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Equi-exponential attraction and rate of convergence of attractors for singularly perturbed evolution equations

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In this paper we consider a family of bounded dissipative asymptotically compact semigroups depending on a parameter and study continuity properties of the corresponding family of its global attractors. We exploit the idea of uniform exponential attraction property to provide the rate of convergence of the approximating attractors to the limit attractor. Within this abstract framework we then discuss the examples concerning finite and infinite dimensional dynamical systems originating in applications, like a second order dissipative ordinary differential equation, the viscous Cahn-Hilliard equation and a singularly perturbed damped wave equation. In this latter case our results involve nonlinearities with critical exponents, for which continuity of the family of attractors has not been proved yet due to inapplicability of a technique based on the continuity property of local unstable manifolds of equilibria, which for such problem remains unknown. May, 2010 ICMC-USP

1. INTRODUCTION

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In this paper we consider a family of semigroups $\{S_\eta(t) : t \geq 0\}$ depending on a parameter $\eta \in [0,1] =: I$ and study continuity properties of the corresponding family of its global attractors $\{\mathbf{A}_\eta, \eta \in I\}$.

This consideration has been already carried out by other authors, see e.g. [2, 14, 11, 16], who used different techniques to achieve this goal. In [11] the gradient semigroups have been considered and continuity properties of the attractors as well as the rate of convergence to the limit attractor have been obtained with the aid of continuity properties of local unstable manifolds of hyperbolic equilibria. Also for gradient semigroups, with all equilibria hyperbolic, another approach has been developed in [2] (see also [14]), where the authors took advantage of the uniform (with respect to η) attracting properties of the attractors $\mathbf{A}_\eta$ (called equi-attraction in [14]) to obtain continuity of attractors.

In general the exponential attracting property of global attractors are usually rather uneasy to achieve in problems originating in applications. Even though, as one can see from [6], the dynamical systems that have these kind of attractors are very abundant. Just to support this claim, note that all gradient exponentially Lipschitz continuous semigroups which possess a global attractor with finitely many equilibria, all of them hyperbolic are in this class as well as all exponentially Lipschitz continuous gradient-like semigroups with all equilibria being hyperbolic. In [6] there are also several examples of systems which are not gradient-like, or for which hyperbolicity of certain equilibria fails, or even a local bifurcation occurs, and there is still a global attractor that attracts exponentially bounded sets of the phase space.

In this paper we show that, with a similar technique as in [6], the exponential attracting properties of the attractors can be exhibited also for systems that undergo perturbations with respect to the parameter $\eta \in I$ and that these properties can be described uniformly for $\eta \to 0$. While this is known, then using some standard argument one obtains a strong conclusion concerning continuity of attractors and can evaluate as well the rate of convergence of 'approximating' attractors to the 'limit' one.

In what follows we will describe these ideas briefly considering a family of semigroups $\{S_\eta(t), t \geq 0\}_{\eta \in I}$ of nonlinear operators $S(t) : X \to X$ in a Banach space X such that

$$S_\eta(0) = I, \quad \text{and} \quad S_\eta(t)S_\eta(\tau) = S_\eta(t+\tau), \text{ for each } t, \tau \geq 0.$$

It becomes important to deal with some singular perturbation problems which requires us sometimes not to assume the continuity of the semigroup at the initial time. Instead we assume that the semigroup becomes continuous (resp. convergent) after a certain time $T \geq 0$, which can be viewed as a time in which the semigroup suitably regularizes.

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DEFINITION 1.1. We say that the family of semigroups $\{S_\eta(t), t \geq 0\}_{\eta\in I}$ is eventually continuous with respect to the 'time' variable if and only if there exists $T \geq 0$ such that for each $\eta \in I$ and $x \in X$

$$[T, \infty) \ni t \to S_\eta(t)x \in X \text{ is a continuous map.} \tag{1.1}$$

We say that a family of semigroups $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ in a Banach space X is eventually convergent at $\eta = 0$ if and only if a certain $T \geq 0$ exists such that

$$\text{if } \eta_n \to 0,\ x_n \to x_0 \text{ and } T \leq s_n \to s_0, \text{ then } S_{\eta_n}(s_n)x_n \to S_0(s_0)x_0. \tag{1.2}$$

If $T = 0$ we say that $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ is convergent at $\eta = 0$.

We denote by $\mathrm{dist}(x, B)$ the distance of a point $x \in X$ to a set $B \subset X$ and by

$$\mathrm{dist}_H(B_1, B_2) = \sup_{b_1\in B_1} \inf_{b_2\in B_2} \|b_1 - b_2\|$$

the Hausdorff semidistance between two sets $B_1, B_2 \subset X$. We say that A is invariant under $\{S_\eta : t \geq 0\}$ if $S_\eta(t)A = A$ for all $t \geq 0$. We also say that A attracts B under $\{S_\eta(t) : t \geq 0\}$ provided that $\mathrm{dist}_H(S_\eta(t)B, A) \to 0$ as $t \to \infty$. Following [10], the set $\mathbf{A}_\eta$, $\eta \in I$, is then called a global attractor for $\{S_\eta(t) : t \geq 0\}$ if and only if $\mathbf{A}_\eta$ is a maximal compact invariant subset of X which attracts bounded sets of X under $\{S_\eta(t) : t \geq 0\}$.

Our main concern in this paper will be the situation when the family of attractors is uniformly bounded and uniformly exponentially attracting.

DEFINITION 1.2. We say that the family $\{\mathbf{A}_\eta\}_{\eta\in I}$ is uniformly bounded in a neighborhood of $\eta = 0$ if and only if for a certain $\eta_0 \in (0, 1]$ the union $\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta$ is bounded in X.

DEFINITION 1.3. The family $\{\mathbf{A}_\eta\}_{\eta\in I}$ is eventually uniformly exponentially attracting if and only if there are $\eta_0 \in (0, 1]$, bounded set B containing $\cup_{\eta\in(0,\eta_0]}\mathbf{A}_\eta$, $t_0 \geq 0$, $c > 0$ and $\varrho > 0$ such that

$$\forall_{t\geq t_0}\ \forall_{\eta\in[0,\eta_0]}\ \mathrm{dist}_H(S_\eta(t)B, \mathbf{A}_\eta) \leqslant ce^{-\varrho t}. \tag{1.3}$$

It is known that for perturbed systems with a suitable modulus of continuity at $\eta = 0$ the uniform attracting property translates into the continuity property of the family of attractors (see e.g. [14, 11, 16]).

DEFINITION 1.4. Let $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ be a family of semigroups in a Banach space X and assume that, for each $\eta \in I$, $\{S_\eta(t) : t \geq 0\}$ has a global attractor $\mathbf{A}_\eta$. If $\kappa : [0,1] \to [0,\infty)$ is a continuous function with $\kappa(0) = 0$, we say that the family of semigroups $\{S_\eta(t) : t \geq 0\}_{\eta\geq 0}$ has a $\kappa-$modulus of continuity at $\eta = 0$ if and only if

$$\begin{aligned}&\text{there are certain } c, L > 0,\ \eta_0 \in (0,1],\ t_0 \geq 0, \text{ for which}\\ &\forall_{u\in\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta}\ \forall_{\eta\in[0,\eta_0]}\ \forall_{t\geq t_0}\ \|S_\eta(t)u - S_0(t)u\| \leqslant ce^{Lt}\kappa(\eta).\end{aligned} \tag{1.4}$$

We say that the family of semigroups $\{S_\eta(t) : t \geq 0\}_{\eta\geq 0}$ has a degenerated modulus of continuity at $\eta = 0$ if and only if

$$\begin{aligned}&\text{There exist constants } c, L > 0 \text{ such that for each } \delta > 0 \text{ and}\\ &\text{for any } \tau > 0 \text{ there is a certain } \eta_{\delta,\tau} > 0 \text{ for which we have}\\ &\|S_\eta(t)x - S_0(t)x\| \leq \delta ce^{Lt},\ \eta \in [0, \eta_{\delta,\tau}],\ t \in [0,\tau],\ x \in \cup_{\eta\in I}\mathbf{A}_\eta.\end{aligned} \tag{1.5}$$

We state the following result concerning continuity of attractors. It says that, if the conditions in the above two definitions are satisfied, the continuity of attractors hold. Additionally, when (1.4) is satisfied, the continuity and the rate of convergence to the limit attractor may be determined in terms of ϱ and L only (see [16] and references therein).

PROPOSITION 1.1. *If the family $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ has a degenerated modulus of continuity and the family $\{\mathbf{A}_\eta\}_{\eta\in I}$ is eventually uniformly exponentially attracting, then*

$$\operatorname{dist}_H(\mathbf{A}_\eta, \mathbf{A}_0) + \operatorname{dist}_H(\mathbf{A}_0, \mathbf{A}_\eta) \to 0 \quad as \quad \eta \to 0^+. \tag{1.6}$$

If the family $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ has $\kappa-$modulus of continuity and the family $\{\mathbf{A}_\eta\}_{\eta\in I}$ is eventually uniformly exponentially attracting, then

$$\operatorname{dist}_H(\mathbf{A}_\eta, \mathbf{A}_0) + \operatorname{dist}_H(\mathbf{A}_0, \mathbf{A}_\eta) \leqslant \hat{c}\kappa^{\frac{\varrho}{\varrho+L}}(\eta),\ \eta \in [0, \hat{\eta}_0], \tag{1.7}$$

where $\hat{c} = 2c\big((\varrho/L)^{\frac{L}{\varrho+L}} + (L/\varrho)^{\frac{\varrho}{\varrho+L}}\big)$, constants $c, L, \varrho, \eta_0, t_0$ and function κ are the same as in conditions (1.3)-(1.4) and $\hat{\eta}_0 \in (0, \eta_0]$ is chosen such that $\kappa(\eta) \leq (\varrho/L)e^{-(\varrho+L)t_0}$ for all $\eta \in [0, \hat{\eta}_0]$.

For convenience of the reader and completeness of the presentation, this proposition will be proved at the end of Section 2.

It is seen from (1.3) and from Proposition 1.1 above that, for studying continuity properties of the attractors at $\eta = 0$ and their rate of convergence, it is sufficient to establish that the attractors $\mathbf{A}_\eta$ actually attract uniformly for η the set of the form $\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta$ with $\eta_0 \in (0,1]$ arbitrary.

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We observe that the equation $\dot{u} = -u^2(u-1-\eta)$ defines a semigroup $\{S_\eta(t) : t \geq 0\}$ in $X = \mathbb{R}$ that has a global attractor $\mathbf{A}_\eta = [0, 1+\eta]$. The attractor $\mathbf{A}_\eta$ attracts exponentially any set of the form $[0, a]$, $a > 1$, under the action of $\{S_\eta(t) : t \geq 0\}$, for any $\eta \in I$. Nonetheless $\{S_\eta(t) : t \geq 0\}$ does not attracts bounded subsets of $\mathbb{R}$ exponentially for any value of $\eta \in I$.

It is thus reasonable to give some suitable assumptions on $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ which imply validity of (1.3) and which are simultaneously flexible for many applications.

To achieve our goal we will need some more definitions concerning the perturbed semigroups and their attractors.

Definition 1.5. A family of semigroups $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ is collectively asymptotically compact at $\eta = 0$ if and only if any sequence of the form $\{S_{\eta_n}(t_n)x_n\}$, where $\eta_n \overset{n\to\infty}{\longrightarrow} 0$, $t_n \overset{n\to\infty}{\longrightarrow} \infty$ and $\{x_n\}$ is bounded in X, has a convergent subsequence.

Definition 1.6. Let $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ be a family of semigroups in a Banach space X and assume that, for each $\eta \in I$, $\{S_\eta(t) : t \geq 0\}$ has a global attractor $\mathbf{A}_\eta$. The family $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ is eventually exponentially Lipschitz continuous relatively to $\{\mathbf{A}_\eta\}_{\eta\in I}$ if and only if there are constants $c, L > 0$ and $\eta_0 \in (0, 1]$, $\tau \geq 0$ such that

$$\|S_\eta(t)x - S_\eta(t)y\| \leq ce^{Lt}\|x - y\|, \tag{1.8}$$

for all $\eta \in [0, \eta_0]$, $x, y \in \gamma_\eta^+(S_\eta(\tau) \cup_{\eta\in[0,\eta_0]} \mathbf{A}_\eta)$ and $t \geq 0$

In what follows a ball centered at $x \in X$ of radius $\epsilon > 0$ is denoted by $\mathcal{O}_\epsilon(x)$ and $\mathcal{O}_\epsilon(B) = \cup_{b\in B}\mathcal{O}_\epsilon(b)$ is an ϵ-neighborhood of a set $B \subset X$. Also $\gamma_\eta^+(B) = \cup_{t\geq 0} S_\eta(t)B$ is the positive orbit of $B \subset X$ for $\{S_\eta(t), t \geq 0\}$ and $\mathcal{GS}_\eta$ is the set of all global solutions for $\{S_\eta(t), t \geq 0\}$, that is maps $\phi_\eta : \mathbb{R} \to V$ such that $S_\eta(t)\phi_\eta(s) = \phi_\eta(t+s)$ for all $t \geq 0$, $s \in \mathbb{R}$.

We next recall the notions of the unstable set and a local unstable set of an invariant set.

Definition 1.7. The unstable set of an invariant set Y_η^* for the semigroup $\{S_\eta(t) : t \geq 0\}$ is defined by

$$W^u(Y_\eta^*) = \{x \in X : \exists\ y_\eta(t) \in \mathcal{GS}_\eta, y_\eta(0) = x \text{ and } \lim_{t\to-\infty} \operatorname{dist}(y_\eta(t), Y_\eta^*) = 0\}.$$

Given $\epsilon > 0$, the local unstable manifold $W^u_{\text{loc}}(Y_\eta^*)$ of Y_η^* is defined as

$$\begin{aligned} W^u_{\text{loc}}(Y^*) = \{&x \in W^u(Y_\eta^*) \cap \mathcal{O}_\epsilon(Y^*) : \exists\ y_\eta(t) \in \mathcal{GS}_\eta \text{ such that } y(0) = x, \\ &\lim_{t\to-\infty} \operatorname{dist}(y_\eta(t), Y_\eta^*) = 0 \text{ and } y_\eta(t) \in \mathcal{O}_\epsilon(Y_\eta^*) \text{ for all } t \leq 0\}. \end{aligned}$$

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As for the family of the global attractors we say that

Definition 1.8. The family $\{\mathbf{A}_\eta\}_{\eta\geq 0}$ of the global attractors is collectively compact at $\eta = 0$ if and only if each sequence of the form $\{a_{\eta_n}\}$, where $a_{\eta_n} \in \mathbf{A}_{\eta_n}$ and $\eta_n \to 0^+$, has a convergent subsequence.

For $\eta \in I$, by $\mathcal{S}_\eta$ we denote a nonempty family of compact disjoint invariant sets for the semigroup $\{S_\eta(t) : t \geq 0\}$. In the simplest case the collection $\mathcal{S}_\eta$ will consist of a finite number of one-point-set equilibria for $\{S_\eta(t) : t \geq 0\}$, although more involved situations are known to fall in this approach (see [6]).

Below we introduce uniform for the parameter pointwise exponential attracting property of local unstable sets of the elements of $\mathcal{S}_\eta$.

Definition 1.9. We say that a family $\{\mathcal{S}_\eta\}_{\eta\in I}$, where $\mathcal{S}_\eta$, $\eta \in I$, consists of compact disjoint invariant sets for $\{S_\eta(t) : t \geq 0\}$, has uniformly pointwise exponentially attracting local unstable sets if and only if there are positive constants c, γ and ϵ such that, for each $\eta \in I$ and every invariant set $Y^*_\eta \in \mathcal{S}_\eta$, we have

$$\operatorname{dist}(S_\eta(t)u_0, W^u_{\mathrm{loc}}(Y^*_\eta)) \leqslant ce^{-\gamma t}, \tag{1.9}$$

whenever $u_0 \in \mathcal{O}_\epsilon(Y^*_\eta)$ and $t \in \mathbb{R}^+$ are such that $\{S_\eta(s)(u_0) : s \in [0, t]\} \subset \mathcal{O}_\epsilon(Y^*_\eta)$.

We will also need a point dissipativeness property that will be specified here in term of the family $\mathcal{S}_0$ of compact disjoint invariant sets for a limit semigroup $\{S_0(t) : t \geq 0\}$.

Definition 1.10. We say that a family $\mathcal{S}_0$ of compact disjoint invariant sets for the semigroup $\{S_0(t) : t \geq 0\}$ attracts points of X if and only if

$$\mathcal{Y}_0 := \cup_{Y^*_0\in\mathcal{S}_0} Y^*_0 \text{ attracts each point of } X, \tag{1.10}$$

that is,

$$\forall_{x\in X} \ \operatorname{dist}(S_0(t)x, \mathcal{Y}_0) \to 0 \ \text{ as } \ t \to +\infty.$$

We will consider the semigroups that fall into some more general class of semigroups than gradient semigroups. These will be semigroups for which there is no ϵ-homoclinic structure.

Definition 1.11. Let $\{S(t) : t \geq 0\}$ be a semigroup in a Banach space X with a finite number of compact disjoint invariant sets $\{Y^*_1, \cdots Y^*_k\} =: \mathcal{S}$.

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We say that there is an ϵ-homoclinic structure for $\{S(t) : t \geq 0\}$ relatively to $\mathcal{S}$ if and only if for each $\epsilon > 0$ there exist $m \leq k$, a set $\{Y^*_{\ell_1}, \cdots, Y^*_{\ell_m}\} \subset \mathcal{S}$, a set of global solutions $\{\phi^\epsilon_i : \mathbb{R} \to X,\ 1 \leq i \leq m\}$ and a time $t_\epsilon \geq 0$ such that, letting $Y^*_{\ell_{k+1}} := Y^*_{\ell_1}$, we have

$$\{\phi^\epsilon_i(t) : t \leq -t_\epsilon\} \subset \mathcal{O}_\epsilon(Y^*_{\ell_i}) \text{ and } \phi^\epsilon_i(t) \stackrel{t\to\infty}{\longrightarrow} Y^*_{\ell_{i+1}} \text{ for each } 1 \leq i \leq m.$$

We remark that the semigroups without ϵ-homoclinic structure are an important class of semigroups, which includes both the gradient semigroups (as e.g. in [2] or [10]) and the gradient-like semigroups. The latter semigroups have been defined in [8] in the following way.

Definition 1.12. Let $\{S(t) : t \geq 0\}$ be an eventually continuous semigroup and assume that it has a global attractor $\mathbf{A}$ and that its set of equilibria $\mathcal{E}$ is finite. We say that $\{S(t) : t \geq 0\}$ is a gradient-like semigroup if and only if the following two conditions are satisfied:

(G1) Given a global solution $\phi : \mathbb{R} \to X$ in $\mathbf{A}$, there are $y^*, z^* \in \mathcal{E}$ such that

$$\lim_{t\to-\infty} ||\phi(t) - y^*|| = 0 \text{ and } \lim_{t\to\infty} ||\phi(t) - z^*|| = 0.$$

(G2) $\mathbf{A}$ does not contain any homoclinic structure.

In this context we simultaneously emphasize that we will not assume nor use in the proofs backward in time convergence of global solutions to a compact invariant set although it is actually the case in a number of applications.

With the above set-up we can now state the main abstract result of the paper which will be the eventual uniform exponential attracting property of the family of attractors.

Theorem 1.1. *Suppose that $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ is a family of semigroups in a Banach space X which is eventually continuous with respect to 'time' variable, collectively asymptotically compact at $\eta = 0$ and eventually convergent at $\eta = 0$.*

Suppose also that given $\eta \in I$, there exists a global attractor $\mathbf{A}_\eta$ for $\{S_\eta(t) : t \geq 0\}$, that the corresponding family of attractors $\{\mathbf{A}_\eta\}_{\eta\in I}$ is collectively compact at $\eta = 0$ and that $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ is eventually exponentially Lipschitz continuous relatively to $\{\mathbf{A}_\eta\}_{\eta\in I}$.

Assume next that there is a family $\{\mathcal{S}_\eta\}_{\eta\in I}$, where $\mathcal{S}_\eta = \{Y^*_{1\eta}, \ldots, Y^*_{k\eta}\}$ consists of the same number $k \in \mathbb{N}$ of compact disjoint invariant sets $Y^*_{j\eta}$

for $\{S_\eta(t) : t \geq 0\}$*, such that*

$$\lim_{\eta\to 0^+} \max_{1\leq j\leq k} \left\{\mathrm{dist}_H(Y^*_{0j}, Y^*_{\eta j}) + \mathrm{dist}_H(Y^*_{\eta j}, Y^*_{0j})\right\} = 0.$$

Assume finally that the family $\{\mathcal{S}_0\} = \{Y^*_{10}, \ldots, Y^*_{k0}\}$ *of compact invariant sets for* $\{S_0(t) : t \geq 0\}$ *attracts points of* X *and that the family* $\{\mathcal{S}_\eta\}_{\eta\in I}$*, where* $\mathcal{S}_\eta = \{Y^*_{1\eta}, \ldots, Y^*_{k\eta}\}$ *for* $\eta \in I$*, has uniformly pointwise exponentially attracting local unstable sets.*

Under these assumptions, if the limit semigroup $\{S_0(t) : t \geq 0\}$ *does not have an* ϵ*-homoclinic structure relative to* $\mathcal{S}_0$*, then the family of attractors* $\{\mathbf{A}_\eta\}_{\eta\in I}$ *is eventually uniformly exponentially attracting.*

The proof of Theorem 1.1 will be given in Section 2. It is based on the asymptotic properties of orbits of points near a compact invariant set, which idea originally comes back to [2] and has been recently exploited in [6] in a more general context.

The main idea that drives the argument in the proof of Theorem 1.1 is that, for each value of $\eta > 0$ which is sufficiently close to zero, one can shadow the positive orbit of $S_\eta(t)v$ through a given point v with the aid of a certain auxiliary map $\tilde{u}_\eta$ having values on the attractor $\mathbf{A}_\eta$. This can be done in such a way that the corresponding distance between the point on the orbit and $\tilde{u}_\eta(t)$ will remain exponentially small when $t \to \infty$ and the resulting exponential estimate will neither depend on the choice of a parameter nor on the choice of v varying in a certain subset of X.

The results of the abstract consideration carried out in Section 2 are then used in Section 3 where we give examples of finite and infinite dimensional dynamical systems for which the uniform exponential attracting property (1.3) can be ensured via Theorem 1.1. As this allows us to apply Proposition 1.1, we will then obtain suitable results concerning continuity and the rate of convergence of attractors. The mentioned applications include a second order ordinary differential equation, the viscous Cahn-Hilliard model and a singularly perturbed semilinear damped wave equation.

Concerning continuity of the family of attractors for a perturbed semilinear damped wave equation the present consideration essentially improves the results previously obtained in [3, 5] as we consider here a nonlinearity with the critical exponent. This 'critical' case was out of scope of the approach in [3] and, in [5], it was considered only from the point of view of an upper semicontinuity property. The continuity enables us to transfer regularity properties of the perturbed attractors to the limit attractor and as a consequence of the results obtained here we prove that the attractor of a damped wave equation (see Section 3.2) with a critical dissipative nonlinearity is of class $C^{1+\mu}(\Omega) \times C^\mu(\Omega)$, $\mu < \frac{1}{2}$, when $\Omega \subset \mathbb{R}^3$ is a

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bounded smooth domain and the nonlinearity is a C^2 map from $\mathbb{R} \to \mathbb{R}$ and $C^{2+\mu}(\Omega) \times C^{1+\mu}(\Omega)$, $\mu < \frac{1}{2}$, if f is C^3.

In [12, Theorem 2.21] the authors obtained sharp time and space regularity results concerning the limit attractor $\mathbf{A_0}$ using *Galerkin method.* In particular they have proved that, if $f \in C^{2+\mu}$ and satisfies a suitable growth assumption, then $\mathbf{A_0}$ is bounded in $H^3(\Omega) \times H^2(\Omega)$. In Corollary 3.2 of Section 3 this result is improved as we have shown that for the 3-dimensional problem with a general cubic nonlinearity we obtain that

$$f \in C^{1+j}(\mathbb{R}, \mathbb{R}) \ \text{ imply } \ \mathbf{A_0} \subset H^{2+j}(\Omega) \times H^{1+j}(\Omega), \ \ j = 1, 2,$$

for which we do not assume Hölder continuity of f'' (resp. f''').

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2. UNIFORM EXPONENTIAL ATTRACTING PROPERTY

This section is devoted to the proof of Theorem 1.1, which will be given in a number of lemmas. The first two contain merely certain observations related to the assumptions of the theorem.

LEMMA 2.1. *The collective compactness of the family of global attractors* $\{\mathbf{A}_\eta\}_{\eta \in I}$ *at* $\eta = 0$ *implies that the latter family is uniformly bounded in a certain neighborhood of* $\eta = 0$*, that is, for a certain* $\eta_0 \in (0, 1]$ *the union* $\cup_{\eta \in [0, \eta_0]} \mathbf{A}_\eta$ *is a bounded set.*

LEMMA 2.2. *The collective asymptotic compactness of the family of semigroups at* $\eta = 0$ *implies that, given a bounded set* B*, there are certain* $\eta_0 \in (0, 1]$ *and* $t_0 \geq 0$ *such that the union* $\cup_{\eta \in [0, \eta_0]} \gamma_\eta^+(S_\eta(t_0)B)$ *is bounded in* X*.*

In the next lemma we generalize previous consideration of [6, Lemma 2.2] to semigroups depending on a parameter.

LEMMA 2.3. *Suppose that* $\{S_\eta(t) : t \geq 0\}_{\eta \in I}$ *is a family of semigroups in a Banach space* X *collectively asymptotically compact at* $\eta = 0$*, and eventually convergent at* $\eta = 0$*. Assume that, for each* $\eta \in I$ *there is a* *global attractor* $\mathbf{A}_\eta$ *for* $\{S_\eta(t) : t \geq 0\}$ *and the corresponding family of attractors* $\{\mathbf{A}_\eta\}_{\eta\in I}$ *is collectively compact at* $\eta = 0$*. Suppose finally that there exists a certain family* $\{\mathcal{S}_\eta\}_{\eta\in I}$*, such that all* $\mathcal{S}_\eta = \{Y^*_{1\eta}, \ldots, Y^*_{k\eta}\}$ *consist of the same number* $k \in \mathbb{N}$ *of compact disjoint invariant sets* $Y^*_{j\eta}$ *for* $\{S_\eta(t) : t \geq 0\}$*, satisfying*

$$\lim_{\eta\to 0^+} \max_{1\leq j\leq k} \operatorname{dist}_H(Y^*_{0j}, Y^*_{\eta j}) = 0,$$

and, in addition, $\mathcal{S}_0 = \{Y^*_{01}, \ldots, Y^*_{0k}\}$ *attracts points of* X *under* $\{S_0(t) : t \geq 0\}$*.*

Then, given any number $\delta > 0$*, there exist* $T_0 > 0$ *and* $\eta_0 \in (0, 1]$ *such that for every* $\eta \in [0, \eta_0]$ *and* $x \in \gamma^+_\eta(S_\eta(T) \cup_{\eta\in[0,\eta_0]} \mathbf{A}_\eta) =: B(\eta, \eta_0)$ *we have*

$$\{S_\eta(t)x : t \in [0, T_0]\} \cap \mathcal{O}_\delta(\mathcal{Y}_\eta) \neq \emptyset,$$

where $\mathcal{Y}_\eta := \cup_{j=1}^k Y^*_{k\eta}$ *and* T *is as in the Definition 1.1.*

Proof: This lemma is proved by contradiction. Assume that there exist $\delta > 0$ and sequences $\eta_n \to 0$, $t_n \overset{n\to\infty}{\longrightarrow} \infty$, $x_n \in B(\eta_n, \frac{1}{n})$, such that

$$\{S_{\eta_n}(t)x_n : t \in [0, t_n]\} \cap \mathcal{O}_\delta(\mathcal{Y}_{\eta_n}) = \emptyset.$$

Since, by assumption, for almost all $n \in \mathbb{N}$ we have $\mathcal{O}_{\frac{\delta}{2}}(\mathcal{Y}_0) \subset \mathcal{O}_\delta(\mathcal{Y}_{\eta_n})$ then, for almost all $n \in \mathbb{N}$,

$$\{S_{\eta_n}(t)x_n : t \in [0, t_n]\} \cap \mathcal{O}_{\frac{\delta}{2}}(\mathcal{Y}_0) = \emptyset.$$

Furthermore, there are also sequences $\{y_n\} \subset \cup_{\eta\in[0,\frac{1}{n}]}\mathbf{A}_\eta$, $\eta_{k_n} \to 0$ and $\{\tau_n\} \subset [0, \infty)$ such that $x_n = S_{\eta_n}(T + \tau_n)y_n$ and hence, for almost all $n \in \mathbb{N}$,

$$\{S_{\eta_n}(t)S_{\eta_n}(T + \tau_n)y_n : t \in [0, t_n]\} \cap \mathcal{O}_{\frac{\delta}{2}}(\mathcal{Y}_0) = \emptyset. \tag{2.1}$$

By assumption we infer that there is a subsequence of $\{S_{\eta_n}(T + \tau_n)y_n\}$ (which we denote the same) and a certain $z \in X$ such that

$$S_{\eta_n}(T + \tau_n)y_n \to z. \tag{2.2}$$

Indeed, if $\{\tau_n\}$ is bounded, we use that the family of attractors is collectively compact at $\eta = 0$ to ensure that $\{y_n\}$ has a subsequence (denoted the same) convergent to a certain $z_0 \in X$ and hence we have (2.2) with $z = S_0(T + \tau_0)z_0$. If $\{\tau_n\}$ is unbounded, (2.2) is an immediate consequence of the collective asymptotic compactness assumption at $\eta = 0$.

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Thus (2.1) implies next that $\{S_0(t)z : t \in [T,\infty)\} \cap \mathcal{O}_{\frac{\delta}{2}}(\mathcal{Y}_0) = \emptyset$, which is in contradiction with the assumption that $\{\mathcal{S}_0\}$ attracts points. ▌

The lemma below exhibits the importance of the property that the limit semigroup does not possess an ϵ-homoclinic structure.

LEMMA 2.4. *Suppose that all the assumptions of Lemma 2.3 hold. Assume additionally that the family of semigroups $\{S_\eta(t) : t \geq 0\}_{\eta\in I}$ is also eventually continuous with respect to 'time' variable and that*

$$\lim_{\eta\to 0^+} \max_{1\leq j\leq k} \mathrm{dist}_H(Y^*_{\eta j}, Y^*_{0j}) = 0. \tag{2.3}$$

Assume finally that there is no ϵ-homoclinic structure for $\{S_0(t) : t \geq 0\}$ relative to $\mathcal{S}_0$ and let $T \geq 0$ be as in the Definition 1.1.

Under these assumptions given $\epsilon > 0$ there are $\delta \in (0,\epsilon)$ and $\eta_0 \in (0,1]$ such that the property (p_j) is satisfied for every $j = 1,\dots,k$:

(p_j) if $\eta \in [0,\eta_0]$, $x \in \mathcal{O}_\delta(Y^*_{j\eta}) \cap \gamma^+_\eta(S_\eta(2T) \cup_{\eta\in[0,\eta_0]} \mathbf{A}_\eta)$ and if for some $t_1 > 0$, $S_\eta(t_1)x \notin \mathcal{O}_\epsilon(Y^*_{j\eta})$, then $S_\eta(t)x \notin \mathcal{O}_\delta(Y^*_{j\eta})$ for all $t \geq t_1$.

Proof: Suppose that the result fails. Then, there is a certain $\epsilon > 0$ and there are sequences $\delta_n \to 0$, $\eta_n \in [0,\frac{1}{n}]$, $v_n \in \mathcal{O}_{\delta_n}(Y^*_{j\eta_n}) \cap \gamma^+_{\eta_n}(S_{\eta_n}(2T) \cup_{\eta\in[0,\frac{1}{n}]} \mathbf{A}_\eta)$, $\{t_n\} \subset (0,\infty)$, $\{\tau_n\} \subset (0,\infty)$ satisfying $\tau_n < t_n$ and also $j \in \{1,\dots,k\}$ such that for every $n \in \mathbb{N}$

$$\mathrm{dist}_H(S_{\eta_n}(\tau_n)\{v_n\}, \mathcal{O}_\epsilon(Y^*_{j\eta_n})) \geq \epsilon \quad \text{and} \quad S_{\eta_n}(t_n)v_n \in \mathcal{O}_{\delta_n}(Y^*_{j\eta_n}).$$

Note that, by assumption, not loosing generality we can have that

$$v_n \to y^* \in Y^*_{j0}, \quad S_{\eta_n}(t_n)v_n \to z^* \in Y^*_{j0} \;\text{ and }\; \mathcal{O}_{\frac{\epsilon}{2}}(Y^*_{j0}) \subset \mathcal{O}_\epsilon(Y^*_{j\eta_n}).$$

Define next a sequence of 'times' $\{s_n\}$, for which

$$\{S_{\eta_n}(t)v_n : t \in [0,s_n)\} \subset \mathcal{O}_{\frac{\epsilon}{2}}(Y^*_{j0}) \;\text{ and }\; \mathrm{dist}_H(S_{\eta_n}(s_n)\{v_n\}, Y^*_{j0}) = \frac{\epsilon}{2}.$$

Note that for a certain subsequence (which we denote the same) we have $s_n \to \infty$. Otherwise, choosing a subsequence which we denote the same, we would have that

$$S_{\eta_n}(s_n)v_n \to S_0(s_0)y^* \in Y^*_{j0}, \tag{2.4}$$

which is absurd.

We remark that the condition (2.4) is proved as follows. There exist sequences $\{T_n\} \subset [0,\infty)$, $\eta_{k_n} < \frac{1}{n}$ and $y_{k_n} \in \mathbf{A}_{\eta_{k_n}}$ such that $v_n =$

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$S_{\eta_n}(T)S_{\eta_n}(T+T_n)y_{k_n}$. Without loss of generality we may assume that there is a $\bar{T} \in [0,\infty]$ such that $T_n \to \bar{T}$. Using that the family $\{\mathbf{A}_\eta : \eta \in I\}$ is collectively compact at $\eta = 0$ and that $\{S_\eta(t) : t \geq 0\}$ is either collectively asymptotically compact (case $\bar{T} = \infty$) or eventually convergent at $\eta = 0$ (case $\bar{T} < \infty$), taking subsequence if necessary, we have that $S_{\eta_n}(T+T_n)y_{k_n} \to w$. Consequently,

$$y^* \leftarrow v_n = S_{\eta_n}(T)S_{\eta_n}(T+T_n)y_{k_n} \to S_0(T)w.$$

From this we conclude that

$$S_{\eta_n}(s_n)v_n = S_{\eta_n}(T+s_n)S_{\eta_n}(T+T_n)y_{k_n} \to S_0(T+s_0)w = S_0(s_0)y^*.$$

Below we will prove that there is an ϵ-homoclinic structure for the semigroup $\{S_0(t) : t \geq 0\}$ related to $\mathcal{S}_0$.

Let $\phi_n : [-s_n,\infty) \to X$ be defined by $\phi_n(t) = S_{\eta_n}(t+s_n)v_n$, $t \geq -s_n$ and $s_n \geq T$. Note that $\phi_n(t) = S_{\eta_n}(t+T)S_{\eta_n}(s_n - T)v_n$ and that, by assumption, $S_{\eta_n}(s_n - T)v_n$ is then precompact. Taking subsequences if necessary we can define $\phi(t) := \lim_{n\to\infty} \phi_n(t)$ for $0 \leq t < \infty$.

To extend ϕ onto larger interval and define $\phi : [-1,\infty) \to X$ let us take a subsequence $\{\phi_{n_{k(1)}}\}$ of $\{\phi_n\}$ such that $s_{n_{k(1)}} \geq 1+T$ and $S_{\eta_{n_{k(1)}}}(s_{n_{k(1)}} - 1 - T)v_{n_{k(1)}}$ is convergent. Then the sequence of maps

$$\phi_{n_{k(1)}}(t) = S_{\eta_{n_{k(1)}}}(t+1+T)S_{\eta_{n_{k(1)}}}(T)S_{\eta_{n_{k(1)}}}(s_{n_{k(1)}} - 1 - T)v_{n_{k(1)}}$$

is a subsequence of $\{\phi_n(t)\}$ and is convergent for all $-1 \leq t < \infty$ and the limit agrees with ϕ on $[0,\infty)$, which allows us to define $\phi(t)$ for $t \in [-1,\infty)$ as the limit of $\phi_{n_{k(1)}}(t)$.

It is now evident that continuing this process we obtain a global solution $\phi : \mathbb{R} \to X$ as a limit of the diagonal type subsequence of $\{\phi_{n_{k(n)}}(t)\}$ and that ϕ has the property that $\mathrm{dist}_H(\phi(t), Y^*_{j0}) \leq \epsilon$ for all $t \leq 0$.

We remark that, by construction, $\phi(t)$ is continuous with respect to $t \in \mathbb{R}$.

By assumption, there exists $Y^*_{l0} \subset \mathcal{S}_0$ such that $\phi(t) \to Y^*_{l0}$ as $t \to \infty$. Hence, if $l = j$, the proof is complete.

If $l \neq j$, we continue our construction. Then there is a neighborhood $\mathcal{O}_\theta(Y^*_{j0}) \subset \mathcal{O}_\epsilon(Y^*_{j0})$ so small that $\phi(t)$ does not enter it for $t \geq 0$. This can be justified using that $\phi(t)$ converges forward to Y^*_{l0}, that Y^*_{j0} is a compact invariant set and that $\phi(t)$ is a global solution which is continuous for each $t \in \mathbb{R}$.

By assumption we also have $\phi(t_m) \to Y^*_{l0}$ and, recalling the construction of ϕ, we know that for each $m \geq m_0$ the point $\phi(t_m)$ can be approximated by an appropriately chosen subsequence $\{S_{\eta_{n'}}(t+s_{n'})v_{n'}\}$ of $\{S_{\eta_n}(t+s_n)v_n\}$ at 'time' t_m where $\eta_{n'} \to 0$. Therefore, there is a sequence of points of the

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form $z_{n'_m} := S_{\eta_{n'_m}}(t_m + s_{n'})v_{n'_m} \in \gamma^+_{\eta_{n'_m}}(S_{\eta_{n'_m}}(T) \cup_{\eta\in[0,\eta_0]} \mathbf{A}_\eta)$ convergent to Y^*_{l0} and such that $\eta_{n'_m} \to 0$. Also, having fixed $m \in N$ and taking then a look at the set $G_m = \{S_{\eta_{n'_m}}(t + s_{n'})v_{n'_m} : t \in [0, t_m]\}$ we observe that $G_m \cap \mathcal{O}_{\frac{\theta}{2}}(Y^*_{j0}) = \emptyset$. This is because $S_{\eta_{n'_m}}(t + s_{n'})v_{n'_m}$ approximates the global solution $\phi(t)$ in the interval $t \in [0, t_m]$ and $\phi(t)$ is separated from $\mathcal{O}_{\frac{\theta}{2}}(Y^*_{j0})$ for all $t \geq 0$. Consequently the sequence of points $\{z_{n'_m}\}$ has an additional property that the orbits $S_{\eta_{n'_m}}(t)z_{n'_m}$ actually need to visit the 'starting' neighborhood $\mathcal{O}_\epsilon(Y^*_{j0})$ for a certain $t_{n'_m} \geq t_m$, which means that all $S_{\eta_{n'_m}}(t)z_{n'_m}$ actually need to leave $\mathcal{O}_\epsilon(Y^*_{l0})$ in 'times', respectively, bigger than t_m.

What was said above also indicates that the procedure described earlier to define the global solution $\phi(t)$ going from $\mathcal{O}_\epsilon(Y^*_{j0})$ to $\mathcal{O}_\epsilon(Y^*_{l0})$ can be repeated again so that we will construct a continuous global solution $\psi(t)$ going now from $\mathcal{O}_\epsilon(Y^*_{l0})$ to $\mathcal{O}_\epsilon(Y^*_{i0})$ for a certain $i \in \{1, \ldots, k\}$.

Proceeding like this, after no more that k-steps, we will 'close the loop'. However the latter means that there is an ϵ-homoclinic structure for $\{S_0(t) : t \geq 0\}$, which completes the proof. ▮

In the final lemma, for each value of $\eta > 0$ which is sufficiently close to zero, we will shadow the positive orbit of a corresponding semigroup $\{S_\eta(t) : t \geq 0\}$ through a given initial data v with the aid of a certain auxiliary map $\tilde{u}_\eta$ with values on the attractor $\mathbf{A}_\eta$.

LEMMA 2.5. *Under the assumptions of Theorem 1.1 there exist* $\eta_0 \in (0, 1]$ *and* $\tau_0 \geq 0$ *such that for each* $(\eta, v) \in [0, \eta_0] \times S_\eta(\tau_0)\left(\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta\right)$ *a certain map* $\tilde{u}_\eta$ *can be constructed,*

$$\tilde{u}_\eta : [0, \infty) \to \mathbf{A}_\eta,$$

such that

$$\mathrm{dist}_H(S_\eta(t)\{v\}, \mathbf{A}_\eta) \leq \|S_\eta(t)v - \tilde{u}_\eta(t)\| \leq \tilde{c}e^{-\tilde{\gamma}t} \quad \textit{for each} \quad t \geq 0. \tag{2.5}$$

Here $\tau_0 = 2T + t_0 + \tau$, *where* t_0 *is as in Lemma 2.2,* T *is as in the Definition 1.1 and* τ *is as in Definition 1.6. Constant* $\tilde{\gamma}$ *depends only on* γ *and* L *whereas constant* $\tilde{c}$ *depends only on* $c, \gamma, L, T_0, T, t_0$ *and* $\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta$, *where* c, γ, L *are as in* (1.8)-(1.9) *and* T_0 *is as in Lemma 2.3.*

Also, $\tilde{\gamma}$ *and* $\tilde{c}$ *are independent of* $(\eta, v) \in [0, \eta_0] \times S_\eta(\tau_0)\left(\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta\right)$.

Proof: Choose $\epsilon > 0$ and $\eta_\epsilon > 0$ so small that, given $\eta \in [0, \eta_\epsilon]$, all k neighborhoods $\mathcal{O}_\epsilon(Y^*_{j\eta})$ are disjoint and that Lemma 2.1 applies; i.e. $\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta$ is a bounded set. Choose also $\delta > 0$ and $\eta_0 \in (0, \eta_\epsilon]$ to the above ϵ according to Lemma 2.4 so that the properties (p_j), $j = 1, \ldots, k$ of this latter lemma are satisfied.

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Due to Lemma 2.2, there is $t_0 \geq 0$ such that, making η_0 smaller if necessary, the union $\cup_{\eta\in[0,\eta_0]}\gamma^+(S_\eta(t_0)\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta)$ is bounded.

According to the Lipschitz property (1.8) there is a suitable $\eta_0 \in (0,1]$, which not loosing generality we denote the same, and there are constants $\tau, c, L > 0$ such that for each $\eta \in [0,\eta_0]$ and every $(x,y) \in \gamma^+_\eta(S_\eta(\tau)\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta) \times \mathbf{A}_\eta$ we have

$$\|S_\eta(t)x - S_\eta(t)y\| \leq ce^{Lt}\|x-y\|,\ t \geq 0.$$

Recall now that, by Lemma 2.3, there exist $T_0 > 0$ and $\eta_0 > 0$ (we again denote here η_0 the same as above) such that, for each $\eta \in [0,\eta_0]$ and $w \in \gamma^+_\eta(S_\eta(T)\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta)$ we have

$$S_\eta(t)w \in \mathcal{O}_\delta(\mathcal{Y}_\eta) = \cup_{j=1}^k \mathcal{O}_\delta(Y^*_{\eta j}) \text{ for some } t \leqslant T_0,$$

where T is as in the Definition 1.1.

Thus, given any $(\eta, v) \in [0,\eta_0] \times S_\eta(2T + t_0 + \tau)(\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta)$, there are sequences $\{t^-_{i(j)}\}_{j=0}^m$, $\{t^+_{i(j)}\}_{j=0}^m$ and $\{Y^*_{\eta i(j)}\}_{j=0}^m$ such that $i(j) \in \{1,\ldots,k\}$, $1 \leq j \leq m \leqslant k$,

$$t^-_{i(1)} \leqslant T_0, \qquad 0 < t^-_{i(j)} - t^+_{i(j-1)} \leqslant T_0,\quad 1 \leqslant j \leqslant m, \qquad t^+_{i(m)} = +\infty,$$

where $t^-_{i(j)}$ is the time $S_\eta(t)v$ enters $\mathcal{O}_\delta(Y^*_{\eta i(j)})$ and $t^+_{i(j)}$ is (for $j < m$) the time $S_\eta(t)v$ leaves $\mathcal{O}_\epsilon(Y^*_{\eta i(j)})$ crossing its boundary never to return to $\mathcal{O}_\delta(Y^*_{\eta i(j)})$; in particular,

$$S_\eta(t)v \in \mathcal{O}_\epsilon(Y^*_{\eta i(j)}) \ \text{ for all } \ t \in [t^-_{i(j)}, t^+_{i(j)}) \ \text{ and } \ j = 1,\ldots,m.$$

For simplicity we will next assume that $i(j) = j$; that is, we enumerate the compact disjoint invariant sets whose neighborhoods are visited by the considered orbit as $Y^*_1,\ldots,Y^*_m$, where $m \leq k$ is a number of those elements of the family $\mathcal{S}_\eta = \{Y^*_{1\eta},\ldots Y^*_{k\eta}\}$ whose neighborhoods are actually visited (see above).

It is true that this enumeration, the number m and the 'entrance/exit times' $t^\mp_l$ actually depend on v and η; however, for simplicity of the notation we will not exhibit this dependence explicitly.

We now consider $(\eta, v) \in [0,\eta_0] \times S_\eta(2T + t_0 + \tau)(\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta)$ and fix an arbitrary point $y_{\eta 1} \in \mathbf{A}_\eta$. With the aid of [6, Lemma 2.1] we then select a map

$$\psi_{\eta,1,t_1^-,t_1^+} : [t_1^-, t_1^+] \to clW^u_{loc}(Y^*_{1\eta}) \subset \mathbf{A}_\eta$$

so that

$$\|S_\eta(t)v - \psi_{\eta,1,t_1^-,t_1^+}(t)\| = \operatorname{dist}(S_\eta(t)v, W^u_{loc}(Y^*_{1\eta}))$$

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and define

$$\tilde{u}_\eta(t) = \begin{cases} y_{\eta 1}, \ t \in [0, t_1^-), \\ \psi_{\eta,1,t_1^-,t_1^+}(t), \ t \in [t_1^-, t_1^+]. \end{cases}$$

By assumption we have

$$\begin{aligned} \|S_\eta(t)v - \tilde{u}(t)\| &\le ce^{LT_0}\mathrm{dist}_H(\mathbf{A}_\eta, \{v\})e^{\gamma T_0}e^{-\gamma t} \\ &\le ce^{LT_0} \sup_{\eta\in[0,\eta_0]} \sup_{v\in S_\eta(2T+t_0+\tau)\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta} \mathrm{dist}_H(\mathbf{A}_\eta, \{v\})e^{\gamma T_0}e^{-\gamma t} \\ &=: \tilde{c}_1 e^{-\gamma t}, \ t \in [0, t_1^-), \\ \|S_\eta(t)v - \tilde{u}(t)\| &\le ce^{-\gamma(t-t_1^-)} \le ce^{\gamma T_0}e^{-\gamma t} =: \hat{c}_1 e^{-\gamma t}, \ t \in [t_1^-, t_1^+], \end{aligned}$$

and we next denote

$$c_1 = \max\{\tilde{c}_1, \hat{c}_1\}, \quad t_0^+ := 0, \quad t_0^1 := t_1^+, \quad \gamma_1 := \gamma. \tag{2.6}$$

Having this done, we define step by step

$$\gamma_j = \frac{\gamma_{j-1}^2}{L + 2\gamma_{j-1}}, \quad t_j^0 = \min\left\{\frac{L+2\gamma_{j-1}}{L+\gamma_{j-1}}t_j^-, t_j^+\right\}, \ j = 2, \ldots, m. \tag{2.7}$$

and we extend $\tilde{u}_\eta$ onto the whole half line $[0,\infty)$ letting

$$\tilde{u}_\eta(t) = \begin{cases} S_\eta(t - t_{j-1}^+)\tilde{u}_\eta(t_{j-1}^+), \ t \in [t_{j-1}^+, t_j^-), \\ S_\eta(t - t_j^-)S(t_j^- - t_{j-1}^+)\tilde{u}_\eta(t_{j-1}^+), \ t \in [t_j^-, t_j^0], \\ \psi_{\eta,j,t_j^-,t_j^+}(t), \ t \in (t_j^0, t_j^+], \end{cases}$$

where applying again [6, Lemma 2.1] we have selected a map

$$\psi_{\eta,j,t_1^-,t_1^+} : [t_j^-, t_j^+] \to clW_{loc}^u(Y_{j\eta}^*) \subset \mathbf{A}_\eta$$

satisfying

$$\|S_\eta(t)v - \psi_{\eta,j,t_1^-,t_1^+}(t)\| = \mathrm{dist}(S_\eta(t)v, W_{loc}^u(Y_{j\eta}^*)).$$

As in [6, Lemma 2.1] we now show that, for each $j = 2, \ldots, m$, the following implication holds:

if (i) $\|S_\eta(t)v - \tilde{u}_\eta(t)\| \le c_{j-1}e^{-\gamma_{j-1}t}, \ t \in [t_{j-2}^+, t_{j-1}^+]$ with some $c_{j-1} > 0$,
then (ii) $\|S_\eta(t)v - \tilde{u}_\eta(t)\| \le c_j e^{-\gamma_j t}, \ t \in [t_{j-1}^+, t_j^+]$ with some $c_j > 0$,

where for $j = m$ we actually set $[t^+_{m-1}, t^+_m] := [t^+_{m-1}, \infty)$. Although it is true that this part of the proof is analogous to the consideration in [6, Lemma 2.2] we include it below for completeness of the presentation.

Using (i) we first note that, for $t \in [t^+_{j-1}, t^-_j]$,

$$\begin{aligned} \|S_\eta(t)v - \tilde{u}_\eta(t)\| &\le ce^{L(t-t^+_{j-1})}\|S_\eta(t^+_{j-1})v - \tilde{u}_\eta(t^+_{j-1})\| \\ &\le cc_{j-1}e^{L(t-t^+_{j-1})-\gamma_{j-1}t^+_{j-1}}. \end{aligned} \tag{2.8}$$

Secondly, due to the above construction, if t^0_j is considered for a while as a variable element of the interval $(t^-_j, t^+_j]$ then by assumption and (2.8) we have

$$\begin{aligned} &\|(S_\eta(t)v - \tilde{u}_\eta(t)\| \le ce^{L(t-t^-_j)}\|(S_\eta(t^-_j)v - S(t^-_j - t^+_{j-1})\tilde{u}_\eta(t^+_{j-1})\| \\ &\le cc_{j-1}e^{LT_0}e^{L(t-t^-_j)-\gamma_{j-1}t^+_{j-1}} \le cc_{j-1}e^{(L+\gamma)T_0}e^{L(t-t^-_j)-\gamma_{j-1}t^-_j} \end{aligned} \tag{2.9}$$

forall $t \in [t^-_j, t^0_j]$ and

$$\forall_{t\in[t^-_j,t^+_j]}\ \|S_\eta(t)v - \psi_{\eta,j,t^-_j,t^+_j}(t)\| \le ce^{-\gamma(t-t^-_j)} \le ce^{-\gamma_{j-1}(t-t^-_j)}. \tag{2.10}$$

Evidently, whenever the 'time' t^0_j lies inside the interval (t^-_j, t^+_j), it is such that the exponents on the right hand sides of (2.9) and (2.10) become equal; namely

$$[L(t-t^-_j) - \gamma_{j-1}t^-_j]_{|t=t^0_j} = [-\gamma_{j-1}(t-t^-_j)]_{|t=t^0_j}.$$

From what was said above one can infer that

$$L(t-t^-_j) - \gamma_{j-1}t^-_j \le -\gamma_j t, \quad t \in [t^-_j, t^0_j]. \tag{2.11}$$

$$-\gamma_{j-1}(t-t^-_j) \le -\gamma_j t, \quad t \in (t^0_j, t^+_j], \tag{2.12}$$

to get from (2.9) and (2.11)

$$\forall_{t\in[t^-_j,t^0_j]}\ \|S_\eta(t)v - \tilde{u}_\eta(t))\| \le cc_{j-1}e^{(L+\gamma)T_0}e^{-\gamma_j t},$$

and from (2.10) and (2.12)

$$\forall_{t\in(t^0_j,t^+_j]}\ \|S_\eta(t)v - \tilde{u}_\eta(t)\| \le ce^{-\gamma_{j-1}(t-t^-_j)} \le ce^{-\gamma_j t}.$$

Since (2.8) extends to the estimate

$$\begin{aligned} \forall_{t\in[t^+_{j-1},t^-_j]}\ \|S_\eta(t)v - \tilde{u}_\eta(t)\| &\le cc_{j-1}e^{L(t-t^+_{j-1})-\gamma_{j-1}(t^+_{j-1}-t+t)} \\ &\le cc_{j-1}e^{(L+\gamma)T_0}e^{-\gamma_j t}, \end{aligned} \tag{2.13}$$

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condition (ii) thus holds with

$$c_j = \max\{c, cc_{j-1}e^{(L+\gamma)T_0}\}. \tag{2.14}$$

From what was said above, for each $(\eta, v) \in [0, \eta_0] \times S_\eta(2T + t_0 + \tau)(\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta)$, we infer that

$$\text{dist}_H(S_\eta(t)\{v\}, \mathbf{A}_\eta) \leq \|(S_\eta(t)v - \tilde{u}_\eta(t)\| \leq c_m e^{-\gamma_m t}, \ t \geq 0.$$

with c_m depending merely on $c, \gamma, L, T_0, T, t_0$ and $\cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta$. The proof is thus complete. ▮

Theorem 1.1 follows easily from the above lemma. Below, we include the proof of Proposition 1.1.

Proof of Proposition 1.1. The proof of the second part follows the proof of a similar result in [2] and is included here for completeness. For $\epsilon \in (0, ce^{-\varrho t_0}]$, $B_{\eta_0} = \cup_{\eta\in[0,\eta_0]}\mathbf{A}_\eta$ and $t_\epsilon = \frac{1}{\varrho}\ln\frac{c}{\epsilon}$ we have $t_\epsilon \geq t_0$ so that for every $\eta \in [0, \eta_0]$ conditions (1.3) and (1.4) imply

$$\text{dist}_H(S_\eta(t_\epsilon)B_{\eta_0}, \mathbf{A}_\eta) < \epsilon, \tag{2.15}$$

and

$$\begin{aligned}\text{dist}_H(S_\eta(t_\epsilon)B_{\eta_0}, S_0(t_\epsilon)B_{\eta_0}) &\leqslant \sup_{x\in B_{\eta_0}} \|S_\eta(t_\epsilon)x - S_0(t_\epsilon)x\| \leqslant c\,\kappa(\eta)\, e^{t_\epsilon L}\\ \text{dist}_H(S_0(t_\epsilon)B_{\eta_0}, S_\eta(t_\epsilon)B_{\eta_0}) &\leqslant \sup_{x\in B_{\eta_0}} \|S_0(t_\epsilon)x - S_\eta(t_\epsilon)x\| \leqslant c\,\kappa(\eta)\, e^{t_\epsilon L}.\end{aligned} \tag{2.16}$$

Since $\mathbf{A}_\eta = S_\eta(t_\epsilon)\mathbf{A}_\eta \subset S_\eta(t_\epsilon)B_{\eta_0}$ and $\mathbf{A}_0 = S_0(t_\epsilon)\mathbf{A}_0 \subset S_0(t_\epsilon)B_{\eta_0}$, we then get

$$\begin{aligned}\text{dist}_H(\mathbf{A}_\eta, \mathbf{A}_0) &\leqslant \text{dist}_H(\mathbf{A}_\eta, S_\eta(t_\epsilon)B_{\eta_0}) + \text{dist}_H(S_\eta(t_\epsilon)B_{\eta_0}, S_0(t_\epsilon)B_{\eta_0})\\ &\quad + \text{dist}_H(S_0(t_\epsilon)B_{\eta_0}, \mathbf{A}_0)\\ &\leqslant \text{dist}_H(S_\eta(t_\epsilon)B_{\eta_0}, S_0(t_\epsilon)B_{\eta_0}) + \epsilon \leqslant c\kappa(\eta)e^{t_\epsilon L} + \epsilon\\ &\leqslant c^{1+\frac{L}{\varrho}}\epsilon^{-\frac{L}{\varrho}}\kappa(\eta) + \epsilon\end{aligned}$$

and similarly

$$\begin{aligned}\text{dist}_H(\mathbf{A}_0, \mathcal{A}_\eta) &\leqslant \text{dist}_H(\mathbf{A}_0, S_0(t_\epsilon)B_{\eta_0}) + \text{dist}_H(S_0(t_\epsilon)B_{\eta_0}, S_\eta(t_\epsilon)B_{\eta_0})\\ &\quad + \text{dist}_H(S_\eta(t_\epsilon)B_{\eta_0}, \mathbf{A}_\eta)\\ &\leqslant \text{dist}_H(S_0(t_\epsilon)B_{\eta_0}, S_\eta(t_\epsilon)B_{\eta_0}) + \epsilon\\ &\leqslant c\kappa(\eta)e^{t_\epsilon L} + \epsilon \leqslant c^{1+\frac{L}{\varrho}}\epsilon^{-\frac{L}{\varrho}}\kappa(\eta) + \epsilon.\end{aligned}$$

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Consequently, for any $\epsilon \in (0, ce^{-\varrho t_0}]$ we have

$$\mathrm{dist}_H(\mathbf{A}_\eta, \mathbf{A}_0) + \mathrm{dist}_H(\mathbf{A}_0, \mathbf{A}_\eta) \leqslant 2(c^{1+\frac{L}{\varrho}} \epsilon^{-\frac{L}{\varrho}} \kappa(\eta) + \epsilon) \tag{2.17}$$

and minimizing the right hand side of (2.17) with respect to $\epsilon \in (0, ce^{-\varrho t_0}]$ we obtain (1.7), which completes the proof of the second part of the proposition.

The first part of the proposition is proved as follows. Fix $\epsilon > 0$ and choose $\eta_0 > 0$, $t_\epsilon \geq t_0$ according to (1.3) so that for every $\eta \in [0, \eta_0]$ and for $B_{\eta_0} = \cup_{\eta \in [0,\eta_0]} \mathbf{A}_\eta$ we have

$$\mathrm{dist}_H(S_\eta(t_\epsilon)B_0, \mathbf{A}_\eta) < \frac{\epsilon}{4}. \tag{2.18}$$

Next, for $\tau := t_\epsilon$ and $\delta := \frac{\epsilon}{8} c^{-1} e^{-L_0 t_\epsilon}$ choose $\eta_{\delta(\epsilon),\tau(\epsilon)} \in (0, \eta_0]$ according to (1.5) to get with the aid of this condition that

$$\begin{aligned} \mathrm{dist}_H(S_\eta(t_\epsilon)B_{\eta_0}, S_0(t_\epsilon)B_{\eta_0}) &\leqslant \sup_{x \in B_{\eta_0}} \|S_\eta(t_\epsilon)x - S_0(t_\epsilon)x\| \leqslant \frac{\epsilon}{8},, \\ \mathrm{dist}_H(S_0(t_\epsilon)B_{\eta_0}, S_\eta(t_\epsilon)B_{\eta_0}) &\leqslant \sup_{x \in B_{\eta_0}} \|S_0(t_\epsilon)x - S_\eta(t_\epsilon)x\| \leqslant \frac{\epsilon}{8}, \end{aligned} \tag{2.19}$$

for all $\eta \in [0, \eta_{\delta(\epsilon),\tau(\epsilon)}]$. Hence we then obtain

$$\mathrm{dist}_H(\mathbf{A}_\eta, \mathbf{A}_0) + \mathrm{dist}_H(\mathbf{A}_0, \mathbf{A}_\eta) < \epsilon, \ \eta \in [0, \eta_{\delta(\epsilon),\tau(\epsilon)}], \tag{2.20}$$

which completes the proof of the first part of the proposition. ∎

3. EXAMPLES

In this section we give applications of the abstract scheme developed in the previous sections to certain evolutionary problems including a second order ordinary differential equation, a singularly perturbed damped wave equation and the viscous Cahn-Hilliard model.

We emphasize that in the case of the singularly perturbed damped wave equation our consideration gives rise to new results concerning continuity of attractors and the regularity property of the global attractor for the 'limit' hyperbolic damped wave equation. Here we consider nonlinearities involving critical exponents which was out of the scope of techniques used in the previous references.

3.1. Second order dissipative ode

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Consider the Cauchy problem for the following second order ordinary differential equation

$$\begin{aligned} &\epsilon\ddot{x} + \dot{x} = -\mu x + f(x),\ x \in \mathbb{R}^N \\ &x(0) = x_0 \in \mathbb{R}^n,\ x_t(0) = v_0 \in \mathbb{R}^N. \end{aligned} \tag{3.1}$$

assume that $\epsilon \in I$, $\mu \geq 1$, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 function which is globally Lipschitz, globally bounded and with symmetric Jacobian matrix at every point. If we rewrite the above equation in the form of a system with variables x and $v = \epsilon\dot{x}$ we have that

$$\begin{aligned} &\frac{d}{dt}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = \begin{bmatrix} 0 & I/\epsilon \\ -\mu & -I/\epsilon \end{bmatrix}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ f(x) \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathbb{R}^N \times \mathbb{R}^N \\ &\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} x_0 \\ \epsilon v_0 \end{smallmatrix}\right] \in \mathbb{R}^N \times \mathbb{R}^N. \end{aligned} \tag{3.2}$$

Clearly, the solutions for (3.2) are globally defined and the solution operator family $\{S_\epsilon(t) : t \geqslant 0\}$ defines a semigroup in $\mathcal{Z} = \mathbb{R}^N \times \mathbb{R}^N$. Furthermore, $\{S_\epsilon(t) : t \geqslant 0\}$ has a global attractor $\mathbf{A}_\epsilon$. The above system can be rewritten as

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} I & I \\ -\epsilon\mu I & 0 \end{bmatrix}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = -\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] + \begin{bmatrix} I & I \\ -\epsilon\mu I & 0 \end{bmatrix}\begin{bmatrix} 0 \\ f(x) \end{bmatrix}, \quad \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathbb{R}^N \times \mathbb{R}^N \\ &\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} x_0 \\ \epsilon v_0 \end{smallmatrix}\right] \in \mathbb{R}^N \times \mathbb{R}^N. \end{aligned} \tag{3.3}$$

As the parameter ϵ tends to zero, one would expect that the dynamical properties of (3.2) are given by

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = -\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] + \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ f(x) \end{bmatrix}, \quad \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathbb{R}^N \times \mathbb{R}^N \\ &\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right] \in \mathbb{R}^N \times \mathbb{R}^N, \end{aligned} \tag{3.4}$$

which corresponds to $v = 0$ and

$$\frac{d}{dt}x = -\mu x + f(x), \quad x_0 \in \mathbb{R}^N. \tag{3.5}$$

Note that the solutions for (3.5) are globally defined and the solution operator family $\{R_0(t) : t \geqslant 0\}$ defines a semigroup in $\mathbb{R}^N$. To compare the dynamics of these two problems we should find a way to see the dynamics of (3.5) in $\mathcal{Z}$. That is done simply defining

$$S_0(t)\left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} R_0(t)x_0 \\ 0 \end{smallmatrix}\right],\ t > 0, \quad S_0(0) = I,$$

and noting that

1) $S_0(t+\tau)\left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} R_0(t+\tau)x_0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} R_0(t)R_0(\tau)x_0 \\ 0 \end{smallmatrix}\right] = S_0(t)\left[\begin{smallmatrix} R_0(\tau)x_0 \\ 0 \end{smallmatrix}\right] = S_0(t)S_0(\tau)$ for all $t,\tau \geqslant 0$;
2) $(0,\infty)\times\mathcal{Z} \ni (t,\left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right]) \mapsto S_0(t)\left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right] \in \mathcal{Z}$ is continuous.

Consequently, $\{S_0(t) : t \geqslant 0\}$ is a semigroup (singular at zero) having a global attractor $\mathbf{A}_0$.

The linear semigroup associated to

$$\begin{aligned} &\frac{d}{dt}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = \begin{bmatrix} 0 & I/\epsilon \\ -\mu I & -I/\epsilon \end{bmatrix}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathbb{R}^N\times\mathbb{R}^N \\ &\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} x_0 \\ \epsilon v_0 \end{smallmatrix}\right] \in \mathbb{R}^N\times\mathbb{R}^N \end{aligned} \tag{3.6}$$

is given by

$$T_\epsilon(t) = \begin{bmatrix} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2} & \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\epsilon(\lambda_1-\lambda_2)} \\ \frac{\epsilon\lambda_1\lambda_2(e^{\lambda_2 t} - e^{\lambda_1 t})}{\lambda_1-\lambda_2} & \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2} \end{bmatrix},$$

where $\lambda_1 = \frac{-1+\sqrt{1-4\epsilon\mu}}{2\epsilon}$ and $\lambda_2 = \frac{-1-\sqrt{1-4\epsilon\mu}}{2\epsilon}$, $\epsilon(\lambda_1-\lambda_2) = \sqrt{1-4\epsilon\mu}$. Note that $\lambda_1 \overset{\epsilon\to 0}{\longrightarrow} -\mu$, $\lambda_2 \overset{\epsilon\to 0}{\longrightarrow} -\infty$, $\epsilon\lambda_1 \overset{\epsilon\to 0}{\longrightarrow} 0$, $\epsilon\lambda_2 \overset{\epsilon\to 0}{\longrightarrow} -1$.

Define the linear semigroup (singular at zero) $\{T_0(t) : t \geqslant 0\}$ by $T_0(0) = I$ and

$$T_0(t) = \begin{bmatrix} e^{-\mu t} & e^{-\mu t} \\ 0 & 0 \end{bmatrix}, \ t > 0.$$

THEOREM 3.1. *Let $\mathcal{Z} := \mathbb{R}^N\times\mathbb{R}^N$ with the norm $\|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\|^2_{\mathcal{Z}} = \|v\|^2_{\mathbb{R}^N} + \mu\|u\|^2_{\mathbb{R}^N}$. There exists a constant $M \geqslant 1$, independent of $\epsilon > 0$ and of $\mu \geqslant 1$, such that*

$$\left\| \begin{bmatrix} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2} & \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\epsilon(\lambda_1-\lambda_2)} \\ \frac{\epsilon\lambda_1\lambda_2(e^{\lambda_2 t} - e^{\lambda_1 t})}{\lambda_1-\lambda_2} & \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2} \end{bmatrix} - \begin{bmatrix} e^{-\mu t} & e^{-\mu t} \\ 0 & 0 \end{bmatrix} \right\|_{L(\mathcal{Z})} \leqslant M t^{-1}\mu^{\frac{1}{2}}\epsilon, \tag{3.7}$$

for all $t > 0$.

Proof: Let us estimate each element of the matrix separately. First note that

$$\mu + \lambda_1 = -\mu^2\epsilon\frac{4}{(1+\sqrt{1-4\epsilon\mu})^2}, \quad \lambda_2 - \lambda_1 = -\frac{\sqrt{1-4\epsilon\mu}}{\epsilon}.$$

Now, if $\frac{1}{2} \leqslant |\epsilon(\lambda_1-\lambda_2)|$ we have that

$$\left|\lambda_1\, t\, e^{\lambda_1 t}\ \frac{e^{(\lambda_2-\lambda_1)t}-1}{(\lambda_1-\lambda_2)t}\right| \leqslant \left|\lambda_1\, t\, e^{\lambda_1 t}\ \frac{e^{(\lambda_2-\lambda_1)t}-1}{\epsilon(\lambda_1-\lambda_2)t}\right|\epsilon \leqslant ct^{-1}\epsilon.$$

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On the other hand, if $|\sqrt{1-4\epsilon\mu}| \leqslant \frac{1}{2}$, then

$$\left|\lambda_1\, t\, e^{\lambda_1 t}\ \frac{e^{(\lambda_2-\lambda_1)t}-1}{(\lambda_1-\lambda_2)t}\right| \leqslant ct^{-1}|\lambda_1|^{-1}|\lambda_1 t|^2 e^{\lambda_1 t} \leqslant ct^{-1}\epsilon.$$

Also, if $\mathrm{Re}(\mu+\lambda_1) < 0$, then

$$|e^{\lambda_1 t} - e^{-\mu t}| \leqslant \left|t(\lambda_1+\mu)e^{-\mu t}\frac{e^{(\lambda_1+\mu)t}-1}{(\lambda_1+\mu)t}\right| \leqslant c\epsilon\mu^2 t e^{-\mu t} \leqslant c\epsilon t^{-1}$$

and, if $\mathrm{Re}(\mu+\lambda_1) > 0$, then

$$|e^{\lambda_1 t} - e^{-\mu t}| \leqslant \left|e^{\lambda_1 t}\right| \left|1 - e^{(-\lambda_1-\mu)t}\right| \leqslant 2e^{-\frac{t}{2\epsilon}} \leqslant c\epsilon t^{-1}.$$

With this, the upper left corner of the matrix being estimated as follows

$$\left|\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2} - e^{-\mu t}\right| = \left|\lambda_1\, t\, e^{\lambda_1 t}\ \frac{e^{(\lambda_2-\lambda_1)t}-1}{(\lambda_1-\lambda_2)t}\right| + |e^{\lambda_1 t} - e^{-\mu t}| \leqslant c\epsilon t^{-1}.$$

For the upper right corner, note that

$$\begin{aligned}\left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\epsilon(\lambda_1-\lambda_2)} - e^{-\mu t}\right| &= \left|e^{\lambda_1 t} - e^{-\mu t} + 2t\lambda_1 e^{\lambda_1 t}\frac{e^{(\lambda_2-\lambda_1)t}-1}{(\lambda_1-\lambda_2)t} - e^{\lambda_2 t}\right| \\ &\leqslant \left|e^{\lambda_1 t} - e^{-\mu t}\right| + 2\left|\lambda_1\, t\, e^{\lambda_1 t}\frac{e^{(\lambda_2-\lambda_1)t}-1}{(\lambda_1-\lambda_2)t}\right| + \left|e^{\lambda_2 t}\right| \\ &\leqslant c\epsilon t^{-1}.\end{aligned}$$

For the lower left corner, note that

$$\left|\epsilon^2\lambda_1\lambda_2\frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{\epsilon(\lambda_1-\lambda_2)}\right| = \epsilon\mu\left|\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\epsilon(\lambda_1-\lambda_2)}\right| \leqslant c\mu\epsilon t^{-1}.$$

Finally, for the lower right corner, note that

$$\left|\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right| = \left|\frac{\lambda_1(e^{\lambda_1 t} - e^{\lambda_2 t})}{\lambda_1-\lambda_2} + e^{\lambda_2 t}\right| \leqslant c\epsilon t^{-1}.$$

This completes the proof. ▮

THEOREM 3.2. *Consider* $\mathcal{Z} := \mathbb{R}^N \times \mathbb{R}^N$ *with the norm* $\|[\begin{smallmatrix} u \\ v \end{smallmatrix}]\|_{\mathcal{Z}}^2 = \|v\|_{\mathbb{R}^N}^2 + \mu\|u\|_{\mathbb{R}^N}^2$. *There exists a constant* $M \geqslant 1$, *independent of* $\epsilon \in I$ *and of* $\mu \geqslant 1$, *such that*

$$\|T_\epsilon(t) - T_0(t)\|_{L(\mathcal{Z})} \leqslant c\mu^{\frac{1+\alpha}{4}}\epsilon^\alpha t^{-\frac{1+\alpha}{2}} e^{-t\frac{1-\alpha}{6}},\ \alpha \in (0,1),\ t > 0. \tag{3.8}$$

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Proof: Note that $T_\epsilon(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]=\left[\begin{smallmatrix}u\\\epsilon v\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]$ is the solution of the initial value problem (3.1). Denote by

$$\left[\begin{smallmatrix}U\\\epsilon V\end{smallmatrix}\right]=[T_\epsilon(t)-T_0(t)]\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]$$

Now consider the function $\mathcal{L}(u,v)=\frac{\epsilon}{2}\|v\|^2_{\mathbb{R}^N}+\frac{\epsilon}{2}\langle u,v\rangle_{\mathbb{R}^N}+\frac{\mu}{2}\|u\|^2_{\mathbb{R}^N}$. Clearly, for $\epsilon\in I$, $\mu\geqslant 1$

$$4\mathcal{L}(u,v)\geqslant \epsilon\|v\|^2_{\mathbb{R}^N}+\mu\|u\|^2_{\mathbb{R}^N}\geqslant \frac{4}{3}\mathcal{L}(u,v)$$

Using (3.1) we have that

$$\dot{\mathcal{L}}(u,u_t)\leqslant -\frac{1}{3}\mathcal{L}(u,u_t).$$

Consequently,

$$\epsilon\|V\|_{\mathbb{R}^N}+\mu^{\frac12}\|U\|_{\mathbb{R}^N}\leqslant \epsilon^{\frac12}\|V\|_{\mathbb{R}^N}+\mu^{\frac12}\|U\|_{\mathbb{R}^N}\leqslant 6\,e^{-\frac{t}{3}}(\epsilon^{\frac12}\|V_0\|_{\mathbb{R}^N}+\mu^{\frac12}\|U_0\|_{\mathbb{R}^N}) \tag{3.9}$$

and from (3.7) we have that

$$\begin{aligned}\epsilon\|V\|_{\mathbb{R}^N}+\mu^{\frac12}\|U\|_{\mathbb{R}^N}&\leqslant c\mu^{\frac12}\epsilon\, t^{-1}(\epsilon\|V_0\|_{\mathbb{R}^N}+\mu^{\frac12}\|U_0\|_{\mathbb{R}^N})\\&\leqslant c\mu^{\frac12}\epsilon\, t^{-1}(\epsilon^{\frac12}\|V_0\|_{\mathbb{R}^N}+\mu^{\frac12}\|U_0\|_{\mathbb{R}^N}),\end{aligned} \tag{3.10}$$

for all $\epsilon\in I$ and $\mu\geqslant 1$. Interpolating the expressions in (3.9) and (3.10) we obtain that

$$\begin{aligned}\epsilon\|V\|_{\mathbb{R}^N}+\mu^{\frac12}\|U\|_{\mathbb{R}^N}&\leqslant \tilde{c}\mu^{\frac14}\epsilon^{\frac12}e^{-\frac{t}{6}}t^{-\frac12}(\epsilon^{\frac12}\|V_0\|_{\mathbb{R}^N}+\mu^{\frac12}\|U_0\|_{\mathbb{R}^N})\\&\leqslant \tilde{c}\mu^{\frac14}e^{-\frac{t}{6}}t^{-\frac12}(\epsilon\|V_0\|_{\mathbb{R}^N}+\mu^{\frac12}\|U_0\|_{\mathbb{R}^N}).\end{aligned} \tag{3.11}$$

Now, interpolating (3.10) and (3.11) we obtain that, for any $\alpha\in[0,1]$

$$\epsilon\|V\|_{\mathbb{R}^N}+\mu^{\frac12}\|U\|_{\mathbb{R}^N}\leqslant \hat{c}\mu^{\frac{1+\alpha}{4}}\epsilon^{\alpha}t^{-\frac{1+\alpha}{2}}e^{-t\frac{1-\alpha}{6}}(\epsilon\|V_0\|_{\mathbb{R}^N}+\mu^{\frac12}\|U_0\|_{\mathbb{R}^N}). \tag{3.12}$$

This translates into estimate (3.8) and completes the proof. ∎

Now define $S_\epsilon(t)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]=\begin{bmatrix}S^1_\epsilon(t)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]\\S^2_\epsilon(t)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]\end{bmatrix}$ by

$$S_\epsilon(t)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]=T_\epsilon(t)\left[\begin{smallmatrix}x_0\\\epsilon v_0\end{smallmatrix}\right]+\int_0^t T_\epsilon(t-s)\left[\begin{smallmatrix}0\\f(S^1_\epsilon(s)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right])\end{smallmatrix}\right]ds,$$

$$S_0(t)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]=T_0(t)\left[\begin{smallmatrix}x_0\\0\end{smallmatrix}\right]+\int_0^t T_0(t-s)\left[\begin{smallmatrix}0\\f(R_0(s)x_0)\end{smallmatrix}\right]ds.$$

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It follows from this and from singular Gronwall's lemma (see Lemma 7.1.2 in [13]) that, for any $0 < \alpha < 1$, bounded subset B of $\mathcal{Z}$ and $T > 1$ we have that

$$\epsilon^{-\alpha} \sup \{\|S_\epsilon(t) \left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right] - S_0(t) \left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right]\|_{\mathcal{Z}} : t \in [1,T], \ \left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right] \in B\} \xrightarrow{\epsilon \to 0} 0.$$

In particular $\{S_\epsilon(t) : t \geq 0\}_{\epsilon \in I}$ is eventually convergent at $\epsilon = 0$.

Denote by $\mathcal{E}$ the set of solutions of the stationary problem

$$0 = -\mu x + f(x). \tag{3.13}$$

Let $u_0^* \in \mathcal{E}$ and assume that $\sigma(\mu I - f'(u_0^*))$ is disjoint from the imaginary axis. Let $\{\nu_i\}_{i=1}^N$ be the eigenvalues of $\bar{A} = \mu I - f'(u_0^*)$, $\{\bar{\phi}_i\}_{i=1}^N$ be a corresponding sequence of orthonormal eigenfunctions and let $0 \leqslant k \leqslant N$ be such that $\nu_i < 0$ for $0 \leqslant i \leqslant k$ and $\nu_i > 0$ for $k < i \leqslant N$.

Let $\bar{\lambda}_1^i = \frac{-1+\sqrt{1-4\epsilon\nu_i}}{2\epsilon}$ and $\bar{\lambda}_2^i = \frac{-1-\sqrt{1-4\epsilon\nu_i}}{2\epsilon}$, $1 \leqslant i \leqslant N$ be the eigenvalues of

$$\bar{\mathbf{A}}_\epsilon := \begin{bmatrix} 0 & -\epsilon^{-1}I \\ \bar{A} & \epsilon^{-1}I \end{bmatrix}.$$

Denote by $\{\bar{T}_\epsilon(t) : t \geqslant 0\}$ the semigroup generated by $-\bar{\mathbf{A}}_\epsilon$

Consider the orthogonal decomposition $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ obtained in the following way.

$$\mathcal{Z}_2 = \text{span}\{\bar{\phi}_1, \cdots, \bar{\phi}_k\} \times \text{span}\{\bar{\phi}_1, \cdots, \bar{\phi}_k\}$$

and let P be the orthogonal projection defined in $\mathcal{Z}$ associated to this decomposition with $R(P) = \mathcal{Z}_2$. Then, there exists $\omega > 0$ such that

$$\|\bar{T}_\epsilon(t)(I-P)\|_{\mathcal{L}(\mathcal{Z})} \leqslant Me^{-\omega t}$$

Now let us consider the restriction of $\bar{T}_\epsilon(t)$ to the range of P. If $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in R(P)$, then

$$\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \sum_{i=1}^{k} \left[\begin{smallmatrix} u_{0i}\bar{x}_i \\ v_{0i}\bar{x}_i \end{smallmatrix}\right]$$

and

$$\bar{T}_\epsilon(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \sum_{i=1}^{k} \bar{T}_\epsilon^i(t) \left[\begin{smallmatrix} u_{0i}\bar{x}_i \\ v_{0i}\bar{x}_i \end{smallmatrix}\right]$$

where

$$\bar{T}_\epsilon^i(t) := \begin{bmatrix} \frac{\epsilon\bar{\lambda}_1^i e^{\bar{\lambda}_2^i t} - \epsilon\bar{\lambda}_2^n e^{\bar{\lambda}_1^n t}}{\epsilon(\bar{\lambda}_1^i - \bar{\lambda}_2^i)} & \frac{e^{\bar{\lambda}_1^n t} - e^{\bar{\lambda}_2^i t}}{\epsilon(\bar{\lambda}_1^i - \bar{\lambda}_2^i)} \\ \frac{\epsilon^2\bar{\lambda}_1^i\bar{\lambda}_2^i(e^{\bar{\lambda}_2^i t} - e^{\bar{\lambda}_1^i t})}{\epsilon(\bar{\lambda}_1^i - \bar{\lambda}_2^i)} & \frac{\epsilon\bar{\lambda}_1^n e^{\bar{\lambda}_1^i t} - \epsilon\bar{\lambda}_2^n e^{\bar{\lambda}_2^n t}}{\epsilon(\bar{\lambda}_1^i - \bar{\lambda}_2^i)} \end{bmatrix}.$$

Consider now the projection orthogonal to $I-P$ with $Q^\epsilon : R(P) \to R(P)$ given by $Q^\epsilon = \mathrm{diag}\{Q^\epsilon_1, \cdots, Q^\epsilon_k\}$, where $Q^\epsilon_i : \mathcal{Z}^i_2 \to \mathcal{Z}^i_2$, $\mathcal{Z}^i_2 = \mathrm{span}\{\bar{x}_i\} \times \mathrm{span}\{\bar{x}_i\}$ and

$$Q^\epsilon_i = \begin{bmatrix} \frac{-\bar\lambda^i_2}{\bar\lambda^i_1 - \bar\lambda^i_2} & \frac{1}{\epsilon(\bar\lambda^i_1 - \bar\lambda^i_2)} \\ -\frac{\epsilon\bar\lambda^i_1\bar\lambda^i_2}{\bar\lambda^i_1 - \bar\lambda^i_2} & \frac{\bar\lambda^i_1}{\bar\lambda^i_1 - \bar\lambda^i_2} \end{bmatrix},$$

$$\bar T^i_\epsilon(t)PQ^\epsilon_i := \begin{bmatrix} \frac{-\bar\lambda^n_2 e^{\bar\lambda^n_1 t}}{\bar\lambda^i_1 - \bar\lambda^i_2} & \frac{e^{\bar\lambda^n_1 t}}{\epsilon(\bar\lambda^i_1 - \bar\lambda^i_2)} \\ -\frac{\epsilon\bar\lambda^i_1\bar\lambda^n_2 e^{\bar\lambda^i_1 t}}{\bar\lambda^i_1 - \bar\lambda^i_2} & \frac{\bar\lambda^n_1 e^{\bar\lambda^i_1 t}}{\bar\lambda^i_1 - \bar\lambda^i_2} \end{bmatrix},$$

$$\bar T^i_\epsilon(t)P(I - Q^\epsilon_i) := \begin{bmatrix} \frac{\bar\lambda^i_1 e^{\bar\lambda^i_2 t}}{\bar\lambda^i_1 - \bar\lambda^i_2} & -\frac{e^{\bar\lambda^i_2 t}}{\epsilon(\bar\lambda^i_1 - \bar\lambda^i_2)} \\ \frac{\epsilon\bar\lambda^i_1\bar\lambda^i_2 e^{\bar\lambda^i_2 t}}{\bar\lambda^i_1 - \bar\lambda^i_2} & -\frac{\bar\lambda^n_2 e^{\bar\lambda^n_2 t}}{\bar\lambda^i_1 - \bar\lambda^i_2} \end{bmatrix}.$$

It is easy to see that

$$\|\bar T_\epsilon(t)(I - Q^\epsilon)\|_{L(\mathcal{Z})} \leqslant Me^{-\omega t},\ t \geqslant 0,$$

and that

$$\|\bar T_\epsilon(t)Q^\epsilon\|_{L(\mathcal{Z})} \geqslant Me^{\omega t}, \quad t \leqslant 0,$$

for some M and ω independent of ϵ. In addition to that,

$$\|Q^\epsilon\|_{L(\mathcal{Z})} \leqslant C$$

with C independent of ϵ. Also note that, if $F([\begin{smallmatrix} u \\ v \end{smallmatrix}]) = \left[\begin{smallmatrix} 0 \\ f(u) \end{smallmatrix}\right]$, then F is differentiable from $\mathcal{Z}$ into itself. It follows that $U^*_0 = \left[\begin{smallmatrix} u^*_0 \\ 0 \end{smallmatrix}\right]$ is a hyperbolic equilibrium point for $\{S_\epsilon(t) : t \geq 0\}$ and that there is a neighborhood $\mathcal{O}$ of U^*_0, $\rho > 0$, independent of ϵ, and a function $\Sigma_\epsilon : R(Q_\epsilon) \to \mathrm{Ker}(Q_\epsilon)$ such that

$$W^{u,\epsilon}_{\mathrm{loc}}(U^*_0) = \{(Q_\epsilon U, \Sigma(Q_\epsilon U)) : U \in \mathcal{O}\}$$

and, for each $U_0 \in V$ and $\tau > 0$ such that $\{S_\epsilon(t)U_0 : 0 \leqslant t \leqslant \tau\} \subset \mathcal{O}$

$$\mathrm{dist}(S_\epsilon(t)U_0, W^{u,\epsilon}(U^*_0)) \leqslant Ce^{-\rho t},\ 0 \leqslant t \leqslant \tau.$$

From this we have that

Proposition 3.1. *$\{S_\epsilon(t) : t \geq 0\}$ is a gradient semigroup with a global attractor $\mathbf{A}_\epsilon$, the union $\cup_{\epsilon\in I}\mathbf{A}_\epsilon$ is bounded, $\{S_\epsilon(t) : t \geq 0\}$ is collectively asymptotically compact at $\epsilon = 0$, and if all equilibria for* (3.5) *are hyperbolic, then the set of equilibria for $\{S_\epsilon(t) : t \geq 0\}$ is finite, independent of*

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ϵ, attracts points under the action of $\{S_\epsilon(t) : t \geq 0\}$ and the local unstable sets are uniformly exponentially attracting.

Assume that $\{S_0(t) : t \geqslant 0\}$ is a gradient-like semigroup and that all its equilibria are hyperbolic. It follows that $\{S_\epsilon(t) : t \geqslant 0\}$ is a gradient-like semigroup for suitably small ϵ (see [8]). In addition, there are constants $\epsilon_0 > 0$, $\rho > 0$ and for each bounded subset B of $\mathcal{Z}$, $t_0 > 0$, a constant $C(B, t_0) > 0$ independent of ϵ such that, via Theorem 1.1,

$$\mathrm{dist}(S_\epsilon(t)B, \mathbf{A}_\epsilon) \leqslant C(B, t_0)e^{-\varrho t}, \ t \geqslant t_0.$$

Since for any $\alpha < 1$ and for all $\epsilon \in [0, \epsilon_0]$,

$$\begin{aligned} &\|S_\epsilon(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - S_0(t)\left[\begin{smallmatrix} y_0 \\ z_0 \end{smallmatrix}\right]\|_{\mathcal{Z}} \leqslant ce^{Lt}\left(\|\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} y_0 \\ z_0 \end{smallmatrix}\right]\|_{\mathcal{Z}} + \epsilon^\alpha\right), \\ &\|S_\epsilon(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - S_\epsilon(t)\left[\begin{smallmatrix} y_0 \\ z_0 \end{smallmatrix}\right]\|_{\mathcal{Z}} \leqslant ce^{Lt}\|\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} y_0 \\ z_0 \end{smallmatrix}\right]\|_{\mathcal{Z}}, \end{aligned} \tag{3.14}$$

for all $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_0 \\ z_0 \end{smallmatrix}\right] \in B$ and $t \geq 1$. the assumptions of Proposition 1.1 are satisfied and we have that there is a constant $\bar{c} > 0$ such that for all sufficiently small $\epsilon > 0$

$$\mathrm{dist}_H(\mathbf{A}_\epsilon, \mathbf{A}_0) + \mathrm{dist}_H(\mathbf{A}_0, \mathbf{A}_\epsilon) \leqslant \bar{c}\ \epsilon^{\frac{\alpha\varrho}{\varrho + L}}.$$

3.2. Damped Wave Equations

For $\eta \geq 0$, in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ consider a family of damped wave equations

$$\begin{cases} u_{tt} + \eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u), \ t > 0, \ x \in \Omega, \\ u(0, x) = u_0(x), \ u_t(0, x) = v_0(x), \ x \in \Omega, \\ u(t, x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{3.15}$$

where $\eta \in I$, $a > 0$, $f : \mathbb{R} \to \mathbb{R}$, and $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ denotes the negative Laplacian in $L^2(\Omega)$ with homogeneous Dirichlet boundary condition and the domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$.

With this set-up we rewrite (3.15) in the form

$$\tfrac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right), \ t > 0, \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y, \tag{3.16}$$

where $Y := H^1_0(\Omega) \times L^2(\Omega)$, $\eta \in I$,

$$\mathcal{A}_\eta : D(A_\eta) \subset Y \to Y,$$

$$\mathcal{A}_\eta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} + aI \end{bmatrix}, \quad D(\mathcal{A}_\eta) = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) =: Y^1,$$

and

$$F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}, \ f^e(u)(x) = f(u(x)) \ \text{ for all } \ u \in H^1_0(\Omega), \ x \in \Omega.$$

Assume that $N = 3$,

$$f \in C^2(\mathbb{R}, \mathbb{R}), \quad |f''(s)| \leqslant c(1 + |s|), \ s \in \mathbb{R}, \tag{3.17}$$

and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \tag{3.18}$$

where λ_1 is the first positive eigenvalue of A.

If (3.17) is satisfied, then $F : Y \to Y$ is a C^1 map and F is Lipschitz continuous in bounded subsets of Y. Hence, for each $\begin{bmatrix} w_0 \\ z_0 \end{bmatrix} \in Y$, there are constants $r > 0$ and $\tau = \tau_{\begin{bmatrix} w_0 \\ z_0 \end{bmatrix}} > 0$ such that for any $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ in the ball $B_r(\begin{bmatrix} w_0 \\ z_0 \end{bmatrix}) \subset Y$ there is a unique mild solution

$$T_\eta(\cdot) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} u \\ v \end{bmatrix} (\cdot, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \eta) \in C([0, \tau], Y)$$

of (3.16) and the map

$$B_r(\begin{bmatrix} w_0 \\ z_0 \end{bmatrix}) \ni \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \mapsto T_\eta(\cdot) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in C([0, \tau], Y)$$

is continuously differentiable.

Note that the problem (3.16) can also be viewed as

$$\begin{aligned} &\frac{d}{dt} \mathcal{A}_\eta^{-1} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{A}_\eta^{-1} \begin{bmatrix} 0 \\ f(u) \end{bmatrix}, \ t > 0, \\ &\begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \end{aligned} \tag{3.19}$$

where

$$\mathcal{A}_\eta^{-1} = \begin{bmatrix} \eta A^{-\frac{1}{2}} + aA^{-1} & A^{-1} \\ -I & 0 \end{bmatrix}, \ \eta \geq 0. \tag{3.20}$$

As shown in [3, Lemma 2.4], for any compact set $K \subset \rho(-A_0)$, there exists $\hat{\eta} > 0$ and $C_K > 0$ such that

$$K \subset \rho(-A_\eta), \ \eta \in (0, \hat{\eta}) \tag{3.21}$$

and

$$\|(\lambda + \mathcal{A}_0)^{-1} - (\lambda + \mathcal{A}_\eta)^{-1}\|_{L(Y)} \leqslant \eta C_K, \ \eta \in (0, \hat{\eta}). \tag{3.22}$$

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Consequently, for η suitably small, we expect that the dynamical properties of (3.16) are given by

$$\begin{aligned}&\frac{d}{dt}\mathcal{A}_0^{-1}\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]+\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]=\mathcal{A}_0^{-1}\left[\begin{smallmatrix}0\\f(u)\end{smallmatrix}\right],\ t>0,\\&\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]_{t=0}=\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\end{aligned}\tag{3.23}$$

(see [3, Lemma 3.1.(iii)]) which is the counterpart of the initial boundary value problem

$$\begin{cases}u_{tt}+au_t+Au=f(u),\ t>0,\ x\in\Omega,\\u(0,x)=u_0(x),\ u_t(0,x)=v_0(x),\ x\in\Omega,\\u(t,x)=0,\ t\geq 0,\ x\in\partial\Omega.\end{cases}\tag{3.24}$$

Proposition 3.2. *Under the assumptions* (3.17)-(3.18) *the solutions of* (3.16) *exist globally in time. For every* $\eta\in I$*, a nonlinear gradient semigroup* $\{T_\eta(t):t\geq 0\}$ *is associated to* (3.16)*, where*

$$[0,\infty)\times Y\ni(t,u_0)\to T_\eta(t)u_0\in Y \text{ is a continuous map,}\tag{3.25}$$

the Lyapunov functional associated to it is given by

$$\mathcal{L}(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right])=\frac{1}{2}\|w_1\|^2_{X_2^{\frac12}}+\frac{1}{2}\|w_2\|^2_{X_2}-\int_\Omega\int_0^{w_1}f(s)dsdx,\ \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y.\tag{3.26}$$

and for any B *bounded in* Y *there is a constant* $C_B>0$ *such that*

$$\text{the union } \cup_{\eta\in I}\cup_{t\geq 0}T_\eta(t)B \text{ is bounded in } Y \text{ whenever } B \text{ is bounded in } Y.\tag{3.27}$$

Furthermore, the set of equilibria $\mathcal{E}=\{\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y:\ u_0=A^{-1}f^e(u_0),\ v_0=0\}$ *is independent of* $\eta\in I$ *and bounded in* $H^2(\Omega)$.

We remark that the asymptotic compactness property of $\{T_\eta(t)\}$ has already been proved in [1] for the case $\eta=0$ and follows from compactness of the resolvent of $-\mathcal{A}_\eta$ in the case $0<\eta\in I$ (see [4]). Therefore,

Proposition 3.3. *If* $\eta\in I$ *and* (3.17), (3.18) *hold, then the* C^1*-semigroup* $\{T_\eta(t)\}$ *of global solutions of* (3.16) *in* Y *possesses a global attractor* $\mathbf{A}_\eta$.

In what follows our first concern will be to extend to nonlinearities satisfying (3.17)-(3.18) continuity properties of the family of global attractors in Hölder norms and regularity results of the limit attractor established in

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[5] under the additional assumption that $\lim_{|s|\to\infty}\frac{f''(s)}{s}=0$. This will be done by proving that the family of the global attractors is continuous at $\eta=0$, in which proof we will use the uniform estimates on the approximating attractors established in [5]. After this is done we will be ready to estimate the actual rate of convergence of the approximating attractors to $\mathbf{A_0}$, as we will have enough regularity for the union of attractors $\cup_{\eta\in I}\mathbf{A}_\eta$ to verify that (1.4) is satisfied.

Lemma 3.1. *Suppose that* (3.17) *and* (3.18) *hold.*
Then,

i) for every $\eta\in I$ and each bounded set $B\subset Y$ the semigroup $\{T_\eta(t):t\geq 0\}$ satisfies the exponential Lipschitz condition of the form

$$\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]-T_\eta(t)\left[\begin{smallmatrix}\tilde u_0\\\tilde v_0\end{smallmatrix}\right]\|\leq c_Be^{L_Bt}\|\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]-\left[\begin{smallmatrix}\tilde u_0\\\tilde v_0\end{smallmatrix}\right]\|,\ t\geq 0,\ \left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\left[\begin{smallmatrix}\tilde u_0\\\tilde v_0\end{smallmatrix}\right]\in B,$$

ii) for each $\delta>0$, $\tau>0$ and any precompact subset B of Y there is $\eta_{\delta,\tau,B}>0$ such that

$$\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]-T_0(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y\leq\delta(1+L_0^{-1})e^{L_0t},$$

for $\eta\in[0,\eta_{\delta,\tau,B}]$, $t\in[0,\tau]$ and $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in B$, where $L_0>0$ is a Lipschitz constant for F in a bounded set $\cup_{\eta\in I}\cup_{t\geq 0}T_\eta(t)B$,

iii) the family of semigroups $\{T_\eta(t):t\geq 0\}_{\eta\in I}$ is convergent at $\eta=0$; namely,

if $\eta_n\to 0$, $Y\ni\left[\begin{smallmatrix}u_{0n}\\v_{0n}\end{smallmatrix}\right]\to\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]$ and $t_n\to t_0$, then $T_{\eta_n}(t_n)\left[\begin{smallmatrix}u_{0n}\\v_{0n}\end{smallmatrix}\right]\to T_0(t_0)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]$,

iv) $\cup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is a bounded subset of $H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)$,

v) property ii) is satisfied in particular for the precompact set $B=\cup_{\eta\in I}\mathbf{A}_\eta$.

Proof: Part i) is a consequence of the variation of constants formula

$$T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])=e^{-\mathcal{A}_\eta t}\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]+\int_0^t e^{-\mathcal{A}_\eta(t-s)}F(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])ds,\ t\geq 0,\tag{3.28}$$

(3.27), and Gronwall's inequality.

Proving part ii) we first use (3.28) to get

$$\begin{aligned}&\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])-T_0(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_Y\leq\|(e^{-\mathcal{A}_\eta t}-e^{-\mathcal{A}_0t})\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y\\&+\int_0^t\|(e^{-\mathcal{A}_\eta t}-e^{-\mathcal{A}_0t})F(T_0(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_Yds\\&+\int_0^t\|e^{-\mathcal{A}_\eta t}(F(T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])-F(T_0(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_Yds=:J_1+J_2+J_3.\end{aligned}\tag{3.29}$$

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Precompactness of B in Y now leads via Trotter-Kato theorem (see [3, Lemma 3.1]) to the relations

$$J_1 \leq \delta,\ \eta \in [0, \eta_{\delta,\tau,B}],\ t \in [0,\tau],\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B,$$

$$\|(e^{-\mathcal{A}_\eta t} - e^{-\mathcal{A}_0 t})F(T_0(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\|_Y \leq \delta,\ \eta \in [0, \eta_{\epsilon,\tau,B}],\ t \in [0,\tau],\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B,$$

so that

$$J_2 \leq \delta t,\ \eta \in [0, \eta_{\delta,\tau,B}],\ t \in [0,\tau],\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B.$$

Using (3.27), Lipschitz continuity of $F : Y \to Y$ on bounded sets and contracting property of the linear semigroups we also obtain

$$J_3 \leq L_0 \int_0^t \|T_\eta(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - T_0(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y ds.$$

Connecting the above estimates and using Gronwall's inequality we get the result.

Part iii) is a consequence of i)-ii) and (3.25).

For the proof of part iv) we refer the reader to [5, Theorem 3.1].

Finally, part v) follows from ii) and iv). ▮

Remark 3. 1. Note that part (ii) of Lemma 3.1 does not imply (1.4). We will be able to prove that latter condition only in Lemma 3.5, after we obtain some stronger information about the perturbed problem (3.15).

Lemma 3.2. *Under the assumptions* (3.17) *and* (3.18) *the family of semigroups of global solutions of* (3.15) *is collectively asymptotically compact at* $\eta = 0$.

Proof. In the proof we will apply the nonlinear variation of constant formula (see [1, 5]).

Following [5] we decompose f as $f(s) = \tilde{f}(s) + \hat{f}(s)$ where

$$\tilde{f}(s) = f(s) - f(0) - K_f s, \quad \hat{f}(s) = f(0) + K_f s, \quad s \in \mathbb{R},$$

and

$$s\tilde{f}(s) \leqslant 0 \text{ for } s \in \mathbb{R}. \tag{3.30}$$

We then denote by $\tilde{T}_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ the solution of

$$\tfrac{d}{dt}\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] + A_\eta \left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] = \begin{bmatrix} 0 \\ \tilde{f}^e(\tilde{u}) \end{bmatrix} =: \tilde{F}(\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]),\ t > 0, \quad \left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \tag{3.31}$$

to write a nonlinear variation of constants formula (see [4, Lemma 7]; also [1]),

$$T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \tilde{T}_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t (\partial\tilde{T}_\eta)(t-s, T_\eta(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\hat{F}(T_\eta(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])ds, \quad (3.32)$$

where $\partial\tilde{T}_\eta$ is partial derivative of the map $(0,\infty)\times Y \ni (t,V_0) \to \tilde{T}_\eta(t)V_0 \in Y$ w.r.t. V_0, $\hat{F}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 \\ K_f w_1 + f(0) \end{smallmatrix}\right]$ for $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in Y$ and $(\partial\tilde{T}_\eta)(\cdot, V_0)\hat{F}(V_0)$ is a solution of

$$\frac{d}{dt}W + A_\eta W = \tilde{F}'(\tilde{T}_\eta(t)V_0)W,\ t>0, \quad W(0) = \hat{F}(V_0). \quad (3.33)$$

As shown in [5], for each $r>0$, there are positive constants c_r and ω_r such that

$$\sup_{\eta\in(0,1]} \sup_{\|\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \le r} \|\tilde{T}_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \le c_r e^{-\omega_r t}, \quad t \ge 0. \quad (3.34)$$

It also follows from the consideration of [5, Lemma 3.4] that, for each $\eta \in (0,1]$, $\alpha \in (0,\frac{1}{2})$ and for any B bounded in Y, if $\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in B$ and $W = \left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]$ is the solution of (3.33) with $W(0) = \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])$, then there are positive constants d'' and ω'' such that

$$\|\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\|_{D(A^{\frac{1+\alpha}{2}})\times D(A^{\frac{\alpha}{2}})} \leqslant d'' e^{-\omega'' t}, \quad t \geqslant 0. \quad (3.35)$$

Thanks to (3.34)-(3.35) and to the decomposition of the semigroups $\{T_\eta(t)\}$ as in (3.32), any sequence of the form $\{T_{\eta_k}(t_k)(\left[\begin{smallmatrix} u_k \\ v_k \end{smallmatrix}\right])\}$ with $\{\eta_k\} \subset I$, $t_k \nearrow \infty$ and with $\{\left[\begin{smallmatrix} u_k \\ v_k \end{smallmatrix}\right]\}$ being bounded in Y has a subsequence convergent in Y. ▮

LEMMA 3.3. *In addition to the assumptions* (3.17) *and* (3.18) *suppose that all the equilibria solutions of* (3.15) *are hyperbolic; namely,*

$$0 \notin \sigma(-A + f'(u_0)) \quad \text{whenever} \quad u_0 \in H_0^1(\Omega) \quad \text{and} \quad u_0 = A^{-1}f(u_0). \quad (3.36)$$

Then,

i) the set $\mathcal{E}$ of all equilibria is finite and does not depend on η; i.e.

$$\mathcal{E}_\eta = \mathcal{E} := \{\left[\begin{smallmatrix} \phi_i \\ 0 \end{smallmatrix}\right],\ i = 1,\dots,k\},\ \eta \in I,$$

ii) for each $\eta \in I$ the family $\mathcal{S}_\eta$ consisting of all one-point-set equilibria $\{\left[\begin{smallmatrix} \phi_i \\ 0 \end{smallmatrix}\right]\}$, $i = 1,\dots,k$, attracts points of Y and has uniformly pointwise exponentially attracting local unstable sets, *iii) there is no ϵ-homoclinic structure related to $\mathcal{S}_0$.*

Proof: Needless to say that the equilibria does not depend on a value of the parameter and that $\mathcal{E}$ attracts points. Condition (3.36) implies next that they are isolated (see e.g. [3, Proposition 4.2]). Since they are all in the attractor, there are finitely many of them.

The remaining part of the proof builds upon the results obtained in [3]. We recall from [3, Appendix] that under our assumptions the local unstable set of each equilibrium solution can be viewed locally, and for any $\eta \in I$, as a graph of a Lipschitz continuous function $\Sigma_\eta^* : Q_\eta(Y) \to P_\eta(Y)$, where Q_η is defined in [3, (4.14)] and $P_\eta = Id - Q_\eta$. We also recall from [3, (A.8)] that $T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, which can be viewed in the form $(Q_\eta T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], P_\eta T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])$, satisfies the estimate:

$$\|P_\eta(T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \phi_i \\ 0 \end{smallmatrix}\right]) - \Sigma_\eta^* Q_\eta(T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \phi_i \\ 0 \end{smallmatrix}\right])\|_Y \leqslant M e^{-\gamma t}$$

for $0 \leqslant t$, $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \mathcal{O}_{\delta_0}(\left[\begin{smallmatrix} \phi_i \\ 0 \end{smallmatrix}\right])$ with certain constants $M \geqslant 1$, $\gamma > 0$ and as long as $T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ remains in a certain ball $\mathcal{O}_{\delta_0}(\left[\begin{smallmatrix} \phi_i \\ 0 \end{smallmatrix}\right])$ for any $i \in \{1, \ldots, k\}$.

We now emphasize that the constants M and γ are chosen independently of $\eta \in I$ based on the uniform exponential dichotomy property reported in [3, Theorem 2.14] and that (1.9) is thus a counterpart of (1.9).

The remaining part of the proof follows easily as there is a Lyapunov functional (3.26) for (3.15). ∎

What was said above ensures that Theorem 1.1 applies; namely, we have the following result.

COROLLARY 3.1. *Suppose that the assumptions of Lemma 3.3 hold. Then the family of attractors is eventually uniformly exponentially attracting.*

The theorem below is now a consequence of Corollary 3.1, Proposition 1.1 and part (ii) of Lemma 3.1.

THEOREM 3.3. *Suppose that* (3.17), (3.18) *and* (3.36) *hold. Then the family of attractors* $\{\mathbf{A}_\eta, \eta \in I\}$ *is continuous at* $\eta = 0$ *with respect to the Hausdorff distance* dist_H^Y *in* $Y = H_0^1(\Omega) \times L^2(\Omega)$, *that is*

$$\mathrm{dist}_H^Y(\mathbf{A}_\eta, \mathbf{A}_0) + \mathrm{dist}_H^Y(\mathbf{A}_0, \mathbf{A}_\eta) \to 0 \quad as \quad \eta \to 0^+.$$

Remark 3. 2. Theorem 3.3 generalizes results of [3] concerning continuity of the attractors for 3-D singularly perturbed damped wave equation (3.15) as in the present paper we consider a nonlinearity with a critical exponent.

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From Theorem 3.3 and from [5] we next obtain the following conclusions.

COROLLARY 3.2. *Suppose that the assumptions of Lemma 3.3 hold. Then*

$$\mathbf{A_0} \textit{ is bounded in } H^3(\Omega)\times H^2(\Omega)$$

and, for each $\mu\in(0,\frac{1}{2})$,

$$\sup_{a_\eta\in\boldsymbol{A}_\eta}\inf_{a_0\in\boldsymbol{A}_0}\|a_0-a_\eta\|_{C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)}+\sup_{a_0\in\boldsymbol{A}_0}\inf_{a_\eta\in\boldsymbol{A}_\eta}\|a_0-a_\eta\|_{C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)}$$

tends to zero as $\eta\to 0$.

If, in addition, $f\in C^3(\mathbb{R},\mathbb{R})$, *then*

$$\mathbf{A_0} \textit{ is also bounded in } H^4(\Omega)\times H^3(\Omega)$$

and, for each $\mu\in(0,\frac{1}{2})$,

$$\sup_{a_\eta\in\boldsymbol{A}_\eta}\inf_{a_0\in\boldsymbol{A}_0}\|a_0-a_\eta\|_{C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)}+\sup_{a_0\in\boldsymbol{A}_0}\inf_{a_\eta\in\boldsymbol{A}_\eta}\|a_0-a_\eta\|_{C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)}$$

tends to zero as $\eta\to 0$.

In Lemma 3.1 we have proved that the approximating semigroups converge in Y uniformly on compact sets to the limit semigroup as $\eta\to 0^+$. Below we will show that, for the initial data from the set $\cup_{\eta\in I}\boldsymbol{A}_\eta$, the rate of convergence of $T_\eta(t)$ to the limit semigroup can be 'measured' by η.

We will first show that the linear semigroups converge with the rate η in a suitable norm.

LEMMA 3.4. *There is a constant* $c>0$ *such that, for every* $\eta\in I$ *and each* $t>0$,

$$\|(e^{-\mathcal{A}_\eta t}-e^{-\mathcal{A}_0 t})\mathcal{A}_\eta^{-2}\|_{L(Y)}\le c\eta.$$

Proof. For $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\in Y$ and $t>0$ we have that

$$\begin{aligned}\|(e^{-\mathcal{A}_0 t}-e^{-\mathcal{A}_\eta t})\mathcal{A}_\eta^{-2}\|_{L(Y)}&\le\|e^{-\mathcal{A}_0 t}(\mathcal{A}_\eta^{-1}-\mathcal{A}_0^{-1})\mathcal{A}_\eta^{-1}\|_{L(Y)}\\&+\|(\mathcal{A}_0^{-1}-\mathcal{A}_\eta^{-1})e^{-\mathcal{A}_\eta t}\mathcal{A}_\eta^{-1}\|_{L(Y)}+\|\mathcal{A}_0^{-1}(e^{-\mathcal{A}_0 t}-e^{-\mathcal{A}_\eta t})\mathcal{A}_\eta^{-1}\|_{L(Y)}.\end{aligned}$$

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For $\begin{bmatrix}\phi\\ \psi\end{bmatrix} \in Y$ and $\eta > 0$ we also have

$$\begin{aligned}\|\mathcal{A}_\eta^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y &= \|\begin{bmatrix}\eta A^{-\frac{1}{2}}\phi + aA^{-1}\phi + A^{-1}\psi\\ -\phi\end{bmatrix}\|_Y\\ &\leq \eta\|\phi\|_{L^2(\Omega)} + a\|A^{-\frac{1}{2}}\phi\|_{L^2(\Omega)} + \|A^{-\frac{1}{2}}\psi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)}\\ &\leq \lambda_1^{-\frac{1}{2}}(a\lambda_1^{-\frac{1}{2}} + 1 + \eta)\|A^{\frac{1}{2}}\phi\|_{L^2(\Omega)} + \lambda_1^{-\frac{1}{2}}\|\psi\|_{L^2(\Omega)}\\ &= \lambda_1^{-\frac{1}{2}}(1 + \eta + a\lambda_1^{-\frac{1}{2}})\ \|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y,\end{aligned}$$

which implies that

$$\|\mathcal{A}_\eta^{-1}\|_{L(Y)} \leq \lambda_1^{-\frac{1}{2}}(1 + \eta + a\lambda_1^{-\frac{1}{2}}). \tag{3.37}$$

Since all the semigroups operators are the contractions it follows from (3.22), (3.37) that

$$\begin{aligned}&\|e^{-\mathcal{A}_0 t}(\mathcal{A}_\eta^{-1} - \mathcal{A}_0^{-1})\mathcal{A}_\eta^{-1}\|_{L(Y)} \leq C_0\eta,\ t \geq 0,\ \eta \in I\\ &\|(\mathcal{A}_0^{-1} - \mathcal{A}_\eta^{-1})e^{-\mathcal{A}_\eta t}\mathcal{A}_\eta^{-1}\|_{L(Y)} \leq C_0\eta,\ \ t \geq 0,\ \eta \in I.\end{aligned}$$

We also recall from [3, Corollary 2.19] that, for a certain $\omega > 0$ and $c \geq 1$ we have

$$\|e^{-\mathcal{A}_\eta t}\|_{L(Y)} \leq ce^{-\omega t},\ t \geq 0. \tag{3.38}$$

From [15, Section 3.4, Lemma 4.1], (3.22) and (3.38) we finally have

$$\|\mathcal{A}_0^{-1}(e^{-\mathcal{A}_\eta t} - e^{-\mathcal{A}_0 t})\mathcal{A}_\eta^{-1}\|_{L(Y)} \leq \hat{C}_0\eta te^{-\omega t} \leq \tilde{C}_0\eta, \quad t \geq 0,\ \eta \in I,$$

which completes the proof. ▮

With the aid of Lemma 3.4 we will next prove that, for the initial data from $\cup_{\eta\in I}\mathbf{A}_\eta$, nonlinear semigroups also converge with the rate η.

Lemma 3.5. *Suppose that the assumptions of Lemma 3.3 hold and let* $B_0 := \cup_{\eta\in I}\mathbf{A}_\eta$. *Then,*

i) $\begin{bmatrix}-A^{-1}f(0)\\ 0\end{bmatrix} + B_0$ *is a bounded subset of* $Y^2 = H^3_{\{I,\Delta\}}(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))$, *where* $H^3_{\{Id,\Delta\}}(\Omega) := \{\phi \in H^1_0(\Omega) : A\phi \in H^1_0(\Omega)\}$ *and, consequently,* B_0 *is bounded in* $H^3(\Omega) \times H^2(\Omega)-$*norm,*

ii) there is a constant $D > 0$ *such that, for each* $t \geq 0$, $\eta \in I$ *and every* $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right] \in B_0$,

$$\|T_\eta(t)\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\|_{Y^1} \leq De^{Dt},$$

iii) there is a constant $\hat{D} > 0$ *such that, for each* $t \geq 0$, $\eta \in I$ *and every* $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right] \in B_0$,

$$\|T_\eta(t)\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right] - T_0(t)\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\|_Y \leq \eta\hat{D}e^{\hat{D}t}.$$

Proof. Boundedness of $\begin{bmatrix} -A^{-1}f(0) \\ 0 \end{bmatrix} + \bigcup_{\eta\in(0,1]} \mathbf{A}_\eta$ in Y^2 has been obtained in [5, Lemma 4.1]. The latter property and Theorem 3.3 imply now boundedness of $\begin{bmatrix} -A^{-1}f(0) \\ 0 \end{bmatrix} + \mathbf{A_0}$ in Y^2. Consequently i) holds and there is a positive constant C such that

$$\sup_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\in B_0} \| \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \begin{bmatrix} -A^{-1}f(0) \\ 0 \end{bmatrix} \|_{Y^2} \le C. \tag{3.39}$$

Proving ii) consider next, for $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B_0$ and $\eta \in I$, the solution of

$$\tfrac{d}{dt} \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \ t > 0, \quad \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]_{t=0} = \begin{bmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{bmatrix}, \tag{3.40}$$

and the solution $T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right]$ of

$$\begin{gathered} \tfrac{d}{dt} \left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right] = F(\left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right]), \ F(\left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 \\ f(u^\eta) \end{smallmatrix}\right], \ t > 0, \\ \left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]. \end{gathered} \tag{3.41}$$

Define

$$\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] (t, T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) := \left[\begin{smallmatrix} u^\eta \\ v^\eta \end{smallmatrix}\right] - e^{-\mathcal{A}_\eta t} \begin{bmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{bmatrix} - \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix}. \tag{3.42}$$

Then $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] (t, \eta)$ solves

$$\tfrac{d}{dt} \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = F(T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]), \ t > 0, \quad \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \tag{3.43}$$

so that we have

$$\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] (t, u^\eta) = \int_0^t e^{-\mathcal{A}_\eta (t-s)} (F(T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])) ds, \ t \ge 0. \tag{3.44}$$

From the properties of the linear operators $\mathcal{A}_\eta$ we infer that

$$d^{-1} \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1} \le \| \mathcal{A}_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_Y \le d \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1}, \ \eta \in I, \ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y^1. \tag{3.45}$$

$$d_1^{-1} \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^2} \le \| \mathcal{A}_\eta^2 \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_Y \le d_1 \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^2}, \ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y^2, \tag{3.46}$$

where d, d_1 are positive constants independent of $\eta \in I$ (see [5]).

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What was said above leads to the following estimate

$$\begin{aligned}
&\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_{Y^1}\\
&\quad=\|e^{-\mathcal{A}_\eta t}\begin{bmatrix}u_0-A^{-1}f(0)\\v_0\end{bmatrix}+\begin{bmatrix}A^{-1}f(0)\\0\end{bmatrix}+\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right](t,T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_{Y^1}\\
&\quad\leq d\|e^{-\mathcal{A}_\eta t}\mathcal{A}_\eta\begin{bmatrix}u_0-A^{-1}f(0)\\v_0\end{bmatrix}\|_Y+\|\begin{bmatrix}A^{-1}f(0)\\0\end{bmatrix}\|_{Y^1}\\
&\quad+d\int_0^t\|e^{-\mathcal{A}_\eta(t-s)}\|_Y\|\mathcal{A}_\eta(F(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])-F\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)\|_Y ds,
\end{aligned}\tag{3.47}$$

where, for each $\eta\in I$, we have

$$\begin{aligned}
&\|\mathcal{A}_\eta\big(F\left(\left[\begin{smallmatrix}u^\eta\\v^\eta\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)\big)\|_Y\leq d\|\left[\begin{smallmatrix}0\\f(u^\eta)-f(0)\end{smallmatrix}\right]\|_{Y^1}\\
&\leqslant\tilde{d}\|f'(u^\eta)\nabla u_\eta\|_{L^2(\Omega)}\leq\hat{d}\|(1+|u^\eta|^2)|\nabla u^\eta|\|_{L^2(\Omega)}\\
&\leq g(\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y)\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_{Y^1},
\end{aligned}\tag{3.48}$$

with a certain continuous nondecreasing function $g:\mathbb{R}\to\mathbb{R}$.

Since the linear semigroups are contractions, from (3.27), (3.39), (3.47), (3.48) we conclude, via Gronwall's inequality, that

$$\sup_{\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in B_0}\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_{Y^1}\leq De^{Dt},\ t\geq 0,\tag{3.49}$$

where c is independent of $\eta\in I$. Part ii) is thus proved.

Now we return to (3.42) and with the aid of (3.46), for $\eta\in I$ and $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in B_0$, we get

$$\begin{aligned}
&\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]-T_0(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y\leq\|(e^{-\mathcal{A}_\eta t}-e^{-\mathcal{A}_0 t})\mathcal{A}_\eta^{-2}\|_{L(Y)}d_1\|\begin{bmatrix}u_0-A^{-1}f(0)\\v_0\end{bmatrix}\|_{Y^2}\\
&+\int_0^t\|(e^{-\mathcal{A}_\eta(t-s)}-e^{-\mathcal{A}_0(t-s)})\mathcal{A}_\eta^{-2}\|_{L(Y)}\|\mathcal{A}_\eta^2(F(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]))\|_Y ds\\
&+\int_0^t\|e^{-\mathcal{A}_0(t-s)}\|_{L(Y)}\|F(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])-F(T_0(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_Y ds,\ t\geq 0.
\end{aligned}$$

Recalling that $N=3$ and $H^2(\Omega)\subset W^{1,4}(\Omega)\cap L^\infty(\Omega)$, we have

$$\begin{aligned}
&\|\mathcal{A}_\eta{}^2\left[\begin{smallmatrix}0\\f(\phi)-f(0)\end{smallmatrix}\right]\|_Y\leq d_1\|\left[\begin{smallmatrix}0\\f(\phi)-f(0)\end{smallmatrix}\right]\|_{Y^2}\\
&\quad\leq\tilde{d}_1\|\left[\begin{smallmatrix}0\\f(\phi)-f(0)\end{smallmatrix}\right]\|_{H^3_{\{I,\Delta\}}(\Omega)\times(H^2(\Omega)\cap H^1_0(\Omega))}\\
&\quad\leq\hat{d}_1\|A(f(\phi)-f(0))\|_{L^2(\Omega)}\\
&\quad=\hat{d}_1\|f''(\phi)|\nabla\phi|^2+f'(\phi)\Delta\phi\|_{L^2(\Omega)}\\
&\quad\leq c\hat{d}_1\|(1+|\phi|)|\nabla\phi|^2+(1+|\phi|^2)|\Delta\phi|\|_{L^2(\Omega)}\\
&\quad\leq\mathbf{D}(1+\|\Delta\phi\|^3_{L^2(\Omega)})\leq\mathbf{D}(1+\|\begin{bmatrix}\phi\\\psi\end{bmatrix}\|^3_{Y^1}),\ \begin{bmatrix}\phi\\\psi\end{bmatrix}\in Y^1.
\end{aligned}$$

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From the above inequality, Lemma 3.4 and (3.49) we now obtain

$$\int_0^t \|(e^{-\mathcal{A}_\eta(t-s)} - e^{-\mathcal{A}_0(t-s)})\mathcal{A}_\eta^{-2}\|_{L(Y)}\|\mathcal{A}_\eta^2(F(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] - F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]))\|_Y ds \leq \eta\tilde{\mathbf{D}}\int_0^t (1+e^{3Ds})ds,\ t \geq 0,$$

whereas, from Lemma 3.4 and (3.39), we infer

$$\|(e^{-\mathcal{A}_\eta t} - e^{-\mathcal{A}_0 t})\mathcal{A}_\eta^{-2}\|_{L(Y)} d_1 \| \begin{bmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{bmatrix} \|_{Y^2} \leq cCd_1\eta,\ t \geq 0. \quad (3.50)$$

Since $F : Y \to Y$ is Lipschitz continuous on bounded sets, with the aid of (3.27) and part (iv) of Lemma 3.1 we also get

$$\int_0^t \|e^{-\mathcal{A}_0(t-s)}\|_{L(Y)}\|F(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]) - F(T_0(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_Y ds \leq L\int_0^t \|T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] - T_0(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y ds,$$

$t \geq 0$, with a certain constant $L > 0$.

Connecting the above estimates, for $t \geq 0$ and $\eta \in I$ we have

$$\|T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] - T_0(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y \leq cCd_1\eta + \eta\tilde{\mathbf{D}}\int_0^t (1+e^{3Ds})ds + L\int_0^t \|T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] - T_0(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_Y ds \quad (3.51)$$

and part iii) now follows easily with the aid of Gronwall's inequality. ∎

Part iii) of Lemma 3.5 ensures that the family of semigroups corresponding to (3.15) has a κ-modulus of continuity at $\eta = 0$ with κ being the identity function on a unit time interval. Since the family of global attractors $\{\mathbf{A}\}_{\eta\in I}$ is eventually uniformly exponentially attracting (see Corollary 3.1 above) Theorem 3.3 can be now strengthened to the following result.

Theorem 3.4. *Suppose that the assumptions* (3.17), (3.18) *and* (3.36) *hold. Then there are constants* $\bar{c}, \hat{D}, \rho > 0$ *such that*

$$\operatorname{dist}_H^Y(\mathbf{A}_\eta, \mathbf{A}_0) + \operatorname{dist}_H^Y(\mathbf{A}_0, \mathbf{A}_\eta) \leqslant \bar{c}\ \eta^{\frac{\varrho}{\varrho+\hat{D}}},$$

for all sufficiently small $\eta > 0$.

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3.3. Viscous Cahn-Hilliard equation

In this subsection we discuss briefly the viscous Cahn-Hilliard model (see [7, 9])

$$\begin{aligned}&(1-\nu)u_t = -\Delta(\Delta u + f(u) - \nu u_t), \text{ in } \Omega,\\ &u(t,x) = \Delta u(t,x) = 0 \text{ in } \partial\Omega,\\ &u(0,x) = u_0(x),\end{aligned} \tag{3.52}$$

where $\nu \in I$, $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies suitable growth and dissipation conditions and Ω is a bounded smooth domain in $\mathbb{R}^N, N \geq 3$.

In compare with previous references some development here lies in a possibility of evaluation of the convergence rate of the attractors.

Following [7] we consider (3.52) on the phase space $H^1_0(\Omega)$ and denote by $A = -\Delta$ the negative Dirichlet Laplacian with the domain $D(A) = H^1_0(\Omega)$.

As in [7] we also define $A_\nu = A^2((1-\nu)I + \nu A)^{-1}$ and $B_\nu = A((1-\nu)I + \nu A)^{-1}$ to view (3.52) in an abstract form

$$u_t = -A_\nu u + B_\nu f(u), \tag{3.53}$$

or, since $A_\nu^{-1} B_\nu = A^{-1}$, as

$$\frac{d}{dt}\left(A_\nu^{-1} u\right) = -u + A^{-1} f(u). \tag{3.54}$$

This gives rise to some intuition about the 'limiting behavior' of the system. Namely, since

$$\begin{aligned}&A_\nu^{-1} = ((1-\nu)I + \nu A)A^{-2} \to A^{-1} \quad \text{as } \nu \to 1^-,\\ &A_\nu^{-1} \to A^{-2} \quad \text{as } \nu \to 0^+\end{aligned}$$

the limit problem $(3.54)_{\nu=1}$ will be the semilinear heat equation, while $(3.54)_{\nu=0}$ will be the Cahn-Hilliard equation. Below we will concentrate on this latter case.

Throughout the rest of this subsection we assume the condition;

$(\mathcal{H})$ $N = 3$, $f \in C^1(\mathbb{R},\mathbb{R})$, $f(0) = 0$, $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of A and for some $\bar{\epsilon}, c > 0$

$$|f''(s)| \leq c(1 + |s|^{3-\bar{\epsilon}}). \tag{3.55}$$

From [7] we then have the following result.

Proposition 3.4. *Under the hypothesis $(\mathcal{H})$ there is a family of gradient semigroups $\{T_\nu(t) : t \geq 0\}_{\nu\in I}$ associated to (3.52) in $H^1_0(\Omega)$, such that for* *each $\nu \in I$*

$$[0,\infty) \times H_0^1(\Omega) \ni (t,v) \to T_\nu(t)v \in H_0^1(\Omega) \quad \text{is a continuous map,}$$

and there exists a global attractor $\mathbf{A}_\nu$ for $\{T_\nu(t) : t \geq 0\}$.

Furthermore, the union $\cup_{\nu\in I}\gamma_\nu^+(\cup_{\nu\in I}\mathbf{A}_\nu)$ is bounded in $H^2(\Omega)$ and thus also in $L^\infty(\Omega)$. In particular, the family of attractors $\{\mathbf{A}_\nu\}_{\nu\in I}$ is collectively asymptotically compact at $\nu = 0$.

Remark 3. 3. Indeed, in [7] the authors consider the almost critical case $\lim_{|s|\to\infty} \frac{|f''(s)|}{s^3} = 0$ and prove that the attractors are uniformly bounded in $L^\infty(\Omega)$. Since the criticality is out of our interests in this example we chose to work with the subcritical case $\epsilon > 0$.

The steady states of (3.52) are independent of the parameter and attract points. Thus note that in this example all the collections of compact disjoint invariant sets $\mathcal{S}_\nu$ can be chosen the same for all $\nu \in I$ and will consist of one-point-set equilibria for (3.52). Referring then to [7] we obtain below the results concerning the existence and suitable attracting properties of local unstable manifolds of equilibria.

Proposition 3.5. *Assume $(\mathcal{H})$ and that all equilibria of (3.52) are hyperbolic. Then, the set $\mathcal{E}$ of equilibria of (3.52) is finite and for each $\nu \in I$ there is a family $\mathcal{S}_\nu$, consisting of all one-point-sets equilibria of (3.52), such that $\{\mathcal{S}_\nu\}_{\nu\in I}$ has uniformly pointwise exponentially attracting local unstable sets.*

We now devote more attention to the convergence of the family $\{T_\nu(t) : \nu \in I\}_{\nu\in I}$ at $\nu = 0$ as a singularity appears there at the initial time. We remark that a singularity also occurs when one tries to derive a κ-modulus of continuity in this example. This proves that the approach of our paper is indeed flexible as both Theorem 1.1 and Proposition 1.1 do allow the system to possess singularities before a certain $T \geq 0$.

We need the following three lemmas, for which we refer the reader to [7].

Lemma 3.6. *The following estimates hold with a constant $M > 0$ independent of $\nu \in I$.*

$$\|A_\nu^{-1} - A^{-2}\| \leq M\nu, \quad \|A_\nu^{-1} - A_\mu^{-1}\| \leq M|\nu - \mu|, \quad \|B_\nu A_\nu^{-\frac{1}{2}}\| \leq M \quad (3.56)$$

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and, for $t > 0$,

$$\|B_\nu e^{-A_\nu t} - A e^{-A^2 t}\| \le M t^{-\frac{1}{2}}, \ \|B_\nu e^{-A_\nu t} - B_\mu e^{-A_\mu t}\| \le M t^{-\frac{1}{2}}. \quad (3.57)$$

LEMMA 3.7. *For any* $0 < \epsilon < 1$ *and* $\nu \in I$,

$$\|A^{1-\epsilon} B_\nu e^{-A_\nu t}\| \le M t^{-1+\frac{\epsilon}{2}}.$$

LEMMA 3.8. *If* $0 < \epsilon < 1$ *then for* $\alpha < \frac{\epsilon}{2(1-\epsilon)}$ *we have* $\beta = 1 + \alpha(1-\epsilon) - \frac{\epsilon}{2} < 1$ *and*

$$\|A^{1-\epsilon} B_\nu e^{-A_\nu t} - A^{2-\epsilon} e^{-A^2 t}\| \le M t^{-\beta} \|A_\nu^{-1} - A^{-2}\|^{\alpha(1-\epsilon)}.$$

Also,

$$\|A^{1-\epsilon} B_\nu e^{-A_\nu t} - A^{1-\epsilon} B_\mu e^{-A_\mu t}\| \le M t^{-\beta} \|A_\nu^{-1} - A^{-2}\|^{\alpha(1-\epsilon)}, \ t > 0, \ \mu, \nu \in I.$$

Consider next $\nu \in I$, $t \ge 0$ and the semigroups operators

$$\begin{aligned} T_\nu(t) u_0 &= e^{-A_\nu t} u_0 + \int_0^t e^{-A_\nu (t-s)} B_\nu f(T_\nu(s) u_0) ds, \\ T_0(t) u_0 &= e^{-A^2 t} u_0 + \int_0^t e^{-A^2 (t-s)} A f(T_0(s) u_0) ds. \end{aligned} \quad (3.58)$$

Note that $f : H_0^1(\Omega) \to H^{-1+\delta}(\Omega)$, for suitably chosen $\delta = \delta(\bar{\epsilon}) > 0$ ($\bar{\epsilon}$ as in (3.55)), is Lipschitz continuous in bounded sets and that orbits of bounded subsets of $H_0^1(\Omega)$ are bounded in $H_0^1(\Omega)$. We also remark that, since A_ν, $\nu \in (0,1]$, and $A_0 = A^2$ are self-adjoint and positive elliptic operators, the linear semigroups we deal with are the semigroups of contractions.

Next, using the variation of constants formula and Gronwall's type lemma, we prove that the family $\{T_\nu(t) : t \ge 0\}$ is exponentially Lipschitz continuous. In fact, for fixed $\epsilon \in (0, \frac{\delta}{2}]$ a simple application of Gronwall's type lemma (see [13], p. 188) to the estimate:

$$\begin{aligned} &\|T_\nu(t) u - T_\nu(t) v\|_{H_0^1(\Omega)} \le c \|u - v\|_{H_0^1(\Omega)} \\ &\quad + \int_0^t \|A^{1-\epsilon} B_\nu e^{-A_\nu (t-s)}\| \|f(T_\nu(s) u) - f(T_\nu(s) v)\|_{H^{-1+2\epsilon}(\Omega)} ds, \end{aligned} \quad (3.59)$$

implies that, given a bounded subset B of $H_0^1(\Omega)$ there is a positive number $L = L(B)$ and a constant $c > 0$ such that

$$\|T_\nu(t)u - T_\nu(t)v\|_{H_0^1(\Omega)} \leq c\|u - v\|_{H_0^1(\Omega)} \frac{2}{\epsilon} e^{\left(ML\Gamma(\frac{\epsilon}{2})\right)^{\frac{2}{\epsilon}} t}, \tag{3.60}$$

for all $t \geq 0$, $\nu \in I$, and $u, v \in B$. As for the eventual convergence, from [7, p. 718] for $\alpha < \frac{\epsilon}{2(1-\epsilon)}$ we have that (here $\epsilon \in (0, \frac{\delta}{2}], \beta = 1 + \alpha(1-\epsilon) - \frac{\epsilon}{2}$),

$$\begin{aligned} &\|T_\nu(t)v - T_0(t)v\|_{H_0^1(\Omega)} \\ &\leq M\|A_\nu^{-1} - A^{-2}\|^{\alpha(1-\epsilon)} \left(t^{-\alpha(1-\epsilon)} \|v\|_{H_0^1(\Omega)} + c\int_0^t (t-s)^{-\beta} ds \right) \\ &+ ML \int_0^t (t-s)^{-1+\frac{\epsilon}{2}} \|T_\nu(s)v - T_0(s)v\|_{H_0^1(\Omega)} ds, \end{aligned} \tag{3.61}$$

for all $t > 0$, $\nu \in I$, and $u, v \in B$. Using Lemma 3.6, we obtain that for all $v \in B$

$$\begin{aligned} \|T_\nu(t)v - T_0(t)v\|_{H_0^1(\Omega)} &\leq C(\|v\|_{H_0^1(\Omega)})\nu^{\alpha(1-\epsilon)}(t^{-\alpha(1-\epsilon)} + t^{-\alpha(1-\epsilon)+\frac{\epsilon}{2}}) \\ &+ ML \int_0^t (t-s)^{-1+\frac{\epsilon}{2}} \|T_\nu(s)v - T_0(s)v\|_{H_0^1(\Omega)} ds. \end{aligned}$$

Applying Gronwall's type lemma (see [13], p. 188), for $\theta = (ML\Gamma(\frac{\epsilon}{2}))^{\frac{2}{\epsilon}}$, we have that

$$\begin{aligned} &\|T_\nu(t)v - T_0(t)v\|_{H_0^1(\Omega)} \\ &\leq C(\|v\|_{H_0^1(\Omega)})\nu^{\alpha(1-\epsilon)} \left[(t^{-\alpha(1-\epsilon)} + t^{-\alpha(1-\epsilon)+\frac{\epsilon}{2}}) + \frac{2}{\epsilon}\theta \right] e^{\theta t}. \end{aligned} \tag{3.62}$$

Using (3.60) and (3.62), for $t \geq 1$, $\nu \in I$ and $u, v \in B$ we finally obtain

$$\|T_\nu(t)u - T_0(t)v\|_{H_0^1(\Omega)} \leq \frac{\bar{c}}{\epsilon} e^{\theta t} \left(\|u - v\|_{H_0^1(\Omega)} + C(\epsilon, \|v\|_{H_0^1(\Omega)})\nu^{\alpha(1-\epsilon)} \right).$$

In conclusion: both Theorem 1.1 and Proposition 1.1 apply to the viscous Cahn-Hilliard model (3.52) and the attractors converge as $\nu \to 0$ with a rate given by a power of ν.

REFERENCES

1. J. M. Arietta, A. N. Carvalho, J. K. Hale, A damped hyperbolic equation with critical exponent, *Comm. Partial Differential Equations* **17** (1992), 841-866.

2. A. V. Babin, M. I. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.

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3. S. M. Bruschi, A. N. Carvalho, J. W. Cholewa, T. Dlotko, Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations, *J. Dynam. Differential Equations* **18** (2006), 767-814.
4. A. N. Carvalho, J. W. Cholewa, Attractors for strongly damped wave equations with critical nonlinearities, *Pacific J. Math.* **207** (2002), 287-310.
5. A. N. Carvalho, J. W. Cholewa, Regularity of solutions on the global attractor for a semilinear damped wave equation, *J. Math. Anal. Appl.* **337** (2008), 932-948.
6. A. N. Carvalho, J. W. Cholewa, Exponential global attractors for semigroups in metric spaces with applications to differential equations, submitted.
7. A. N. Carvalho, T. Dlotko, Dynamics of the viscous Cahn-Hilliard equation, *J. Math. Anal. Appl.* **344** (2008), 703-725.
8. A. N. Carvalho, J. A. Langa, An extension of the concept of gradient semigroups which is stable under perturbation, *J. Differential Equations* **246** (2009), 2646-2668.
9. C.M. Elliott, A.M. Stuart, Viscous Cahn-Hilliard equation II. Analysis, *J. Differential Equations* **128** (1996), 387-414.
10. J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, RI, 1988.
11. J.K. Hale, G. Raugel, Lower semicontinuity of attractors of gradient systems and applications. *Ann. Mat. Pura Appl.* **154** (1989), 281–326.
12. J. K. Hale, G. Raugel, Regularity, determining modes and Galerkin methods, *J. Math. Pures Appl.* **82** (2003), 1075-1136.
13. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.
14. Desheng Li, P. E. Kloeden, Equi-attraction and the continuous dependence of attractors on parameters, *Glasg. Math. J.* **46** (2004), 131-141.
15. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.
16. G. Raugel, Global attractors in partial differential equations; in: *Handbook of Dynamical Systems*, Vol. 2, 885–982, North-Holland, Amsterdam, 2002.

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