CADERNOS DE MATEMÁTICA **10**, 99–104 May (2009)
ARTIGO NÚMERO SMA#308

Stable types and the minimal Whitney stratification of discriminants

Grazielle F. Barbosa

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: gbarbosa@icmc.usp.br

Nivaldo G. Grulha Jr

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: njunior@icmc.usp.br

Marcelo J. Saia

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: mjsaia@icmc.usp.br

In this article we give relations between the minimal Whitney stratification, as described by Teissier and Lê and the stratification given by stable types for discriminants of finitely determined map germs. May, 2009 ICMC-USP

Mathematics Subject Classification: 14B05, 32B10, 32S60.

1. INTRODUCTION

The minimal Whitney stratification for any reduced equidimensional complex analytic space V is a consequence of Teissier's work ([8]) and was defined in [6] in terms of the invariance of a set of multiplicities. The first multiplicity considered in this definition is the multiplicity of the space V at the point and in the sequel one considers the invariance of the local polar multiplicities.

On the other side, it is well known in Singularity Theory the Thom-Boardman stratification by stable types of the discriminant of any finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. This stratification is done as a consequence of the Geometric Criterion for finite determinacy given by Mather-Gaffney see [1], where it is shown that out of the origin, any

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

point of the critical set is stable. Therefore the strata of the stratification by the stable types is obtained by the type of singularity which appears in the discriminant set, out of the origin.

The first work that relates polar multiplicities and the stable types in the discriminant is [2], we see also such relations in [4], or in [5].

In this article we study when this stratification of the discriminant, which is Whitney, is minimal. We show that this is not always true, and the answer depends of the pair of dimensions (n, p).

2. THE MINIMAL WHITNEY STRATIFICATION

We follow [6] and [8] and consider $(V, 0)$ the germ of a reduced equidimensional analytic complex space of complex dimension d. We suppose that $(V, 0)$ is embedded in $(\mathbb{C}^{N+1}, 0)$. Let us take V a sufficiently small representative of the germ.

The procedure to construct the minimal Whitney stratification that we describe below, is given in [6] in terms of the multiplicities of the absolute local polar varieties at each point.

Let $\nu : \widetilde{V} \to V$ be the proper analytic Nash modification of V, here $\widetilde{V} \subset V \times G$, where G is the Grassmannian manifold of d planes in $\mathbb{C}^{N+1}$, for details see [6] and [8]. Let $\mathfrak{D} = \{D_d \subset D_{d-1} \subset \ldots \subset D_0 \subset \mathbb{C}^{N+1}\}$ be one flag, we associate $\mathfrak{D}$ with the Schubert variety $c_k(\mathfrak{D}) = \{E \in G / \dim E \cap D_{d-k} \geq k\}$.

The morphism $\gamma : \widetilde{V} \to G$ induced by the second projection $V \times G \to G$ allow us to define the absolute polar varieties $P_k(\mathfrak{D}) = \nu(\gamma^{-1}(c_k(\mathfrak{D})))$. If $\mathfrak{D}$ is generic we just denote the polar variety by $P_k(V, y)$.

For any germ $(Z, 0)$ of a reduced equidimensional analytic complex space of complex dimension d, denote by $\Gamma_x(Z)$ the sequence of multiplicities of generic polar varieties of Z in x.

In [6] it is defined a decreasing sequence of algebraic subvarieties of V

$$F_0 \supset F_1 \supset \ldots \supset F_k \supset \ldots$$

where $F_0 = V$, $F_1 = \Sigma(V)$, the singular set of V. To obtain the other F_j, let us denote $(F_{1,j_1})_{j_1 \in J_1}$ the irreducible components of F_1, then F_2 is the union of the singular set of F_1 with a finite number of closed algebraic sets given by the points $x \in (F_{1,j})$ such that the sequence $\Gamma_x(V)$ or $\Gamma_x(F_{1,j_1})$, $j_1 \in J_1$, is different of one computed on a generic point. Let us denote $(F_{2,j_2})_{j_2 \in J_2}$, the irreducible components of F_2. In general, let F_k be the union of the singular set of F_{k-1} with a finite number of closed algebraic sets given by the points x such that the sequence of multiplicities $\Gamma_x(V)$, $\Gamma_x(F_{1,j_1})$ $(j_1 \in J_1)$, …, $\Gamma_x(F_{k-1,j_{k-1}})$, $(j_{k-1} \in J_{k-1})$, does not coincide with the sequence on a generic point of $F_{k-1,j_{k-1}}$.

In [8] Teissier proved the following result:

THEOREM 2.1. *Let V be a reduced equidimensional analytic space of dimension d and Y a non-singular analytic subspace of V such that $0 \in Y$, then the following conditions are equivalent:*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

(1) *The multiplicity of the local polar varieties* $P_k(V,y)$ *of the germ* (V,y) *for* $y \in Y$ *is locally constant on* Y *in a neighborhood of the origin, for* $0 \leq k \leq d-1$*;*

(2) *The pair* (V_{reg}, Y) *satisfy the Whitney conditions* (a) *and* (b) *at the origin, and* V_{reg} *denotes the regular part of* V*.*

The next corollary was introduced in [6].

Corollary 2.2. *Let* V *be a sufficiently small representative of an analytic germ as above, the stratification of* V *such that the strata are given by* $F_{i,j} \setminus \cup_{k>i} F_{k,l}$ *satisfies the Whitney conditions and it is called canonical or minimal stratification.*

3. THE STRATIFICATION BY THE STABLE TYPES

Here we follow Gaffney in [2] and denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$. A map germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable in a finite set S if, by composition with families of holomorphic diffeomorphisms in source and target, every deformation is $\mathcal{A}$-trivial, where $\mathcal{A}$ denotes the usual Mather group of germs of holomorphic diffeomorphisms in the source and in the target. We call stable type, an equivalence class of stable map germs.

A germ is k-$\mathcal{A}$-determined if any $g \in \mathcal{O}(n,p)$ with the same k-jet as f, i.e. $j^k g = j^k f$, is $\mathcal{A}$-equivalent to f. The germ f is said to be finitely $\mathcal{A}$-determined if it is k-$\mathcal{A}$-determined for some k.

Mather and Gaffney showed the characterization of finitely determined map germs in terms of stable germs. See [1] for a proof.

Proposition 3.3. *Suppose* $f \in \mathcal{O}(n,p)$*. Then* f *is finitely determined if and only if for each representative* $\tilde{f}$ *of* f*, there exist neighborhoods of the origin* $U \subset \mathbb{C}^n$*,* $V \subset \mathbb{C}^p$ *such that* $\tilde{f}^{-1}(0) \cap U \cap \Sigma(\tilde{f}) = 0$ *and for each* $y \in V$*,* $y \neq 0$*, the germ* $\tilde{f}_y : (\mathbb{C}^n, S_y) \to (\mathbb{C}^p, y)$ *is stable, where* $S_y = \tilde{f}^{-1}(y) \cap U \cap \Sigma(\tilde{f})$ *and* $\Sigma(\tilde{f})$ *denotes the critical set of* $\tilde{f}$*.*

Remark 3.4. (1) The critical set $\Sigma(f)$ of any map germ f is defined as the set of points $x \in \mathbb{C}^n$ such that the differential of f at x is not a submersion.

(2) When $n < p$, we have $\Sigma(f) = \mathbb{C}^n$.

(3) We shall use the same symbol Σ to denote different objects, the singular set of any complex analytic space V is denoted by $\Sigma(V)$ and also the critical set $\Sigma(f)$ of a map germ f, however this does not cause any confusion.

The stable types which appear in the critical set of any finitely determined germ f are given in any versal unfolding of f. We say that f has discrete stable type if there exist a versal unfolding of f in which only a finite number of stable types occur. If the numbers (n,p) are in Mather's "nice dimensions" ([7]) or on the boundary thereof, then every finitely determined germ $f \in \mathcal{O}(n,p)$ has discrete stable type.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

The main point now is to describe for each of such pairs (n, p) all stable types which appear in any finitely determined map germ.

In general, for any pair of dimensions (n, p) the description of the stable types is done in terms of sub-schemes of multiple points sets of a germ f, as we can see in [4] for the case $n = p$ or in [5] for the case $n < p$. This description leads to the Thom-Boardman stratification by the stable types.

We shall describe directly in each case considered in this work how to obtain the Thom-Boardman stratification by the stable types, see Theorems 4.5, 4.6 and 4.7 below.

First we decompose the critical set $\Sigma(f)$ by the mono-germs and then we obtain a first stratification in the discriminant by the images of the strata of the critical set. We remark that Gibson shows in [3], a description of the singularities which appear as stable mono-germs.

Now, to obtain the final stratification in the discriminant $\Delta(f) = f(\Sigma(f))$ we need to consider the images of these strata and refine this stratification by the strata composed by the stable multi-germs.

For any Boardman symbol $i = (i_1, \ldots, i_r)$, we denote by $\Sigma^i(f)$ the set of points in $\Sigma(f)$ of type i. Then in $\Sigma(f)$ we first consider the stratification done by the smooth parts of the sets $\Sigma^i(f)$ for all Boardman symbol i which $\Sigma^i(f) \neq \emptyset$.

Now, to obtain the stratification in the discriminant $\Delta(f) = f(\Sigma(f))$ we consider the images of these strata and refine this stratification by the strata composed by the stable multi-germs.

We shall denote by r-stable type any stable class of singularities which is r-dimensional in the discriminant of f. We can have $0 \leq r \leq d - 1$ if the discriminant is d-dimensional.

4. RESULTS

In this section we prove the main results of this paper.

THEOREM 4.5. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ. Let us consider V as the discriminant of f, then the minimal Whitney stratification of V coincides with the stratification of V given by stable types.*

Proof. In this case we have in the discriminant of f, which is two dimensional, only one type of 0-stable mono-germ, the cross-caps: images of points of type $\Sigma^1(f)$, one 0-stable multi-germ, the triple points: given as the transversal crossing of the images of three points of type $\Sigma^0(f)$, and one 1-stable bi-germ, the curves of double points given as the transversal crossing of the images of two points of type $\Sigma^0(f)$, here the stratification by the stable types of the discriminant is the minimal Whitney stratification because the multiplicities of the 0-stable singularities in the discriminant are different, here the multiplicity of the cross-cap is two while the multiplicity of the triple point is three, so they belong to "different strata" in the minimal stratification and are also in different strata in the stratification by stable types. As there is only one type of 1-stable singularity in the singular set of $\Delta(f)$, this is the 1-dimensional stratum in both cases.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

We remark here that it is needed a special care when the germ is not stable at the origin, in this case this point is an isolated stratum in both situations. ▮

THEOREM 4.6. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ. Let us consider V as the discriminant of f, then the minimal Whitney stratification of V coincides with the stratification of V given by stable types.*

Proof. Whitney showed in [10] and [9] that the discriminant of f, which is n-dimensional has only one type of 0-stable mono-germ, the generalized cross-caps: which is the image of one point of type $\Sigma^1(f)$, one 0-stable multi-germ, the points given as the transversal crossing of the images of $2n-1$ points of type $\Sigma^0(f)$, and for each r with $1 \le r \le n-1$ one type of r-stable multi-germ, the r-dimensional sets of points given as the transversal crossing of the images of $(2n-1)-r$ points of type $\Sigma^0(f)$.

Again here we have the stratification by stable types of the discriminant as the minimal Whitney stratification because the multiplicities of the 0-stable singularities in the discriminant are different, the multiplicity of the cross-cap is two while the multiplicity of the $(2n-1)$-uple point is $2n-1$, so they belong to "different strata" in Teissier's stratification and are also in different strata in the stratification by stable types. As there is only one type of r-stable singularity in the singular set of $\Delta(f)$ for all r, they form the r-dimensional strata of the discriminant.

Here the stratification by stable types of the discriminant is the minimal Whitney stratification because all the multiplicities of the 0-stable singularities considered by Teissier coincide and there is only one type of 1-stable singularity.

We remark here that when the germ is not stable at the origin, this point is an isolated stratum in both situations. ▮

The cases described above are quite special, since in general this stratifications does not coincide, as we shall show next.

THEOREM 4.7. *Let $f : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$, $n \ge 0$, be a finitely determined corank 1 map germ. Let us consider V as the discriminant of f, then the stratification of V given by stable types is finer then the minimal Whitney stratification of V.*

Proof. In this case the stable singularities that appear in the discriminant are suspensions of the stable singularities of corank 1 map germs from $(\mathbb{C}^p, 0)$ to $(\mathbb{C}^p, 0)$. So we only need to describe these, since the multiplicities of the suspensions are equal. We follow Arnold's notation and say that a singularity is of type A_k if it is the image of a point of type $\Sigma^i(f)$, with $i = 1, \ldots, 1$ with the number 1 appearing k times. The stable multiple points sets in the discriminant appear as normal crossings of these germs.

We shall show that in this case, there are at least two different stable singularities in the discriminant, so in different strata of the stratification by the stable types, but the associated sequence of multiplicities coincide, therefore the minimal stratification does not distinguish these strata.

We show it for the map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$ and for the other cases the strata appear in an analogous way. Here, as the discriminant is 2-dimensional, the possible 0-stable singularities are: swallowtails, the A_3 singularities, normal crossings between cuspidal edges and planes, the $A_{2,1} = A_2 \cap A_1$ singularities, and the triple points $A_{1,1,1} = A_1 \cap A_1 \cap A_1$. The 1-stable singularities which can appear are: the cuspidal edge, or points A_2 and the double points curves $A_{1,1} = A_1 \cap A_1$.

In this case, the double points curves and the cuspidal edges belong to different strata in the stratification by stable types, but all multiplicities considered in the minimal stratification are equal for all points in these curves, so they are in the same strata. In both cases, the multiplicity of $\Delta(f)$ in any point of these strata is two and the local polar curve of $\Delta(f)$ at any point of these strata is empty, therefore the polar multiplicity at these points is 0.

In the general case of corank 1 map germs from $\mathbb{C}^p$ to $\mathbb{C}^p$, there are at least two $(p-2)$-dimensional strata which are in different strata in the stratification by stable types, but are in the same strata in the minimal Whitney stratification. ∎

REFERENCES

1. T. Gaffney, *Properties of finitely determined germs*, Ph.D. Thesis, Brandeis University, 1975.
2. T. Gaffney, Polar multiplicities and equisingularity of map germs, *Topology*, **32** (1993), n.1, 185–223.
3. C. G. Gibson, *Singular points of smooth mappings*, Research Notes in Mathematics, **25**, Pitman, Boston, Mass. London, 1979.
4. V. H. Jorge-Pérez, D. Levcovitz and M. J. Saia, Invariants, equisingularity and Euler obstruction of map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$, *Journal für Die Reine und Angewandte Mathematik*, **587** (2005), 145–167.
5. V. H. Jorge-Pérez and M. J. Saia, Euler Obstruction, polar multiplicities and equisingularity of map germs in $O(n,p)$, $n < p$, *International Journal of Mathematics*, **17** (2006), 1–17.
6. D. T. Lê and B. Teissier, Variétés polaires locales et classes de Chern de variétés singulères, *Annals of Math.*, **114** (1981), 457–491.
7. J. N. Mather, *Stability of $\mathbb{C}^\infty$ mappings. VI: The nice dimensions*, Proceedings of Liverpool Singularities-Symposium, I (1969/70), Lecture Notes in Math., **192**, Springer, Berlim, 1971, 207–253.
8. B. Teissier, Variétés polaires locales et conditions de Whitney, *Note aux C. R. Acad. Sc.*, **290** (1980), 799–802.
9. H. Whitney, The singularities of a smooth n-manifold in $(2n-1)$-space, *Ann. of Math.*, **45**(2) (1944), 247–293.
10. H. Whitney, he general type of singularity of a set of $2n-1$ smooth functions of n variables, *Duke Math. J.*, **10** (1943), 161–172.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC