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A new continuous dependence result for impulsive retarded functional differential equations

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We consider a large class of impulsive retarded functional differential equations and we present a new continuous dependence result for this class of equations. May, 2010 ICMC-USP

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1. INTRODUCTION

We consider a class of impulsive retarded differential equations (we write impulsive RFDEs for short) with Lebesgue integrable righthand sides whose indefinite integrals satisfy Carathéodory- and Lypschitz-type conditions. We also consider that the impulse operators are Lypschitzian functions and that the initial conditions are regulated functions. Then we prove an existence and uniqueness theorem for this class of equations as well as a new continuous dependence type result.

In the above setting, the following continuous dependence result for impulsive RFDEs is well-known (see [1], Theorem 4.1). Consider a sequence of impulsive retarded initial value problems (we write IVPs for short) whose righthand sides converge to the righthand side of an impulsive RFDE and whose initial data also converge. Let the sequence of impulse operators be convergent as well. Suppose each element of the sequence of impulsive retarded IVPs admits a unique solution and that the sequence of solutions is uniformly convergent.

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Consider also the limiting IVP with limiting righthand side, limiting impulse operators and limiting initial condition. Then the limit of the sequence of solutions is a solution of the limiting IVP, provided certain conditions are fulfilled.

In the present paper, we state and prove a certain reciprocal of this result. Roughly speaking, we consider a sequence of IVPs for impulsive RFDEs, in the above setting, with convergent righthand sides, convergent impulse operators and uniformly convergent initial data. We assume the limiting equation is an impulsive RFDE whose initial condition is the uniform limit of the sequence of initial data and whose solution exists and is unique. Then, under certain conditions, for sufficient large indexes, the elements of the sequence of impulsive retarded IVPs admit a unique solution and such a sequence of solutions converges to the solution of the limiting IVP.

2. THE SETTING OF IMPULSIVE RFDES

Let X be a Banach space. A function $f : [a, b] \to X$ is called *regulated*, if the following limits exist

$$\lim_{s\to t-} f(s) = f(t-) \in X, \quad t \in (a, b], \quad \text{and} \quad \lim_{s\to t+} f(s) = f(t+) \in X, \quad t \in [a, b).$$

In this case, we write $f \in G([a, b], X)$ and we endow $G([a, b], X)$ with the usual supremum norm $\|f\|_\infty = \sup_{a\leq t\leq b} \|f(t)\|$. Then $(G([a, b], X), \|\cdot\|_\infty)$ is a Banach space. Also, any function in $G([a, b], X)$ is the uniform limit of step functions (see [2]).

Define

$$G^-([a, b], X) = \{u \in G([a, b], X) : u \text{ is left continuous at every } t \in (a, b]\}.$$

In $G^-([a, b], X)$, we consider the norm induced by $G([a, b], X)$.

Given a function $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, with $r > 0$ and $\sigma > 0$, we consider $y_t : [-r, 0] \to \mathbb{R}^n$ given by

$$y_t(\theta) = y(t + \theta), \qquad \theta \in [-r, 0],\ t \in [t_0, t_0 + \sigma].$$

Then it is clear that for a function $y \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, we have $y_t \in G^-([-r, 0], \mathbb{R}^n)$ for all $t \in [t_0, t_0 + \sigma]$.

We consider the following initial value problem for a retarded functional differential equation (RFDE) with impulses

$$\begin{cases} \dot{y}(t) = f(y_t, t),\ t \neq t_k \\ \Delta y(t_k) = I_k(y(t_k)),\ k = 0, 1, \ldots, m, \\ y_{t_0} = \phi, \end{cases} \tag{1}$$

where $t_k,\ k = 0, 1, \ldots, m$, with $t_0 < t_1 < \ldots < t_k < \ldots < t_m \leq t_0 + \sigma$, $\sigma > 0$, are pre-assigned moments of impulse, for $k = 0, 1, \ldots, m$, $y \mapsto I_k(y)$ maps $\mathbb{R}^n$ into itself and

$$\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k),$$

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that is, y is left continuous at $t = t_k$ and the lateral limit $y(t_k+)$ exists, for $k = 0, 1, 2, \ldots, m$. We also consider $\phi \in G^-([-r, 0], \mathbb{R}^n)$.

