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Analytic torsion for manifolds with totally geodesic boundary

L. Hartmann*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: hartmann@icmc.usp.br

M. Spreafico

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: mauros@icmc.usp.br

We prove that the extension of the Cheeger Müller theorem, [3] and [12], for manifolds with boundary and product metric near the boundary, due to Lück [10], generalizes further under the weaker condition that the boundary is totally geodesic. Our proof is based on recent results of Brüning and Ma [2] on the anomaly boundary term (see also the work of Dai and Fang [5]). May, 2009 ICMC-USP

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1. INTRODUCTION

Let (W, g) be a closed connected Riemannian manifold with metric g. Let $\tau(M)$ denote the Reidemeister Franz torsion of M [14] [6], and $T(M)$ the analytic torsion of M [13]. If M is not acyclic, assume the base for the homology is fixed by the choice of an orthonormal base of harmonic forms, as in [13]. The theorem of Cheeger [3] and Müller [12] affirms the equivalence of these two torsions, proving a conjecture of Ray and Singer [13]. When W is no longer closed, the Cheeger Müller theorem still holds but boundary terms appear. Namely, the results reads

$$\log T(W) = \log \tau(W) + f(\partial W).$$

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A first result for the boundary term $f(\partial W)$ was given by Lück [10] in the case of a product metric near the boundary. In this case, the boundary term is purely topological and is proportional to the Euler characteristic of the boundary. More recently, the boundary term has been investigated in the general case by Dai and Fang [5], and by Brüning and Ma [2]. They proved the existence of a further non topological contribution when the metric is not a product near the boundary, called anomaly boundary term.

In this work we show that the result of Lück extends to the more general case of manifolds with totally geodesic boundary. Our main result is the following theorem, where we denote by $T_{\rm abs}(W)$ the analytic torsion of W with absolute (Neumann) boundary conditions, and by $T_{\rm rel}(W)$ the analytic torsion of W with relative (Dirichlet) boundary conditions (see Section 3 for the precise definitions).

Theorem 1.1. *Let W be a compact connected Riemannian manifold of dimension m with totally geodesic boundary ∂W, then*

$$(-1)^{m-1}\log T_{\rm rel}(W)=\log T_{\rm abs}(W)=\log\tau(W)+\frac{1}{4}\chi(\partial W)\log 2,$$

where $\chi(\partial W)$ is the Euler characteristic of the boundary.

Our proof exploit the results of Brüning and Ma [2], and is given in Section 4. In Section 2, we recall the definition of totally geodesic submanifolds, and in Section 3 the definitions of Reidemeister torsion and analytic torsion. In Section 5, we apply Theorem 1.1 to deduce the analytic torsion of an half sphere. In the last Section 6, we exhibit an explicit calculation of the analytic torsion in the particular case of the half sphere of dimension two.

2. MANIFOLDS WITH TOTALLY GEODESIC BOUNDARY

Let (W,g) be a connected Riemannian manifold of dimension m, and metric g. A submanifold M of W is said to be *totally geodesic* if the geodesics of M are geodesics in W. In other words, if $i:M\to W$ denotes the inclusion, M is totally geodesic if and only if the isometric immersion $i:(M,i^*g)\to(W,g)$ is *totally geodesic*: namely if any geodesic of (M,i^*g) is carried under i into a geodesic of (W,g). If $M=\partial W$ is the boundary of W, and ∂W is totally geodesic, we said that W has *totally geodesic boundary.* We give a local metric condition for a manifold to have a totally geodesic boundary. Let ∂_x denotes the inward pointing unit normal vector to the boundary, i.e. an orthonormal local base for $N\partial W$, the normal bundle. Near the boundary we have the collar decomposition $Coll(\partial W)=[0,\epsilon)\times\partial W$, obtained using the inward geodesic flow. If $y=(y_1,\dots,y_{m-1})$ is a system of local coordinates on the boundary, then (x,y) is a system of local coordinates on $Coll(\partial W)$. Here the curves $x\to(x,y)$ are unit speed geodesics perpendicular to the boundary. In this system of local coordinates, the metric tensor reads

$$g=dx\otimes dx+\tilde{g}(x),$$

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where $\tilde{g}(x)$ is a family of metric on ∂W such that $\tilde{g}(0) = i^* g$. Let $\Gamma_{\alpha\ \beta}^{\ \gamma}$ denote the Christoffel symbols relative to the coordinate base associated to the local coordinates (x, y) on (W, g), and $\tilde{\Gamma}_{j\ k}^{\ l}$ denote the Christoffel symbols relative to the coordinate base associated to the local coordinates (y) on $(\partial W, \tilde{g}(0))$. Then, it is easy to see that the geodesic equation on (W, g)

$$d_t^2 f^\gamma + \Gamma_{\alpha\ \beta}^{\ \gamma} d_t f^\alpha d_t f^\beta = 0,$$

corresponds to the system of equations

$$\begin{cases} d_t^2 f^x - \frac{1}{2}\partial_x \tilde{g}_{y_j y_k} d_t f^{y_j} d_t f^{y_k} = 0, \\ d_t^2 f^{y_l} + \tilde{\Gamma}_{j\ k}^{\ l} d_t f^{y_j} d_t f^{y_k} + \tilde{g}^{y_l y_k}(x) \partial_x \tilde{g}_{y_k y_h}(x) d_t f^x d_t f^{y_h} = 0. \end{cases}$$

This shows that a geodesic of $(\partial W, i^* g)$ is a geodesic in (W, g) if and only if $\partial_x \tilde{g}(x) = 0$, and hence proves the following result.

