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Global inverse mapping theorems

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In this paper we prove a Global Inverse Mapping Theorem for differenciable (not necessarily C^1) maps between smooth connected Riemannian manifolds. As applications, we extend to the case of local diffeomorphisms, some C^1 results about global inversion (or global injectivity) of self-maps of $\mathbb{R}^n$. May, 2009 ICMC-USP

1. INTRODUCTION

The local and global inversion of maps is one of the most important subjects in Topology and Analysis. With respect to local problem we have the fundamental Inverse Mapping Theorem for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ which requires regularity and continuity of the derivative, of the map, around the given point. The stronger Černavskii Theorem [6], [7] (see also [1] and [19]), under weaker conditions of differentiable, state that

Theorem A [1, Theorem 4.4]. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be differentiable (not necessarily C^1) map without critical points. Then*

(*a*) *f is a local homeomorphism; and*

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(b) *if for some open subset* V *of* U, $f|_V : V \to W$ *is a homeomorphism then* $(f|_V)^{-1}: W \to V$ *is also a differentiable homeomorphism.*

Let us mention some other results that, similarly to the case of Theorem A, use weaker assumptions. F. H. Clarke [9], using the concept of generalized gradient, proves the Inverse Mapping Theorem under a condition that need the map be differentiable. In [18], S. Radulescu and M. Radulescu, state some Inverse Mapping Theorems for mappings in Banach spaces, where the derivative is regular, around the given point, but may no vary continuously.

In this paper, by using Theorem A, we proved the following Global Inverse Mapping Theorem:

THEOREM 1.1. *Let* M *and* N *be smooth connected Riemannian manifolds of the same finite dimension* n*, where* M *is complete. Let* $f: M \to N$ *be a differentiable map (which may or may not be of class* C^1*) without critical points. Let* $x_0 \in M$*; given a non-negative real number* s*, denote by* $\Gamma_{x_0}(s) = \sup\{||(Df(x))^{-1}|| : d(x, x_0) \leq s\}$*, where* $d(\cdot, \cdot)$ *denotes the intrinsic metric of* M*. Suppose that*

$$\int_0^\infty \frac{du}{\Gamma_{x_0}(u)} = \infty. \tag{1.1}$$

Then

(a) f *is a covering map (in particular,* f *is onto).*
(b) *If* $f_* : \pi_1(M) \to \pi_1(N)$ *is an epimorphism,* f *is a (global) homeomorphism.*

Our main result, Theorem 1.1, generalizes the following well known theorem:

Theorem B (Hadamard-Plastock Theorem) [16, Thm. 4.2]. *A local* C^1 *diffeomorphism* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is bijective if*

$$\int_0^\infty \inf_{|x|=r} \|DF(x)^{-1}\|^{-1} dr = \infty.$$

In the sections 3 and 4, as applications of Theorem A and Theorem 1.1, we extend to the case of local diffeomorphisms, the C^1 results of [11] about global inversion (or global injectivity) of self-maps of $\mathbb{R}^n$. In this sense, some articles somehow related to this work are those of [4], [5], [8], [10], [12], [13], [14], [15], [17] and [20].

REMARK 1.1. Let us observe that, by using Theorem A (see also [2] and [3]), we may obtain the following Implicit Function Theorem:

THEOREM 1.2. *Let* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$ *be open sets and let* $f : U \times V \to \mathbb{R}^n$ *be a differentiable map. Suppose that for all* $(x, y) \in U \times V$, $\frac{\partial f}{dx}(x, y) : \mathbb{R}^n \to \mathbb{R}^n$ *is non-singular. Given* $(x_0, y_0) \in U \times V$ *there exists an open neighborhood* V_1 *of* y_0 *and a differentiable map* $h : V_1 \to U$ *such that* $h(y_0) = x_0$ *and* $f(h(y), y) = f(x_0, y_0)$*, for all* $y \in V_1$.

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Throughout this paper we shall use the following definitions. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map. We say that F is a *diffeomorphism* if it is homeomorphism and both F and F^{-1} are differentiable (not necessarily C^1) maps. We say that F is a *local diffeomorphism at* $p \in \mathbb{R}^n$ if there exist neighborhoods U of p and V of $F(p)$ such that $F|_U : U \to V$ is a diffeomorphism. When F is a local diffeomorphism everywhere, we simply say that F is a *local diffeomorphism*.

Note: Carlos Gutiérrez passed away on December 3, 2008. The main ideas of this work were developed while he was still in reasonable good health condition. We, Carlos Biasi and Edivaldo dos Santos, were fortunate to have worked closely with him and dedicate this paper to Carlos Gutiérrez with friendship.

2. A GLOBAL INVERSE MAPPING THEOREM

In this section, we will prove Theorem 1.1. For this, we need prove the following lemmas, under assumptions of Theorem 1.1.

LEMMA 2.1. *Let $\alpha : [0, b] \to M$ and $\beta = f \circ \alpha : [0, b] \to N$ be differentiable paths such that $\alpha(0) = x$. Then*

$$\int_0^b \parallel \beta'(u) \parallel du \geq \int_{d(x_0,x)}^{d(x_0,x)+s(t)} \frac{du}{\Gamma_{x_0}(u)},$$

where $s(t) = \int_0^t \parallel \alpha'(u) \parallel du$.

Proof. It follows that $\beta'(t) = Df(\alpha(t)) \cdot \alpha'(t)$ and so $Df(\alpha(t))^{-1} \cdot \beta'(t) = \alpha'(t)$. Hence

$$||Df(\alpha(t))^{-1}|| \cdot ||\beta'(t)|| \geq ||\alpha'(t)||.$$

Given $t \in [0, b]$, let

$$s(t) = \int_0^t ||\alpha'(u)||du \geq d(\alpha(0), \alpha(t)) = d(x, \alpha(t)).$$

Then we obtain that

$$d(x_0, \alpha(t)) \leq d(x_0, x) + d(x, \alpha(t)) = d(x_0, x) + s(t),$$

which implies that

$$\Gamma_{x_0}(s(t) + d(x_0, x))\, ||\beta'(t)|| \geq ||\alpha'(t)||,$$

and so

$$||\beta'(t)|| \geq \frac{||\alpha'(t)||}{\Gamma_{x_0}(s(t) + d(x_0, x))}.$$

In this way, we obtain that

$$\int_0^b ||\beta'(t)||dt \geq \int_0^b \frac{||\alpha'(t)||}{\Gamma_{x_0}(s(t)+d(x_0,x))}dt.$$

By considering, in the right integral, the following change of variables $u = s(t) + d(x_0, x)$ where $du = s'(t)dt = ||\alpha'(t)||dt$, we obtain that

$$\int_0^b ||\beta'(t)||dt \geq \int_{d(x_0,x)}^{d(x_0,x)+s(b)} \frac{du}{\Gamma_{x_0}(u)} \geq \int_{d(x_0,x)}^{d(x_0,x)+s(t)} \frac{du}{\Gamma_{x_0}(u)},$$

which proves the lemma. ▮

LEMMA 2.2. *Suppose $x \in M$ and $\beta\colon [0,b] \to N$ be differentiable path such that $\beta(0) = f(x)$. Then, there exists a unique continuous path $\alpha\colon [0,b] \to M$ such that $\alpha(0) = x$ and $f \circ \alpha = \beta$. Moreover, α is differentiable.*

Proof. By Theorem A, we have that f is a local homeomorphism. Then, there exists $c \in (0,b]$ and a path $\alpha : [0,c) \to M$ such that $\beta = f \circ \alpha$, for all $t \in [0,c)$. We will show that α can be continuously extended for $[0,c]$ such that $\beta = f \circ \alpha$ for all $t \in [0,c]$. In fact, let (x_n) be a sequence such that $c_n < c$, $c_n < c_{n+1}$ and $\lim c_n = c$. Let us consider the sequence $(\alpha(c_n)) \subset M$. Now, by hypothesis (condition (1.1)), we have that

$$\infty = \int_{d(x_0,x)}^{\infty} \frac{du}{\Gamma_{x_0}(u)} = \lim_{s\to\infty} \int_{d(x_0,x)}^{d(x_0,x)+s} \frac{du}{\Gamma_{x_0}(u)}.$$