It is known that the impulsive system (1) is equivalent to the integral equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)ds + \sum_{t_0 \leq t_k < t} I_k(y(t_k)) \\ y_{t_0} = \phi, \end{cases} \tag{2}$$

whenever the integral exists in some sense. We will consider Lebesgue integration in (2).

Let $PC_1 \subset G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be an open set (in the topology of locally uniform convergence in $G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ with the following property: if y is an element of PC_1 and $\bar{t} \in [t_0 - r, t_0 + \sigma]$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \leq t \leq \bar{t}, \\ y(\bar{t}), & \bar{t} < t \leq \infty, \end{cases} \tag{3}$$

is also an element of PC_1. In particular, any open ball in $G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ has this property.

We assume that $f : (\psi, t) \in G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ and that the mapping $t \mapsto f(y_t, t)$ is Lebesgue integrable. We also assume the following conditions:

(A) There is a Lebesgue integrable function $M : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$,

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \leq \int_{u_1}^{u_2} M(s)\, ds;$$

(B) There is a Lebesgue integrable function $L : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$,

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \leq \int_{u_1}^{u_2} L(s) \|x_s - y_s\|\, ds.$$

For the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots$, we assume the following conditions:

(A') there is a constant $K_1 > 0$ such that for all $k = 0, 1, 2, \ldots$, and all $x \in \mathbb{R}^n$,

$$|I_k(x)| \leq K_1;$$

(B') there is a constant $K_2 > 0$ such that for all $k = 0, 1, 2, \ldots$, and all $x, y \in \mathbb{R}^n$,

$$|I_k(x) - I_k(y)| \leq K_2 |x - y|.$$

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Note that the Carathéodory- and Lipschitz-type conditions (A) and (B) are required for the indefinite integral of f only and not for the function f itself. Thus the standard requirement that $f(\psi, t)$ is continuous in ψ does not need to be fulfilled. Also, the mapping $t \mapsto f(y_t, t)$ does not need to be piecewise continuous.

Let us recall the concept of a solution of problem (1).

Definition 2.1. A function $y \in PC_1$ for which $(y_t, t) \in G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma]$ for $t_0 \leq t < t_0 + \sigma$ and the conditions

(i) $\dot{y}(t) = f(y_t, t)$, almost everywhere (in Lebesgue's sense), whenever $t \neq t_k$.
(ii) $y(t_k+) = y(t_k) + I_k(y(t_k))$, $k = 0, 1, 2, \ldots, m$.
(iii) $y_{t_0} = \phi$

are satisfied is called a *solution of* (1) *in* $[t_0, t_0 + \sigma]$ (or sometimes also in $[t_0 - r, t_0 + \sigma]$) *with initial condition* (ϕ, t_0).

In [1], it was proved that under the conditions (A), (B), (A') and (B'), system (1) can be identified in a one-to-one correspondence with a system of generalized ordinary differential equations taking values in a Banach space. Local existence and uniqueness of solutions are guaranteed by Theorems 2.15, 3.4 and 3.5 from [1].

In what follows, we give a direct proof of an existence and uniqueness theorem for the impulsive RFDE (1) without employing the theory of generalized ODEs. Nevertheless, our proof is inspired in the proof of [1], Theorem 2.15.

Theorem 2.1. *Consider problem (1) and suppose conditions* (A), (B), (A') *and* (B') *are fulfilled. Then there is a* $\Delta > 0$ *such that on the interval* $[t_0, t_0 + \Delta]$ *there exists a unique solution* $y : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ *of problem (1) for which* $y_{t_0} = \phi$.

Proof. For $t \in [t_0, t_0 + \sigma]$, define

$$h_1(t) = \int_{t_0}^{t} [M(s) + L(s)] ds \qquad \text{and} \qquad h_2(t) = \max(K_1, K_2) \sum_{k=0}^{m} H_{t_k}(t),$$

where H_{t_k} denotes the left continuous Heaviside function concentrated at t_k, that is,

$$H_{t_k}(t) = \begin{cases} 0, & \text{for } \ t \leq t_k \\ 1, & \text{for } \ t_k > t. \end{cases}$$

Let $h = h_1 + h_2$. Then h is nondecreasing and left continuous. Besides, for $s \in [t_0, t_0 + \sigma]$, we have

$$\left| \int_{t_0}^{s} f(y_t, t) dt + \sum_{t_0 \leq t_j < s} I_j(y(t_j)) \right| = \left| \int_{t_0}^{s} f(y_t, t) dt + \sum_{j=0}^{m} I_j(y(t_j)) H_{t_j}(s) \right| \leq$$

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$$\leq \int_{t_0}^{s} M(t)dt + K_1 \sum_{j=0}^{m} H_{t_j}(s) \leq h(s) - h(t_0),$$

by conditions (A) and (A').

We will consider two cases: when t_0 is a point of continuity of $h : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ and otherwise.

At first, let t_0 be a point of continuity of h. Assume that $\Delta > 0$ is such that $[t_0, t_0 + \Delta] \subset [t_0, t_0 + \sigma)$ and $h(t_0 + \Delta) - h(t_0) < \frac{1}{2}$.

Let Q be the set of functions $z : [t_0, t_0 + \Delta] \to \mathbb{R}^n$ such that $z \in BV([t_0, t_0 + \Delta], \mathbb{R}^n)$ and $|z(t) - y(t_0)| \leq h(t) - h(t_0)$ for $t \in [t_0, t_0 + \Delta]$. It is easy to show that the set $Q \subset BV([t_0, t_0 + \Delta], \mathbb{R}^n)$ is closed.

For $s \in [t_0, t_0 + \Delta]$ and $z \in Q$, define

$$Tz(s) \;=\; y(t_0) + \int_{t_0}^{s} f(z_t, t)dt + \sum_{t_0 < t_j \leq s} I_j(z(t_j)).$$

Then,

$$|Tz(s) - y(t_0)| = \left| \int_{t_0}^{s} f(z_t, t)dt + \sum_{t_0 \leq t_j < s} I_j(z(t_j)) \right| \leq h(s) - h(t_0), \quad s \in [t_0, t_0 + \Delta].$$

Also, the fact that Tz belongs to $BV([t_0, t_0 + \Delta], \mathbb{R}^n)$ is not difficult to prove. Thus, it follows that T maps Q into itself.

Take $t_0 \leq s_1 < s_2 \leq t_0 + \Delta$ and $z_1, z_2 \in Q$. Using conditions (B) and (B'), we obtain

$$\begin{aligned}
&|Tz_2(s_2) - Tz_1(s_2) - [Tz_2(s_1) - Tz_1(s_1)]| = \\
&= \left| \int_{s_1}^{s_2} [f(z_2, t) - f(z_1, t)]dt + \sum_{s_1 \leq t_j < s_2} [I_j(z_2(t_j)) - I_j(z_1(t_j))] \right| \\
&\leq \int_{s_1}^{s_2} L(t)|z_2(t) - z_1(t)|dt + K_2 \sum_{j=0}^{m} |z_2(t_j) - z_1(t_j)| \left[H_{t_j}(s_2) - H_{t_j}(s_1)\right] \\
&\leq \sup_{s \in [t_0, t_0+\Delta]} |z_2(s) - z_1(s)| \left(\int_{s_1}^{s_2} L(t)dt + K_2 \sum_{j=0}^{m} \left[H_{t_j}(s_2) - H_{t_j}(s_1)\right] \right) \\
&\leq \|z_2 - z_1\|_{BV([t_0, t_0+\Delta])} [h(s_2) - h(s_1)].
\end{aligned}$$

Recall that $\|z\|_{BV([t_0,t_0+\Delta])} = \|z(t_0)\| + var_{t_0}^{t_0+\Delta}(z)$ defines a norm in $BV([t_0, t_0+\Delta], \mathbb{R}^n)$, where $var_{t_0}^{t_0+\Delta}(z)$ denotes the variation of z on the interval $[t_0, t_0 + \Delta]$. Therefore

$$\begin{aligned}
\|Tz_2 - Tz_1\|_{BV([t_0,t_0+\Delta])} &\leq \|z_2 - z_1\|_{BV([t_0,t_0+\Delta])} [h(t_0 + \Delta) - h(t_0)] \\
&< \frac{1}{2} \|z_2 - z_1\|_{BV([t_0,t_0+\Delta])}
\end{aligned}$$

and hence T is a contraction. Then, by the Banach fixed-point theorem, the result follows.

Now, we consider the case where t_0 is not a point of continuity of h (or of h_2). Define

$$\overline{h}_2(t) = \begin{cases} h_2(t), & t = t_0 \\ h_2(t) - h_2(t_0+) + h_2(t_0) = h_2(t) - h_2(t_0+), & t_0 \leq t \leq t_0 + \sigma, \end{cases}$$

and $\overline{h} = h_1 + \overline{h}_2$. Then $\overline{h}$ is continuous at $t = t_0$, left continuous and nondecreasing.

Define an impulse operator $\overline{I}_0$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that, for $y \in G^-([t_0, t_0 + \sigma], \mathbb{R}^n)$,

$$\overline{I}_0(y(t)) = \begin{cases} I_0(y(t)), & t = t_0 \\ I_0(y(t)) - I_0(y(t_0+)) + I_0(y(t_0)), & t_0 \leq t \leq t_0 + \sigma \end{cases}$$

and consider the impulsive RFDE

$$\begin{cases} \dot{z}\,(t) = f\,(z_t, t)\,,\ t \neq t_k \\ \Delta z\,(t_k) = I_k\,(z\,(t_k))\,,\ k = 1, \ldots, m, \\ \Delta z\,(t_0) = \overline{I}_0\,(z\,(t_0))\,, \\ z_{t_0} = \overline{\phi}, \end{cases} \tag{4}$$

where

$$\overline{\phi}(\theta) = \begin{cases} \phi(\theta), & -r \leq \theta < 0 \\ y(t_0+), & \theta = 0. \end{cases}$$

Then, for $s \in [t_0, t_0 + \sigma]$, we have

$$\left| \int_{t_0}^{s} f(z_t, t)dt + \overline{I}_0(z(t_0)) + \sum_{t_0 < t_j < s} I_j(z(t_j)) \right| \leq \overline{h}(s) - \overline{h}(t_0).$$

As in the previous case, let $\Delta > 0$ be such that $[t_0, t_0 + \Delta] \subset [t_0, t_0 + \sigma)$ and $\overline{h}(t_0 + \Delta) - \overline{h}(t_0) < \frac{1}{2}$. Then, define $\overline{Q}$ as the set of functions $z : [t_0, t_0 + \Delta] \to \mathbb{R}^n$ such that $z \in BV([t_0, t_0 + \Delta], \mathbb{R}^n)$ and $|z(t) - y(t_0+)| \leq \overline{h}(t) - \overline{h}(t_0)$ for $t \in [t_0, t_0 + \Delta]$ and consider the operator $\overline{T}$ defined on $\overline{Q}$ and given by

$$Tz(s) = y(t_0+) + \int_{t_0}^{s} f(z_t, t)dt + \overline{I}_0(z(t_0)) + \sum_{t_0 < t_j \leq s} I_j(z(t_j)).$$

Following the procedure of the previous case, it can be shown that equation (4) admits a unique solution on $[t_0, t_0 + \Delta]$. Then, defining $y_{t_0} = \phi$ and $y(t) = z(t)$, for $t > t_0$, we obtain a unique solution of (1), for which $y_{t_0} = \phi$ in $[t_0, t_0 + \Delta]$. ∎

For the next theorem, we consider the following sequence of initial value problems:

$$\begin{cases} \dot{y}\,(t) = f_p\,(y_t, t)\,,\ t \neq t_k \\ \Delta y\,(t_k) = I_k^p\,(y\,(t_k))\,,\ k = 0, 1, \ldots, m, \\ y_{t_0} = \phi_p, \end{cases} \tag{5}$$

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where $t_0 < t_1 < \ldots < t_k < \ldots < t_m \leq t_0 + \sigma$, and for each $p = 1, 2, \ldots,$ $x \mapsto I_k^p(x)$ maps $\mathbb{R}^n$ into itself and $\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k)$, $k = 0, 1, 2, \ldots, m$. We will show that, under conditions (A), (B), (A') and (B') for f_p and I_k^p, $k = 0, 1, 2, \ldots, m$, the sequence $\{y_p\}_{p\geq 1}$ of solutions of (5) is equibounded and of uniformly bounded variation on some closed subinterval of $[t_0, t_0 + \sigma]$.

Theorem 2.2. *Assume that for* $p = 0, 1, \ldots,$ $\phi_p \in G^-([-r, 0], \mathbb{R}^n)$ *and moreover* $f_p : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \mapsto \mathbb{R}^n$ *and* $I_k^p : \mathbb{R}^n \mapsto \mathbb{R}^n$ *satisfy conditions* (A), (B), (A') *and* (B') *for the same functions* M, L *and the same constants* K_1, K_2. *Then there is a* $\Delta > 0$ *such that* $y_p : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ *is a solution of* (5)*, for each* p*, and the sequence* $\{y_p\}_{p\geq 1}$ *is equibounded and of uniformly bounded variation on* $[t_0, t_0 + \Delta]$.

Proof. ¿From the proof of Theorem 2.1, it is clear that a unique $\Delta > 0$ can be obtained such that a solution $y_p : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ of (5) exists and is unique, independently of p. Thus, for $p = 0, 1, \ldots,$ we have

$$y_p(s) = y_p(t_0) + \int_{t_0}^{s} f_p((y_p)_t, t)dt + \sum_{t_0 < t_j \leq s} I_j^p(y_p(t_j)), \quad s \in [t_0, t_0 + \Delta].$$

Consider a partition $t_0 = s_0 < s_1 < \ldots < s_n = t_0 + \Delta$ of $[t_0, t_0 + \Delta]$. Then

$$y_p(s_i) - y_p(s_{i-1}) = \int_{s_{i-1}}^{s_i} f_p((y_p)_t, t)dt + \sum_{s_{i-1} < t_j \leq s_i} I_j(y_p(t_j)), \quad i = 1, 2, \ldots, n.$$

Therefore, conditions (A) and (A') imply

$$\begin{aligned}\sum_{i=1}^{n} |y_p(s_i) - y_p(s_{i-1})| &\leq \sum_{i=1}^{n} \left| \int_{s_{i-1}}^{s_i} f_p((y_p)_t, t)dt \right| + \sum_{t_0 < t_j \leq t_0+\Delta} |I_j(y_p(t_j))| \\ &\leq \int_{t_0}^{t_0+\Delta} M(s)ds + K_1 l,\end{aligned}$$

where l is the number of impulse moments in the interval $[t_0, t_0 + \Delta]$. Then

$$var_{t_0}^{t_0+\Delta}(y_p) \leq \int_{t_0}^{t_0+\Delta} M(s)ds + K_1 l, \quad \text{for all } p \in \mathbb{N}. \tag{6}$$

where $var_{t_0}^{t_0+\Delta}(y_p)$ denotes the variation of $y_p \in [t_0, t_0 + \Delta]$, and hence $\{y_p\}_{p\geq 1}$ is of uniformly bounded variation on $[t_0, t_0 + \Delta]$.

Now, we are going to show that $\{y_p\}_{p\geq 1}$ is equibounded. Again, since

$$y_p(t) = y_p(t_0) + \int_{t_0}^{t} f((y_p)_s, s)ds + \sum_{t_0 < t_j \leq t} I_j(y_p(t_j))$$

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for each $t \in [t_0, t_0 + \Delta]$ and each $p = 1, 2, 3, \ldots$, we have

$$
\begin{aligned}
|y_p(t)| &\leq |y_p(t_0)| + \left| \int_{t_0}^{t} f_p((y_p)_s, s) ds \right| + \left| \sum_{t_0 < t_j \leq t_0 + \Delta} I_j(y_p(t_j)) \right| \\
&\leq |y_p(t_0)| + \int_{t_0}^{t} M(s) ds + K_1 l \\
&\leq |y_p(t_0)| + \int_{t_0}^{t_0+\sigma} M(s) ds + K_1 l
\end{aligned}
$$

Therefore $\{y_p\}_{p \geq 1}$ is equibounded on $[t_0, t_0 + \Delta]$ and the result follows. ▌

3. CONTINUOUS DEPENDENCE FOR IMPULSIVE RFDES

In general, one cannot expect that an impulsive RFDE depends on the initial data. We mention [3] for an elucidative discussion on the continuous dependence of solutions of an impulsive RFDE whose impulse operators also involve delays.

The next theorem is a continuous dependence result which, together with Theorem 2.2, are important to prove the theorem following it. A proof of it can be found in [1], Theorem 4.1.