LEMMA 2.1. *Let (W, g) be a connected Riemannian manifold with boundary ∂W. Let (x, y) be a sytem of local coordinates near the boundary, where x is the geodesic distance from the boundary, and the metric reads $g = dx \otimes dx + \tilde{g}(x)$. Then, ∂W is totally geodesic if and only if $\partial_x \tilde{g}(x) = 0$.*

We interpret this result using the second fundamental form of ∂W, whose definition we briefly recall now. Let $i : M \to W$ be a submanifold of (W, g). The restriction of the tangent bundle TW_M decomposes as the Whitney sum $TM \oplus NM$, where NM is the normal bundle to M. For a vector v denote by $v_{\tan}$ and $v_{\rm norm}$ the components. Let $\nabla \in \Gamma(W, T^*W \otimes End(TW))$ be the Levi-Civita connection of (W, g). Given two sections $u, v \in \Gamma(M, TM)$ of TM, the second fundamental form S of M is defined by $S_M(u, v) = (\nabla_u v)_{\rm norm}$. It is easy to see that M is totally geodesic if and only if its second fundamental form vanishes identicaly. Recalling the local description of the second fundamental form as a the symmetric (0,2)-tensor $S_{y_j y_k} = \Gamma_{y_j x y_k} = -\frac{1}{2}\partial_x \tilde{g}_{y_j y_k}$, Lemma 2.1 follows.

3. REIDEMEISTER TORSION AND ANALYTIC TORSION

We recall the definitions of Reidemeister and analytic torsion. Main reference are [11] and [13]. We follow the notation introduced in [7], and we refer to that work (or to the original ones) for further details. Let

$$C: \qquad C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0, \tag{3.1}$$

be a chain complex of real vector spaces. Denote by $Z_q = \ker \partial_q$, by $B_q = \Im \partial_{q+1}$, and by $H_q(C) = Z_q / B_q$ as usual. For two bases $x = \{x_1, \ldots, x_k\}$ and $y = \{y_1, \ldots, y_k\}$ of a vector space V, denote by (y/x) the matrix defined by the change of base. For each q, fix a base c_q for C_q, and a base h_q for $H_q(C)$. Let b_q be a set of (independent) elements of

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C_q such that $\partial_q(b_q)$ is a base for B_{q-1}. Then the set of elements $\{\partial_{q+1}(b_{q+1}), h_q, b_q\}$ is a base for C_q. In this situation, the *Reidemeister torsion* of the complex C with respect to the graded base $h = \{h_q\}$ is the positive real number

$$\tau_{\mathrm{R}}(C; h) = \prod_{q=0}^{m} |\det(\partial_{q+1}(b_{q+1}), h_q, b_q/c_q)|^{(-1)^q} . \tag{3.2}$$

Let (K, L) be a pair of connected finite cell complexes of dimension m, $(\tilde{K}, \tilde{L})$ its universal covering complex pair, and identify the fundamental group of K with the group of the covering transformations of $\tilde{K}$. Let $C((\tilde{K}, \tilde{L}); \mathbb{R})$ be the real chain complex of $(\tilde{K}, \tilde{L})$. The action of the group of covering transformations makes each chain group $C_q((\tilde{K}, \tilde{L}); \mathbb{R})$ into a module over the group algebra $\mathbb{R}\pi_1(K)$, and each of these modules is $\mathbb{R}\pi_1(K)$-free and finitely generated by the natural choice of the q-cells of K. We have got the complex $C((\tilde{K}, \tilde{L}); \mathbb{R}\pi_1(K))$ of free finitely generated modules over $\mathbb{R}\pi_1(K)$. Let $\rho : \pi_1(K) \to G$ be a representation of the fundamental group in some group G (typically $G = O(k, \mathbb{R})$, and consider the twisted complex $C((K, L); (\mathbb{R}\pi_1(K))_\rho)$. Assume $H_q(C((K, L); (\mathbb{R}\pi_1(K))_\rho))$ are free finitely generated modules over $\mathbb{R}\pi_1(K)$. The Reidemeister torsion of K with respect to the representation ρ and to the graded base h is defined applying the previous construction to the twisted complex $C((K, L); (\mathbb{R}\pi_1(K))_\rho)$, namely

$$\tau_{\mathrm{R}}((K, L); h, \rho) = \tau_{\mathrm{R}}(C((K, L); (\mathbb{R}\pi_1(K))_\rho); h),$$

in $\mathbb{R}^+$. If (K, L) is the cellular (or simplicial) decomposition of a pair of spaces (X, A), the Reidemeister torsion of (X, A) is defined accordingly, and denoted by $\tau_{\mathrm{R}}((X, A); h, \rho)$.

Let $(W, \partial W, g)$ be a closed connected orientable Riemannian manifold of dimension m, with boundary ∂W and Riemannian metric g. Then, all the previous assumptions are satisfied, and the R torsions $\tau_{\mathrm{R}}(W; h, \rho)$ and $\tau_{\mathrm{R}}((W, \partial W); h, \rho)$ are well defined for each fixed graded base h for the homology of W, and each representation ρ of the fundamental group. Moreover, it is possible to prove that $\tau_{\mathrm{R}}((W, \partial W); h, \rho)$ does not depend on the cellular decomposition. In this context, Ray and Singer suggest a natural geometric invariant object, by fixing an appropriate base h for the homology using the geometric structure, as follows. Let $E_\rho \to W$ be the real vector bundle associated to the representation $\rho : \pi_1(W) \to O(k, \mathbb{R})$. Let $\Omega(W, E_\rho)$ be the graded linear space of smooth forms on W with values in E_ρ. The exterior differential on W defines the exterior differential on $\Omega^q(W, E_\rho)$, $d : \Omega^q(W, E_\rho) \to \Omega^{q+1}(W, E_\rho)$. The metric g defines an Hodge operator on W and hence on $\Omega^q(W, E_\rho)$, $* : \Omega^q(W, E_\rho) \to \Omega^{m-q}(W, E_\rho)$, and, using the inner product in E_ρ, an inner product on $\Omega^q(W, E_\rho)$.

In order to deal with the boundary, we need suitable boundary conditions. Consider the natural splitting of ΛW as direct sum of vector bundles $\Lambda \partial W \oplus N^* W$. The smooth forms on W near the boundary decompose as $\omega = \omega_{\tan} + \omega_{\mathrm{norm}}$. Let $B_{\mathrm{abs}}(\omega) = \omega_{\mathrm{norm}}|_{\partial W}$ and $B_{\mathrm{rel}}(\omega) = \omega_{\tan}|_{\partial W}$. Absolute boundary conditions are defined by

$$\mathcal{B}_{\mathrm{abs}}(\omega) = B_{\mathrm{abs}}(\omega) \oplus B_{\mathrm{abs}}((d + d^\dagger)(\omega)) = 0, \tag{3.3}$$

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and relative boundary conditions by

$$\mathcal{B}_{\rm rel}(\omega) = B_{\rm rel}(\omega) \oplus B_{\rm rel}((d+d^\dagger)(\omega)) = 0. \tag{3.4}$$

Then the operator $\Delta = (d+d^\dagger)^2$ with boundary conditions $\mathcal{B}(\omega) = 0$ is self adjoint, and the relevant spaces of harmonic forms are

$$\begin{aligned} \mathcal{H}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)}\omega = 0\}, \\ \mathcal{H}^q_{\rm abs}(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)}\omega = 0, B_{\rm abs}(\omega) = 0\}, \\ \mathcal{H}^q_{\rm rel}(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)}\omega = 0, B_{\rm rel}(\omega) = 0\}. \end{aligned}$$

Following Ray and Singer, we introduce the de Rham maps $\mathcal{A}_q$:

$$\begin{aligned} \mathcal{A}^{\rm rel}_q &: \mathcal{H}^q(W, E_\rho) \to C_q((W, \partial W); E_\rho), \\ \mathcal{A}^{\rm abs}_q &: \mathcal{H}^q_{\rm abs}(W, E_\rho) \to C_q(W; E_\rho), \end{aligned}$$

both defined by (see [7] for details on the contruction)

$$\mathcal{A}^{\rm abs}_q(\omega) = \mathcal{A}^{\rm rel}_q(\omega) = (-1)^{(m-1)q} \sum_{j,i} \left(\int_{\hat c_{q,j}} (*\omega, e_i) \right) c_{q,j} \otimes_\rho e_i, \tag{3.5}$$

where the sum runs over all q-simplices $c_{q,j}$ of W in the absolute case, but runs over all q-simplices $c_{q,j}$ of $W - \partial W$ in the relative case, and $\hat c$ denotes the Poincaré dual of c.

In this situation, let a be a graded orthonormal base for the space of the harmonic forms in $\Lambda W \otimes_\rho \mathbb{R}^k$. Then, we call the positive real number

$$\tau_{\rm R}((W, g); \rho) = \tau_{\rm R}(W; \mathcal{A}^{\rm abs}(a), \rho), \tag{3.6}$$

the *Reidemeister torsion* of (W, g) with respect to the representation ρ, and

$$\tau_{\rm R}((W, \partial W, g); \rho) = \tau_{\rm R}((W, \partial W); \mathcal{A}^{\rm rel}(a), \rho), \tag{3.7}$$

the *Reidemeister torsion of the pair* $(W, \partial W, g)$ with respect to the representation ρ. It is possible to prove that both $\tau_{\rm R}((W, g); \rho)$ and $\tau_{\rm R}((W, \partial W, g); \rho)$ do not depend on the choice of the orthonormal base a.

Next, we define the analytic torsion. First assume W has no boundary. With the inner product previously defined $\Omega(W, E_\rho)$ is an Hilbert space. Let $d^\dagger = (-1)^{mq+m+1} * d *$ be the formal adjoint of d, then the Laplacian $\Delta = (d^\dagger d + d d^\dagger)$ is a symmetric non negative definite operator in $\Omega(W, E_\rho)$, and has pure point spectrum $\operatorname{Sp}\Delta$. Let $\Delta^{(q)}$ be the restriction of Δ to $\Omega^q(W, E_\rho)$. Then we define the *zeta function* of $\Delta^{(q)}$ by the series

$$\zeta(s, \Delta^{(q)}) = \sum_{\lambda \in \operatorname{Sp}_+ \Delta^{(q)}} \lambda^{-s},$$

for $\Re(s) > \frac{m}{2}$, and where Sp_+ denotes the positive part of the spectrum. The above series converges uniformly for $\Re(s) > \frac{m}{2}$, and extends to a meromorphic function analytic at $s = 0$. Following Ray and Singer [13], we define the *analytic torsion* of (W, g) with respect to the representation ρ by

$$\log T((W,g);\rho) = \frac{1}{2}\sum_{q=1}^{m}(-1)^q q\zeta'(0,\Delta^{(q)}). \tag{3.8}$$

If W has a boundary, we denote by $T_{\mathrm{abs}}((W,g);\rho)$ the number defined by equation (3.8) with Δ satisfying absolute boundary conditions, and by $T_{\mathrm{rel}}((W,g);\rho)$ the number defined by the same equation with Δ satisfying relative boundary conditions.

4. THE PROOF OF THEOREM 1.1

We proceed by taking $(W, \partial W, g)$ a fixed compact connected Riemannian manifold with boundary ∂W and metric g. We start by assuming absolute boundary conditions on ∂W for the Laplacian operator Δ on forms associated to g, as defined in equation (3.3). Since our result does not depend on the representation of the fundamental group of W, we will use the simplified notations $\tau(W)$ and $T_{\mathrm{abs}}(W)$ for the torsions $\tau_{\mathrm{R}}((W,g);\rho)$ and $T_{\mathrm{abs}}((W,g);\rho)$ defined in the previous Section 3, respectively. With this notation, Cheeger [3] proves that

$$\log T_{\mathrm{abs}}(W) = \log \tau(W) + f(\partial W),$$

where the last term only depends on the boundary. When the metric g is a product near the bundary, Lück [10] proves that

$$\log T_{\mathrm{abs}}(W) = \log \tau(W) + \frac{1}{4}\chi(\partial W)\log 2,$$

where $\chi(X)$ is the Euler characteristic of X. In the general case, a further contribution appears, that measures how the metric is *far* from a product metric:

$$\log T_{\mathrm{abs}}(W) = \log \tau(W) + \frac{1}{4}\chi(\partial W)\log 2 + A(\partial W).$$

A formula for this new *anomaly* contribution has been recently given by Brüning and Ma [2]. More precisely, in [2] (equation (0.6)) is given a formula for the ratio of the analytic torsion of two metrics, g_0 and g_1,

$$\log \frac{T_{\mathrm{abs}}(W,g_1)}{T_{\mathrm{abs}}(W,g_0)} = \frac{1}{2}\int_{\partial W}\left(B(\nabla_1^{TW}) - B(\nabla_0^{TW})\right),$$

where ∇_j^{TW} is the connection of the metric g_j, and the forms $B(\nabla_j^{TW})$ are defined in equation (4.3) below (see Section 1 of [2]), and note that we use the formula of [2] in the particular case of a flat trivial bundle F. Taking $g_1 = g$, and g_0 an opportune deformation of g, that is a product metric near the boundary,

$$A(\partial W) = \log \frac{T_{\rm abs}(W, g_1)}{T_{\rm abs}(W, g_0)},$$

and therefore

$$\log T_{\rm abs}(W) = \log \tau(W) + \frac{1}{4}\chi(\partial W)\log 2 + \frac{1}{2}\int_{\partial W}\left(B(\nabla_1^{TW}) - B(\nabla_0^{TW})\right). \tag{4.1}$$

We recall now the definition of the forms $B(\nabla_j^{TW})$. First, we need some notation from [1] Chapter III and [2] Section 1.1. For two $\mathbb{Z}/2$-graded algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A}\hat{\otimes}\mathcal{B} = \mathcal{A}\wedge\hat{\mathcal{B}}$ denotes the $\mathbb{Z}/2$-graded tensor product. For two real finite dimensional vector spaces V and E, of dimension k and n, with E Euclidean and oriented, the Berezin integral is the linear map

$$\int^B : \Lambda V^* \hat{\otimes} \Lambda E^* \to \Lambda V^*,$$

$$\int^B : \alpha\hat{\otimes}\beta \mapsto \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^{\frac{n}{2}}}\beta(e_1, \ldots, e_n)\alpha,$$

where $\{e_j\}_{j=1}^n$ is an orthonormal base of E. Let A be an skew symmetric endomorphism of E. Consider the map

$$\hat{}: A \mapsto \hat{A} = \frac{1}{2}\sum_{j,l=1}^{n}(e_j, Ae_l)\hat{e}^j \wedge \hat{e}^l.$$

For example,

$$\int^B \mathrm{e}^{-\frac{\hat{A}}{2}} = Pf\left(\frac{A}{2\pi}\right),$$

and this vanishes if $\dim E = n$ is odd. Second, recalling the splitting of the tangent bundle TW on the collar $Coll(\partial W) = [0, \epsilon) \times \partial W$ described in Section 2, we define the section s of one forms with values in the skew-adjoint endomorphisms of TW

$$s = (\nabla P_{\rm tg})_{\rm norm} + (\nabla P_{\rm norm})_{\rm tg},$$

where ∇ is the Levi-Civita connection on (W, g), and P denotes the projection associated to the splitting $TColl(\partial W) = N\partial W \otimes T\partial W$. With this notation, we set

$$\begin{aligned}\mathcal{S} &= \frac{1}{2}\sum_{k=1}^{m-1}(i^* s(e_k))_{\rm norm}\hat{e}^k,\\ \mathcal{R} &= \hat{\hat{R}},\end{aligned} \tag{4.2}$$

where $(e_{y_1}, \ldots, e_{y_{m-1}})$ is an orthonormal basis in $T\partial W$, and $\tilde{R}$ is the curvature of $i^*\nabla$, on ∂W. These quantities permit to define the forms appearing in the boundary anomaly term

$$B(\nabla^{TW}) = \frac{1}{2}\int_0^1 \int^B \mathrm{e}^{-\frac{1}{2}\mathcal{R}-u^2\mathcal{S}^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} u^{k-1}\mathcal{S}^k du. \tag{4.3}$$

Thus, in order to obtain the anomaly boundary term, by using equation (4.1), we consider the metrics

$$\begin{aligned} g_0 &= dx \otimes dx + \tilde{g}(0), \\ g_1 &= dx \otimes dx + \tilde{g}(x). \end{aligned}$$

in the system (x, y) of local coordinates introduced in Section 3 on the collar $Coll(\partial W)$ of the boundary of W. Note that in the language of bundles and conections (used in [7]), if ω denotes the connection one form associated to the metric g, and Θ be the curvature two form of the boundary, using the notation ${M^a}_b$ for the entry with line a and column b of the matrix M,

$$\begin{aligned} \mathcal{S} &= \frac{1}{2}\sum_{k=1}^{m-1} {(i^*\omega - i^*\omega_0)^x}_{y_k} \hat{e}^{y_k}, \\ \mathcal{R} = \hat{\Theta} &= \frac{1}{2}\sum_{k,l=1}^{m-1} {\Theta^{y_k}}_{y_l} \hat{e}^{y_k} \wedge \hat{e}^{\theta_l}. \end{aligned} \tag{4.4}$$

This shows that $\mathcal{S}_0$ vanishes identically. However, also $\mathcal{S}_1$ vanishes when the boundary is totally geodesic. For, by the definition in equation (4.2), $\mathcal{S}_1$ is the restriction of the form s on the boundary, and this is precisely the second fundamental form S_M of the boundary. This follows as well using the formula in equation (4.4), recalling that ${(i^*\omega)^x}_{y_k} = {\Gamma_{y_j}}^x{}_{y_k} e^{y_j}$ vanishes by Lemma 2.1. This completes the proof of Theorem 1.1, in case of absolute BC. In case of relative BC, we apply Poincaré duality for analytic torsion, as given in Proposition 2.10 of [10].

5. REIDEMEISTER TORSION AND ANALYTIC TORSION OF HALF SPHERES

We compute in this section the torsion of the half spheres. Since these are simply conected manifolds, the representation of the fundamental group only enters as dimension of the representation space. Let $\mathbb{S}^m_l$ denote the half sphere of dimension m and radius l, $\mathbb{S}^m_l = \{x \in \mathbb{R}^{m+1} \mid |x| = l, x_{m+1} \geq 0\}$. The boundary of $\mathbb{S}^m_l$ is the sphere $S^{m-1}_l = \{x \in$

$\mathbb{R}^{m+1} \mid |x| = l, x_{m+1} = 0\}$. We parameterize $\mathbb{S}^m_l$ by

$$\mathbb{S}^m_l = \begin{cases} x_1 &= l\sin\theta_m \sin\theta_{m-1}\cdots\sin\theta_3\sin\theta_2\cos\theta_1 \\ x_2 &= l\sin\theta_m \sin\theta_{m-1}\cdots\sin\theta_3\sin\theta_2\sin\theta_1 \\ x_3 &= l\sin\theta_m \sin\theta_{m-1}\cdots\sin\theta_3\cos\theta_2 \\ &\vdots \\ x_m &= l\sin\theta_m\cos\theta_{m-1} \\ x_{m+1} &= l\cos\theta_m \end{cases}$$

with $\theta_1 \in [0, 2\pi]$, $\theta_2, \ldots, \theta_{m-1} \in [0, \pi]$ and $\theta_m \in [0, \frac{\pi}{2}]$. The induced metric is

$$\begin{aligned} g &= \sin^2\theta_m g_{S^{m-1}_l} + l^2 d\theta_m \otimes d\theta_m \\ &= l^2 \left(\sum_{j=1}^{m-1} \left(\prod_{j=i+1}^{m} \sin^2\theta_j \right) d\theta_k \otimes d\theta_k + d\theta_m \otimes d\theta_m \right), \end{aligned}$$

and $\sqrt{|\det g|} = l^m (\sin\theta_m)^{m-1} (\sin\theta_{m-1})^{m-2} \cdots (\sin\theta_3)^2 (\sin\theta_2)$.

Let K be the cellular decomposition of $\mathbb{S}^m_l$, with one top cell, one $m-1$-cell and one 0-cell, $K = c^1_m \cup c^1_{m-1} \cup c^1_0$. Let the subcomplex L of K be the cellular decomposition of $\partial\mathbb{S}^m_l$, $L = c^1_{m-1} \cup c^1_0$.

We consider first the case of relative boundary conditions. Then the complex of real vector spaces of equation (3.1) reads

$$C_{\rm rel}: \ 0 \longrightarrow \mathbb{R}[c^1_m] \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \ ,$$

with preferred base $c_m = \{c^1_m\}$. To fix the base for the homology, we need a graded orthonormal base a for the harmonic forms. Since a base for $\Omega^m(\mathbb{S}^m_l)$ is $\{\sqrt{|\det g|} d\theta_1 \wedge \cdots \wedge d\theta_m\}$, we get $a_m = \left\{ \frac{\sqrt{|\det g|} d\theta_1 \wedge \cdots \wedge d\theta_m}{\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}} \right\}$. Applying the formula in equation (3.5) for the de Rham map, we obtain $h_m = \{h^1_m\}$, with

$$\begin{aligned} h^1_m = \mathcal{A}^{\rm rel}_m(a^1_m) &= \frac{1}{\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}} \int_{\rm pt} * \sqrt{|\det g|} d\theta_1 \wedge \cdots \wedge d\theta_m c^1_m \\ &= \frac{1}{\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}} c^1_m. \end{aligned}$$

As $b_q = \emptyset$, for all q, we have that

$$|\det(h_m/c_m)| = \frac{1}{\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}}, \qquad |\det(b_q/c_q)| = 1, \ 0 \le q \le m-1.$$

Applying the definition in equations (3.7) and (3.2), this proves the following result.

PROPOSITION 5.1. *Assume the notation introduced above.*

$$\begin{aligned}\tau_{\rm R}((\mathbb{S}^m_l, S^{m-1}_l, g); \rho) &= \left(\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}\right)^{(-1)^{m-1}{\rm rk}(\rho)}\\ &= \left(\frac{l^m \pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}\right)^{\frac{(-1)^{m-1}{\rm rk}(\rho)}{2}}.\end{aligned}$$

Next, we consider the case of absolute boundary conditions. By equation (3.1), the relevant complex is

$$C_{\rm abs}: \ 0 \longrightarrow \mathbb{R}[c^1_m] \longrightarrow \mathbb{R}[c^1_{m-1}] \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{R}[c^1_0] \longrightarrow 0\ ,$$

with preferred bases $c_m = \{c^1_m\}$, $c_{m-1} = \{c^1_{m-1}\}$ and $c_0 = \{c^1_0\}$. Hence, $H_p(K) = 0$, for $p > 1$, and $H_0(K) = \mathbb{R}[c^1_0]$. Since a base for $\Omega^0(\mathbb{S}^m_l)$ is the constant form $\{1\}$, we have $a_0 = \left\{\frac{1}{\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}}\right\}$. Applying the formula in equation (3.5) for the de Rham map, we obtain $h_0 = \{h^1_0\}$, with

$$\begin{aligned}h^1_0 = \mathcal{A}^{\rm abs}_0(a^1_0) &= \frac{1}{\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}}\int_{\mathbb{S}^m_l} *1 c^1_0\\ &= \sqrt{{\rm Vol}_g(\mathbb{S}^m_l)} c^1_0.\end{aligned}$$

As $b_q = \emptyset$ for $q = 0, \ldots, m-1$, $b^1_m = c^1_m$ and $\partial(b^1_m) = c_{m-1}$, we have that

$$\begin{aligned}&|\det(h_0/c_0)| = \sqrt{{\rm Vol}_g(\mathbb{S}^m_l)},\\ &|\det(\partial(b^1_m)/c_{m-1})| = 1, \qquad |\det(b_m/c_m)| = 1.\end{aligned}$$

Applying the definition in equations (3.6) and (3.2), this proves the following result.

PROPOSITION 5.2. *Assume the notation introduced above.*

$$\begin{aligned}\tau_{\rm R}((\mathbb{S}^m_l, g); \rho) &= \left(\sqrt{{\rm Vol}_g(\mathbb{S}^m_l)}\right)^{{\rm rk}(\rho)}\\ &= \left(\frac{l^m \pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}\right)^{\frac{{\rm rk}(\rho)}{2}}.\end{aligned}$$

For the analytic torsion, in the absolute case, using Theorem 1.1, we have:

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Corollary 5.1. *Assume the notation introduced above.*

$$\begin{aligned}T_{\rm abs}((\mathbb{S}^m_l,g);\rho) &= \frac{{\rm rk}(\rho)}{2}\log {\rm Vol}_g(\mathbb{S}^m_l)+\frac{1}{4}\chi(S^{m-1})\log 2\\ &= \frac{{\rm rk}(\rho)}{2}\log\frac{l^m\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}+\frac{1}{4}((m-1){\rm mod}2)\log 2.\end{aligned}$$

6. THE ANALYTIC TORSION OF THE HALF SPHERE IN DIMENSION TWO

In this section we investigate the problem of obtaining the analytic torsion by direct application of its definition. This is of some interest as an example of applicability of analytic tools in geometry (indeed this was the original aim of Ray and Singer), and also due to the few appearences in the literature of works where similar explicit calculations are performed (see for example [16] [8]).

Let $\mathbb{S}^2_l$ the hemisphere of radius l in $\mathbb{R}^3$

$$\mathbb{S}^2_l=\begin{cases} x_1 &= l\sin\theta_2\cos\theta_1\\ x_2 &= l\sin\theta_2\sin\theta_1\\ x_3 &= l\cos\theta_2\end{cases}$$

with $\theta_1\in[0,2\pi]$, $\theta_2\in[0,\frac{\pi}{2}]$. The induced metric is

$$g=l^2d\theta_2\otimes d\theta_k+l^2\sin^2\theta_2 d\theta_1\otimes d\theta_1.$$

We just need the Laplace operator on function, that reads:

$$\Delta=-\left(\frac{1}{l^2\sin^2\theta_2}\partial^2_{\theta_1}+\frac{\cos\theta_2}{l^2\sin\theta_2}\partial_{\theta_2}+\frac{1}{l^2}\partial^2_{\theta_2}\right)$$

The boundary conditions follow applying equations (3.3) and (3.4). We obtain the following absolute boundary condition

$$\text{absolute BC}\qquad \begin{array}{ll} 0-\text{forms}: & \partial_{\theta_2}f(\theta_1,\pi/2)=0,\\ 1-\text{forms}: & \begin{cases} f_{\theta_2}(\theta_1,\frac{\pi}{2})=0,\\ \partial_{\theta_2}f_{\theta_1}(\theta_1,\frac{\pi}{2})=0,\end{cases}\\ 2-\text{forms}: & f_{\theta_1\theta_2}(\theta_1,\pi/2)=0.\end{array} \tag{6.1}$$

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and relative boundary conditions

$$\text{relative BC} \qquad \begin{array}{ll} 0-\text{forms}: & f(\theta_1,\pi/2)=0, \\ 1-\text{forms}: & \left\{ \begin{array}{l} f_{\theta_1}(\theta_1,\frac{\pi}{2})=0, \\ \partial_{\theta_2} f_{\theta_2}(\theta_1,\frac{\pi}{2})=0, \end{array}\right. \\ 2-\text{forms}: & \partial_{\theta_2} f_{\theta_1\theta_2}(\theta_1,\pi/2)=0. \end{array} \tag{6.2}$$

In order to obtain a spectral resolution for the Laplace operator on functions on $\mathbb{S}^m_1$, we proceed as follows. It is well known that a spectral resolution for the Laplace operator on function on the sphere S^m_l can be obtained by using harmonic polynomials. More precisely, denote by $\mathcal{H}^m_j$ the vector space of polynomials $P(x)\in\mathbb{C}[x_0,\dots,x_m]$ that are homogeneous of degree j and harmonic with respect to the Laplace operator $\Delta=-\partial^2_{x_0}-\cdots-\partial^2_{x_m}$ in $\mathbb{R}^{m+1}$. Then, the subspace $\mathcal{H}=\sum_{n=0}^\infty \mathcal{H}^m_n$ is dense in $C^\infty(S^m_1)$ by the Stone Weierstrass theorem, and therefore $\overline{\mathcal{H}}=L^2(S^m_1)$. Moreover, taking polar coordinates $x=(r,\theta)$,

$$\Delta=-\partial_r^2-\frac{m}{r}\partial_r+\frac{1}{r^2}\Delta_{S^m_1},$$

and therefore

$$\Delta_{S^m_1}P(\theta)=n(n+m-1)P(\theta).$$

This shows that $L^2(S^m_1)=\sum_{n=0}^\infty \mathcal{H}^m_n$, and that the eigenvalue relative to the eigenspace $\mathcal{H}^m_n$ is $\lambda_n=n(n+m-1)$. It is also well know that

$$\dim\mathcal{H}^m_n=\binom{m+n}{m}-\binom{m+n-2}{m}. \tag{6.3}$$

Now, let denote by S the reflection in $\mathbb{R}^{m+1}$ obtained by changing the sign of the last coordinate. Then, S defines $\mathbb{Z}/2$-grading on $\mathcal{H}$, by setting $P_\pm=P\pm SP$, for $P\in\mathcal{H}$. It is clear that $P=\frac{1}{2}(P_+ + P_-)$, for each P, and that $SP_\pm=\pm P_\pm$.

With this notation, and recalling equations (6.1) and (6.2) for the boundary conditions, we see that a spectral resolution for $\Delta_{\mathbb{S}^m_l}$ is given by the family of eigenspaces and eigenvalues displayed below:

$$\begin{array}{ll} \text{absolute BC} & \left\{\lambda_n=\dfrac{n(n+m-1)}{l^2},(\mathcal{H}^m_n)_+\right\}, \\ \text{relative BC} & \left\{\lambda_n=\dfrac{n(n+m-1)}{l^2},(\mathcal{H}^m_n)_-\right\}. \end{array}$$

We need the dimension of these subspaces. We consider first the case $m=2$. Then, $\dim\mathcal{H}^2_n=2n+1$. Since obviously $(\mathcal{H}^2_0)_+$ consists of the constant map, the unique possible partition is $n+(n+1)$, and hence

$$\begin{aligned} \dim(\mathcal{H}^2_n)_+ &= n+1 \\ \dim(\mathcal{H}^2_n)_- &= n. \end{aligned}$$

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This can also be proved using induction, as follows. Assume we have $\dim(\mathcal{H}_n^2)_+ = n+1$, and $\dim(\mathcal{H}_n^2)_- = n$. Let P be any one of the $n+1$ generators of $(\mathcal{H}_n^2)_+$. Let $\int_z$ denote the inverse operation of ∂_z, namely $\int_z \partial_z P = P$. Then, $Q = \int_z P \in (\mathcal{H}_{n+1}^2)_-$. For Q is obviously homogeneous of degree $n+1$ and odd, by definition. Moreover, $\Delta Q = \Delta \int_z P = \int_z \Delta P = 0$, since P is harmonic.

Next, using properties of homogeneous harmonics polynomial, we can prove the following result for the general m, that is of some independent interest, and we were not able to find in the existing literature.

Proposition 6.1. *The spectrum of the Laplace operator on the m-dimensional hemisphere of radius l with Dirichlet boundary condition is $\left\{\lambda_n = \frac{n(n+m-1)}{l^2}\right\}_{n=0}^{\infty}$, and the eigenvalue λ_n has multiplicity $\binom{m+n-2}{m-1}$. The spectrum with Newmann boundary conditions is the same, but the multiplicity is $\binom{m+n-1}{m-1}$.*

Proof. Let $\mathcal{P}_n^m$ denotes the set of the homogeneous polynomial of degree n in $\mathbb{R}^{m+1}$. It is known that for each $P \in \mathcal{P}_n^m$, we have

$$P(x_0, \dots, x_m) = \sum_{j=0}^{n} Q_{n-j}(x_0, \dots, x_{m-1}) x_m^j,$$

with $Q_l \in \mathcal{P}_l^{m-1}$. Moreover, if $P \in \mathcal{H}_n^m$, then P is completely determined by Q_n and Q_{n-1}. It follows that if $P \in (\mathcal{H}_n^m)_+$, then P is completely determined by Q_n, and therefore

$$\dim(\mathcal{H}_n^m)_+ = \dim P_n^{m-1} = \binom{m+n-1}{m-1}.$$

Recalling the dimension of $\mathcal{H}_n^m$ given in equation (6.3), we conclude the proof. ∎

We point out that this result can be proved also by explicit analysis of the eigenvalues differential equation $\Delta f = \lambda f$. For, decomposing on the spectral resolution of the restriction of the Laplacian on the boundary $\{m^2, e^{im\theta_1}\}_{m\in\mathbb{Z}}$, the eigenvalues equation decomposes in the sum of the following differential equations

$$\left(\partial_{\theta_2}^2 + \frac{\cos\theta_2}{\sin\theta_2}\partial_{\theta_2} + l^2\lambda - \frac{m^2}{\sin^2\theta_2}\right) g(\theta_2) = 0.$$

With the substitution $x = \cos\theta_2$, this reduces to the Legendre equation

$$(1-x^2)\frac{d^2 g(x)}{dx^2} - 2x\frac{dg(x)}{dx} + \left(\nu(\nu+1) - \frac{m^2}{1-x^2}\right) g(x) = 0, \tag{6.4}$$

with $\lambda = \frac{\nu(\nu+1)}{l^2}$. Different pairs of linear independent solutions of equation (6.4) can be given in terms of pairs of Legendre functions, depending on the values of ν. Looking for solutions in $L^2([0,1])$, the unique solution is $P_n^m(x)$, with $\nu = n \in \mathbb{N}$, and $|m| \leq n$. Imposing the absolute boundary condition given in equation (3.3), we obtain

$$\frac{2^{m+1}}{\sqrt{\pi}} \sin\left(\frac{n+m}{2}\pi\right) \frac{\Gamma\left(\frac{n+m}{2}+1\right)}{\Gamma\left(\frac{n-m+1}{2}\right)} = 0,$$

with solutions $n+m = 2k \geq 0$. This means that in this case $\lambda = \lambda_n = \frac{n(n+1)}{l^2}$, with multiplicity $n+1$. In a similar way, imposing the relative boundary condition in equation (3.4), we obtain

$$\frac{2^m\sqrt{\pi}}{\Gamma\left(\frac{n-m}{2}+1\right)\Gamma\left(\frac{-n-m+1}{2}\right)} = 0,$$

with solutions $n+m = 2k+1 \geq 1$. This means that in this case $\lambda = \lambda_n = \frac{n(n+1)}{l^2}$, with multiplicity n.

Next, we obtain the spectrum of the Laplacian on forms. For we use the Hodge decompositions in the absolute case:

$$\begin{aligned}
\Omega^0_{\rm abs}(H^2_l) &= \mathcal{H}^0_{\rm abs}(H^2_l) \oplus d^\dagger(\Omega^1_{\rm abs}(H^2_l)),\\
\Omega^1_{\rm abs}(H^2_l) &= d(\Omega^0_{\rm abs}(H^2_l)) \oplus d^\dagger(\Omega^2_{\rm abs}(H^2_l)),\\
\Omega^2_{\rm abs}(H^2_l) &= d(\Omega^1_{\rm abs}(H^2_l)),
\end{aligned}$$

and in the relative case:

$$\begin{aligned}
\Omega^0_{\rm rel}(H^2_l) &= d^\dagger(\Omega^1_{\rm rel}(H^2_l)),\\
\Omega^1_{\rm rel}(H^2_l) &= d(\Omega^0_{\rm rel}(H^2_l)) \oplus d^\dagger(\Omega^2_{\rm rel}(H^2_l)),\\
\Omega^2_{\rm rel}(H^2_l) &= \mathcal{H}^2_{\rm rel}(H^2_l) \oplus d(\Omega^1_{\rm rel}(H^2_l)),
\end{aligned}$$

the isomorphisms

$$* : \Omega^0_{\rm abs}(H^2_l) \to \Omega^2_{\rm rel}(H^2_l), \quad * : \Omega^0_{\rm rel}(H^2_l) \to \Omega^2_{\rm abs}(H^2_l),$$

and the fact that, for any eigenfunction f, with $\Delta f = \lambda f$,

$$\begin{aligned}
\Delta^{(1)} df &= \lambda df,\\
\Delta^{(1)} d^\dagger(*f) &= \lambda d^\dagger(*f),\\
\Delta^{(2)}(*f) &= \lambda(*f).
\end{aligned}$$

Using these facts, we easily obtain that the eigenvalues of $\Delta^{(q)}$, $q = 1, 2$, are the same of Δ, and the multiplicty are given as follows. With absolute BC, 0-eigenforms have multiplicty $n+1$, closed 1-eigenforms have multiplicity $n+1$, coclosed 1-eigenforms have multiplicty n, 2-eigenforms have multiplicity n. With relative BC, 0-eigenforms have multiplicty n, closed 1-eigenforms have multiplicity n, coclosed 1-eigenforms have multiplicty $n+1$, 2-eigenforms have multiplicity $n+1$.

We can now compute the analytic torsion. Let define the following function

$$t(s) = \frac{1}{2}\sum_{q=0}^{2}(-1)^q q\zeta(s, \Delta^{(q)}),$$

such that $\log T((\mathbb{S}^2_l, g), \rho) = t'(0)$, by the definition in equation (3.8). Then,

$$t_{\rm abs}(s) = -t_{\rm rel}(s) = -\frac{l^{2s}}{2}\sum_{n=1}^{\infty}\frac{1}{(n(n+1))^s}.$$

We compute $t'_{\rm abs}(0)$ as follows. Let $t_{\rm abs}(s) = -\frac{l^{2s}}{2}v(s)$. Then,

$$\begin{aligned} v(s) &= 4^s\sum_{n=2}^{\infty}(n^2-1)^{-s} - \sum_{n=1}^{\infty}\left(n^2-\frac{1}{4}\right)^{-s} \\ &= 4^s\zeta(s,i) - z(s,i/2), \end{aligned}$$

where

$$\begin{aligned} z(s,a) &= \sum_{n=1}^{\infty}(n^2+a^2)^{-s}, \\ \zeta(s,a) &= \sum_{n=2}^{\infty}(n^2+a^2)^{-s} = z(s,a) - (1+a^2)^{-s}, \end{aligned}$$

The function $z(s,a^2) = \sum_{n=1}^{\infty}(n^2+a^2)^{-s}$ has been studied in [15] 3.1. We have (where we take the principal value of the square root)

$$\begin{aligned} z(0,a) &= -\frac{1}{2}, \\ z'(0,a) &= -\log\frac{2\sinh\pi a}{a}. \end{aligned}$$

Thus,

$$\begin{aligned} \zeta(0,a) &= -\frac{3}{2}, \\ \zeta'(0,i) &= -\log\pi, \end{aligned}$$

and this gives

$$\begin{aligned} v'(0) &= \zeta(0,i)\log 4 + \zeta'(0,i) - z'(0,i/2) \\ &= -\log 2\pi, \end{aligned}$$

and hence

$$\log T_{\rm abs}((\mathbb{S}^2_l, g), \rho) = t'_{\rm abs}(0) = -\frac{1}{2}\left(2v(0)\log l + v'(0)\right)$$
$$= \frac{1}{2}\log 2\pi l^2,$$

in agreement with the corollary of Proposition 5.2.

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