that whenever $d^k \overset{k\to\infty}{\longrightarrow} \infty$ there is a subsequence (which we denote the same) and a certain $\tilde{\mathbf{s}} \in X_\infty$ such that

$$\sum_{i=0}^{n} \|u_i^{d^k} - \tilde{s}_i\|_{L^2(\Omega_i)} \overset{d^k\to\infty}{\to} 0 \tag{3.53}$$

and consequently, since also $\sum_{i=0}^{n}\int_{\Omega_i}|\nabla(u_i^{d^k}-\tilde{s}_i)|^2 \overset{d^k\to\infty}{\to} 0$ then

$$\sum_{i=1}^{n}\|u_i^{d^k}-\tilde{s}_i\|_{L^2(\Gamma_i)} \overset{d^k\to\infty}{\to} 0. \tag{3.54}$$

Hence $\liminf_{d\to\infty}\lambda_d \geq \lambda_\infty$ and (3.51) is proved.

¿From what we said above we infer that

$$\frac{1}{2}a_{d^k}(\mathbf{u},\mathbf{u})-\langle\mathbf{g},\mathbf{u}\rangle_{\mathbf{L}^2}\to\frac{1}{2}\tilde{a}_\infty(\tilde{\mathbf{s}},\tilde{\mathbf{s}})-\langle\mathbf{m},\tilde{\mathbf{s}}\rangle_\infty = l_\infty.$$

This implies that $\tilde{\mathbf{s}}$ coincides with the unique minimizer $\mathbf{r}$ of $\tilde{a}_\infty$ so that (3.53) actually reads

$$\sum_{i=0}^{n}\int_{\Omega_i}(u_i^{d^k}-r_i)^2 \overset{d^k\to\infty}{\longrightarrow} 0. \tag{3.55}$$

Consequently, recalling that P and $\mathbf{m}$ are as in (3.49) and (3.51),

$$\left\|A_{d^k}^{-1}\mathbf{g}-A_\infty^{-1}P\mathbf{g}\right\|_{\mathbf{L}^2}=\left\|A_{d^k}^{-1}\mathbf{g}-A_\infty^{-1}\mathbf{m}\right\|_{\mathbf{L}^2}=\|\mathbf{u}^{d^k}-\mathbf{r}\|_{\mathbf{L}^2} \overset{d^k\to\infty}{\longrightarrow} 0. \tag{3.56}$$

We will now show the convergence $A_d^{-1} \dashrightarrow A_\infty^{-1}$. Consider $\mathbf{h}_d\in X_d$ and $\mathbf{h}_\infty\in X_\infty$ such that $\mathbf{h}_d \dashrightarrow \mathbf{h}_\infty$. Since $E_d=E$ behaves like identity; that is $EA_\infty^{-1}\mathbf{h}_\infty$ is just a constant function $A_\infty^{-1}\mathbf{h}_\infty$ we want to prove that

$$\|A_d^{-1}\mathbf{h}_d-A_\infty^{-1}\mathbf{h}_\infty\|_{\mathbf{L}^2}\to 0. \tag{3.57}$$

Since $\|A_d^{-1}\|_{L(X_d)}\leqslant C$, where C is a constant which does not depend on d, we have

$$\|A_d^{-1}\mathbf{h}_d-A_d^{-1}E\mathbf{h}_\infty\|_{\mathbf{L}^2}\leqslant C\|\mathbf{h}_d-E\mathbf{h}_\infty\|_{\mathbf{L}^2} \overset{d\to\infty}{\longrightarrow} 0. \tag{3.58}$$

On the other hand $PE\mathbf{h}_\infty=\mathbf{h}_\infty$ so that using (3.56) with $\mathbf{g}=E\mathbf{h}_\infty$

$$\|A_d^{-1}E\mathbf{h}_\infty-A_\infty^{-1}\mathbf{h}_\infty\|_{\mathbf{L}^2}=\|A_d^{-1}\mathbf{g}-A_\infty^{-1}P\mathbf{g}\|_{\mathbf{L}^2} \overset{d\to\infty}{\longrightarrow} 0. \tag{3.59}$$

Thus (3.57) follows from (3.58) and (3.59).