Therefore, given $\epsilon > 0$ there exists $s_0 > 0$ such that for all $s > s_0$ we have

$$\epsilon < \int_{d(x_0,x)}^{d(x_0,x)+s} \frac{du}{\Gamma_{x_0}(u)}. \tag{2.1}$$

From (2.1) and Lemma 2.1 we conclude that $s[0,c) = \{s(t) = \int_0^t \| \alpha'(u) \| du, t \in [0,c)\}$ must be a bounded set. Hence,

$$0 = \lim_{\substack{m,n\to+\infty \\ m>n}} \left(\int_{c_n}^{c_m} \| \alpha'(u) \| du \right) \geq \lim_{\substack{m,n\to+\infty \\ m>n}} d(\alpha(c_n), \alpha(c_m)). \tag{2.2}$$

Thus, $(\alpha(c_n))$ is a Cauchy sequence in M and since M is complete, $\lim \alpha(c_n) = x_1 \in M$. Define $\alpha(c) = x_1$. Then $f \circ \alpha(c) = f \circ (\alpha(\lim c_n)) = \lim(f \circ (\alpha(c_n))) = \lim(\beta(c_n)) = \beta(\lim c_n) = \beta(c)$ and we conclude that $\alpha : [0,c] \to M$ is such that $\beta = f \circ \alpha$ for all $t \in [0,c]$. Since $[0,b]$ is compact, we have that α can be extended for $[0,b]$ such that $\beta = f \circ \alpha$ for all $t \in [0,b]$. Certainly α is uniquely defined. Moreover, since $\beta = f \circ \alpha\colon [0,b] \to N$ is a differentiable path and $f\colon M \to N$ is a differentiable map without critical points we have that α is a differentiable path. ▮

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Proof of Theorem 1.1. It follows from Lemma 2.2 that $f : M \to N$ is a proper map and we conclude that f is a covering map. This proves (a). Therefore, $f_* : \pi_1(M) \to \pi_1(N)$ is monomorphism and by hypothesis $f_* : \pi_1(M) \to \pi_1(N)$ is an epimorphism, so f_* is an isomorphism which implies that $f : M \to N$ is a global homeomorphism. This proves (b). ▮

3. ON THE WEAK MARKUS-YAMABE CONJECTURE FOR LIPSCHITZ MAPS

In this section we extend, to the case of local diffeomorphisms, the C^1 results of [11] about global inversion (or global injectivity) of self-maps of $\mathbb{R}^n$. In many cases, proofs are, mutis mutandis, the same as those of [11]. In this section we are suppose to use the Theorems A, 1.1 and 1.2 wherever they are necessary.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. We denote by $\mathrm{Spec}(F)$ the set of (complex) eigenvalues of the derivative DF_p, as p varies in $\mathbb{R}^n$.

Lemma 3.1 (Main Lemma). *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that $\det(F'(x)) \neq 0$ for all x in $\mathbb{R}^n$. Given $t \in \mathbb{R}$, let $F_t \colon \mathbb{R}^n \to \mathbb{R}^n$ denote the map $F_t(x) = F(x) - tx$. If there exists a sequence $\{t_m\}$ of real numbers converging to 0 such that every map $F_{t_m} \colon \mathbb{R}^n \to \mathbb{R}^n$ is injective, then F is injective.*

Proof. Choose $x_1, x_2 \in \mathbb{R}^n$ such that $F(x_1) = y = F(x_2)$. We will prove $x_1 = x_2$. By the Inverse Mapping Theorem, we may find neighborhoods U_1, U_2, V of x_1, x_2, y, respectively, such that, for $i = 1, 2$, $F|_{U_i} : U_i \to V$ is a homeomorphism and $U_1 \cap U_2 = \emptyset$. If m is large enough, then $F_{t_m}(U_1) \cap F_{t_m}(U_2)$ will contain a neighborhood W of y. In this way, for all $w \in W$, $\#(F_{t_m}^{-1}(w)) \geq 2$. This contradiction with the assumptions, proves the lemma. ▮

Remark 3.1. Even if $n = 1$ and the maps F_{tm} in Main Lemma are diffeomorphisms, we cannot conclude that F is a diffeomorphism. For instance, if $F : \mathbb{R} \to (0, 1)$ is an orientation reversing diffeomorphism, then for every $t > 0$, the map $F_t : \mathbb{R} \to \mathbb{R}$ (defined by $F_t(x) = F(x) - tx$) will be an orientation reversing global diffeomorphism.

The following theorem is in the opposite direction from the Smyth and Xavier example [17, Theorem 4]:

Theorem 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism such that $\mathrm{Spec}(F)$ is disjoint of a sequence $\{t_m\}$ of real numbers which converges to 0 as $m \to \infty$. If there exist $R > 0$ and $0 < \alpha < 1$ such that, for all x in $\mathbb{R}^n$ with $|x| > R$, $|F(x)| \leq |x|^\alpha$, then F is injective.*

Proof. Define $F_{t_m} \colon \mathbb{R}^n \to \mathbb{R}^n$ by $F_{t_m}(x) = F(x) - t_m x$. Since every t_m is not in $\mathrm{Spec}(F)$, we have that every F_{t_m} is a local diffeomorphism. By the assumptions, $F_{t_m}(x) \to \infty$ as

$x \to \infty$, which implies that F_{t_m} is proper. It follows from Theorem 1.1 that F_{t_m} is injective, for every t_m. Therefore, we conclude from Main Lemma that F is injective. ∎

Next result proves the Chamberland Conjecture for Lipschitz maps. We shall need the following

LEMMA 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable Lipschitz map. If $K > 0$ is a Lipschitz constant for F then, for all $x \in \mathbb{R}^n$,*

$$\|DF(x)\| := \sup\{|DF(x)v| \ : \ |v| = 1\} \leq K,$$

Proof. Suppose by contradiction that there exists $x \in \mathbb{R}^n$ and a unitary vector $v \in \mathbb{R}^n$ such that, for some $\varepsilon > 0$, $|DF(x)v| > K + \varepsilon$. Take $\delta > 0$ so small that for all $|t| < \delta$,

$$|DF(x)v - \frac{F(x+tv) - F(x)}{t}| < \frac{\varepsilon}{2}.$$

It follows that, if $|t| < \delta$ then

$$\frac{\varepsilon}{2} > |DF(x)v| - |\frac{F(x+tv) - F(x)}{t}| > K + \varepsilon - K = \varepsilon.$$

This contradiction proves the lemma. ∎

THEOREM 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable Lipschitz map. Suppose that for some $\epsilon > 0$, $\mathrm{Spec}(F) \cap \{z \in \mathbb{C} : |z| \leq \epsilon\} = \emptyset$. Then F is bijective.*

Proof. Let $K > 0$ be such that, for all $x, y \in \mathbb{R}^n$, $|F(x) - F(y)| \leq K|x - y|$. By Lemma 3.2, for all $x \in \mathbb{R}^n$,

$$\|DF(x)\| := \sup\{|DF(x)v| \ : \ |v| = 1\} \leq K,$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$. Let $\|\cdot\|_M$ be the norm, on the space of real matrices $n \times n$, given by

$$\|A\|_M := \sup\{|a_{ij}| \ : \ 1 \leq i, j \leq n\},$$

where $A = \{a_{ij}\}$. As the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent, there exists $K_1 > 0$ such that, for all $x \in \mathbb{R}^n$, $\|DF(x)\|_M \leq K_1$. Therefore, there exists a positive constant $K_2 > 0$ such that the classical adjoint matrix $A(x)$ of $DF(x)$ satisfies, for all $x \in \mathbb{R}^n$,

$$\|A(x)\|_M \leq K_2.$$

By the assumptions on $\mathrm{Spec}(F)$, we have that for all $x \in \mathbb{R}^n$, $|\det(DF(x))| \geq \epsilon^n$. Therefore, for all $x \in \mathbb{R}^n$,

$$\|DF(x)^{-1}\|_M \leq K_3,$$

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where $K_3 = K_2/\epsilon^n > 0$ is constant. Again since the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent, there exists $K_4 > 0$ such that for all $x \in \mathbb{R}^n$,

$$\|DF(x)^{-1}\| \leq K_4.$$

This theorem follows after applying Theorem 1.1. ▮

We are grateful to L. A. Campbell and N. Van Chau who, among other helpful comments, let us know that the proof of Theorem 3.2 could also be applied to the following case:

THEOREM 3.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz local diffeomorphism. Suppose that for some $\epsilon > 0$,*

$$\liminf_{|x|\to\infty} (|x| \cdot |\det(DF(x))|) > \epsilon.$$

Then F is a global diffeomorphism.

Proof. Before the last statement involving an inequality in the proof of Theorem 3.2, this proof proceeds the same way. By the current assumptions, such last statement takes this time the following form: there exists $K_4 > 0$ such that, for all $x \in \mathbb{R}^n$ large enough,

$$\|DF(x)^{-1}\| \leq K_4|x|.$$

This theorem follows, again as above, after applying Theorem 1.1. ▮

We now prove that the Weak Markus-Yamabe Conjecture is true for Lipschitz maps.

THEOREM 3.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz local diffeomorphism. Suppose that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbb{C}$ (with non-empty interior), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and*

$$\mathrm{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

Then, F is injective.

Proof. Let $Id : \mathbb{R}^n \to \mathbb{R}^n$ denote the Identity Map. Then, for every t_m,

$$\mathrm{Spec}(F - t_m Id) \cap (D_m - t_m) = \emptyset,$$

where $D_m - t_m = \{z \in \mathbb{C} : z + t_m \in D_m\}$ is a compact disc centered at 0. Since $F - t_m Id : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable Lipschitz map, applying Theorem 3.2, we obtain that, for all t_m, the map $F - t_m Id$ is a (global) diffeomorphism. Main Lemma allow us to conclude that F is injective. ▮

COROLLARY 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map. Suppose that* $\mathrm{Spec}(F) \cap \{z \in \mathbb{C} : \mathbb{R}(z) \geq 0\} = \emptyset$. *Then F is injective.*

REMARK 3.2. Corollary 3.1 is stronger than Vidossich's Theorem 2,a [20] which has the additional assumption that, for some constant $A > 0$ and for all $x \in \mathbb{R}^n$, $Trace(DF(x)) \leq -A$. The proof of Vidossich fails when he claims that the linear map h is invertible ([20], page 972). Nevertheless, his proof still works if the assumptions are strengthened, for instance, to the following one: "for some real constant $A > 0$, $\text{Spec}(F) \cap \{z \in \mathbb{C} : \mathbb{R}(z) \geq -A\} = \emptyset$."

4. EQUIVALENT STATEMENTS

It is interesting to know if these conjectures are true for other families of maps. We prove the equivalence of conjectures in another class of maps.

Let $\mathcal{L}$ be a subset of the differentiable self-maps of $\mathbb{R}^n$ which contains the identity map Id of $\mathbb{R}^n$ and such that, for all $(F, s, t) \in \mathcal{L} \times \mathbb{R} \times \mathbb{R}$, $sF + tId \in \mathcal{L}$. Let $A > 0$ be a real constant. We state

(1) A-WEAK MARKUS-YAMABE CONJECTURE FOR $\mathcal{L}$: If $F \in \mathcal{L}$ satisfies $\text{Spec}(F) \subset \{z \in \mathbb{C} : \mathbb{R}(z) < -A\}$, then F is injective.

(2) STRONG CHAMBERLAND CONJECTURE FOR $\mathcal{L}$: If $F \in \mathcal{L}$ is a local diffeomorphism which has the property that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbb{C}$ (not reduced to points), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and

$$\text{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

then, F is injective.

PROPOSITION 4.1. *The Weak Markus-Yamabe Conjecture for $\mathcal{L}$ is true if, and only if, A-Weak Markus-Yamabe Conjecture for $\mathcal{L}$ is true.*

Proof. Suppose that the A-Weak Markus-Yamabe Conjecture for $\mathcal{L}$ is true. Let t be any positive real constant. Choose $F \in \mathcal{L}$ such that $\text{Spec}(F) \subset \{z \in \mathbb{C} : \mathbb{R}(z) < -t\}$. Then $(A/t)F = G \in \mathcal{L}$ and $\text{Spec}(G) \subset \{z \in \mathbb{C} : \mathbb{R}(z) < -A\}$. G is injective and so F is injective. This implies the t-Weak Markus-Yamabe Conjecture for $\mathcal{L}$. Since for all $(F, t) \in \mathcal{L} \times \mathbb{R}$, $F + tId \in \mathcal{L}$, we apply Main Lemma to show that the Weak Markus Yamabe Conjecture for $\mathcal{L}$ is also true. The converse is obvious. ▌

PROPOSITION 4.2. *The Strong Chamberland Conjecture for $\mathcal{L}$ and the Chamberland Conjecture for $\mathcal{L}$ are equivalent.*

Proof. It is obvious that the Strong Chamberland Conjecture for $\mathcal{L}$ implies the Chamberland Conjecture for $\mathcal{L}$. The proof of the converse follows the lines of the proof of Theorem 3.3. ▌

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It is obvious that the Chamberland Conjecture for $\mathcal{L}$ implies the A-Weak Markus-Yamabe Conjecture for $\mathcal{L}$. Therefore,

COROLLARY 4.1. *The Chamberland Conjecture for $\mathcal{L}$ implies the Weak Markus-Yamabe Conjecture for $\mathcal{L}$.*

Let $\mathcal{L}_1$ be the set of differentiable maps $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that if $n \geq 2$, there exist $K > 0$ and a compact subset of $\mathbb{R}^n$ outside of which

$$||DF(x)|| \leq K|x|^{\frac{1}{n-1}}.$$

THEOREM 4.1. *Let $F \in \mathcal{L}_1$. The following statements hold true:*

(a) If there is $\epsilon > 0$ such that $|\det(DF(x))| > \epsilon$, for all $x \in \mathbb{R}^n$, then F is a global diffeomorphism.
(b) If $\mathrm{Spec}(F) \subset \{z \in \mathbb{C} : \mathbb{R}(z) < 0\}$, then F is injective.

Proof. Item (a) follows by using the same argument as that of the proof of Theorem 3.3. Item (b) follows immediately from item (a) and Corollary 4.1 applied to the set $\mathcal{L}_1$. ∎

Note that, by the Mean Value Theorem, $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable Lipschitz map if, and only if, there is a constant $K > 0$ such that, for all $x \in \mathbb{R}^n$, $||DF(x)|| \leq K$. Therefore, Theorem 4.1 is stronger than Corollary 3.1.

Now we state a result which follows at once from [5, Theorem 3.2] and Corollary 4.1 applied to the set $\mathcal{L}_2$ of the differentiable maps $F : \mathbb{R}^n \to \mathbb{R}^n$ such that there is a compact subset of $\mathbb{R}^n$ outside of which the Jacobian matrix $JF(x)$ commutes with its transpose

THEOREM 4.2. *If $F \in \mathcal{L}_2$ and $\mathrm{Spec}(F) \subset \{z \in \mathbb{C} : \mathbb{R}(z) < 0\}$, then F is injective.*

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