CADERNOS DE MATEMÁTICA **11**, 71–82 May (2010)
ARTIGO NÚMERO SMA#328

Invariants of relative right and contact equivalences

Imran Ahmed*

Department of Mathematics, COMSATS Institute of Information Technology, M.A. Jinnah Campus, Defence Road, off Raiwind Road Lahore, PAKISTAN.
E-mail: drimranahmed@ciitlahore.edu.pk

Maria Aparecida Soares Ruas†

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: maasruas@icmc.usp.br

Key Words: relative Milnor algebra, relative Tjurina algebra, $\mathcal{R}_V$-equivalence, $\mathcal{K}_V$-equivalence, quasihomogeneous polynomial, function germ.

1. INTRODUCTION

Let $\mathcal{O}_n$ be the ring of germs of analytic functions $h : \mathbb{C}^n, 0 \to \mathbb{C}$. Consider the analytic variety $V = \{x : f_1(x) = \ldots = f_r(x) = 0\} \subset \mathbb{C}^n, 0$, where $f_1, \ldots, f_r$ are germs of analytic functions. In this note we study function germs $h : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ under the equivalence relation that preserves the analytic variety $V, 0$.

We say that two function germs h_1 and h_2 :$\mathbb{C}^n, 0 \to \mathbb{C}, 0$ are $\mathcal{R}_V$-equivalent if there exists a germ of a diffeomorphism $\psi : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ with $\psi(V) = V$ and $h_1 \circ \psi = h_2$. That is,

$$\mathcal{R}_V = \{\psi \in \mathcal{R} : \psi(V) = V\}$$

where $\mathcal{R}$ is the group of germs of diffeomorphisms of $\mathbb{C}^n, 0$.

Two function germs h_1 and h_2 :$\mathbb{C}^n, 0 \to \mathbb{C}, 0$ are $\mathcal{K}_V$-equivalent if there exists a germ of a diffeomorphism $\psi : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ and a unit $u \in \mathcal{O}_n^*$ such that $\psi(V) = V$ and $h_1 = u.(h_2 \circ \psi)$.

We denote by θ_n the set of germs of tangent vector fields in $\mathbb{C}^n, 0$; θ_n is a free $\mathcal{O}_n$ module of rank n. Let $I(V)$ be the ideal in $\mathcal{O}_n$ consisting of germs of analytic functions vanishing on V. We denote by $\Theta_V = \{\eta \in \theta_n : \eta(I(V)) \subseteq I(V)\}$, the submodule of germs of vector fields tangent to V.

* Supported by CNPq-TWAS, grant # FR 3240188100.
† Partially supported by FAPESP, grant # 08/54222-6, and CNPq, grant # 301474/2005-2.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

The tangent space to the action of the group $\mathcal{R}_V$ is $T\mathcal{R}_V(h) = dh(\Theta_V^0) = \mathcal{J}_h(\Theta_V^0)$, where Θ_V^0 is the submodule of Θ_V given by the vector fields that are zero at zero. When the point $x = 0$ is a stratum in the logarithmic stratification of the analytic variety, this is the case when V has an isolated singularity at the origin, see [2] for details, both spaces Θ_V and Θ_V^0 coincide.

The tangent space to the action of the group $\mathcal{K}_V$ is $T\mathcal{K}_V(h) = \langle h, dh(\Theta_V^0)\rangle = \langle h, \mathcal{J}_h(\Theta_V^0)\rangle$.

We fix a system of local coordinates x of $\mathbb{C}^n$. Due to the identification between $\mathcal{O}_n$ and the ring of convergent power series $\mathbb{C}\{x_1, \ldots, x_n\}$ we identify a germ $f \in \mathcal{O}_n$ with its power series $f(x) = \sum a_\alpha x^\alpha$, where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

The relative Milnor and Tjurina algebras, $M_V(h)$ and $T_V(h)$, of h are defined respectively, by

$$M_V(h) = \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\mathcal{J}_h(\Theta_V^0)} \text{ and } T_V(h) = \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\langle h, \mathcal{J}_h(\Theta_V^0)\rangle}.$$

When V is a weighted homogeneous variety, we can always choose weighted homogeneous generators for Θ_V. Moreover, it is finitely generated [3], Lemma 3.2, p.41.

We recall first Mather's Lemma 4.1 providing effective necessary and sufficient conditions for a connected submanifold to be contained in an orbit. In Theorem 4.1 we show that two arbitrary (i.e. not necessary with isolated singularities) quasihomogeneous polynomials f and g having isomorphic relative Milnor algebras $M_V(f)$ and $M_V(g)$ are $\mathcal{R}_V$-equivalent.

In Theorem 4.2 we prove that two arbitrary complex-analytic hypersurfaces (i.e. not necessary with isolated singularities), one is quasihomogeneous and other is arbitrary, are determined by isomorphism of Jacobean ideals.

The Example of Gaffney and Hauser, in [5], suggests us that we can not extend our results for arbitrary analytic germs.

In Theorem 4.4 we show that two arbitrary (i.e. not necessary with isolated singularities) function germs f and g having isomorphic relative Tjurina algebras $T_V(f)$ and $T_V(g)$ are $\mathcal{K}_V$-equivalent, where $V = \Phi^{-1}(0)$ be a variety in $\mathbb{C}^n, 0$ such that $\Phi \in \mathbb{C}\{x_1, \ldots, x_n\}$ and Θ_V is finitely generated.

This is the relative version of the celebrated theorem by Mather and Yau [9], saying that the isolated hypersurface singularities are determined by their Tjurina algebras. For arbitrary hypersurface singularities, the Mather-Yau Theorem (even a more general version) has been proved by Greuel, Lossen and Shustin [6].

2. QUASIHOMOGENEOUS FUNCTIONS AND FILTRATIONS

We recall first some basic facts on quasihomogeneous functions and filtrations in the ring A of formal power series. We introduce, in the next section, their analogues for quasihomogeneous diffeomorphisms and vector fields. For a more complete introduction see [1], Chap. 1, §3.

A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ (defined on the complex space $\mathbb{C}^n$) is a quasihomogeneous function of degree d with weights $w_1, \ldots, w_n$ if

$$f(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^d f(x_1, \ldots, x_n) \,\forall\, \lambda > 0.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

In terms of the Taylor series $\sum f_{\underline{k}} x^{\underline{k}}$ of f, the quasihomogeneity condition means that the exponents of the nonzero terms of the series lie in the hyperplane

$$L = \{\underline{k} : w_1 k_1 + \ldots + w_n k_n = d\}.$$

Any quasihomogeneous function f of degree d satisfies Euler's identity

$$\sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} = d.f \tag{1}$$

It implies that a quasihomogeneous function f belongs to its Jacobean ideal $\mathcal{J}_f$. The following is the well known result of Saito [10].

THEOREM 2.1. *A function-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivalent to a quasihomogeneous function-germ if and only if $f \in \mathcal{J}_f$.*

Consider $\mathbb{C}^n$ with a fixed coordinate system $x_1, \ldots, x_n$. The algebra of formal power series in the coordinates will be denoted by $A = \mathbb{C}[[x_1, \ldots, x_n]]$. We assume that a quasihomogeneity type $\underline{w} = (w_1, \ldots, w_n)$ is fixed. With each such $\underline{w}$ there is associated a filtration of the ring A, defined as follows.

The monomial $\mathbf{x}^{\underline{k}}$ is said to have degree d if $< \underline{w}, \underline{k} >= w_1 k_1 + \ldots + w_n k_n = d$.

The order d of a series (resp. polynomial) is the smallest of the degrees of the monomials that appear in that series (resp. polynomial).

The series of order larger than or equal to d form a subspace $A_d \subset A$. The order of a product is equal to the sum of the orders of the factors. Consequently, A_d is an ideal in the ring A. The family of ideals A_d constitutes a decreasing filtration of A: $A_{\acute{d}} \subset A_d$ whenever $\acute{d} > d$. We let A_{d+} denote the ideal in A formed by the series of order higher than d.

The quotient algebra A/A_{d+} is called the algebra of d-quasijets, and its elements are called d-quasijets.

3. QUASIHOMOGENEOUS DIFFEOMORPHISMS AND VECTOR FIELDS

Several Lie groups and algebras are associated with the filtration defined in the ring A of power series by the type of quasihomogeneity $\underline{w}$. In the case of ordinary homogeneity these are the general linear group, the group of k-jets of diffeomorphisms, its subgroup of k-jets with $(k-1)$-jet equal to the identity, and their quotient groups. Their analogues for the case of a quasihomogeneous filtration are defined as follows.

A formal diffeomorphism $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a set of n power series $g_i \in A$ without constant terms for which the map $g^* : A \to A$ given by the rule $g^* f = f \circ g$ is an algebra isomorphism.

The diffeomorphism g is said to have order d if for every s

$$(g^* - 1) A_s \subset A_{s+d}.$$

The set of all diffeomorphisms of order $d \geq 0$ is a group G_d. The family of groups G_d yields a decreasing filtration of the group G of formal diffeomorphisms; indeed, for $\acute{d} > d \geq 0$, $G_{\acute{d}} \subset G_d$ and is a normal subgroup in G_d.

The group G_0 plays the role in the quasihomogeneous case that the full group of formal diffeomorphisms plays in the homogeneous case. We should emphasize that in the quasihomogeneous case $G_0 \neq G$, since certain diffeomorphisms have negative orders and do not belong to G_0.

The group of d-quasijets of type $\underline{w}$ is the quotient group of the group of diffeomorphisms G_0 by the subgroup G_{d+} of diffeomorphisms of order higher than d: $J_d = G_0/G_{d+}$.

Note that in the ordinary homogeneous case our numbering differs from the standard one by 1: for us J_0 is the group of 1-jets and so on.

J_d acts as a group of linear transformations on the space A/A_{d+} of d-quasijets of functions. A special importance is attached to the group J_0, which is the quasihomogeneous generalization of the general linear group.

A diffeomorphism $g \in G_0$ is said to be quasihomogeneous of type $\underline{w}$ if each of the spaces of quasihomogeneous functions of degree d (and type $\underline{\mathrm{w}}$) is invariant under the action of g^*.

The set of all quasihomogeneous diffeomorphisms is a subgroup of G_0. This subgroup is canonically isomorphic to J_0, the isomorphism being provided by the restriction of the canonical projection $G_0 \to J_0$.

The infinitesimal analogues of the concepts introduced above look as follows.

A formal vector field $v = \sum v_i \partial_i$, where $\partial_i = \partial/\partial x_i$, is said to have order d if differentiation in the direction of v raises the degree of any function by at least d: $L_v A_s \subset A_{s+d}$.

We let $\mathfrak{g}_d$ denote the set of all vector fields of order d. The filtration arising in this way in the Lie algebra $\mathfrak{g}$ of vector fields (i.e., of derivations of the algebra A) is compatible with the filtrations in A and in the group of diffeomorphisms G:

1. $f \in A_d, v \in \mathfrak{g}_s \Rightarrow fv \in \mathfrak{g}_{d+s}, L_v f \in A_{d+s}$
2. The module $\mathfrak{g}_d$, $d \geq 0$, is a Lie algebra w.r.t. the Poisson bracket of vector fields.
3. The Lie algebra $\mathfrak{g}_d$ is an ideal in the Lie algebra $\mathfrak{g}_0$.
4. The Lie algebra $\mathfrak{j}_d$ of the Lie group J_d of d-quasijets of diffeomorphisms is equal to the quotient algebra $\mathfrak{g}_0/\mathfrak{g}_{d+}$.
5. The quasihomogeneous vector fields of degree 0 form a finite-dimensional Lie subalgebra of the Lie algebra $\mathfrak{g}_0$; this subalgebra is canonically isomorphic to the Lie algebra $\mathfrak{j}_0$ of the group of 0-jets of diffeomorphisms.

The support of a quasihomogeneous function of degree d and type $\underline{w}$ is the set of all points $\underline{k}$ with nonnegative integer coordinates on the diagonal

$$L = \{\underline{k} : \langle \underline{k}, \underline{w} \rangle = d\}.$$

Quasihomogeneous functions can be regarded as functions given on their supports: $\sum f_{\underline{k}} x^{\underline{k}}$ assumes at $\underline{k}$ the value $f_{\underline{k}}$. The set of all such functions is a linear space $\mathbb{C}^r$, where r is the number of points in the support. Both the group of quasihomogeneous diffeomorphisms (of type $\underline{\mathrm{w}}$) and its Lie algebra $\mathfrak{a}$ act on this space.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

The Lie algebra $\mathfrak{a}$ of a quasihomogeneous vector field of degree 0 is spanned, as a $\mathbb{C}$-linear space, by all monomial fields $x^{\underline{P}}\partial_i$ for which $< \underline{P}, \underline{w} >= w_i$. For example, the n fields $x_i\partial_i$ belong to $\mathfrak{a}$ for any $\underline{w}$.

Example 3.1. Consider the quasihomogeneous polynomial $f = x^2y + z^2$ of degree $d = 6$ w.r.t. weights $(2, 2, 3)$. Note that the Lie algebra of quasihomogeneous vector fields of degree 0 is spanned by

$$\mathfrak{a} = \langle x^{\underline{P}}\partial_i :< \underline{P}, \underline{w} >= w_i, i = 1, 2, 3\rangle = \langle x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\rangle$$

4. FUNCTION GERMS WITH ISOMORPHIC LOCAL ALGEBRAS

We recall first Mather's lemma providing effective necessary and sufficient conditions for a connected submanifold (in our case the path P) to be contained in an orbit.

Lemma 4.1. *([8]) Let $m : G \times M \to M$ be a smooth action and $P \subset M$ a connected smooth submanifold. Then P is contained in a single G-orbit if and only if the following conditions are fulfilled:*
(a) $T_x(G.x) \supset T_xP$, for any $x \in P$.
(b) $\dim T_x(G.x)$ is constant for $x \in P$.

For arbitrary (i.e. not necessary with isolated singularities) quasihomogeneous polynomials we establish the following results.

Lemma 4.2. *Let $f, g \in H^d_{\underline{w}}(n, 1; \mathbb{C}) = H^d_{\underline{w}}$ be two quasihomogeneous polynomials of degree d and $\Phi \in H^r_{\underline{w}}(n, 1; \mathbb{C})$ be a quasihomogeneous polynomial of degree r w.r.t. the same weights $\underline{w} = (w_1, \ldots, w_n)$ such that $\mathcal{J}_f(\Theta^0_V) = \mathcal{J}_g(\Theta^0_V)$, where $\Phi^{-1}(0) = V$ is a hypersurface in $(\mathbb{C}^n, 0)$. Then $f \overset{\mathcal{R}_V}{\sim} g$, where $\overset{\mathcal{R}_V}{\sim}$ denotes the relative right equivalence.*

Proof.
To prove this claim choose an appropriate submanifold of $H^d_{\underline{w}}(n, 1; \mathbb{C})$ containing f and g and then apply Mather's lemma to get the result.

Let $f, g \in H^d_{\underline{w}}(n, 1; \mathbb{C})$ such that $\mathcal{J}_f(\Theta^0_V) = \mathcal{J}_g(\Theta^0_V)$. Set $f_t = (1-t)f + tg \in H^d_{\underline{w}}(n, 1; \mathbb{C})$. Consider the $\mathcal{R}_V$-equivalence action on $H^d_{\underline{w}}(n, 1; \mathbb{C})$ under the group $\mathcal{R}^0_V = \mathcal{R}_V \cap J_0$, we have

$$T_{f_t}(\mathcal{R}^0_V.f_t) = \mathcal{J}_{f_t}(\Theta^0_V) \cap H^d_{\underline{w}} = \langle df_t(\xi_i) : i = 1, \ldots, p\rangle \cap H^d_{\underline{w}} \subset T_{f_t}(J_0.f_t) \qquad (2)$$

where $df_t(\xi_i) = \sum_{j=1}^n a_{ij}x^{\underline{P}}\frac{\partial f_t}{\partial x_j} = \sum_{j=1}^n a_{ij}x^{\underline{P}}[(1-t)\frac{\partial f}{\partial x_j} + t\frac{\partial g}{\partial x_j}]$, $< \underline{P}, \underline{w} >= w_j$.
Note that we have the inclusion of finite dimensional $\mathbb{C}$-vector spaces

$$T_{f_t}(\mathcal{R}^0_V.f_t) = \langle df_t(\xi_i)\rangle \cap H^d_{\underline{w}} \subset \mathcal{J}_f(\Theta^0_V) \cap H^d_{\underline{w}} \qquad (3)$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

with equality for $t=0$ and $t=1$. The spaces $\mathcal{J}_{f_t}(\Theta_V^0)\cap H^d_{\underline{w}}$ and $\mathcal{J}_f(\Theta_V^0)\cap H^d_{\underline{w}}$ are not trivial by Euler identity 1.

Let's show that we have equality for all $t\in[0,1]$ except finitely many values.
Take $\dim(\mathcal{J}_{f_t}(\Theta_V^0)\cap H^d_{\underline{w}})=\dim(\mathcal{J}_f(\Theta_V^0)\cap H^d_{\underline{w}})=s$ (say). Let's fix $\{e_1,\ldots,e_s\}$ a basis of $\mathcal{J}_f(\Theta_V^0)\cap H^d_{\underline{w}}$. Consider the s polynomials corresponding to the generators of the space (2):

$$\alpha_i(t)=df_t(\xi_i)=\sum_{j=1}^n a_{ij}x^{\underline{P}}\frac{\partial f_t}{\partial x_j}=\sum_{j=1}^n a_{ij}x^{\underline{P}}[(1-t)\frac{\partial f}{\partial x_j}+t\frac{\partial g}{\partial x_j}],\ <\underline{P},\underline{w}>=w_j$$

We can express each $\alpha_i(t)$, $i=1,\ldots,s$ in terms of above mentioned fixed basis as

$$\alpha_i(t)=\phi_{i1}(t)e_1+\ldots+\phi_{is}(t)e_s,\ \forall\ i=1,\ldots,s \tag{4}$$

where each $\phi_{ij}(t)$ is linear in t. Consider the matrix of transformation corresponding to the eqs. (4)

$$(\phi_{ij}(t))_{s\times s}=\begin{pmatrix}\phi_{11}(t) & \phi_{12}(t) & \ldots & \phi_{1s}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \phi_{s1}(t) & \phi_{s2}(t) & \ldots & \phi_{ss}(t)\end{pmatrix}$$

having rank at most s. Note that the equality

$$\mathcal{J}_{f_t}(\Theta_V^0)\cap H^d_{\underline{w}}=\mathcal{J}_f(\Theta_V^0)\cap H^d_{\underline{w}}$$

holds for those values of t in $\mathbb{C}$ for which the rank of above matrix is precisely s. We have the $s\times s$-matrix whose determinant is a polynomial of degree s in t and by the fundamental theorem of algebra it has at most s roots in $\mathbb{C}$ for which rank of the matrix of transformation will be less than s. Therefore, the above-mentioned equality does not hold for at most finitely many values, say $t_1,\ldots,t_q$ where $1\le q\le s$.

It follows that the dimension of the space (2) is constant for all $t\in\mathbb{C}$ except finitely many values $\{t_1,\ldots,t_q\}$.
For an arbitrary smooth path

$$\alpha:\mathbb{C}\longrightarrow\mathbb{C}\backslash\{t_1,\ldots,t_q\}$$

with $\alpha(0)=0$ and $\alpha(1)=1$, we have the connected smooth submanifold

$$P=\{f_t=(1-\alpha(t))f(x)+\alpha(t)g(x):\ t\in\mathbb{C}\}$$

of $H^d_{\underline{w}}$. By the above, it follows $\dim T_{f_t}(\mathcal{R}_V^0.f_t)$ is constant for $f_t\in P$.

Now, to apply Mather's lemma, we need to show that the tangent space to the submanifold P is contained in that to the orbit $\mathcal{R}_V^0.f_t$ for any $f_t\in P$. One clearly has

$$T_{f_t}P=\{\dot{f}_t=-\dot{\alpha}(t)f(x)+\dot{\alpha}(t)g(x):\ \forall\, t\in\mathbb{C}\}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Therefore, by Euler formula 1, we have

$$T_{f_t}P \subset T_{f_t}(\mathcal{R}^0_V.f_t)$$

By Mather's lemma the submanifold P is contained in a single orbit. Hence the result. ▌

THEOREM 4.1. *Let $f, g \in H^d_{\underline{w}}(n, 1; \mathbb{C}) = H^d_{\underline{w}}$ be two quasihomogeneous polynomials of degree d and $\Phi \in H^r_{\underline{w}}(n, 1; \mathbb{C})$ be a quasihomogeneous polynomial of degree r w.r.t. the same weights $\underline{w} = (w_1, \ldots, w_n)$. If $M_V(f) \simeq M_V(g)$ (isomorphism of graded $\mathbb{C}$-algebra) then $f \overset{\mathcal{R}_V}{\sim} g$, where $\Phi^{-1}(0) = V$ is a hypersurface in $(\mathbb{C}^n, 0)$.*

Proof.
We show firstly that an isomorphism of graded $\mathbb{C}$-algebras

$$\varphi : (M_V(g))_l = (\frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{J}_g(\Theta^0_V)})_l \overset{\simeq}{\longrightarrow} (M_V(f))_l = (\frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{J}_f(\Theta^0_V)})_l$$

is induced by an isomorphism $u : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $u^*(\mathcal{J}_g(\Theta^0_V)) = \mathcal{J}_f(\Theta^0_V)$.
Consider the following commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
(\mathcal{J}_g(\Theta^0_V))_{d+l} & \overset{u^*}{\dashrightarrow} & (\mathcal{J}_f(\Theta^0_V))_{d+l} \\
\downarrow i & & \downarrow j \\
J^{d+l}_{\underline{w}} & \overset{u^*}{\dashrightarrow} & J^{d+l}_{\underline{w}} \\
\downarrow p & & \downarrow q \\
(M_V(g))_l & \underset{\simeq}{\overset{\varphi}{\longrightarrow}} & (M_V(f))_l \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Define the morphism $u^* : J^{d+l}_{\underline{w}} \to J^{d+l}_{\underline{w}}$ by

$$u^*(x_i) = L_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_{ij} x_j^{\alpha_j} + \sum a_{ik_1\ldots k_n} x_{k_1}^{\beta_1} \ldots x_{k_n}^{\beta_n} \,;\; i = 1, \ldots, n \qquad (5)$$

where $k_m \in \{1, \ldots, n\}$ & $w_{k_1}\beta_1 + \ldots + w_{k_n}\beta_n = deg_{\underline{w}}(x_i) = w_j\alpha_j$, which is well defined by commutativity of diagram below.

$$\begin{array}{ccc} x_i & \overset{u^*}{\longmapsto} & L_i \\ \downarrow p & & \downarrow q \\ \widehat{x_i} & \underset{\simeq}{\overset{\varphi}{\longmapsto}} & \widehat{L_i} \end{array}$$

Note that the isomorphism φ is a degree preserving map and is also given by the same morphism u^*. Therefore, u^* is an isomorphism.
Now, we show that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$. For every $G \in (\mathcal{J}_g(\Theta_V^0))_{d+l}$, we have $u^*(G) \in (\mathcal{J}_f(\Theta_V^0))_{d+l}$ by commutative diagram below.

$$\begin{array}{ccc} G & \overset{u^*}{\longmapsto} & F = u^*(G) \\ \downarrow p & & \downarrow q \\ \widehat{0} & \overset{\varphi}{\longmapsto} & \widehat{F} = \widehat{0} \end{array}$$

It implies that $u^*((\mathcal{J}_g(\Theta_V^0))_{d+l}) \subset (\mathcal{J}_f(\Theta_V^0))_{d+l}$. As u^* is an isomorphism, therefore it is invertible and by repeating the above argument for its inverse, we have $u^*((\mathcal{J}_g(\Theta_V^0))_{d+l}) \supset (\mathcal{J}_f(\Theta_V^0))_{d+l}$. Therefore, $u^*((\mathcal{J}_g(\Theta_V^0))_{d+l}) = (\mathcal{J}_f(\Theta_V^0))_{d+l}$. It follows that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$. Thus, u^* is an isomorphism with $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$.

By eq. (5), the map $u : \mathbb{C}^n \to \mathbb{C}^n$ can be defined by

$$u(z_1, \ldots, z_n) = (L_1(z_1, \ldots, z_n), \ldots, L_n(z_1, \ldots, z_n))$$

where $L_i(z_1, \ldots, z_n) = \sum_{j=1}^n a_{ij}x_j^{\alpha_j} + \sum a_{ik_1\ldots k_n} x_{k_1}^{\beta_1} \ldots x_{k_n}^{\beta_n}$; $i = 1, \ldots, n, k_m \in \{1, \ldots, n\}$ & $w_{k_1}\beta_1 + \ldots + w_{k_n}\beta_n = deg_{\underline{w}}(x_i) = w_j\alpha_j$. Note that u is an isomorphism by Prop. 3.16 [4], p.23.

In this way, we have shown that the isomorphism φ is induced by the isomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ such that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$.

Consider $u^*(\mathcal{J}_g(\Theta_V^0)) =< g_1 \circ u, \ldots, g_n \circ u >= \mathcal{J}_{g\circ u}(\Theta_V^0)$, where g_j are the generators of $\mathcal{J}_g(\Theta_V^0)$. Therefore, $\mathcal{J}_{g\circ u}(\Theta_V^0) = \mathcal{J}_f(\Theta_V^0) \Rightarrow g \circ u \overset{\mathcal{R}_V}{\sim} f$, by Lemma 4.2. Hence, by definition there exists an analytic isomorphism $h \in \mathcal{R}_V$ such that $g \circ u = f \circ h$. Since $\mathcal{R}_V$ is a group, therefore $h^{-1} \in \mathcal{R}_V$. Taking $u = h^{-1}$ we have $g \circ h \overset{\mathcal{R}_V}{\sim} g$. Thus, $f \overset{\mathcal{R}_V}{\sim} g$. ▌

Remark 4. 1. The converse implication, namely

$$f \overset{\mathcal{R}_V}{\sim} g \Rightarrow M_V(f) \simeq M_V(g)$$

always holds(even for analytic germs f, g defining IHS).

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Proof. Let $f \overset{\mathcal{R}_V}{\sim} g$. Then, by definition, there exists an analytic isomorphism $h \in \mathcal{R}_V$ such that $f \circ h = g$. It follows that $\mathcal{J}_{f\circ h}(\Theta_V^0) = \mathcal{J}_g(\Theta_V^0) \Rightarrow h^*(\mathcal{J}_f(\Theta_V^0)) = \mathcal{J}_g(\Theta_V^0)$.
By Prop. 3.16 [4], p.23 h^* is also analytic isomorphism.
Thus, $M_V(f) \cong M_V(g)$ by the commutativity of the diagram below.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{J}_g(\Theta_V^0) & \xrightarrow{h^*} & \mathcal{J}_f(\Theta_V^0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
\mathbb{C}[x_1, \ldots, x_n] & \xrightarrow{h^*} & \mathbb{C}[x_1, \ldots, x_n] \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
M_V(g) & \xrightarrow{\varphi} & M_V(f) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

∎

THEOREM 4.2. *Let $f \in H_{\underline{w}}^d(n, 1; \mathbb{C}) = H_{\underline{w}}^d$ be a quasihomogeneous polynomials of degree d and $\Phi \in H_{\underline{w}}^r(n, 1; \mathbb{C})$ be a quasihomogeneous polynomial of degree r w.r.t. the same weights $\underline{w} = (w_1, \ldots, w_n)$. Let g be an arbitrary analytic germ such that $\mathcal{J}_f(\Theta_V^0) \cong \mathcal{J}_g(\Theta_V^0)$, where $\Phi^{-1}(0) = V$ is a hypersurface in $(\mathbb{C}^n, 0)$. Then $f \overset{\mathcal{R}_V}{\sim} g$.*

Proof. Let $\mathcal{J}_f(\Theta_V^0) \cong \mathcal{J}_g(\Theta_V^0)$. Then there exists an analytic isomorphism $h \in \mathcal{R}$ such that $h^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$ by Prop. 3.16 [4], p.23.
It follows that $\mathcal{J}_{g\circ h}(\Theta_V^0) = \mathcal{J}_f(\Theta_V^0)$, where $g\circ h$ is quasihomogeneous polynomial. It implies that $g\circ h \overset{\mathcal{R}_V}{\sim} f$ by Lemma 4.2. Hence, by definition there exists an analytic isomorphism $u \in \mathcal{R}_V$ such that $g \circ h = f \circ u$. Since $\mathcal{R}_V$ is a group, therefore $u^{-1} \in \mathcal{R}_V$. Taking $h = u^{-1}$ we have $g \circ h \overset{\mathcal{R}_V}{\sim} g$. Thus, $f \overset{\mathcal{R}_V}{\sim} g$. ∎

The following Example of Gaffney and Hauser, in [5], suggests us that we can not extend the Lemma 4.2 and Theorem 4.2 for arbitrary analytic germs.

EXAMPLE 4.1. Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be any function satisfying $h \notin \mathcal{J}_h \subseteq \mathcal{O}_n$ i.e. $h \notin H_{\underline{w}}^d(n, 1; \mathbb{C})$. Define a family $f_t : (\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ by $f_t(x, y, z) = h(x) + (1 + z + t)h(y)$, and let $(X_t, 0) \subseteq (\mathbb{C}^{2n+1}, 0)$ be the hypersurface defined by f_t. Note that

$$\mathcal{J}_{f_t} = \langle \frac{\partial h}{\partial x_i}(x), \frac{\partial h}{\partial y_j}(y), h(y) \rangle, \ t \in \mathbb{C}.$$

On the other hand, the family $\{(X_t, 0)\}_{t\in\mathbb{C}}$ is not trivial i.e. $(X_t, 0) \not\cong (X_0, 0)$: For, if $\{f_t\}_{t\in\mathbb{C}}$ were trivial, we would have by Prop. 2, §1, [5]

$$\frac{\partial f_t}{\partial t} = h(y) \in (f_t) + m_{2n+1}\mathcal{J}_{f_t} = (f_t) + m_{2n+1}\mathcal{J}_{h(x)} + m_{2n+1}\mathcal{J}_{h(y)} + m_{2n+1}(h(y))$$

Solving for $h(y)$ implies either $h(y) \in \mathcal{J}_{h(y)}$ or $h(x) \in \mathcal{J}_{h(x)}$ contradicting the assumption on h.
It follows that f_t is not $\mathcal{R}$-equivalent to f_0.

Before proceeding further, we state the lifting lemma.

LEMMA 4.3. *Let φ be a morphism of analytic K-algebras*

$$\varphi : A = K\langle x_1, \ldots, x_n\rangle/I \to B = K\langle y_1, \ldots, y_m\rangle/J$$

Then φ has a lifting $\widetilde{\varphi} : K\langle \mathbf{x}\rangle \to K\langle \mathbf{y}\rangle$ which can be chosen as an isomorphism in the case that φ is an isomorphism and $n = m$.

For arbitrary hypersurface singularities (i.e. not necessary with isolated singularities), the following result has been obtained by Greuel, Lossen and Shustin [6], 2007.

THEOREM 4.3. *(Mather-Yau Theorem) Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ be two arbitrary hypersurface singularities having isomorphic Tjurina algebras $T(f) \simeq T(g)$. Then $f \overset{\mathcal{K}}{\sim} g$, where $\overset{\mathcal{K}}{\sim}$ denotes the contact equivalence.*

For arbitrary hypersurface singularities (i.e. not necessary with isolated singularities), we establish now the relative version of Mather-Yau Theorem under the hypothesis that $V = \Phi^{-1}(0)$ be a variety in $\mathbb{C}^n, 0$ such that $\Phi \in \mathbb{C}\{x_1, \ldots, x_n\}$ and Θ_V is finitely generated. We remark that if $\Phi \in \mathbb{C}[[x_1, \ldots, x_n]]$, the ring of formal power series, it is known that Θ_V is always finitely generated [7], Th. 5.4, p.771. However, we do not know whether the result holds for all $\Phi \in \mathbb{C}\{x_1, \ldots, x_n\}$.

THEOREM 4.4. *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ be two arbitrary hypersurface singularities having isomorphic relative Tjurina algebras $T_V(f) \simeq T_V(g)$, where $V = \Phi^{-1}(0)$ be a variety in $\mathbb{C}^n, 0$ such that $\Phi \in \mathbb{C}\{x_1, \ldots, x_n\}$ and Θ_V is finitely generated. Then $f \overset{\mathcal{K}_V}{\sim} g$, where $\overset{\mathcal{K}_V}{\sim}$ denotes the relative contact equivalence.*

Proof. Consider the isomorphism of graded $\mathbb{C}$-algebras

$$\varphi : T_V(f) = \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\langle f, \mathcal{J}_f(\Theta_V^0)\rangle} \xrightarrow{\simeq} T_V(g) = \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\langle g, \mathcal{J}_g(\Theta_V^0)\rangle}.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Note that φ lifts to an isomorphism $\widetilde{\varphi}:\mathbb{C}\{x_1,\ldots,x_n\}\to\mathbb{C}\{x_1,\ldots,x_n\}$ with $\widetilde{\varphi}(\langle f,\mathcal{J}_f(\Theta_V^0)\rangle) = \langle g,\mathcal{J}_g(\Theta_V^0)\rangle$ by Lifting Lemma 4.3. Since $\widetilde{\varphi}(\langle f,\mathcal{J}_f(\Theta_V^0)\rangle) = \langle\widetilde{\varphi}(f),\mathcal{J}_{\widetilde{\varphi}(f)}(\Theta_V^0)\rangle$, we may assume that

$$\langle f,\mathcal{J}_f(\Theta_V^0)\rangle = \langle g,\mathcal{J}_g(\Theta_V^0)\rangle. \tag{6}$$

Let $f,g\in\mathbb{C}\{x_1,\ldots,x_n\}$. Set $F(x,t)=f_t(x)=(1-t)f(x)+tg(x)$ for $t\in\mathbb{C}$. Thus (f_t) is a 1-parameter family of germs with $f_0=f$, $f_1=g$. We intend to show that any two germs in this family satisfying Equation 6 are $\mathcal{K}_V$ equivalent.
Consider the family of ideals

$$I_{f_t}=\langle f_t,\mathcal{J}_{f_t}(\Theta_V^0)\rangle\subset\mathbb{C}\{x_1,\ldots,x_n,t\},\ t\in\mathbb{C}.$$

By Eq. 6, $I_{f_t}\subset I_f=\langle f,\mathcal{J}_f(\Theta_V^0)\rangle$ and $I_f=I_g$.
Now, represent f,g in a neighbourhood $W=W(\mathbf{0})\subset\mathbb{C}^n$ by holomorphic functions and consider the coherent $\mathcal{O}_{W\times\mathbb{C}}$-module

$$\mathfrak{F}=\langle f,\mathcal{J}_f(\Theta_V^0)\rangle/\langle f_t,\mathcal{J}_{f_t}(\Theta_V^0)\rangle,$$

whose support is a closed analytic set in $W\times\mathbb{C}$, see A.7 [6]. Moreover, note that

$$supp(\mathfrak{F})\cap(\{0\}\times\mathbb{C})=\{t\in\mathbb{C}|\mathfrak{F}_{(0,t)}\neq 0\}=\{t\in\mathbb{C}|I_f\neq I_{f_t}\},$$

which is a closed analytic, hence a discrete, set of points in $\mathbb{C}=\{\mathbf{0}\}\times\mathbb{C}$. It follows that the set $U=\{t\in\mathbb{C}|I_{f_t}=I_f\}$ is open and connected and contains 0 and 1. Note that

$$\frac{\partial f_t}{\partial t}=g-f\in I_f=I_{f_t}=\langle f_t,\mathcal{J}_{f_t}(\Theta_V^0)\rangle$$

for all $t\in U$. By Theorem 2.22 [6], p.126, we get that $f_t\overset{\mathcal{K}_V}{\sim}f_{\acute{t}}$ for $t,\acute{t}\in U$ such that $|t-\acute{t}|$ is sufficiently small. Therefore, $f_t\overset{\mathcal{K}_V}{\sim}f$ for all $t\in U$, in particular, $f\overset{\mathcal{K}_V}{\sim}g$. ▮

Note that the converse implication is just an application of the chain rule, as performed in the proof of Lemma 2.10 [6], p.119.

The Theorems 4.1 and 4.2 are particular cases of Theorem 4.4. We dealt with the quasihomogeneous part first to explore the ideas deeply.

REFERENCES

1. Arnol'd, V.I.: Dynamical Systems VI, Singularity Theory I, Springer-Verlag, Berlin Heidelberg (1993).
2. J.W. Bruce and M.Roberts, Critical Points of Functions on Analytic Varieties, Topology, 27 (1988), no. 1, 57-90.
3. J. Damon, On the Freeness of Equisingular Deformations of Plane Curve Singularities, Topology and its Applications, 118 (2002), 31-43.
4. A. Dimca: Topics on Real and Complex Singularities, Vieweg (1987).
5. Terence Gaffney and Herwig Hauser: Charecterizing Singularities of Varities and of Mappings, Invent. math. 81 (1985), 427-447.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

6. G.-M. Greuel, C. Lossen, E. Shustin: Introduction to Singularities and Deformations, Springer Verlag, Berlin, Heidelberg, New York (2007).
7. M. Granger and M. Schulze: On the Formal Structure of Logarithmic Vector Fields, Compositio Math. 142 (2006), 765-778.
8. J.N. Mather: Stability of C^∞-mappings IV: Classification of stable germs by $\mathbb{R}$-algebras, Publ. Math. IHES 37 (1970), 223-248.
9. J.N. Mather: S.S.T. Yau: Classification of Isolated Hypersurface Singularities by their Moduli Algebras, Invent. Math. 69 (1982), 243-251.
10. K.Saito: Quasihomogene Isolierte Singularitäten von Hyperflächen, Invent. Math., 14 (1971), No. 2, 123-142.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC