CADERNOS DE MATEMÁTICA **10**, 133–143 May (2009)
ARTIGO NÚMERO SMA# 313

Homoclinic tangency and variation of entropy

M. Bronzi*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: bronzi@icmc.usp.br

A. Tahzibi†

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: tahzibi@icmc.usp.br

In this paper we study the effect of a homoclinic tangency in the variation of the topological entropy. We prove that a diffeomorphism with a homoclinic tangency associated to a basic hyperbolic set with maximal entropy is a point of entropy variation in the C^∞-topology. We also discuss variational problem in C^1-topology. May, 2009 ICMC-USP

1. INTRODUCTION

Topological entropy is one of the most important invariants of topological conjugacy in dynamical systems. By the $\Omega-$stability of Axiom A diffeomorphisms with no cycle condition, it comes out that the entropy is a C^1-locally constant function among such dynamics. We say that a diffeomorphism f is a point of constancy of topological entropy in C^k topology if there exists a C^k-neighborhood $\mathcal{U}$ of f such that for any diffeomorphism $g \in \mathcal{U}, h(g) = h(f)$. We also call a diffeomorphism as a point of variation of entropy if it is not a point of constancy.

In [11], Pujals and Sambarino proved that surface diffeomorphisms far from homoclinic tangency are the constancy points of topological entropy in C^∞ topology. In this paper we address the reciprocal problem. That is we are interested in the effect of a homoclinic tangency to the variation of the topological entropy for a surface diffeomorphism. Of course after unfolding a homoclinic tangency, new periodic points will emerge, but it is not

* M. Bronzi thanks the support of FAPESP-Brazil with a doctoral fellowship.
† This work was mostly done at the Instituto de Ciências de Matemática e de Computação (ICMC) and finished when A. Tahzibi was on leave from ICMC-USP for a post-doctoral at IMERL-Uruguay supported by CNPq-Brazil.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

clear whether they contribute to the variation of the topological entropy. We mention that Diaz-Rios [3] studied unfolding of critical saddle-node horseshoes and when the saddle-node horseshoe is not an attractor they proved that the entropy may decrease after the bifurcation. In our context, the tangency occurs outside a basic hyperbolic set.

For Axiom A diffeomorphisms, by the spectral decomposition theorem of Smale (see *e.g.* [13])), we have $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$, where each Λ_i is a basic set, i.e, an isolated f-invariant hyperbolic set with a dense orbit. By the definition of topological entropy we have

$$h(f) = \max_{0 \leq i \leq k} h(f|_{\Lambda_i}).$$

So, we conclude that there exists (at least) a set which is *responsible* for the topological entropy of an Axiom A, i.e, there exists some $k_0 \in \{1, \ldots, k\}$ such that $h(f) = h(f|_{\Lambda_{k_0}})$.

We consider a class of diffeomorphisms on the frontier of Axiom A systems which exhibit homoclinic tangency corresponding to a periodic point of Λ_k. We show that the topological entropy increases after small C^∞ perturbations.

More precisely, consider a parametrized family $f_\mu : M \to M$ of diffeomorphisms of a closed surface M unfolding generically a homoclinic tangency at $\mu = 0$ where $\Omega(f_0) = \Lambda_1 \cup \cdots \Lambda_k \cup \mathcal{O}(q)$ where each Λ_i is an isolated hyperbolic set and q is a homoclinic tangency associated to a saddle fixed point p of some Λ_i.

Theorem 1.1. *Let f_μ be a one parameter family of C^2 surface diffeomorphism as above, then*

(1) *If Λ_i is responsible for the entropy then, f_0 is a variation point of the topological entropy in C^∞-topology*

(2) *There exist examples where Λ_i is not responsible for entropy and f_0 is a variation point in C^1-topology.*

Observe that in the first item (which is the main part) of the above theorem we claim the variation of entropy in C^∞ topology. In the second item we give an example which shows that even if the tangency corresponds to a basic set which is not responsible for the entropy, we can increase the entropy after C^1-perturbations. This example is easily made of a horseshoe and a homoclinic tangency corresponding to a hyperbolic fixed point outside the horseshoe.