THEOREM 3.1. *Assume that for* $p = 0, 1, \ldots$, $\phi_p \in G^-([-r, 0], \mathbb{R}^n)$ *and moreover* $f_p : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \mapsto \mathbb{R}^n$ *and* $I_k^p : \mathbb{R}^n \mapsto \mathbb{R}^n$, $k = 0, 1, 2, \ldots$, *satisfy conditions* (A), (B), (A') *and* (B') *for the same functions* M, L *and the same constants* K_1, K_2. *Let the relations*

$$\lim_{p \to \infty} \sup_{\vartheta \in [t_0, t_0 + \sigma]} \left| \int_{t_0}^{\vartheta} [f_p(y_s, s) - f_0(y_s, s)] ds \right| = 0 \tag{7}$$

for every $y \in PC_1$ *and*

$$\lim_{p \to \infty} I_k^p(x) = I_k^0(x) \tag{8}$$

for every $x \in \mathbb{R}^n$, $k = 0, 1, \ldots, m$ *be satisfied. Assume that* $y_p : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, $p = 1, 2, \ldots$, *is a solution on* $[t_0 - r, t_0 + \sigma]$ *of the problem*

$$
\begin{cases}
\dot{y}(t) = f_p(y_t, t), \ t \neq t_k \\
\Delta y(t_k) = I_k^p(y(t_k)), \ k = 1, \ldots, m \\
y_{t_0} = \phi_p,
\end{cases} \tag{9}
$$

such that

$$\lim_{p \to \infty} y_p(s) = y(s) \quad \text{uniformly on } [t_0 - r, t_0 + \sigma]. \tag{10}$$

Then $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *is a solution on* $[t_0 - r, t_0 + \sigma]$ *of the problem*

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), \ t \neq t_k \\ \Delta y(t_k) = I_k^0(y(t_k)), \ k = 1, \ldots, m \\ y_{t_0} = \phi_0. \end{cases} \tag{11}$$

The assumptions (7) and (8) in Theorem 2.2 ensure that if the sequence $\{y_p\}_{p\geq 1}$, $y_p : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$, $p = 1, 2, \ldots$, of solutions of (5) converges uniformly to a function $y : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$, then this limit is a solution of (11).

The next result says that adding an uniqueness condition to the "limiting" equation, then for sufficient large $p \in \mathbb{N}$, $y_p : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ is a solution of (5), provided the sequence of initial data $\{\phi_p\}_{p\geq 1}$ converges uniformly on $[-r, 0]$.

Theorem 3.2. *Assume that* $f_p(\phi, t) : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, $p = 0, 1, 2, \ldots$, *satisfies conditions* (A) *and* (B) *for the same functions* M *and* L. *Let* ${I_k}^p : \mathbb{R}^n \to \mathbb{R}^n$ $k = 1, 2, \ldots$, $p = 0, 1, 2, \ldots$, *be impulse operators which satisfy conditions* (A') *and* (B') *for the same constants* K_1 *and* K_2. *Assume that*

$$\lim_{p\to\infty} \int_{t_0}^{t} [f_p(y_s, s) - f_0(y_s, s)]ds = 0, \quad t \in [t_0, t_0 + \sigma] \tag{12}$$

for every $y \in PC_1$, *and*

$$\lim_{p\to\infty} I_k^p(x) = I_k^0(x) \tag{13}$$

for every $x \in \mathbb{R}^n$ *and* $k = 1, \ldots, m$. *Let* $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *be a solution of*

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), \ t \neq t_k \\ \Delta y(t_k) = I_k^0(y(t_k)), \ k = 1, \ldots, m \\ y_{t_0} = \phi_0, \end{cases} \tag{14}$$

on $[t_0 - r, t_0 + \sigma]$. *Assume that if there exists* $\rho > 0$ *such that* $\sup_{\theta\in[-r,0]} |u(\theta) - \phi_0(\theta)| < \rho$, *then* $u \in G^-([-r, 0], \mathbb{R}^n)$. *Assume further that* $\phi_p \to \phi_0$ *uniformly on* $[-r, 0]$ *as* $p \to \infty$. *Then, for sufficiently large* $p \in \mathbb{N}$, *there exists a solution* y_p *of*

$$\begin{cases} \dot{y}(t) = f_p(y_t, t), \ t \neq t_k \\ \Delta y(t_k) = I_k^p(y(t_k)), \ k = 1, \ldots, m \\ y_{t_0} = \phi_p, \end{cases} \tag{15}$$

on $[t_0 - r, t_0 + \sigma]$ *and*

$$\lim_{p\to\infty} y_p(s) = y(s), \qquad s \in [t_0 - r, t_0 + \sigma] \tag{16}$$

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Proof. The present proof is inspired in the proof of [4], Theorem 8.6, for generalized ODEs. We strongly use the fact that the functions f_p, $p = 0, 1, 2, \ldots$, take values in a finite dimensional space so that we can apply Helly's choice principle.

Because $\phi_p \to \phi_0$ uniformly on $[-r, 0]$ as $p \to \infty$, there exists $\rho > 0$ and $k \in \mathbb{N}$ such that

$$\sup_{\theta \in [-r,0]} |\phi_p(\theta) - \phi_0(\theta)| < \rho, \quad \text{for} \;\; p > k. \tag{17}$$

Thus, by the assumption, $\phi_p \in G^-([-r, 0], \mathbb{R}^n)$, for $p > k$. Then this implies by the existence theorem (Theorem 2.1) that for $p > k$, there exists a $\Delta > 0$ such that y_p is a solution of (15) on $[t_0 - r, t_0 + \Delta]$.

We assert that

$$\lim_{p\to\infty} y_p(s) = y(s), \qquad \text{for } s \in [t_0 - r, t_0 + \sigma]. \tag{18}$$

By Theorem 3.1, if the sequence $\{y_p\}_{p\geq 1}$ admits a convergent subsequence, then since $\phi_p \to \phi_0$, it will follow by the uniqueness of solutions that there is a $\Delta > 0$ such that $\lim_{p\to\infty} y_p(s) = y(s)$, for $s \in [t_0 - r, t_0 + \Delta]$, where $y_{t_0} = \phi_0$.

We will use Helly's choice principle to prove that in fact $\{y_p\}_{p\geq 1}$ admits a convergent subsequence.

By Theorem 2.2 the sequence $\{y_p\}$, $p > k$, of functions on $[t_0, t_0 + \Delta]$ is equibounded and of uniformly bounded variation. Thus, by the Helly's choice principle, the sequence $\{y_p\}$, $p > k$, contains a pointwise convergent subsequence and hence $y(t)$ is the only accumulation point of the sequence $y_k(t)$ for every $t \in [t_0, t_0 + \Delta]$. Therefore the theorem holds on $[t_0, t_0 + \Delta]$, $\Delta > 0$. It also holds on $[t_0 - r, t_0]$, since $\phi_p \to \phi_0$.

Now, let us assume that the convergence result does not hold on the whole interval $[t_0 - r, t_0 + \sigma]$. Thus there exist a Δ', $0 < \Delta' < \sigma$, such that for every $\Delta < \Delta'$ and for $p > k$, there is a solution y_p of (15) on $[t_0 - r, t_0 + \Delta]$, with $(y_p)_{t_0} = \phi_0$, and $\lim_{p\to\infty} y_p(t) = y(t)$ for $t \in [t_0 - r, t_0 + \Delta]$, but this does not hold on $[t_0 - r, t_0 + \Delta]$, whenever $\Delta > \Delta'$.

By the proof of Theorem 2.1, $|y_k(s_2) - y_k(s_1)| < |h(s_2) - h(s_1)|$ for every $s_2, s_1 \in [t_0 - r, t_0 + \Delta')$ and every $p > k$. Hence the limits

$$y_p((t_0 + \Delta')-) = \lim_{\varepsilon\to 0^-} y_p(t_0 + \Delta' + \varepsilon), \quad p > k,$$

exist and $y_p((t_0 + \Delta')-) = y(t_0 + \Delta')$, for $p > k$, since y is left continuous.

Defining $y_p(t_0 + \Delta') = y_p((t_0 + \Delta')-)$, for $p > k$, then $\lim_{p\to\infty} y_p(t_0 + \Delta') = y(t_0 + \Delta')$. Therefore the theorem holds on $[t_0 - r, t_0 + \Delta']$ as well. Then, using $t_0 + \Delta' < t_0 + \sigma$ as the starting point, it can be proved analogously that the theorem holds on the interval $[t_0 + \Delta', t_0 + \Delta' + \eta]$, for some $\eta > 0$, and this contradicts our assumption. Thus the theorem holds on the whole interval $[t_0 - r, t_0 + \sigma]$. ▌

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