Condition (cc1) required in the Definition 2.3 can be now obtained with the similar argument as above because we obtain that $\|\mathbf{g}_k\|_{\mathbf{L}^2}=1$ implies for $\mathbf{u}_k=A_{d^k}^{-1}\mathbf{g}_k$ where $d^k\to\infty$ the estimate

$$\sum_{i=0}^{n}\int_{\Omega_i}(d_i^k|\nabla u_{ki}|^2+|u_{ki}|^2)\leq a_{d^k}(\mathbf{u},\mathbf{u})=\|\mathbf{g}_k\|_{\mathbf{L^2}}^2=1;$$

consequently, the relation (3.52) holds true. ▮

Remark 3. 4. The minimizer $\mathbf{s}^*$ of the functional $\mathbf{s} \mapsto \frac{1}{2}\tilde{a}_\infty(\mathbf{s},\mathbf{s}) - \langle \mathbf{s}, \mathbf{m}\rangle_\infty$ is actually given as

$$s_0^* = \frac{|\Omega_0| - \sum_{i=1}^n (1+|\Gamma_i|)^{-1}|\Gamma_i| m_i}{|\Omega_0| + \sum_{i=1}^n (1+|\Gamma_i|)^{-1}|\Gamma_i|}, \ s_j^* = \frac{|\Gamma_j| s_0^* + m_j}{1+|\Gamma_j|}, \ j = 1, 2, \cdots, n,$$

whereas the eigenvalues of (3.40) (thus the eigenvalues of A_d) are given by

$$\lambda_k^d = \min \left\{ \frac{\displaystyle\sum_{i=0}^n \int_{\Omega_i} (d_i |\nabla \phi_i|^2 + \phi_i^2) + \sum_{i=1}^n \int_{\Gamma_i} (\phi_i - \phi_0)^2}{\sum_{i=0}^n \int_{\Omega_i} \phi_i^2} : (\phi_0, \cdots, \phi_n) \perp V_{k-1} \right\},$$

where V_{k-1} is the subspace generated by the first $k-1$ eigenfunctions.

We will now proceed with the analysis of the nonlinear problem of the form

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(u_i), \ t > 0, \ x \in \Omega_i, \quad i = 0, 1 \ldots, n, \\ d_0 \dfrac{\partial u_0}{\partial \vec{\nu}} = 0, \ t > 0, \ x \in \Gamma, \\ d_0 \dfrac{\partial u_0}{\partial \vec{\nu}_i} = d_i \dfrac{\partial u_i}{\partial(-\vec{\nu}_i)} = (u_i - u_0), \ t > 0, \ x \in \Gamma_i, \quad i = 1, \ldots, n, \\ u_i(0, x) = \chi_i(x), \ x \in \Omega_i, \quad i = 0, \ldots, n, \end{cases} \tag{3.60}$$

where d_i, $i = 0, 1, \ldots, n$, are positive constants and

$$\begin{aligned} &f_i : \mathbb{R} \to \mathbb{R} \text{ are bounded and globally Lipschitz functions} \\ &\text{satisfying dissipativeness conditions } \limsup_{|s|\to\infty} \frac{f_i(s)}{s_i} < \theta_i < 0. \end{aligned} \tag{3.61}$$

With the notation as in this subsection the problem (3.60) can be viewed in the abstract form as

$$\begin{cases} \dot{u} = -A_d u + F(u) + u, \\ u(0) = \chi, \end{cases} \tag{3.62}$$

where $F : \mathbf{L}^2 \to \mathbf{L}^2$ is the Nemitskĭi operator associated to the map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$f(s) = (f_0(s_0), \ldots, f_n(s_n)) \ \text{ for } \ s = (s_0, \ldots, s_n) \in \mathbb{R}^{n+1}, \tag{3.63}$$

and A_d, $d \geq 1$, is the operator defined in Proposition 3.1 (see also Remark 3.3) and $\chi = (\chi_0, \chi_1, \cdots, \chi_n) \in \mathbf{L}^2$.

Remark 3. 5. Note that (3.62) is globally well posed in $\mathbf{L^2}$ as A_d is selfadjoint in $\mathbf{L}^2$ and $F : \mathbf{L}^2 \to \mathbf{L}^2$ is a globally Lipschitz function. Furthermore, multiplying the first equation in (3.62) by u in $\mathbf{L}^2$ and using (3.43) together with dissipativeness conditions (3.61) we obtain

$$\frac{d}{dt}\frac{1}{2}\|u\|^2_{\mathbf{L}^2} = -a_d(u,u) + \langle F(u) + u, u\rangle_{\mathbf{L}^2} \leq \theta\|u\|^2_{\mathbf{H}^1} + C_\theta, \tag{3.64}$$

where $\max_{i=0,\ldots,n}\theta_i < \theta < 0$ and C_θ is a certain positive constant that does not depend on $d \geq 1$. Consequently, the solutions of (3.62) satisfy the estimate

$$\|u\|^2_{\mathbf{L}^2} \leq \|u(0)\|^2_{\mathbf{L}^2} e^{2\theta t} - \frac{C_\theta}{\theta},\ t \geq 0,\ d \geq 1. \tag{3.65}$$

Since the resolvent of A_d is compact Remark 3.5 leads to the following proposition.

PROPOSITION 3.2. *For each $d \geq 1$ the problem* (3.62) *defines a C^0 semigroup in $\mathbf{L}^2$ which has a global attractor $\mathbf{A}_d$. Furthermore, the union of attractors $\cup_{d\geq 1}\mathbf{A}_d$ is contained in a ball $\mathcal{O}_R(0)$ in $\mathbf{L}^2$ of radius $R = -\frac{C_\theta}{\theta}$ around zero.*

Defining $\tilde{F} : \mathbf{L}^2 \to \mathbf{L}^2$ by

$$\tilde{F}(\psi) = \begin{cases} F(\psi) + \psi, \ \|\psi\|_{\mathbf{L}^2} \leq R, \\ F(\psi) + R\|\psi\|^{-1}_{\mathbf{L}^2}\psi, \ \|\psi\|_{\mathbf{L}^2} > R, \end{cases}$$

we will consider next the Cauchy problem

$$\begin{cases} \dot{u} = -A_d u + \tilde{F}(u), \\ u(0) = \chi. \end{cases} \tag{3.66}$$

Now both $\{-A_d\}_{d\in[1,\infty]}$ is of the class $CC(\mathbf{L}^2) \cap \mathcal{H}(\mathbf{L}^2, M, \theta)$ and $\tilde{F}$ is of the class $\mathcal{F}(\mathbf{L}^2, C)$. Also, the solutions of (3.66) satisfy the estimate (3.65) so that there is a global attractor $\tilde{\mathbf{A}}_d$ and $\tilde{\mathbf{A}}_d \subset \mathcal{O}_R(0)$; consequently,

$$\tilde{\mathbf{A}}_d = \mathbf{A}_d \ \text{ for every } \ d \geq 1. \tag{3.67}$$

Applying to the problem (3.66) the abstract Theorem 2.1 of Section 2 and taking into account Proposition 3.2 and (3.67) we obtain the following result.

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THEOREM 3.4. *Given ϕ_j^d, $j = 1, \ldots, n+1$ as in Theorem 3.3 define in X_d, $d \geq 1$, the projected spaces*

$$Y_d = span[\phi_1^d, \phi_2^d, \cdots, \phi_{n+1}^d], \ d \geq 1,$$

and decompose $\chi \in \mathbf{L}^2$ as

$$\chi = v^d + w^d = v_1^d \phi_1^d + \ldots + v_{n+1}^d \phi_{n+1}^d + w^d \in Y_d \oplus Y_d^\perp$$

where

$$v_j^d = \langle \chi, \phi_j^d \rangle_{\mathbf{L}^2} \ \ for \ \ j = 1, \ldots, n+1, \quad w^d = \chi - v_1^d \phi_1^d - \ldots - v_{n+1}^d \phi_{n+1}^d.$$

Then, for each d sufficiently large,
i) the problem (3.62) has a global attractor $\mathbf{A}_d$ and

$$\mathbf{A}_d \subset \{\chi \in \mathbf{L}^2 : \chi = y_d + \Sigma_d^*(y_d), \ y_d \in Y_d\},$$

where $\Sigma_d^* : Y_d \to Y_d^\perp$ is a certain Lipschitz continuous map satisfying

$$\sup_{y_d \in Y_d} \|\Sigma_d^*(y_d)\|_{\mathbf{L}^2} \overset{d\to\infty}{\longrightarrow} 0, \tag{3.68}$$

ii) the flow on $\mathbf{A}_d$ is given by

$$u(t) = v_1^d(t)\phi_1^d + \ldots + v_{n+1}^d(t)\phi_{n+1}^d + \Sigma_d^*(v(t)), \tag{3.69}$$

where

$$\dot{v}_j^d(t) = -\lambda_j^d v_j^d(t) + \langle F(v^d(t) + \Sigma_d^*(v^d(t))) + v^d(t) + \Sigma_d^*(v^d(t)), \phi_j^d \rangle_{\mathbf{L}^2} \tag{3.70}$$

for $j = 1, \ldots, n+1$.

Proof: We merely remark that Σ_d^* actually defines an exponentially attracting invariant manifold $\mathcal{S}_d$ for (3.66), which contains $\tilde{\mathbf{A}}_d = \mathbf{A}_d$ and that $\tilde{F} = F + Id$ on $\tilde{\mathbf{A}}_d$. ▮

We now show that the 'limit' problem can be viewed as the system of ordinary differential equations:

$$\begin{cases} \dot{u}_0 = f_0(u_0) + |\Omega_0|^{-1} \sum_{i=1}^n |\Gamma_i|(u_i - u_0), \\ \dot{u}_1 = f_1(u_1) + |\Gamma_1|(u_0 - u_1), \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \dot{u}_n = f_n(u_n) + |\Gamma_n|(u_0 - u_n). \end{cases} \tag{3.71}$$

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Remark 3. 6. Note that there is a global attractor for (3.71) (denoted by $\mathbf{A}_\infty$) as, multiplying the first equation in (3.71) by $|\Omega_0|u_0$, the others by u_i for $i = 1, \dots, n$ respectively and summing the results we get

$$\frac{d}{dt}\|u\|^2_{X_\infty} = \langle f(u), u\rangle_{X_\infty} - \sum_{i=1}^{n} |\Gamma_i|(u_i - u_0)^2,$$

where $f = (f_0, \dots, f_n)$. Using next dissipativeness condition from (3.61) we obtain

$$\frac{d}{dt}\|u\|^2_{X_\infty} \leq \theta\|u\|^2_{X_\infty} + C_\theta,$$

which is a counterpart of (3.64). Note also that (3.71) can be rewritten as

$$\dot{u} = -A_\infty u + f(u) + u. \tag{3.72}$$

Lemma 3.6. *If $d^k \to \infty$ and $\{u^{d^k}(t)\}$ is a sequence of the solutions of* $(3.62)_{|_{d=d^k}}$ *such that $u^{d^k}(t)$ 'lies' on the global attractor $\mathbf{A}_{d^k}$, then there is a subsequence, denoted the same, which converges in $\mathbf{L}^2$-topology to a certain bounded function $u : \mathbb{R} \to \mathbb{R}^{n+1} \subset \mathbf{L}^2$ which is a solution of* (3.72).

Proof: ¿From Theorem 3.3 we know that $\{\lambda^d_j\}$ and $\{\|\phi^d_j\|_{\mathbf{L}^2}\}$, $j = 1, \dots, n+1$, remain bounded as $d \to \infty$. From Proposition 3.2 we also know that $\|u^d\|_{\mathbf{L}^2} \leq R$. Hence both $v^d_j(t)$ and $\dot{v}^d_j(t)$ are functions from $\mathbb{R}$ to $\mathbb{R}$ bounded uniformly with respect to the parameter $d \geq 1$.

Thus there is a subsequence of the sequence $v^{d^k}(t) = v^{d^k}_1(t)\phi^{d^k}_1 + \dots + v^{d^k}_{n+1}(t)\phi^{d^k}_{n+1}$, $k \in \mathbb{N}$, denoted the same, and there are certain bounded functions $v^\infty_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, n+1$, such that if $v^\infty(t) = v^\infty_1(t)\phi^\infty_1 + \dots + v^\infty_{n+1}(t)\phi^\infty_{n+1}$ then

$$v^{d^k}(t) \to v^\infty(t) \text{ in } \mathbf{L}^2 \text{ uniformly for } t \text{ in compact time intervals.}$$

Since (3.68)-(3.69) hold we actually have that

$$u^{d^k}(t) \to v^\infty(t) \text{ in } \mathbf{L}^2 \text{ uniformly for } t \text{ in compact time intervals}$$

and, due to the Lipschitz continuity of $F : \mathbf{L}^2 \to \mathbf{L}^2$,

$$F(v^d(t) + \Sigma^*_d(v^d(t))) \to F(v^\infty(t))$$

in $\mathbf{L}^2$ uniformly for t in compact time intervals. Writing now (3.70) in the integral form and passing to the limit we get for every $j = 1, \dots, n+1$

$$v^\infty_j = v^\infty_j(0) - \lambda^\infty_j \int_0^t v^\infty_j(s)ds + \int_0^t \langle f(v^\infty(s)) + v^\infty(s), \phi^\infty_j\rangle_{X_\infty} ds;$$

that is v_j^∞ is differentiable and

$$\dot{v}_j^\infty = -\lambda_j^\infty v_j^\infty + \langle f(v^\infty(s)) + v^\infty(s), \phi_j^\infty \rangle_{X_\infty},\ j = 1, \dots, n+1.$$

Consequently we get

$$\langle \dot{v}^\infty, \phi_j^\infty \rangle_{X_\infty} = \langle v^\infty, -A_\infty \phi_j^\infty \rangle_{X_\infty} + \langle f(v^\infty(s)) + v^\infty(s), \phi_j^\infty \rangle_{X_\infty},$$

$j = 1, \dots, n+1$. If $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n+1}}$ denotes the standard scalar product in $\mathbb{R}^{n+1}$ and D is as in (3.48), then

$$\langle w_\infty, z_\infty \rangle_{X_\infty} = \langle w_\infty, D z_\infty \rangle_{\mathbb{R}^{n+1}} = \langle D w_\infty, z_\infty \rangle_{\mathbb{R}^{n+1}} \quad \text{for} \quad w_\infty, z_\infty \in X_\infty$$

so that recalling (3.47) we get

$$\begin{aligned}
&\langle \dot{v}^\infty, \phi_j^\infty \rangle_{X^\infty} - \langle f(v^\infty(s)) + v^\infty(s), \phi_j^\infty \rangle_{X_\infty} = \langle v^\infty, -A_\infty \phi_j^\infty \rangle_{X_\infty} \\
&= \langle v^\infty, -D A_\infty \phi_j^\infty \rangle_{\mathbb{R}^{n+1}} = \langle v^\infty, -\tilde{A}_\infty \phi_j^\infty \rangle_{\mathbb{R}^{n+1}} = \langle -\tilde{A}_\infty v^\infty, \phi_j^\infty \rangle_{\mathbb{R}^{n+1}} \\
&= \langle -D^{-1} \tilde{A}_\infty v^\infty, D\phi_j^\infty \rangle_{\mathbb{R}^{n+1}} = \langle -A_\infty v^\infty, D\phi_j^\infty \rangle_{\mathbb{R}^{n+1}} = \langle -A_\infty v^\infty, \phi_j^\infty \rangle_{X^\infty}
\end{aligned}$$

for each $j = 1, \dots, n+1$, which gives the result. ∎

Corollary 3.4. *The family of attractors $\{\mathbf{A}_d\}_{d \geqslant d_0}$ is upper semicontinuous at infinity relatively to E, where $\mathbf{A}_\infty$ is a global attractor for* (3.72).

Remark 3. 7. One can see that the Nemitskĭi operator F associated to f defined in (3.63) is also of class $\mathcal{F}(X_d^\alpha, C)$ and from the above results the vector field in (3.70) is a C^1 small perturbation of the vector field in

$$\dot{v}_j^\infty = -\lambda_j^\infty v_j^\infty + \langle f(v^\infty(s)) + v^\infty(s), \phi_j^\infty \rangle_{X_\infty},\ j = 1, \dots, n+1,$$

which is a counterpart of (3.71). Thus, if the flow in $\mathbf{A}_\infty$ is structurally stable, then the flow in $\mathbf{A}_d$ is topologically equivalent to the flow in $\mathbf{A}_\infty$.

3.3. An example of compact convergence involving non self-adjoint operators

Let us mention briefly that compact convergence property can also be established in the analysis of equations involving non self-adjoint operators, for which we mention a sample problem concerning a strongly damped wave equation of the form

$$\begin{cases} u_{tt} + B_d^{\frac{1}{2}} u_t + B_d u = f(u), & 0 < x < 1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x), & 0 < x < 1, \\ u_x(0) = u_x(1) = 0, & t > 0, \end{cases} \tag{3.73}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a globally bounded globally Lipschitz continuous function and B_d is the operator defined in (3.9). Let $X_d^\alpha = D(B_d^\alpha)$ with the graph norm $\alpha \in \mathbb{R}^+$ and $X_d^{-\alpha}$ be the completion of $X_d = L^2(0,1)$ with the norm $\|B_d^{-\alpha} \cdot \|_{X_d}$, $\alpha \in \mathbb{R}^+$. We remark that (3.73) can be viewed as the abstract parabolic problem

$$\begin{cases} z_t = \mathcal{L}_{d_{-1}} z + F(z), \\ z(0) = z_0 \end{cases} \tag{3.74}$$

in a Banach space $\mathcal{Y}_d := X_d \times X_d^{-\frac{1}{2}}$. Here $z = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, $z_0 = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, $F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 \\ f^e(u) \end{smallmatrix}\right]$, $f^e : X_d \to X_d \subset X_d^{-\frac{1}{2}}$ denotes the Nemitskĭi operator associated with f and the operator $\mathcal{L}_{d_{-1}}$ is an extension to $\mathcal{Y}_d$ of the operator

$$\mathcal{L}_d : D(\mathcal{L}_d) \subset X_d^{\frac{1}{2}} \times X_d \to X_d^{\frac{1}{2}} \times X_d, \ d \geq 1,$$

given on the domain $D(\mathcal{L}_d) = X_d^1 \times X_d^{\frac{1}{2}}$ in the matrix form

$$\mathcal{L}_d = \begin{bmatrix} 0 & I \\ -B_d & -B_d^{\frac{1}{2}} \end{bmatrix}.$$

We remark that $\mathcal{L}_d$ has compact resolvent, generates a compact C^0 analytic semigroup in $X_d^{\frac{1}{2}} \times X_d$ and its spectrum consists of isolated eigenvalues $\mu_k^{d,\pm}$ given by

$$\mu_k^{d,\pm} = (-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i})\sqrt{\lambda_k^d}, \quad \text{where } \lambda_k^d \in \sigma(B_d), \tag{3.75}$$

(see [18, Lemma A.1] and [8, Lemma 1]). We also remark that the same holds true for $\mathcal{L}_{d_{-1}}$ in $\mathcal{Y}_d$ (see [7, Lemma 2 and Proposition 5]) as $\mathcal{L}_{d_{-1}}$ can be viewed as the closure of $\mathcal{L}_d$ in the extrapolated space generated by $(\mathcal{L}_d, X_d^{\frac{1}{2}} \times X_d)$ (see [1]).

Let us define the 'limit' operator $\mathcal{L}_\infty : X_\infty \times X_\infty \to X_\infty \times X_\infty$ as

$$\mathcal{L}_\infty = \begin{bmatrix} 0 & I \\ -B_\infty & -B_\infty^{\frac{1}{2}} \end{bmatrix},$$

where X_∞ is the Hilbert space defined in Lemma 3.1 and B_∞ is as in (3.6). Then the operators $\mathcal{L}_{d_{-1}}^{-1} : \mathcal{Y}_d \to \mathcal{Y}_d$ and $\mathcal{L}_\infty^{-1} : X_\infty \times X_\infty \to X_\infty \times X_\infty$ can be viewed as

$$\mathcal{L}_{d_{-1}}^{-1} = \begin{bmatrix} -B_d^{-\frac{1}{2}} & -B_d^{-1} \\ I & 0 \end{bmatrix}, \ d \geq 1, \quad \mathcal{L}_\infty^{-1} = \begin{bmatrix} -B_\infty^{-\frac{1}{2}} & -B_\infty^{-1} \\ I & 0 \end{bmatrix}.$$

If $\mathbf{E}_d : \mathcal{Y}_\infty \to \mathcal{Y}_d$ is defined by

$$\mathbf{E}_d(\left[\begin{smallmatrix} u_\infty \\ v_\infty \end{smallmatrix}\right]) = \begin{bmatrix} E_d u_\infty \\ (B_d)_{-1}^{\frac{1}{2}} E_d v_\infty \end{bmatrix}$$

where $E_d : X_\infty \to X_d$ is given by (3.8) then, due to Lemma 3.1, $B_d^{-1} \overset{cc}{\dashrightarrow} B_\infty^{-1}$

$$\mathcal{L}_{d_{-1}}^{-1} \overset{cc}{\dashrightarrow} \mathcal{L}_\infty^{-1},$$

which is the crucial property if one wishes to consider asymptotic properties of the perturbed nonlinear problem (3.73) as $d \to \infty$.

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