We recall also a method for perturbation of dynamics with homoclinical tangency, due to Newhouse, which is so called the "Snake like" perturbation. Although after such perturbation the non wandering set becomes richer, the topological entropy does not necessarily increase. See Theorem 1.2 for the relation between an estimate of entropy after the perturbation and the eigenvalues of the periodic point corresponding to the homoclinic tangency.

Theorem 1.2 ([9]). *Let p be a (conservative) hyperbolic periodic point of a C^1-diffeomorphism f, such that $W^u(\mathcal{O}(p))$ is tangent to $W^s(\mathcal{O}(p))$ in some point. Given $\varepsilon > 0$,*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

for all neighborhood $\mathcal{N}$ of f there exists $g \in \mathcal{N}$ such that

$$h(g) > \frac{1}{\tau(p)} \log|\lambda(p)| - \varepsilon.$$

As a corollary of continuity of topological entropy for the surface C^∞ diffeomorphisms we prove that:

THEOREM 1.3. *It is not possible to substitute C^∞ instead of C^1 in the above theorem.*

Finally, let us also mention a result of Hua, Saghin and Xia [6] where they prove that the topological entropy is locally constant for some partially hyperbolic diffeomorphisms with one dimensional central bundle.

2. MAIN INGREDIENTS

2.1. Topological Entropy

Consider $\Sigma_N = \{1, \ldots, N\}^{\mathbb{Z}}$ and the *shift* $\sigma : \Sigma_N \to \Sigma_N$ given by $\sigma(\mathbf{x}) = \mathbf{y}$ where $y_i = x_{i+1}$, $i \in \mathbb{Z}$. For $A = (a_{ij})_{i,j=1}^N$ a square $0-1$-matrix of order N, the correspondent *subshift of finite type* is the restriction of σ to $\Sigma_A = \{\mathbf{x} \in \Sigma_N \mid a_{x_i x_{i+1}} = 1 \quad \text{for} \quad i \in \mathbb{Z}\}$.

The following proposition is a well known result that can be found for instance in [12].

PROPOSITION 2.1 ([12]). *If $\sigma|_A : \Sigma_A \to \Sigma_A$ is a subshift of finite type, then*

$$h(\sigma_A) = \log(\lambda_{max}),$$

where λ_{max} is the largest eigenvalue of A in modulus.

The main properties of the topological entropy that we use in this text can be found in [5].

2.2. Ω-Homoclinic Explosions and Markov Partitions

The celebrated result of Bowen [2] on the construct of Markov partition for hyperbolic systems is

THEOREM 2.1 ([1]). *Let Λ be a hyperbolic invariant set with a local product structure for a diffeomorphism f. Then, there exists a Markov partition of Λ for f with rectangles of arbitrarily small diameter.*

In particular, for basic set from the spectral decomposition of a Axiom A diffeomorphism there are Markov partitions with arbitrarily small rectangles.

An important consequence of this Bowen's result is the existence of a topological conjugation between the system $f|_\Lambda$ and a subshift of finite type $\sigma|_{\Sigma_A}$.

To prove our main theorem, we focus on $\Omega-$explosion like in the model in Palis-Takens result [10], of course, without any hypothesis on the fractal dimensions.

That is, we are considering a one parameter family f_μ where for the parameter $\mu = 0$ the nonwandering set $\Omega(f_0) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \tilde{\Lambda}_k$ such that $\Lambda_i, i < k$ is a hyperbolic basic and $\tilde{\Lambda}_k = \Lambda_k \cup \mathcal{O}(q)$ where Λ_k is a basic set and $\mathcal{O}(q)$ is the orbit of a homoclinic tangency associated with a saddle fixed point $p \in \Lambda_k$.

For $\mu > 0$ we can consider the basic sets $\Lambda_i(\mu)$ as the continuation of Λ_i. Thereby, we have that $\Lambda_i(\mu)$ is hyperbolic and $f_\mu|_{\Lambda_i(\mu)}$ is conjugated to $f_0|_{\Lambda_i}$. Then, we have

$$h(f_\mu|_{\Lambda_i(\mu)}) = h(f_0|_{\Lambda_i})$$

for all $i = 1, \ldots, k$ and all μ positive or negative.

However, when we unfold the family f_μ new periodic points are created and the entropy of the nonwandering sets may increase for positive parameters μ. We will see that, in fact, the entropy increases for small positive parameters. This can be shown by constructing a subsystem of f_μ which is not topologically conjugated to $f_0|_{\Lambda_k}$. These facts will be important to proof the main theorem.

To construct such a subsystem, we find a subset of $\Omega(f_\mu)$ containing $\Lambda_k(\mu)$ using Markov partitions. Take a parameter μ very close to $\mu = 0$. Since f_μ unfolds generically, the map f_μ has transversal homoclinic intersections close to $\mathcal{O}(q_0)$, the tangency orbit of f_0. We have the situation represented below in the figure 1.

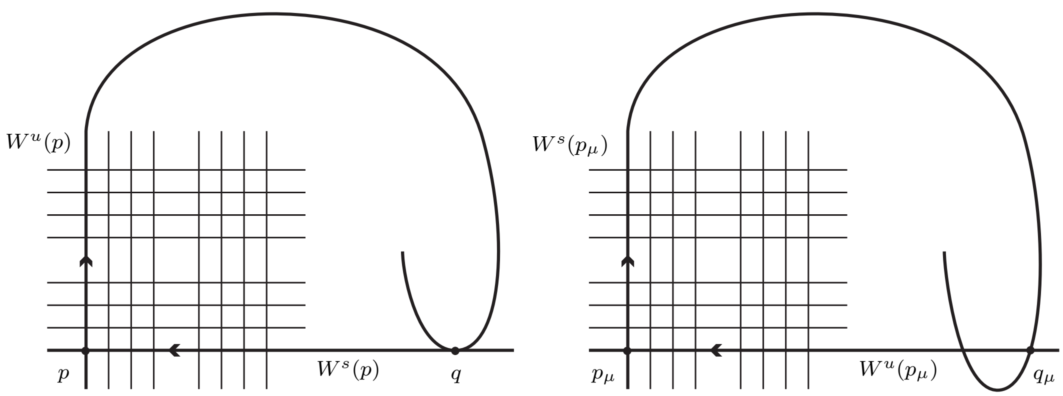

FIG. 1. Unfolding of a homoclinic tangency close to $\mu = 0$.

Consider q_μ a transversal homoclinic intersection point between $W^s(p_\mu)$ and $W^u(p_\mu)$ close to q_0 (the tangency for f_0). Since $\Lambda_k(\mu)$ is hyperbolic and maximal invariant set for f_μ, there exist a isolating neighborhood of $\Lambda_k(\mu)$, say V_k. Suppose that $q_\mu \notin V_k$. Moreover, we can use the Bowen's construction of Markov partition [2]. Consider $\{R_1, \ldots, R_s\}$ a

Markov partition for $\Lambda_k(\mu)$ such that

$$\Lambda_k(\mu) = \bigcup_{j=1}^{s} R_j \subset V_k.$$

Furthermore, as $q_\mu \notin V_k$ we have that a part of $\mathcal{O}(q_\mu)$ remains out of V_k. Take $N_1, N_2 \in \mathbb{N}$ such that $f_\mu^{N_1}(q_\mu) \in R_s$, $f_\mu^{-N_2}(q_\mu) \in R_1$ and $f_\mu^j(q_\mu) \notin \bigcup_{j=1}^s R_j$ for $j = -N_2 + 1, \ldots, 0, \ldots, N_1 - 1$. In other words, R_s is the rectangle containing the first forward iterated of q_μ that belongs to V_k, and R_1 is the rectangle containing the first backward iterated of q_μ that belongs to V_k.

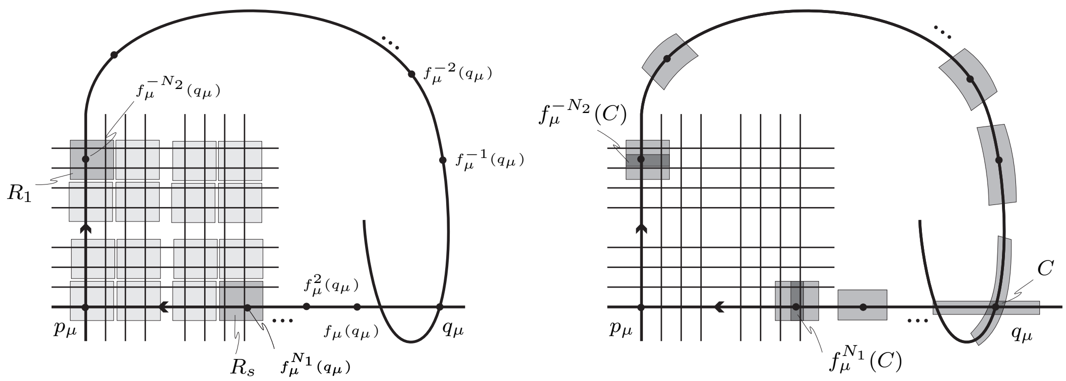

FIG. 2. Construction of the Markov partition.

Given the Markov partition for $\Lambda_k(\mu)$, we extend it for a larger set which contains $\Lambda_k(\mu) \cup \mathcal{O}(q_\mu)$ by constructing other rectangles containing $\{f_\mu^{-N_2+1}(q_\mu), \ldots, q_\mu, \ldots, f_\mu^{N_1+1}(q_\mu)\}$ in the following way: if we iterate R_1 under f^{N_2}, we get a narrow strip around $W^u(p_\mu)$ containing q_μ. And if we iterate R_s under $f_\mu^{-N_1}$, we get a narrow strip around $W^s(p_\mu)$ containing q_μ. We know that $W^s(p_\mu)$ and $W^u(p_\mu)$ have transversal intersection on q_μ. As we could take the diameter of the partition small enough, it comes out that $f_\mu^{-N_1}(R_s)$ and $f_\mu^{N_2}(R_1)$ are transversal. Let $C := f_\mu^{-N_1}(R_s) \cap f_\mu^{N_2}(R_1)$. It is clear that C is disjoint from $\bigcup_{i=1}^s R_i$ and contains q_μ.

Note that $f_\mu^{N_1}(C)$ is a vertical strip of full height contained in R_s and $f_\mu^{-N_2}(C)$ is a horizontal strip of full weight contained in R_1. Consider the disjoint sets S_i defined as

$$S_j = f_\mu^{-N_2+j}(C)$$

for $j = 1, 2, \ldots, N_2, N_2+1, \ldots, N_1+N_2-1$. Note that $S_{N_2} = C$. Now denote $\ell = N_1+N_2-1$ and consider $\mathcal{P} = \{R_1, \ldots, R_s, S_1, \ldots, S_\ell\}$ and

$$R = \bigcup_{i=1}^{s} R_i \cup \bigcup_{j=1}^{\ell} S_j.$$

So $\Lambda_R = \bigcap_{n\in\mathbb{Z}} f_\mu^n(R)$ is an isolated hyperbolic set such that $\Lambda_k(\mu) \subset \Lambda_R \subset \Omega(f_\mu)$. The desired subsystem is the restriction $f_\mu : \Lambda_R \to \Lambda_R$.

Lemma 2.1. *$\mathcal{P}$ is Markov partition for Λ_R.*

Proof. We already know that all R_i's satisfy the definition of Markov partition. It remains to verify the Markov property for S_j's. By construction we have all R_i's and S_j's pairwise disjoint. Furthermore $f_\mu(S_j) = S_{j+1}$ for $j = 1, \ldots, \ell-1$. In particular, we have $f_\mu(S_\ell) \subset R_s$ is a vertical strip of full height. So

$$\begin{aligned} &f_\mu(S_\ell) \cap R_s \neq \varnothing, \\ &f_\mu(S_\ell) \cap R_i = \varnothing, \text{ for } i = 1, \ldots, s \text{ and} \\ &f_\mu(S_\ell) \cap S_j = \varnothing, \text{ for } j = 1, \ldots, \ell. \end{aligned}$$

On the other hand, only R_1 has image by f_μ that intersects some S_j. In fact,

$$f_\mu(R_1) \cap S_1 \neq \varnothing \text{ and } f_\mu(R_1) \cap S_j = \varnothing, \text{ for } j = 2, \ldots, \ell.$$

Note that since $f_\mu^{-1}(S_1) \subset R_1$ is a horizontal strip of full weight in R_1, then $S_1 = f_\mu(f_\mu^{-1}(S_1)) \subset f_\mu(R_1)$. So, $f_\mu(R_1) \cap S_1 \neq \varnothing$ and, by the construction of S_1, this intersection satisfies the transversality condition of Markov partitions. Thus $\mathcal{P}$ is a Markov partition for Λ_R. ∎

We can associate the system $f_\mu : \Lambda_R \to \Lambda_R$ to a subshift of finite type as follows. We consider the Markov partition $\mathcal{P} = \{P_1, \ldots, P_{s+\ell}\}$ as above and we define a transition matrix $A_\mu = (a_{ij})_{(s+\ell)\times(s+\ell)}$ for f_μ taking

$$a_{ij} = \begin{cases} 1, & \text{if } f_\mu(P_i) \cap P_j \neq \varnothing; \\ 0, & \text{if } f_\mu(P_i) \cap P_j = \varnothing \end{cases}$$

for $i, j \in \{1, \ldots, s+\ell\}$. In this way we obtain a topological conjugacy between the systems $f_\mu : \Lambda_R \to \Lambda_R$ and the subshift of the finite type $\sigma_{A_\mu} : \Sigma_{A_\mu} \to \Sigma_{A_\mu}$, where $\Sigma_{A_\mu} \subset \Sigma_{s+\ell}$. The transition matrix A_μ has the following form

$$a_{ij} = \begin{cases} H_{ij}, & \text{if } 1 \leq i, j \leq s; \\ 1, & \text{if } i = 1, j = s+1 \text{ or } i = s+\ell, j = s; \\ 1, & \text{if } j = i+1 \text{ for } s+1 \leq i \leq s+\ell-1; \\ 0, & \text{in other cases.} \end{cases}$$

where $H_\mu = (H_{ij})_{s\times s}$ is the transition matrix of $f_\mu : \Lambda_k(\mu) \to \Lambda_k(\mu)$ which is irreducible, because $f_\mu|_{\Lambda_k(\mu)}$ is topologically transitive (see (3.1) in the next section).

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

3. PROOF OF THEOREMS ?? AND ??

3.1. First statement of Theorem 1.1

Let f_μ be the one parameter family as in the Theorem 1.1. In the previous section, we constructed a Markov partition for the subsystem $f_\mu|_{\Lambda_R}$, for $\mu \geq 0$. By means of this Markov partition, one may give a conjugacy between such invariant subsystem $f_\mu|_{\Lambda_k(\mu)}$ and the dynamic of a subshift of finite type.

Let A_μ be the transition matrix of $f_\mu|_{\Lambda_R}$, for $\mu > 0$ small enough. Recall that $h(f_\mu) = \log \lambda_\mu$ where λ_μ is the largest eigenvalue of A_μ (by Theorem 2.1). By construction of Markov partition in the previous section, we conclude that

$$A_\mu = \begin{pmatrix} \begin{bmatrix} \\ \quad H_\mu \quad \\ \\ \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{pmatrix}. \tag{3.1}$$

The following proposition asserts that the largest eigenvalue of the matrix A_μ is strictly bigger than the largest eigenvalue of the matrix $A_0 = H_0$. From this proposition we can conclude that the entropy of the system $f_\mu|_{\Lambda_k(\mu)}$ is bigger than the entropy of $f_0|_{\Lambda_k}$.

PROPOSITION 3.1. *Let A_μ as defined above. If λ_μ is the largest eingenvalue of A_μ in modulus, then for any $\mu > 0$ near to zero, $\lambda_\mu > \lambda_0$.*

Proof. To proof the proposition we use matricial properties given by the following theorem due to Perron and Frobenius.

THEOREM 3.1 (Perron-Frobenius, [4]). *Every non-negative $s \times s$ matrix A has a non-negative eigenvector, $AU = \lambda U$, with the property that the associated λ is equal to the spectral radius $|\lambda|_{max}$. If the matrix A is irreducible, then there is just one non-negative eigenvector up to multiplication by positive constant, and this eigenvector is strictly positive. Furthermore, the maximal eigenvalue λ' of every principal minor (of order less than s) of A satisfies $\lambda' \leq \lambda$. If A is irreducible, then $\lambda' \lneqq \lambda$.*

To use this theorem in a convenient way, we need the matrix A_μ be irreducible, that is, for any pair i, j there is some power $n(i,j)$ of A_μ such that $A_\mu^{n(i,j)} > 0$. By the definition of the transition matrix A_μ, we get a characterization of irreducibility using Markov partitions.

LEMMA 3.1. *The transition matrix A_μ for a Markov partition $\mathcal{P} = \{P_i\}$ is irreducible if, and only if, for each pair i, j there exists $n = n(i, j)$ such that $f^n(P_i) \cap P_j \neq \varnothing$.*

Proof. It follows directly of the construction of the transition matrix. ∎

Note that the Markov partitions $\mathcal{P} = \{R_1, \dots, R_s, S_1, \dots, S_\ell\}$ for the systems $f_\mu|_{\Lambda_R}$ ($\mu > 0$ small) constructed in the previous section satisfy the conditions of the previous lemma. So, we have that A_μ is irreducible. Indeed, this conditions are satisfied by the rectangles R_i's because the system $f_\mu|_{\Lambda\mu}$ is transitive. Since for each S_j, the iterated $f^{\ell-j}(S_j)$ intersects R_s and the iterated $f^j(R_1)$ intersects S_j, we obtain the desired property for all elements of $\mathcal{P}$.

Now we can apply the Perron-Frobenius Theorem to the sub-matrix $A_{\mu,1}$ of the irreducible transition matrix A_μ, obtained by excluding the last line and the last column of A_μ. So we obtain that the largest eigenvalue λ_μ of A_μ is strictly bigger than the largest eigenvalue $\lambda_{\mu,1}$ of $A_{\mu,1}$. Even though $A_{\mu,1}$ is not necessarily an irreducible matrix, we can use the Perron-Frobenius Theorem again to the sub-matrix $A_{\mu,2}$, whose the largest eigenvalue is $\lambda_{\mu,2}$ and obtain that $\lambda_{\mu,2} \leq \lambda_\mu$. We repeat this step up to obtaining the sub-matrix H_μ, whose largest eigenvalue $\lambda_{\mu,\ell}$ is equal to λ_0, because the systems $f_\mu|_{\Lambda_k(\mu)}$ and $f_0|_{\Lambda_k}$ are topologically conjugated. Thus we have $\lambda_\mu \gneqq \lambda_0$. □

To conclude the proof of the the first statement of the Theorem 1.1 observe that for all C^2-neighborhood $\mathcal{V}$ of $f = f_0$ we can take f_μ with μ very close to 0 such that $f_\mu \in \mathcal{V}$ and the Proposition 3.1 holds. So, since Λ_k is responsible for the entropy of f_0, $h(f_\mu) \geq h(f_\mu|_{\Lambda_k(\mu)}) > h(f_0|_{\Lambda_k}) = h(f_0)$. Then $h(f_\mu) \neq h(f_0)$ and thus f_0 is a point of entropy variation. ∎

3.2. Second statement of Theorem 1.1

To proof of the second statement of the Theorem 1.1 we construct a system with a horseshoe and a homoclinic tangency corresponding to a hyperbolic fixed point outside the horseshoe. Then we perturb the system in a small neighborhood of the tangency to create a transversal intersections (using C^1-perturbations "Snake like" as in Newhouse [9]) to obtain a new system with topological entropy bigger.

Consider the system f on the sphere $\mathbb{S}^2$ whose orbits follows the meridians from p_∞ (the North Pole) to p_0 (the South Pole). Suppose that the system has a horseshoe and a homoclinic loop in two disjointed regions. These regions are delimited by meridians. See the Figure 3.

We suppose this homoclinic loop is associated a fixed hyperbolic point p which has derivative with eigenvalues $\lambda(p) = 3$ and $\lambda(p)^{-1} = 3^{-1}$. The horseshoe Γ in the first region is a two legs horseshoe, p_∞ is the source which send the orbits to a topological disc Q whose interior in a trapping neighborhood for Γ. The sink of this horseshoe coincides with p_0. See the figure 4.

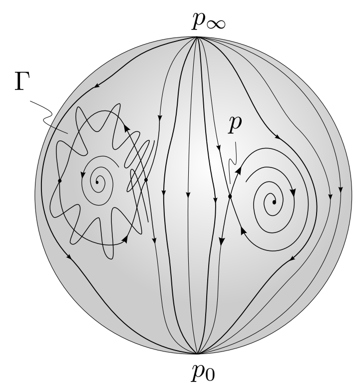

FIG. 3. System with a horseshoe and a homoclinic loop.

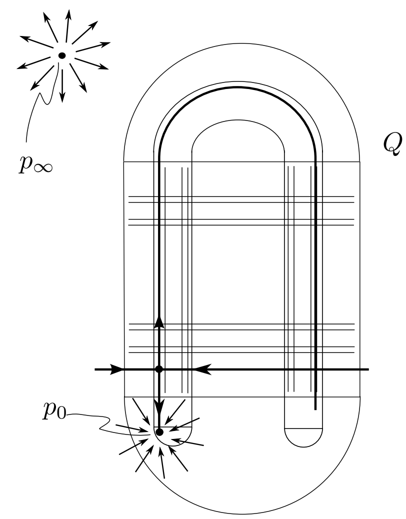

FIG. 4. Region of the horseshoe Γ.

Thus the nonwandering set $\Omega(f)$ consists of three sinks, one source, a (isolated) horseshoe and a hyperbolic point on the homoclinic loop. Then, the topological entropy of f is $h(f|_\Gamma) = \log 2$.

Now we perturb f in the C^1-topology to obtain a new system g, which has a horseshoe Λ instead of the homoclinic loop. Choose a point q in $W^s_{loc}(p)$ and a neighborhood U of q such that U is a fundamental domain. Consider $I = W^s(q) \cap U$. We make a perturbation in U as in Newhouse [9]. Suppose that $\phi : U \to \phi(U) \subset \mathbb{R}^2$ is a linearizing coordinate with $\phi(q) = 0$. Consider $a > 0$ such that $\phi^{-1}([a, a]) \subset I$. For each big $N > 0$ pick $A = A(N) > 0$ such that $A.N \to 0$ when $N \to \infty$. Consider the following function

$$\Phi(x, y) = \left(x, y + A\cos\frac{\pi x N}{2a}\right).$$

Note that Φ sends $[-a, a]$ on the curve γ, where γ is the graphic of the function $x \mapsto y + A\cos(\frac{\pi x N}{2a})$. The maximal distance between γ and the x-axis is A. Furthermore, γ intercepts $[-a, a]$ N times and

$$D\Phi(x, y) = \begin{bmatrix} 1 & 0 \\ -\frac{A\pi N}{2a}\sin(\frac{\pi x N}{2a}) & 1 \end{bmatrix},$$

then Φ is conservative. Take A small enough, then for $\delta > 0$ there exists a diffeomorphism h δ-close to Id in the C^1-topology such that $h = \Phi$ near the origin and $h = Id$ outside of a neighborhood of the origin. Pick $g = h \circ f$. We have that g is δ-C^1-close to f, $g(z) = f(z)$ for $z \in f^{-1}(U)$, $[-a, a] \subset W^s(p, g)$ and $\gamma \subset W^u(p, g)$.

Starting from here, the argument is the same as the one used by Newhouse. Choose a narrow rectangle R close to $W^s_{loc}(p, g)$ that returns, after n iterates by g, close to $W^u(p, g) \cap$

U and it intercepts (transversally) N times. We define Λ' the maximal invariant set for $g^n|_R$ and we have that $h(g^n|_{\Lambda'}) = \log N$. Consider $\Lambda = \bigcup_{j=0}^{N-1} g^j(\Lambda')$ and it turns out $h(g|_\Lambda) = \frac{1}{n}\log N$.

We have that $\Lambda \subset \overline{H(p,g)}$, where $p = p(f) = p(g)$, because since $p \notin f^{-1}(U)$, $g(p) = f(p) = p$. Here $H(p,g)$ denotes the homoclinic class of p for g (see [5] for definition). Therefore, by Newhouse [9], we obtain

$$\lim_{N\to\infty} \frac{1}{n}\log N = \log|\lambda(p)|.$$

Take N large enough such that

$$h(g|_\Lambda) = \frac{1}{n}\log N > \log|\lambda(p)| - \varepsilon.$$

In this case, we have $|\lambda(p)| = 3$ and we can take $\varepsilon < \log(\frac{6}{5})$. So $h(g|_\Lambda) > \log(\frac{5}{2}) > h(g|_\Gamma)$.

Note that g is different from f just inside a region delimited by two meridians. Then, the entropy of g restricted to Γ coincides with the entropy of f restricted to Γ. Thus, the system $g : \mathbb{S}^2 \to \mathbb{S}^2$ has $\overline{H(p,g)} \supset \Lambda$ as the responsible basic set. □

3.3. Proof of Theorem 1.3

It is well known that topological entropy function $f \to h_{top}(f)$ is a continuous function in C^∞ topology. Using this observation, our example in the proof of the second statement of Theorem 1.1 shows that the Newhouse perturbation Theorem 1.2 can not be applied in C^∞ topology.

4. FURTHER REMARKS: RELATION WITH YOMDIN RESULT

Now let us recall a result of Yomdin about the defect of semi-continuity of the entropy function in the space of C^k-diffeomorphisms.

THEOREM 4.1 ([14]). *For $f : M \to M$ of class C^k and $g_n \to f$ in the C^k topology,*

$$\limsup_{n\to\infty} h(g_n) \le h(f) + \frac{2m}{k} R(f), \tag{4.1}$$

where $k \ge 1$, $m = \dim M$ and $R(f) = \lim_{n\to\infty} \frac{1}{n}\log\max_{x\in M}\|Df^n(x)\|$.

QUESTION 4.1. *Let f be a C^1-diffeomorphism with $\Omega(f) = \Lambda_1 \cup \cdots \Lambda_k \cup \mathcal{O}(q)$ where Λ_i are basic pieces and $\mathcal{O}(q)$ is the homoclinic tangency corresponding to a piece not responsible for the entropy, for instance, Λ_1. Is there $\kappa > 0$ such that if $|h(f) - h(f|\Lambda_1)| > \kappa$ then f is a point of constancy in the C^1 topology?*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

It may be conjectured that $k \geq 4R(f)$ for the surface case where $m = 2$. However, using our example we conclude that such k in the question would be greater than $R(f)$.

REFERENCES

1. R. Bowen, Markov partitions for Axiom A diffeomorphisms, *Amer. J. Math.*, **92**(3) (1970), 725–747.
2. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math., **470**, Springer, 1975.
3. L.J. Díaz and I. L. Rios, Critical saddle-node horseshoes: bifurcations and entropy, *Nonlinearity*, **16**(3) (2003), 897–928.
4. F. R. Gantmacher, *The theory of matrices*, Vols. **1**, **2** (Translated by K. A. Hirsch), Chelsea Publishing Co., New York, 1959.
5. B. Hasselblatt and A. Katok, *Introduction to the theory of dynamical systems*, Cambridge University Press, 1995.
6. Y. Hua, R. Saghin and Z. Xia, Topological entropy and partially hyperbolic diffeomorphisms, *Ergodic Theory & Dynamical Systems*, **28**(3) (2008), 843–862.
7. A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publications Mathematiques de l'IHS*, **51** (1980), 137–173.
8. S.E. Newhouse, Continuity Properties of Entropy, *The Annals of Mathematics*, 2a Ser., **129**(1) (1989), 215–235.
9. S.E. Newhouse, Topological entropy and Hausdorff dimension for area preserving diffeomorphisms of surfaces, *Soc. Math. France (Asterisque)*, **51** (1978), 323–334.
10. J. Palis and F. Takens, *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations*, Cambridge: Cambridge University Press, 53–91, 1993.
11. S. Pujls and M. Sambrino, On homoclinic tangencies, hyperbolicity, creation of homoclinic orbits and variation of entropy, (English summary), *Nonlinearity*, **13**(3) (2000), 921–926.
12. C. Robinson, *Introduction to the theory of dynamical systems*, Boca Raton: CRC Press, 1995.
13. S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73** (1967), 747–817.
14. Y. Yomdin, Volume growth and entropy, *Israel J. Math.*, **57**(3) (1987), 285–300.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC