CADERNOS DE MATEMÁTICA **11**, 83–91 May (2010)
ARTIGO NÚMERO SMA#329

Stability of the Euler obstruction on free divisors

Nivaldo G. Grulha Jr.*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação - USP.
Av. Trabalhador São-carlense, 400 - Centro.
Caixa Postal: 668 - CEP: 13560-970 - São Carlos - SP - Brasil.
E-mail: njunior@icmc.usp.br

We study in this paper the stability of the Euler obstruction of a function for families of functions with isolated singularities on weighted homogeneous hypersurfaces and on holonomic free divisors. May, 2010 ICMC-USP

INTRODUCTION

In a series of papers Damon has discussed the importance of free divisors [7, 8, 9]. In those papers he discussed that since the introduction of the notion of free divisors by Saito [15], people have discovered how commonplace they are. For example, discriminants of the versal unfoldings of isolated hypersurface and complete intersection singularities are free divisors, the bifurcation sets associated to the versal unfoldings of isolate hypersurfaces singularities are also free divisors, the discriminant of versal deformation of a space curve singularity is a free divisor. One way to investigate this objects is compute and understand the behavior of some invariants on them, for example we can study the local Euler obstruction.

The local Euler obstruction was first introduced by R. MacPherson in [14] as a key ingredient for his construction of characteristic classes of singular complex algebraic varieties. An equivalent definition was given by J.-P. Brasselet and M.-H. Schwartz in [5] using vector fields. Their viewpoint brings the local Euler obstruction into the framework of "indices of vector fields on singular varieties", though their definition only considers radial vector fields. A survey has been written by Brasselet [1] and another one by Brasselet and the author [2].

In the paper [3], J.-P. Brasselet, Lê D. T. and J. Seade gave a Lefschetz type formula for the local Euler obstruction, which shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms. A natural continuation of the result is the paper by J.-P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade [4], whose aim is to understand what is the obstacle for the local Euler obstruction to satisfy the Euler condition relatively to analytic functions with isolated singularity at

* Research Supported by FAPESP grant 2009/08774-0, Brazil

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

the considered point. That is the role of the so-called the local Euler obstruction of f, denoted by $Eu_{f,V}(0)$.

The relation between local Euler obstruction of f and the number of Morse points of a Morsification of f has been described, for particular germs of singular varieties, in [16] by J. Seade, M. Tibar and A. Verjovsky. They compare $Eu_{f,V}(0)$ with two different generalizations of the Milnor number for functions with isolated singularities on singular varieties. In the case where $(V, 0)$ is a complete intersection with isolated singularities they also study the GSV index [10].

In [11] the author presents some relations between the Euler obstruction of f and the notion of Milnor number of functions with an isolated critical point on singular spaces introduced by J. W. Bruce and R. M. Roberts in [6], denoted in this work by $\mu_{BR}(f)$. The originality of the main formula in [11] is that we use the information of all strata, while in [16] Seade, Tibar and Verjovsky use only the information in the regular stratum.

¿From the relations between the Euler obstruction of f and the Bruce and Roberts' Milnor number the author obtains in [11] relations between the Bruce and Roberts' Milnor number and Lê's Milnor number and Goryunov-Mond's Milnor number.

Applying some results of [11, 12], we study in this paper the stability of the Euler obstruction of f for families of functions with isolated singularities on the germ of holonomic free divisors $(V, 0)$. In this case the *characteristic logarithmic variety* associated to V, denoted by $LC(V)$, is Cohen-Macaulay [6].

1. MORSIFICATION OF A FUNCTION

Let $(V, 0)$ be the germ of a reduced complex analytic space at the origin, embedded in $\mathbb{C}^n$. In this work we will always consider $(V, 0)$ with pure dimension. Let $\{V_\alpha\}$ be a Whitney stratification of a sufficiently small representative V of the germ. We may suppose that V has only a finite number of strata V_α, $\alpha \in \{0, 1, 2, \cdots, d\}$, for some $d \in \mathbb{N}$, such that $0 \in \overline{V_\alpha}$.

To define a stratified Morse function we need the definition of a general point of a function (or general function at a point).

Definition 1.1. Let $(V, 0)$ be the germ of an analytic variety in $\mathbb{C}^n$, endowed with a Whitney stratification, and let $f : (V, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function, restriction of an analytic function $F : U \to (\mathbb{C}, 0)$, where U is an open ball around 0. We say that 0 is a general point of f (or $f : (V, 0) \to (\mathbb{C}, 0)$ is general at 0) if the hyperplane $ker(dF(0))$ is transverse in $\mathbb{C}^n$ to every generalized tangent space at 0; that is, for every sequence $\{x_n\}$ of points in some stratum V_α such that the sequence converges to 0 and the sequence of tangent spaces $\{T_{x_n} V_\alpha\}$ has a limit T, one has that $ker(dF(0))$ is transverse to T.

Definition 1.2. Let $(V, 0)$ be the germ of an analytic variety in $\mathbb{C}^n$, endowed with a Whitney stratification, and let $f : (V, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function. Let V be a sufficiently small representative of the germ $(V, 0)$, we say that $f : V \to \mathbb{C}$ is a stratified Morse function if the following hold:

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

(a) If $0 \in V_\alpha$, $\dim V_\alpha \geq 1$, the restriction of f to the stratum V_α has a Morse point at 0.
(b) If $0 \notin V_\alpha$, f is general at 0 with respect to the strata V_α.

The complex conjugate of the gradient of the extension F projects to the tangent spaces of the strata of V into a vector field, which may not be continuous. One can make it continuous [4], one gets a well-defined continuous stratified vector field, up to stratified homotopy, which we denote by $\overline{\nabla}_V f$. If f is a function on V with an isolated singularity at 0, with respect to the stratification, then $\overline{\nabla}_V f$ has an isolated zero at 0. If $\nu : \widetilde{V} \to V$ is the Nash modification of V and $\widetilde{T}$ is the Nash bundle over $\widetilde{V}$, then $\overline{\nabla}_V f$ lifts canonically to a never-zero section $\widetilde{\overline{\nabla}} f_V$ of $\widetilde{T}$ restricted to $\widetilde{V} \cap \nu^{-1}(V \cap \mathbb{S}_\varepsilon)$, where $\mathbb{S}_\varepsilon$ is a small enough sphere around 0. Following [4], the obstruction to extend $\widetilde{\overline{\nabla}}_V f$ without zeros over $\widetilde{V} \cap \nu^{-1}(V \cap \mathbb{B}_\varepsilon)$, where $\mathbb{B}_\varepsilon$ is a small ball around 0 is denoted by $Eu_{f,V}(0)$ and is called the local Euler obstruction of f at 0.

Lemma 1.1. *(Lemma 4.1, [17]) Let f be the germ of an analytic function on $(V, 0)$ with an isolated Morse singularity at the origin in the stratified sense, V_α the stratum that contains the origin. Then $Eu_{f,V}(0) = 0$ if $\dim V_\alpha < \dim V$, and $Eu_{f,V}(0) = (-1)^{\dim_{\mathbb{C}}(V,0)}$ if V_α is the regular stratum V_{reg}.*

Proposition 1.1 (Proposition 2.3, [16]). *Let $f : (V, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with isolated singularity at the origin. Then*

$$Eu_{f,V}(0) = (-1)^{\dim_{\mathbb{C}}(V,0)} n_{reg},$$

where n_{reg} is the number of Morse points in V_{reg} in a stratified Morsification of f.

2. GENERALIZATIONS OF THE MILNOR NUMBER

Let us recall that $\mu(f)$ denotes the Milnor number of a germ of an analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin, defined as $\dim_{\mathbb{C}} \mathcal{O}_n / J(f)$, where $\mathcal{O}_n$ is the ring of germs of analytic functions at the origin, and $J(f)$ is the Jacobian ideal of f.

Let (V, x) be the germ of a complex analytic space, for simplicity let us take $x = 0$. Bruce and Roberts in [6] defined a Milnor number for functions on singular spaces.

Let V be a sufficiently small representative of the germ $(V, 0)$ and $\mathcal{I}(V)$ denote the ideal in $\mathcal{O}_n$ consisting of the germs of functions vanishing on V.

One of the main goals in [6] is to characterize germs of diffeomorphisms preserving V. The usual technique is the integration of germs of vector fields tangent to V.

Definition 2.1. For $x \in \mathbb{C}^n$, let $Der_x\mathbb{C}^n$ denote the $\mathcal{O}_n$-module of germs of analytic vector fields on $\mathbb{C}^n$ at x. A vector field δ in $Der_x\mathbb{C}^n$ is said to be logarithmic for (V, x) if, when considered as a derivation $\delta : \mathcal{O}_n \to \mathcal{O}_n$, we have $\delta(h) \in \mathcal{I}(V)$ for all $h \in \mathcal{I}(V)$. The $\mathcal{O}_n$-module of such vector fields is denoted by $\Theta_{(V,x)}$. When $x = 0$, we denote it by Θ_V.

LEMMA 2.1 (Lemma 1.5, [6]). *Let V be a sufficiently small representative of the germ $(V, 0)$ and U an open neighborhood of the origin. There is a unique stratification $\{V_\alpha\}$ of U with the following properties:*

(i)Each stratum V_α is a connected manifold embedded in U, and U is the union $\cup V_\alpha$;

(ii)if V_α and V_β are two different strata, such that $V_\alpha \cap \overline{V}_\beta \neq \emptyset$, then $V_\alpha \subset \partial V_\beta$, where ∂V_β denotes the boundary of V_β.

(iii)if $x \in V_\alpha$ then the tangent space $T_x V_\alpha$ of V_α at x coincide with $\Theta_{(V,x)}$;

DEFINITION 2.2. The stratification $\{V_\alpha\}$ as above is called a logarithmic stratification of V, and each V_α is called the logarithmic stratum.

DEFINITION 2.3. The germ $(V, 0)$ is holonomic for some neighborhood U at 0 in $\mathbb{C}^n$ if the logarithmic stratification of U has only a finite number of strata.

PROPOSITION 2.1 (Proposition 1.10, [6]). *Let $(V, 0)$ be a germ of holonomic variety, then in a sufficiently small neighborhood at the origin, the logarithmic stratification is Whitney regular.*

Let $(V, 0)$ be the germ of an analytic variety in $\mathbb{C}^n$ endowed with a Whitney stratification $\{V_\alpha\}$. Let $f : (V, 0) \to (\mathbb{C}, 0)$ be an analytic function, which is the restriction of a analytic function $F : (U, 0) \to (\mathbb{C}, 0)$. Let V be a sufficiently small representative of the germ $(V, 0)$, we say that a critical point of f is a point $x \in V$ such that, if $x \in V_\alpha$ we have $dF(x)(T_x(V_\alpha)) = 0$. We say that f has an isolated singularity at $0 \in V$ relative to the given Whitney stratification, if f has no critical points in a punctured neighborhood of 0 in V.

DEFINITION 2.4. Let $J_V(f)$ be the ideal $\{\delta f : \delta \in \Theta_V\}$ in $\mathcal{O}_{n,0}$. If the germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has an isolated singularity on V at the origin, then $\dim_{\mathbb{C}} \mathcal{O}_n / J_V(f)$ is finite and is called the multiplicity of f over V at 0, or the Bruce-Roberts' Milnor number, and is denoted by $\mu_{BR}(f)$.

The *characteristic logarithmic variety* associated to V, denoted by $LC(V)$ is an important tool in [6], and we will use the information of this variety to obtain the main result of this work.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

DEFINITION 2.5. [6] Let us suppose that the vector fields $\delta_1, \ldots, \delta_m$ generate Θ_V for some neighborhood U at $0 \in \mathbb{C}^n$. Then if $T_U^*\mathbb{C}^n$ is the restriction of the cotangent bundle of $\mathbb{C}^n$ in U, we define $LC_U(V)$ as $\{(x, \xi) \in T_U^*\mathbb{C}^n : \xi(\delta_i(x)) = 0, i = 1 \ldots, m\}$.

$LC(V)$ is the germ of $LC_U(V)$ in $T_0^*\mathbb{C}^n$, and it can be shown that it is independent of the choice of generators of Θ_V.

Let V_α be a stratum of the logarithmic stratification of V. The conormal space of V_α is the subspace of $T_0^*\mathbb{C}^n$ given by all forms vanishing on the tangent bundle TV_α. We denote it by $C(V_\alpha)$.

Then we have,

$$LC(V) = \bigcup_\alpha \overline{C(V_\alpha)}.$$

PROPOSITION 2.2 (Proposition 1.14 (ii), [6]). *Let V be a holonomic space with stratification $\{V_\alpha\}$, then the sets $\overline{C(V_\alpha)}$ are the irreducible components of $LC(V)$.*

THEOREM 2.1 (Corollary 5.8, [6]). *If $f : (V, 0) \to \mathbb{C}$ has an isolated singularity at the origin and n_α is the number of critical points of a stratified Morsification of f on V_α, and m_α denotes the multiplicity of $C(V_\alpha)$ in $LC(V)$, then,*

$$\sum_\alpha m_\alpha n_\alpha \leq \dim_{\mathbb{C}} \mathcal{O}_n / J_V(f).$$

with equality if and only if $LC(V)$ is Cohen-Macaulay at $(0, df(0))$.

THEOREM 2.2 (Theorem 3.14, [11], also see [12]). *Let $(V, 0)$ be as above such that the $LC(V)$ is Cohen-Macaulay and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a function with an isolated singularity at the origin and such that $f : V \to \mathbb{C}$ has also a stratified isolated singularity at the origin,then*

$$\mu_{BR}(f) = \sum_{\alpha=0}^{d} m_\alpha (-1)^{\dim_{\mathbb{C}} V_\alpha} Eu_{f, \overline{V}_\alpha}(0),$$

where m_α denotes the multiplicity of $C(V_\alpha)$ in $LC(V)$.

COROLLARY 2.1. *Let $(V, 0)$ be a weighted homogeneous hypersurface with isolated singularities and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a function with an isolated singularity at the origin and such that $f : V \to \mathbb{C}$ has also a stratified isolated singularity at the origin,then*

$$\mu_{BR}(f) = \sum_{\alpha=0}^{d} (-1)^{\dim_{\mathbb{C}} V_\alpha} Eu_{f, \overline{V}_\alpha}(0).$$

Proof. This follows from the last theorem and by the fact that since V is weighted homogeneous hypersurface withisolated singularity, then $LC(V)$ is Cohen-Macaulay and all of the multiplicities m_α in this case are 1 [6]. ∎

PROPOSITION 2.3 (Proposition 3.16, [11]). *Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be an ICIS, and $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ a family of functions with isolated singularity, we denote $F(x, u) = f_u(x)$. Then, the following are equivalent,*

1.$Eu_{f_u,V}(0)$ is constant for the family.
2.$\mu_L(f_u)$ is constant for the family.

3. STABILITY OF THE EULER OBSTRUCTION OF *F* AND THE BRUCE-ROBERTS' MILNOR NUMBER

In [13], another notion of Milnor number arises, a generalization of the Milnor number for analytic functions defined on singular analytic spaces such that the rectified homotopical depth of V at 0, denoted $\mathrm{rhd}(V, 0)$ satisfies $\mathrm{rhd}(V, 0) = \dim_{\mathbb{C}}(V, 0)$.

Let V be a sufficiently small representative of the germ $(V, 0)$. The Milnor fiber of the complex analytic function f, defined on V, with an isolated singularity at 0 (in the stratified way), has the homotopy type of a bouquet of spheres. The Lê's Milnor number, denoted by $\mu_L(f)$, is defined as the number of spheres in the bouquet.

The relations between this invariant and the local Euler obstruction of f were obtained in [16], in particular for a complete intersection with isolated singularity (ICIS), since in this case we have $\mathrm{rhd}(V, 0) = \dim(V, 0)$, the following holds (Section 3.1, [16]):

THEOREM 3.1. *Let V be a sufficiently small representative of an ICIS germ, $0 \in V$, f an analytic function on V with stratified isolated singularity at 0, and l a generic linear form. Then, we have*

$$Eu_{f,V}(0) = (-1)^{\dim_{\mathbb{C}}(V,0)}[\mu_L(f) - \mu_L(l)].$$

The next result relates these different notions of Milnor number above (and the classical one) and the Euler obstruction of a function. This theorem was presented in [11], but the assumptions of the if $LC(V)$ is Cohen-Macaulay was missing. In fact, this is a open problem. Another version of this result was presented in [12].

THEOREM 3.2 (Theorem 4.18, [11], see also [12]). *Let $V \subset \mathbb{C}^n$ be a hypersurface with isolated singularity such that its $LC(V)$ is Cohen-Macaulay and $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ a family of functions with isolated singularity on V at 0, then:*

(a)$\mu_{BR}(f_u)$ constant for the family implies $\mu(f_u)$, $\mu_L(f_u)$ and $Eu_{f_u,V}(0)$ constant for the family.

(b)When $\mu(f_u)$ is constant for the family, we have that $Eu_{f_u,V}(0)$ or $\mu_L(f_u)$ constant for the family implies $\mu_{BR}(f_u)$ is constant for the family.

In the weighted homogeneous case we have the following result:

PROPOSITION 3.1. *Let $V \subset \mathbb{C}^n$ be a weighted homogeneous hypersurface with isolated singularity and $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ a family of functions with isolated singularity on V at 0, then:*

(a)$\mu_{BR}(f_u)$ constant for the family implies $\mu(f_u)$, $\mu_L(f_u)$ and $Eu_{f_u,V}(0)$ constant for the family.
(b)When $\mu(f_u)$ is constant for the family, we have that $Eu_{f_u,V}(0)$ or $\mu_L(f_u)$ constant for the family implies $\mu_{BR}(f_u)$ is constant for the family.

Proof. Since V is weighted homogeneous hypersurface with isolated singularity, then $LC(V)$ is Cohen-Macaulay [6], and by Corollary 2.1, the following holds

$$\mu_{BR}(f_u) = \mu(f_u) + [\mu_L(f_u) - \mu_L(l)] + 1.$$

Since l is a generic linear form, we know that $[\mu_L(f_u) - \mu_L(l)] \geq 0$, and since μ and μ_L are upper semicontinuous we have by the relations above that $\mu_{BR}(f_u)$ constant for the family implies $\mu(f_u)$ and $\mu_L(f_u)$ are constant for the family. Therefore, by the Proposition 2.3, it follows that $Eu_{f_u,V}(0)$ is constant for the family, so we conclude the item (a).
To prove (b), notice that in the same relation above, if $\mu(f_u)$ is constant for the family, it follows that $\mu_L(f_u)$ constant for the family implies $\mu_{BR}(f_u)$ constant for the family, and by the Proposition 2.3 we conclude the item (b). ▮

Let us now give the definition of a free divisor introduced by Saito [15]. Examples of free divisors are discriminants of the versal unfoldings of isolated hypersurface and complete intersection singularities, the bifurcation sets associated to the versal unfoldings of isolate hypersurfaces singularities are also free divisors and the discriminant of versal deformation of a space curve singularity.

DEFINITION 3.1. A reduced hypersurface $(V, 0) \subset (\mathbb{C}^n, 0)$ is said to be a *free divisor* if $\Theta_{V,0}$ is a free $\mathcal{O}_n$-module.

Remark 3. 1. If $(V, 0) \subset (\mathbb{C}^n, 0)$ is a free divisor, then $\Theta_{V,0}$ is necessarily generated by n elements.

THEOREM 3.3 (Proposition 6.3, [6]). *If $(V, 0) \subset (\mathbb{C}^n, 0)$ is a reduced analytic subvariety, any two of the following properties imply the third.*

(i)V is holonomic;
(ii)$LC(V)$ is a complete intersection;
(iii)$(V,0)$ is a free divisor.

In particular when V is holonomic and a free divisor, $LC(V)$ is a complete intersection, therefore Cohen-Macaulay, and in this case we have the equality in the formula of Theorem 2.1

$$\mu_{BR}(h) = \sum_{\alpha=0}^{d} m_\alpha n_\alpha,$$

where n_α is the number of Morse points of a Morsification f_t on V_α.

Let $(V,0) \subset (\mathbb{C}^n, 0)$ be the germ of a holonomic analytic variety. Let us take V as a sufficiently small representative of the germ, such that the logarithmic stratification $\{V_\alpha\}$, with $\alpha \in \{0,1,2,\cdots,d\}$ for some $d \in \mathbb{N}$, be Whitney. The closure of each stratum V_α is itself an analytic space, with regular part V_α, so it make sense to define the invariant $Eu_{f,\overline{V}_\alpha}(0)$.

THEOREM 3.4. *Let $V \subset \mathbb{C}^n$ be a holonomic free divisor, and*

$$F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$$

a family of functions with isolated singularity on V at 0, then $\mu_{BR}(f_u)$ constant for the family implies $\mu(f_u)$, $\mu_L(f_u)$ and $Eu_{f_u,V}(0)$ constant for the family.

Proof. Since V is a holonomic free divisor, by Proposition 6.3 of [6] the $LC(V)$ is a complete intersection, therefore Cohen-Macaulay, and in this case we have by Theorem 2.1, the following

$$\mu_{BR}(f) = \sum_{\alpha=0}^{d} m_\alpha (-1)^{\dim_{\mathbb{C}} V_\alpha} Eu_{f,\overline{V}_\alpha}(0),$$

where m_α denotes the multiplicity of $C(V_\alpha)$ in $LC(V)$.

Since $g_\alpha(u) = (-1)^{\dim_{\mathbb{C}} V_\alpha} Eu_{f_u,\overline{V}_\alpha}(0)$ are upper semicontinuous, because it is counting the number of morse points, we have by the relations

$$\mu_{BR}(f) = \sum_{\alpha=0}^{d} m_\alpha g_\alpha(u),$$

that $\mu_{BR}(f_u)$ constant for the family implies that all terms $g_\alpha(u)$ and $\mu(f_u)$ are constant for the family in particular it follows that $\mu(f)$ and $Eu_{f_u,V}(0)$ are constant for the family. Therefore, by the Proposition 2.3 $\mu_L(f_u)$ is also constant. ∎

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

REFERENCES

1. J.-P. Brasselet, *Local Euler obstruction, old and new*, in “Proceedings of XI Brazilian Topology Meeting (Rio Claro, 1998)”, World Sci. Publishing, River Edge, NJ (2000), 140-147.
2. J.-P. Brasselet and N. G. Grulha Jr., *Local Euler obstruction, old and new II*, To appear in Proceedings of the London Mathematical Society, 2010.
3. J.-P. Brasselet, Lê D. T. and J. Seade, *Euler obstruction and indices of vector fields*, Topology 39 (2000), no. 6, 1193-1208.
4. J.-P. Brasselet, D. Massey and A. J. Parameswaran and J. Seade, *Euler obstruction and defects of functions on singular varieties*, J. London Math. Soc. (2) 70 (2004), no. 1, 59-76.
5. J.-P. Brasselet et M.-H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Astérisque 82-83 (1981), 93-147.
6. J. W. Bruce and R. M. Roberts, *Critical points of functions on analytic varieties*, Topology Vol. 27, No 1. pp 57-90, 1988.
7. J. Damon, *On the legacy of free divisors: discriminants and Morse-type singularities*, Amer. J. Math. 120 (1998), no. 3, 453–492.
8. J. Damon, *On the legacy of free divisors. II: Free* divisors and complete intersections.* Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 361–395, 742.
9. J. Damon, *On the legacy of free divisors. III: Functions and divisors on complete intersections.* Q. J. Math. 57 (2006), no. 1, 49–79.
10. X. Gómez-Mont, J. Seade and A. Verjovsky, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. 291 (1991), 737-751.
11. N. G. Grulha Jr, *The Euler Obstruction and Bruce-Roberts' Milnor Number*, Quart. J. Math. 60(3) (2009) 291–302.
12. N. G. Grulha Jr, *The Euler Obstruction and Bruce-Roberts' Milnor Number (Erratum)*, 2010.
13. Lê D. T., *Complex analytic functions with isolated singularities*, J. Algebraic Geometry 1 (1992), 83 - 100.
14. R. D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. 100,(1974), 423-432.
15. K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 2, 265–291.
16. J. Seade, M. Tibar and A. Verjovsky, *Milnor Numbers and Euler obstruction*, Bull. Braz. Math. Soc. (N.S.) 36 (2005), no. 2, 275–283.
17. J. Seade, M. Tibar and A. Verjovsky, *Global Euler obstruction and polar invariants*, Math. Ann. 333 (2005), no. 2, 393–403.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC