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# Oscillation for a second-order neutral differential equation with impulses

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We consider a certain type of second-order neutral delay differential systems and we establish two results concerning the oscillation of solutions after the system undergoes controlled abrupt perturbations (called impulses). As a matter of fact, some particular non-impulsive cases of the system are oscillating already. Thus, we are interested in finding adequate impulse controls under which our system remains oscillating. October, 2008 ICMC-USP

## 1. INTRODUCTION

Because systems subject to impulse effects may undergo unusual phenomena such as "beating", "dying", "merging", "noncontinuation of solutions", etc, and because they are widely used to model real-world problems in science and technology, the theory of impulsive differential systems has been attracting the attention of many mathematicians and the interest in the subject is still growing. In the last years, the action of impulses on functional differential systems has been intensively investigated.

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In this paper, we are mainly concerned with oscillating systems which remain oscillating after being perturbed by instantaneous changes of state. We consider a certain type of second-order neutral delay differential system and give sufficient conditions governing the impulse operators acting on the system so that its solutions are oscillatory.

An important application of *second-order differential equations with impulses* appears in impact theory. An impact is an interaction of bodies which happens in a short period of time and can be considered as an impulse. Billiard-type systems, for instance, can be modelled by second-order differential systems with impulses acting on the first derivatives of the solutions. Indeed, the positions of the colliding balls do not change at the moments of impact (impulses), but their velocities gain finite increments. For models describing viscoelastic bodies colliding, *systems with delay and impulses* are more appropriate. See [6].

An application of *second-order neutral delay differential equations* appears, for instance, in problems dealing with vibrating masses attached to an elastic. They also appear, as the Euler equation, in some vibrational problems. See [2, 3, 7, 9, 11, 15], for instance.

In recent years, there has been an increasing interest on the oscillatory behavior of second order nonlinear or quasilinear delay differential equations with impulse action. We refer to the papers [12, 13, 18], for example.

When considering a system subject to impulse effects, one expects that either the impulses act as a control and cease the oscillation of the system, or the impulse operators are somehow "*under control*" so that the system remains oscillating. It is known, for instance, that impulses can make oscillating systems become non-oscillating and, likewise, non-oscillating systems can become oscillating by the imposition of proper impulse controls. In [4], the authors adapt the techniques of [5] and [17] and give conditions so that the solutions of certain second-order delay differential equation oscillates. See also [1, 16, 19, 20, 23, 24, 25].

In the present paper, we consider the second-order neutral delay differential equation

$$\begin{cases} \left[r(t)\left(x(t)+p(t)x(t-\tau)\right)'\right]' + f(t,x(t),x(t-\delta)) = 0, & t \geq t_0, \quad t \neq t_k, \\ x(t_k) = I_k(x(t_k^-)), \qquad x'(t_k) = J_k(x'(t_k^-)), \quad k = 1,2,\ldots, & \\ x(t) = \phi(t), \qquad t_0 - \sigma \leq t \leq t_0, & \end{cases} \tag{1.1}$$

where $p \in PC^1([t_0, +\infty[, \mathbb{R}_+)$, $r(t)$ is a positive continuous function defined in $[t_0, +\infty[$, $\delta$ and $\tau$ are non-negative constants, $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim\limits_{k\to+\infty} t_k = +\infty$ and $t_{k+1} - t_k > \sigma$, where $\sigma := \max\{\delta, \tau\}$, and $\phi, \phi' : [t_0 - \sigma, t_0] \to \mathbb{R}$ have at most a finite number of discontinuities of the first kind and are right continuous at these points. Then we state sufficient conditions so that the solutions of system (1.1) are oscillatory.

In fact, it is known that some particular cases of (1.1) oscillate without the presence of impulses. See [10, 16, 24] for instance. Our main results, namely Theorems 2.1 and 2.2, give conditions under which system (1.1) remains oscillating. In order to obtain such result, we employ some ideas from [4] and specially from [10].

We note that when right continuity is replaced by left continuity, the results of the present paper remain valid (with obvious modifications). For left continuous functions and

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in the absence of impulses, see the results from [10] and [24], for instance. In the absence of delay, see [14] and [21].

In [17], the authors state oscillation results for the impulsive delay differential system

$$\begin{cases} (r(t)(x'(t))^\sigma)' + f(t, x(t), x(t-\delta)) = 0, & t \geq t_0, \quad t \neq t_k, \\ x(t_k) = I_k(x(t_k^-)), \qquad x'(t_k) = J_k(x'(t_k^-)), & k = 1, 2, \ldots, \\ x(t_0^+) = x_0, \quad x'(t_0^+) = x'(t_0), \end{cases}$$

where $0 < \sigma = p/q$, with $p$ and $q$ being odd integers. See also [13] and [12].

For neutral differential systems, we mention [22], where the authors consider the non-impulsive system

$$\left[ r(t) \left|(x(t) + p(t)x(\sigma(t)))'\right|^{\alpha-1} (x(t) + p(t)x(\sigma(t)))' \right]' + q(t)f(x(\sigma(t))) = 0, \qquad t \geq t_0,$$

where $\alpha$ is a positive constant. An oscillation result is proved for this system. When $\alpha = 1$ and $\sigma(t) = t - \sigma$, where $\sigma := \max\{\delta, \tau\}$ as in (1.1), our results generalize the result from [22].

In the case of neutral differential systems with impulses, we mention [18], where the author states some criteria for the oscillation of the solutions of the discrete system

$$\Delta \left( r_{n-1} |\Delta(x_{n-1} - x_{n-\tau-1})|^{\alpha-1} \Delta(x_{n-1} - x_{n-\tau-1}) \right) + f(n, x_n, x_{n-1}) = 0$$

subject to the impulse action

$$\begin{aligned} r_{n_k} &|\Delta(x_{n_k} - x_{n_k-\tau})|^{\alpha-1} \Delta(x_{n_k} - x_{n_k-\tau}) \\ &= M_k \left( r_{n_k-1} |\Delta(x_{n_k-1} - x_{n_k-\tau-1})|^{\alpha-1} \Delta(x_{n_k-1} - x_{n_k-\tau-1}) \right), \end{aligned}$$

where $\Delta x_n = x_{n+1} - x_n$, $\alpha$ is a positive constant and the impulse operator $M_k$ fulfills certain conditions, $k, \tau \in \mathbb{N}$. Thus, up to now, it seems that no result concerning oscillation of solutions for piecewise continuous neutral differential systems subject to impulses have been found yet. Hence our result is a contribution in this direction.

Furthermore, we assume that $p(t)$ in system (1.1) takes any positive value improving the usual assumption that $0 \leq p(t) \leq 1$.

## 2. MAIN RESULTS

By $w \in PC^1([T, +\infty[, \mathbb{R}_+)$ we mean the set of functions $w \in C^1([\lambda_k, \lambda_{k+1}[, \mathbb{R}_+)$, for each $k = 0, 1, 2, ...$, where $\{\lambda_k\}_{k \geq 1}$ is a sequence of positive real numbers, with $\lambda_0 = T$, and the limits $w(\lambda_k^-)$ and $w'(\lambda_k^-)$ exist, for all $k = 0, 1, 2, \ldots$. Consider the second-order neutral delay differential equation

$$\begin{cases} [r(t)(x(t)+p(t)x(t-\tau))']' + f(t,x(t),x(t-\delta)) = 0, & t \geq t_0, \quad t \neq t_k, \\ x(t_k) = I_k(x(t_k^-)), \qquad x'(t_k) = J_k(x'(t_k^-)), & k = 1,2,\ldots, \\ x(t) = \phi(t), \quad t_0 - \sigma \leq t \leq t_0, \end{cases} \tag{2.1}$$

where $\delta$ and $\tau$ are positive real numbers, $\sigma := \max\{\delta, \tau\}$, $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to+\infty} t_k = +\infty$ and $t_{k+1} - t_k > \sigma$, for all $k \in \mathbb{N}$, $p \in PC^1([t_0, +\infty[, \mathbb{R}_+)$ and $\phi, \phi' : [t_0-\sigma, t_0] \to \mathbb{R}$ have at most a finite number of discontinuities of the first kind and are right continuous at these points.

We will state oscillation results for (2.1) in two situations which we will refer to as case $A$ and case $B$.

## 2.1. Case *A*

Throughout this section we assume that

$(H_1)$ $f : [t_0-\sigma, +\infty[\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, $uf(t,u,v) > 0$ for all $uv > 0$,

$$\frac{f(t,u,v)}{\varphi(v)} \geq m(t),$$

for all $v \neq 0$, where $m(t)$ is continuous on $[t_0-\sigma, +\infty[$, $m(t) \geq 0$, and $x\varphi(x) > 0$, for all $x \neq 0$ and $\varphi'(x) \geq 0$;

$(H_2)$ $I_k$, $J_k : \mathbb{R} \to \mathbb{R}$ are continuous, with $I_k(0) = J_k(0) = 0$, and there exist positive numbers $a_k$, $b_k$ and $c_k$ such that

$$a_k \leq \frac{I_k(x)}{x} \leq b_k, \qquad J_k(x) = c_k x, \qquad x \neq 0, \quad k = 1,2,\ldots,$$

for all $k \in \mathbb{N}$.

$(H_3)$ $r$ is a positive continuous function on $[t_0, +\infty[$ and

$$\lim_{t\to+\infty} \int_{t_0}^{t} \left( \frac{1}{r(s)} \prod_{t_0<t_k<s} \frac{c_k}{\max\{b_k,\, c_k\}} \right) ds = +\infty\,.$$

$(H_4)$ $p(t)$ and $p'(t)$ are right continuous on $]t_k, t_{k+1}[$ with left lateral limits $p(t_k^-) = \dfrac{1}{c_k} p(t_k)$, and $p'(t_k^-) = \dfrac{1}{c_k} p'(t_k)$, for each $k \in \mathbb{N}$.

We start by presenting a lemma which is borrowed from [8] (see Theorem 1.4.1 there) replacing the left continuity by the right continuity of $g(t)$ and $g'(t)$ at $t_k$, for all $k \in \mathbb{N}$.

Lemma 2.1. *Suppose*

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*(i) the sequence* $\{t_k\}_{k\in\mathbb{N}}$ *satisfies* $0 \le t_0 < t_1 < \ldots < t_k < \ldots$, *with* $\lim_{k\to+\infty} t_k = +\infty$.

*(ii)* $g, g' : \mathbb{R}_+ \to \mathbb{R}$ *are continuous on* $\mathbb{R}_+ \setminus \{t_k : k \in \mathbb{N}\}$, *there exist the lateral limits* $g(t_k^-), g'(t_k^-), g(t_k^+), g'(t_k^+)$ *and* $g(t_k^+) = g(t_k)$, $k = 1, 2, \ldots$.

*(iii) for* $k = 1, 2, \ldots$ *and* $t \ge t_0$, *we have*

$$g'(t) \le p(t)\, g(t) + q(t), \qquad t \ne t_k, \tag{2.2}$$

$$g(t_k) \le \alpha_k g(t_k^-) + \beta_k, \tag{2.3}$$

*where* $p, q \in C(\mathbb{R}_+, \mathbb{R})$, $\alpha_k$ *and* $\beta_k$ *are real constants with* $\alpha_k \ge 0$.

*Then the following inequality holds*

$$\begin{aligned} g(t) \le g(t_0) \prod_{t_0<t_k<t} \alpha_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \int_{t_0}^{t} \prod_{s<t_k<t} \alpha_k \exp\left(\int_{s}^{t} p(u)du\right) q(s)ds \\ + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} \alpha_j \exp\left(\int_{t_k}^{t} p(s)ds\right) \beta_k, \qquad t \ge t_0. \end{aligned} \tag{2.4}$$

Remark 2.1. If the inequalities (2.2) and (2.3) are reversed, then the inequality (2.4) is also reversed.

For the sake of convenient notation, let $z(t) = x(t) + p(t)x(t-\tau)$.

Lemma 2.2. *Suppose* $(H_1)$ *to* $(H_4)$ *are fulfilled,* $a_k, c_k \ge 1$, $k \in \mathbb{N}$, *and there exists* $T \ge t_0$ *such that* $x(t) > 0$ *for* $t \ge T - \tau - \delta$. *Then* $z(t) > 0$ *on the interval* $[T, +\infty[$ *and* $z'(t) \ge 0$ *for* $t \in [t_k, t_{k+1}[$, *where* $t_k \ge T$ *and* $k \in \mathbb{N}$. *Furthermore,* $z(t)$ *is non-decreasing on* $[T, +\infty[$.

**Proof:** Suppose $x(t) > 0$, for $t \ge T - \tau - \delta$. Then $x(t-\tau) > 0$ for all $t \ge T - \delta$. In particular $x(t-\tau) > 0$ for all $t \ge T$ and hence

$$z(t) = x(t) + p(t)x(t-\tau) > 0, \quad t \ge T \ge t_0.$$

Now we are going to prove that $z'(t_k^-) \ge 0$, $t_k \ge T$ and $k \in \mathbb{N}$. Suppose the opposite, that is, there exists $t_{j_0} \ge T$ such that $z'(t_{j_0}^-) < 0$. Let $z'(t_{j_0}^-) = -\alpha$, with $\alpha > 0$. Since $t_{k+1} - t_k > \sigma \ge \tau$ for each $k \in \mathbb{N}$, we have

$$t_k < t_{k+1} - \tau < t_{k+1} \tag{2.5}$$

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for all $k \in \mathbb{N}$. Thus, from the continuity of $x$ and $x'$ on $[t_{k-1}, t_k[$, inequality (2.5), assumptions $(H_2)$ and $(H_4)$, and equation (2.1), we have

$$\begin{aligned} z'(t_k) &= x'(t_k) + p'(t_k)x(t_k - \tau) + p(t_k)x'(t_k - \tau) \\ &= J_k(x'(t_k^-)) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau) \\ &= c_k x'(t_k^-) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau) \\ &= c_k z'(t_k^-), \end{aligned}$$

that is, $z'(t_k) = c_k z'(t_k^-)$ for all $k \in \mathbb{N}$.

On the other hand, if $t \in ]t_k, t_{k+1}[$, $k \in \mathbb{N}$ and $t_k > T$, it follows by $(H_1)$ that

$$[r(t)z'(t)]' = -f(t, x(t), x(t-\delta)) \leq -m(t)\varphi(x(t-\delta)) \leq 0.$$

Hence $r(t)z'(t)$ is non-increasing on each interval $[t_k, t_{k+1}[$, $k \in \mathbb{N}$, such that $t_k > T$.

We now consider the impulsive differential inequality

$$\begin{aligned} (r(t)z'(t))' &\leq 0, \qquad t > t_{j_0}, \quad t \neq t_k, \quad k = j_0+1, j_0+2, \ldots, \\ z'(t_k) &= c_k z'(t_k^-), \qquad k = j_0+1, j_0+2, \ldots. \end{aligned}$$

Let $g(t) = r(t)z'(t)$. Then

$$\begin{aligned} g'(t) &\leq 0, \qquad t > t_{j_0}, \quad t \neq t_k, \quad k = j_0+1, j_0+2, \ldots, \\ g(t_k) &= c_k g(t_k^-), \qquad k = j_0+1, j_0+2, \ldots. \end{aligned}$$

By Lemma 2.1, we have

$$g(t) \leq g(t_{j_0}^-) \prod_{t_{j_0} < t_k < t} c_k,$$

that is,

$$z'(t) \leq \left(\frac{r(t_{j_0})}{r(t)}\right) z'(t_{j_0}^-) \prod_{t_{j_0} < t_k < t} c_k. \tag{2.6}$$

For $k = j_0+1, j_0+2, \ldots$, we also have

$$\begin{aligned} z(t_k) &= x(t_k) + p(t_k)x(t_k - \tau) \\ &= I_k(x(t_k^-)) + c_k p(t_k^-)x(t_k^- - \tau) \\ &\leq b_k x(t_k^-) + c_k p(t_k^-)x(t_k^- - \tau) \\ &\leq \max\{b_k,\, c_k\} z(t_k^-). \end{aligned}$$

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By (2.6) and since $z(t_k) \leq \max\{b_k,\, c_k\} z(t_k^-)$, $k = j_0+1, j_0+2, \ldots$, it follows from Lemma 2.1 that

$$z(t) \leq z(t_{j_0}^-) \prod_{t_{j_0}<t_k<t} \max\{b_k,\, c_k\} + \int_{t_{j_0}}^{t} \prod_{s<t_k<t} \max\{b_k,\, c_k\} \left[ \left( \frac{r(t_{j_0})}{r(s)} \right) z'(t_{j_0}^-) \prod_{t_{j_0}<t_k<s} c_k \right] ds$$

$$= \prod_{t_{j_0}<t_k<t} \max\{b_k,\, c_k\} \left[ z(t_{j_0}^-) - \alpha r(t_{j_0}) \int_{t_{j_0}}^{t} \left( \frac{1}{r(s)} \prod_{t_{j_0}<t_k<s} \frac{c_k}{\max\{b_k,\, c_k\}} \right) ds \right].$$

And since $z(t) > 0$ for $t \geq T$, the last inequality contradicts $(H_3)$. Therefore $z'(t_k^-) \geq 0$ for all $t_k$, $t_k \geq T$.

Since $r(t)z'(t)$ is non-increasing on $[t_k, t_{k+1}[$, it is clear that

$$z'(t) \geq \frac{r(t_{k+1}^-)}{r(t)} z'(t_{k+1}^-) \geq 0,$$

for $t \in [t_k, t_{k+1}[$, $t_k \geq T$. Finally, take any $t_k$, $k \in \mathbb{N}$, such that $t_k > T$. Then

$$\begin{aligned} z(t_k) &= x(t_k) + p(t_k)x(t_k - \tau) \\ &= I_k(x(t_k^-)) + c_k p(t_k^-) x(t_k^- - \tau) \\ &\geq a_k x(t_k^-) + c_k p(t_k^-) x(t_k^- - \tau) \\ &\geq \min\{a_k,\, c_k\} z(t_k^-) \\ &\geq z(t_k^-). \end{aligned}$$

Hence $z(t)$ is non-decreasing on $[T, +\infty[$ and the proof is complete. □

Remark 2.2. When $x(t)$ is eventually negative and $a_k, c_k \geq 1$, $k \in \mathbb{N}$, then under hypotheses $(H_1)$ to $(H_4)$ one can prove similarly that $z(t) < 0$ on the interval $[T, +\infty[$ and $z'(t) \leq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$. In particular, $z(t)$ is non-increasing on $[T, +\infty[$.

Now we present an auxiliary function whose definition is borrowed from [10] and which will be used in the proofs of the following results.

Let $\Phi \in C^2([t_0, +\infty), \mathbb{R}_+)$ be given and define $h \in C([t_0, +\infty[, \mathbb{R})$ by

$$h(t) = -\frac{\Phi'(t)}{2\Phi(t)}.$$

Now, define the function $\psi$ by

$$\psi(t) = \Phi(t) \left\{ m(t)[1 - p(t-\delta)] + r(t-\delta)h^2(t)c_k - [r(t-\delta)h(t)c_k]' \right\}$$

for each $t_k \leq t < t_{k+1}$, $k = 1, 2, 3, \ldots.$

PROPOSITION 2.1. *Suppose* $(H_1)$ *to* $(H_4)$ *are fulfilled,* $a_k, c_k \geq 1$, $k \in \mathbb{N}$ *and* $\varphi(v) = v$ *in assumption* $(H_1)$. *If equation* (2.1) *is nonoscillatory, then there exist a number* $k_0 \in \mathbb{N}$ *and a function* $w \in PC^1([t_{k_0}, +\infty[, \mathbb{R})$ *satisfying*

$$w'(t) + \psi(t) + \frac{w^2(t)}{r(t-\delta)\Phi(t)} \leq 0, \quad t_k < t < t_{k+1}, \tag{2.7}$$

*for each* $k = k_0, k_0+1, k_0+2, \ldots$.

**Proof:** Let $x(t)$ be a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $x(t) > 0$ on $[T-\tau-\delta, +\infty[$, for some $T \geq t_0$.

Recall that $z(t) = x(t) + p(t)x(t-\tau)$. By Lemma 2.2, $z(t) > 0$, $z'(t) \geq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$ and $k \in \mathbb{N}$ and $z(t)$ is non-decreasing on $[T, +\infty[$.

Let $k_0 = \min\{k : t_k \geq T, k = 1, 2, 3, \ldots\}$. By (2.1) and hypothesis $(H_1)$, we obtain

$$[r(t)z'(t)]' = -f(t, x(t), x(t-\delta)) \leq -m(t)x(t-\delta) < 0,$$

for every $t \geq T$ and $t \neq t_k$, $k \in \mathbb{N}$. Consequently, $r(t)z'(t)$ is a non-increasing function on each interval $[t_k, t_{k+1}[$, $k = k_0, k_0+1, \ldots$.

Now, we assert that

$$r(t)z'(t) \leq c_k r(t-\delta)z'(t-\delta), \tag{2.8}$$

for each $t_k \leq t < t_{k+1}$, $k = k_0, k_0+1, \ldots$. Indeed. First, note that

$$\begin{aligned} r(t_k)z'(t_k) &= r(t_k)[x'(t_k) + p'(t_k)x(t_k-\tau) + p(t_k)x'(t_k-\tau)] \\ &= r(t_k^-)[J_k(x'(t_k^-)) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau)] \\ &= r(t_k^-)[c_k x'(t_k^-) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau)] \\ &= c_k r(t_k^-)z'(t_k^-). \end{aligned}$$

If $t_k + \delta \leq t < t_{k+1}$, $k = k_0, k_0+1, \ldots$, we have $t_k \leq t-\delta < t_{k+1}-\delta < t_{k+1}$, then

$$r(t)z'(t) \leq r(t-\delta)z'(t-\delta) \leq c_k r(t-\delta)z'(t-\delta).$$

If $t_k \leq t < t_k + \delta$, $k = k_0, k_0+1, \ldots$, we have $t_{k-1} < t_k - \delta \leq t-\delta < t_k$, then

$$r(t)z'(t) \leq r(t_k)z'(t_k) = c_k r(t_k^-)z'(t_k^-) \leq c_k r(t-\delta)z'(t-\delta).$$

Thus, the assertion is proved.

Note that

$$f(t, x(t), x(t-\delta)) \geq m(t)x(t-\delta) = m(t)[z(t-\delta) - p(t-\delta)x(t-\delta-\tau)],$$

for $t \neq t_k$, $k \in \mathbb{N}$ and $t \geq T$. Then,

$$[r(t)z'(t)]' + m(t)[z(t-\delta) - p(t-\delta)x(t-\tau-\delta)] \leq [r(t)z'(t)]' + f(t, x(t), x(t-\delta)) = 0,$$

for $t \neq t_k$, $k \in \mathbb{N}$ and $t \geq T$, that is

$$[r(t)z'(t)]' + m(t)[z(t-\delta) - p(t-\delta)x(t-\delta-\tau)] \leq 0.$$

Since $z(t)$ is non-decreasing from Lemma 2.2, we have

$$x(t-\delta-\tau) \leq z(t-\delta-\tau) \leq z(t-\delta), \qquad t \geq T.$$

Then

$$m(t)z(t-\delta)[1-p(t-\delta)] \leq m(t)[z(t-\delta) - p(t-\delta)x(t-\delta-\tau)]$$

and, consequently,

$$[r(t)z'(t)]' + m(t)z(t-\delta)[1-p(t-\delta)] \leq 0,$$

for $t \geq T$, $t \neq t_k$, $k \in \mathbb{N}$.

Now, define

$$w(t) = \Phi(t)\left\{\frac{r(t)z'(t)}{z(t-\delta)} + r(t-\delta)h(t)c_k\right\},$$

for each $t \in [t_k, t_{k+1}[$, $k = k_0, k_0+1, \ldots$. Note that $w \in PC^1([t_{k_0}, +\infty), \mathbb{R})$.

We also have

$$w'(t) \leq -2h(t)w(t) + \Phi(t)\left\{-m(t)[1-p(t-\delta)] - \frac{r(t)z'(t)z'(t-\delta)}{z^2(t-\delta)} + [r(t-\delta)h(t)c_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \ldots$.

Since $r(t)z'(t) \leq c_k r(t-\delta)z'(t-\delta)$ from (2.8), we have

$$\frac{r(t)z'(t)z'(t-\delta)}{z^2(t-\delta)} \geq \frac{1}{c_k r(t-\delta)}\left(\frac{r(t)z'(t)}{z(t-\delta)}\right)^2.$$

Then

$$w'(t) \leq -2h(t)w(t) + \Phi(t)\Bigg\{-m(t)[1-p(t-\delta)] +$$

$$-\frac{1}{c_k r(t-\delta)}\left(\frac{r(t)z'(t)}{z(t-\delta)}\right)^2 + [r(t-\delta)h(t)c_k]'\Bigg\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \ldots$.

Since

$$\frac{r(t)z'(t)}{z(t-\delta)} = \frac{w(t)}{\Phi(t)} - r(t-\delta)h(t)c_k,$$

we have

$$w'(t) \leq -\frac{w^2(t)}{c_k\Phi(t)r(t-\delta)} + \Phi(t)\left\{-m(t)[1-p(t-\delta)] + \right.$$

$$-r(t-\delta)h^2(t)c_k + [r(t-\delta)h(t)c_k]'\big\},$$

Therefore,

$$w'(t) \leq -\psi(t) - \frac{w^2(t)}{r(t-\delta)\Phi(t)c_k}, \quad t_k < t < t_{k+1},$$

$k = k_0, k_0+1, \ldots$.

When $x(t)$ is eventually negative, then proof follows analogously. □

LEMMA 2.3. *If $c_k = 1$ and $a_k \geq 1$ for each $k = 1, 2, 3, \ldots$, then there is $T \geq t_0$ such that $w(t_k) - w(t_k^-) \leq 0$ for each $k \in \mathbb{N}$ with $t_k > T$.*

**Proof:** At first, given $t_k$ for some $k \in \mathbb{N}$, suppose $t_k - \delta - \tau \neq t_{k-1}$. Then,

$$\begin{aligned} w(t_k) &= \Phi(t_k)\left\{\frac{r(t_k)z'(t_k)}{z(t_k-\delta)} + r(t_k-\delta)h(t_k)c_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)c_k z'(t_k^-)}{z(t_k^- -\delta)} + r(t_k^- -\delta)h(t_k^-)c_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)z'(t_k^-)}{z(t_k^- -\delta)} + r(t_k^- -\delta)h(t_k^-)\right\} \\ &= w(t_k^-), \end{aligned}$$

$k = 1, 2, 3, \ldots$.

Now, we need to consider the case when $t_k - \delta - \tau = t_{k-1}$. Without loss of generality, we may assume that $x(t) > 0$ on $[T - \tau - \delta, +\infty[$, for some $T \geq t_0$. Then

$$\begin{aligned} z(t_k-\delta) - z(t_k^- -\delta) &= p(t_k-\delta)[x(t_k-\delta-\tau) - x(t_k^- -\delta-\tau)] \\ &= p(t_k-\delta)[x(t_{k-1}) - x(t_{k-1}^-)]. \end{aligned}$$

Since, $x(t_{k-1}) \geq a_k x(t_{k-1}^-) \geq x(t_{k-1}^-)$, it follows that

$$z(t_k-\delta) - z(t_k^- -\delta) \geq 0.$$

By Lemma 2.2, $z(t) > 0$ on the interval $[T, +\infty[$ and $z'(t) \geq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$ and $k \in \mathbb{N}$. Thus, we can conclude that

$$w(t_k) \leq w(t_k^-),$$

for $t_k > T$.

When $x(t) < 0$ on $[T - \tau - \delta, +\infty[$, for some $T \geq t_0$, the result follows analogously. □

The following theorem is an extension of Horng-Jaan Li's criteria to oscillation. See [10].

Theorem 2.1. *Suppose* $(H_1)$ *to* $(H_4)$ *are fulfilled,* $a_k \geq 1$, $c_k = 1$, $k \in \mathbb{N}$, $\varphi(v) = v$ *in assumption* $(H_1)$ *and*

$$\sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \psi(s)ds = +\infty, \tag{2.9}$$

*for* $n \in \mathbb{N}$. *If there exist sequences* $\{\alpha_n\}_{n\geq 1}$ *and* $\{\xi_n\}_{n\geq 1}$ *of positive real numbers, such that* $\xi_n \in ]t_n,\, t_{n+1}[$, $n \in \mathbb{N}$, $\limsup_{n\to+\infty}(\xi_n - t_{n+1}) > 0$, $\sum_{n=1}^{+\infty} \frac{1}{\alpha_n} < +\infty$ *and*

$$\int_{t_k}^{\xi_k} \frac{ds}{r(s-\delta)\Phi(s)c_k} \geq \alpha_k,$$

$k \in \mathbb{N}$, *then system* (2.1) *is oscillatory.*

**Proof:** Suppose system (2.1) is non-oscillatory. Then it follows from Proposition 2.1 that there exist a number $k_0 \in \mathbb{N}$ and a function $w(t) \in PC^1([t_{k_0},\, +\infty[, \mathbb{R})$ satisfying (2.7) for $t_k < t < t_{k+1}$, $k = k_0, k_0+1, k_0+2, \ldots$.

Integrating (2.7) over $[t_k,\, t_{k+1}]$, $k \in \mathbb{N}$ and $k \geq k_0$, we obtain

$$w(t_{k+1}^-) \leq w(t_k) - \int_{t_k}^{t_{k+1}} \psi(s)ds - \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)c_k} ds. \tag{2.10}$$

For $n \in \mathbb{N}$, we have

$$\sum_{k=k_0}^{k_0+n} w(t_{k+1}^-) \leq \sum_{k=k_0}^{k_0+n} w(t_k) - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \psi(s)ds - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)c_k}\, ds.$$

Consequently,

$$\begin{aligned} w(t_{k_0+n+1}^-) \;\leq\; & w(t_{k_0}) + \sum_{k=k_0+1}^{k_0+n} [w(t_k) - w(t_k^-)] - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \psi(s)ds \; + \\ & - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)c_k}\, ds, \end{aligned}$$

where $n \in \mathbb{N}$.

By Lemma 2.3 and equation (2.9), there exists $N_0 > 0$ such that

$$w(t_{k_0}) + \sum_{k=k_0+1}^{k_0+n} [w(t_k) - w(t_k^-)] - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \psi(s)ds < -1, \quad \text{for} \quad n \geq N_0.$$

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Thus

$$w(t^-_{k_0+n+1}) \leq -1 - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)c_k}\, ds, \quad \text{for } n \geq N_0.$$

Note that for all $t_{k_0+n} < \xi < t_{k_0+n+1}$, we have

$$w(\xi) \leq -1 - \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)c_{k_0+n}}\, ds, \quad \text{for } n \geq N_0. \tag{2.11}$$

Then

$$\int_{t_{k_0+n}}^{\xi} \frac{w(s)}{r(s-\delta)\Phi(s)c_{k_0+n}}\, ds \leq - \int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} +$$

$$- \int_{t_{k_0+n}}^{\xi} \frac{1}{r(s-\delta)\Phi(s)c_{k_0+n}} \left[ \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)c_{k_0+k}}\, d\mu \right] ds, \tag{2.12}$$

for each $t_{k_0+n} < \xi < t_{k_0+n+1}$ and $n \geq N_0$.

Let us consider $\xi_{k_0+n} \in ]t_{k_0+n}, t_{k_0+n+1}[$, $n \geq N_0$, given by the hypotheses, and define

$$v(\xi) = \int_{t_{k_0+n}}^{\xi} \frac{w(s)}{r(s-\delta)\Phi(s)c_{k_0+n}}\, ds, \quad \xi_{k_0+n} \leq \xi < t_{k_0+n+1}, \quad n \geq N_0.$$

Then the Cauchy-Schwartz inequality implies

$$\int_{t_{k_0+n}}^{s} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)c_{k_0+n}}\, d\mu \geq v^2(s) \left[ \int_{t_{k_0+n}}^{s} \frac{d\mu}{r(\mu-\delta)\Phi(\mu)c_{k_0+n}} \right]^{-1},$$

$\xi_{k_0+n} \leq s < t_{k_0+n+1}$.

Since

$$\int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)c_k}\, d\mu \geq \int_{t_{k_0+n}}^{s} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)c_{k_0+n}} d\mu,$$

$\xi_{k_0+n} \leq s < t_{k_0+n+1}$, then by (2.12), we get

$$\begin{aligned} v(\xi) \ \leq \ & - \int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} + \\ & - \int_{t_{k_0+n}}^{\xi} \frac{v^2(s)}{r(s-\delta)\Phi(s)c_{k_0+n}} \left[ \int_{t_{k_0+n}}^{s} \frac{d\mu}{r(\mu-\delta)\Phi(\mu)c_{k_0+n}} \right]^{-1} ds, \end{aligned}$$

where $\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$ and $n \geq N_0$.

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Now, we define $H(\xi)$ by

$$\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} + \int_{t_{k_0+n}}^{\xi} \frac{v^2(s)}{r(s-\delta)\Phi(s)c_{k_0+n}} \left[\int_{t_{k_0+n}}^{s} \frac{d\mu}{r(\mu-\delta)\Phi(\mu)c_{k_0+n}}\right]^{-1} ds,$$

$\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$ and $n \geq N_0$. Then

$$H'(\xi) = \frac{1}{r(\xi-\delta)\Phi(\xi)c_{k_0+n}} + \frac{v^2(\xi)}{r(\xi-\delta)\Phi(\xi)c_{k_0+n}} \left[\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}\right]^{-1}$$

and

$$0 \leq \int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} \leq H(\xi) \leq |v(\xi)|,$$

for $\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$, $n \geq N_0$. Then

$$\frac{H'(\xi)}{H^2(\xi)} \geq \frac{H'(\xi)}{v^2(\xi)} \geq \frac{1}{r(\xi-\delta)\Phi(\xi)c_{k_0+n}} \left[\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}\right]^{-1},$$

$\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$, $n \geq N_0$.

Integrating the above inequality from $\xi_{k_0+n}$ to $t_{k_0+n+1}$, we have

$$-\frac{1}{H(t^-_{k_0+n+1})} + \frac{1}{H(\xi_{k_0+n})} \geq$$

$$\geq \ln \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} - \ln \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}.$$

Thus

$$\frac{1}{H(\xi_{k_0+n})} \geq \ln \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} - \ln \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}},$$

$n \geq N_0$.

Since

$$H(\xi_{k_0+n}) \geq \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}} \geq \alpha_{k_0+n},$$

$n \geq N_0$, we have

$$\sum_{n=N_0}^{+\infty} \frac{1}{H(\xi_{k_0+n})} \leq \sum_{n=N_0}^{+\infty} \frac{1}{\alpha_{k_0+n}} < +\infty.$$

Thus,

$$\sum_{n=N_0}^{+\infty}\left[\ln\int_{t_{k_0+n}}^{t_{k_0+n+1}}\frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}-\ln\int_{t_{k_0+n}}^{\xi_{k_0+n}}\frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}\right]<+\infty.$$

Then,

$$\lim_{n\to+\infty}\left[\ln\int_{t_{k_0+n}}^{t_{k_0+n+1}}\frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}-\ln\int_{t_{k_0+n}}^{\xi_{k_0+n}}\frac{ds}{r(s-\delta)\Phi(s)c_{k_0+n}}\right]=0,$$

and this is a contradiction, because $\limsup_{n\to+\infty}[\xi_{k_0+n}-t_{k_0+n+1}]>0$. Hence, we finished the proof. □

Consider the following neutral delay differential equation of second-order,

$$\begin{cases}\left(x(t)+\frac{1}{t}\,x(t-1)\right)''+(t^3-t^2)x(t-1)\arctan(t)=0, & t\geq 1, \quad t\neq t_k,\\ x(t)=\phi(t), \qquad -1\leq t\leq 0, & \end{cases} \tag{2.13}$$

where $\phi,\phi':[-1,0]\to\mathbb{R}$ are continuous functions. Note that

$$r(t)=1, \qquad \text{and} \qquad p(t)=\frac{1}{t}.$$

By using the notations from [10], let $q(t)=(t^3-t^2)\arctan(t)$, $\gamma=1$ and $f(x)=x$. Choose $\Phi(t)=\frac{1}{t^2}$. Then $h(t)=\frac{1}{t}$ and we have

$$\psi(t)=\frac{(t^3-t^2)\arctan(t)}{t^2}\left(\frac{t-2}{t-1}\right)+\frac{2}{t^4}, \quad \text{for} \quad t>1.$$

Then, by using the software Maple, we obtain

$$\int_t^{+\infty}\frac{ds}{r(s-1)\Phi(s)}=\int_t^{+\infty}\psi(s)ds=+\infty,$$

for all $t\geq 1$. Therefore, from [10], Theorem 2.2, the non-impulsive system (2.13) is oscillatory.

As we did before, we now consider system (2.13) and prove that it remains oscillating after the imposition of proper impulse controls.

EXAMPLE 2.1. Consider the following second-order neutral delay differential equation

$$\begin{cases} \left(x(t) + \dfrac{1}{t}\,x(t-1)\right)'' + (t^3 - t^2)x(t-1)\arctan(t) = 0, \quad t \geq 1, \quad t \neq t_k, \\ x(t_k) = \left(\dfrac{k+1}{k}\right) x(t_k^-), \qquad x'(t_k) = x'(t_k^-), \quad k = 1, 2, \ldots, \\ x(t) = \phi(t), \qquad -1 \leq t \leq 0, \end{cases} \tag{2.14}$$

where $\phi, \phi' : [-1, 0] \to \mathbb{R}$ are continuous functions and $t_k = 2k - 1$, $k = 2, 3, 4, \ldots$. Note that $t_{k+1} - t_k = 2 > 1$, for all $k = 2, 3, 4, \ldots$.

We have

$$r(t) = 1, \quad p(t) = \frac{1}{t}, \quad a_k = b_k = \frac{k+1}{k} \quad \text{and} \quad c_k = 1,\; k = 1, 2, \ldots.$$

Let us consider $m(t) = (t^3 - t^2)\arctan(t)$. Then

$$\lim_{t\to+\infty} \int_{t_0}^{t} \left( \frac{1}{r(s)} \prod_{t_0<t_k<s} \frac{c_k}{\max\{b_k,\, c_k\}} \right) ds =$$

$$\begin{aligned} &= \int_{t_0}^{+\infty} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\ &= \int_{t_0}^{t_1} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \int_{t_1}^{t_2} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\ &\quad + \int_{t_2}^{t_3} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \cdots \\ &= (t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \cdots \\ &\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty. \end{aligned}$$

Thus hypotheses $(H_1)$ to $(H_4)$ are satisfied.

Choose $\Phi(t) = \dfrac{1}{t^2}$. Then $h(t) = \dfrac{1}{t}$ and

$$\psi(t) = \frac{m(t)}{t^2}\left(\frac{t-2}{t-1}\right) + \frac{2}{t^4}, \quad \text{for} \quad t > 1.$$

Then

$$\sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \psi(s)ds = \int_{t_n}^{+\infty} \psi(s)\, ds = +\infty,$$

for each $t_n \geq 1$.

Now, define the sequences $\{\xi_k\}_{k\geq 2}$ and $\{\alpha_k\}_{k\geq 2}$ by

$$\xi_k = 2k \qquad \text{and} \qquad \alpha_k = k^2,$$

for each $k = 2, 3, 4, \ldots.$ Note that

$$t_k < \xi_k < t_{k+1}, \qquad k = 2, 3, 4, \ldots,$$

$$\limsup_{k\to+\infty}[\xi_k - t_{k+1}] = 1,$$

$$\sum_{k=2}^{+\infty} \frac{1}{\alpha_k} = \sum_{k=2}^{+\infty} \frac{1}{k^2} < +\infty,$$

and

$$\int_{t_k}^{\xi_k} \frac{1}{r(s-\delta)\Phi(s)c_k} ds = \int_{t_k}^{\xi_k} \frac{ds}{r(s-\delta)\Phi(s)} = \int_{t_k}^{\xi_k} s^2\, ds =$$

$$= \frac{\xi_k^3}{3} - \frac{t_k^3}{3} = \frac{12k^2 - 6k + 1}{3} > k^2 = \alpha_k,$$

for each $k = 2, 3, 4, \ldots.$ Therefore, it follows from Theorem 2.1 that all solutions $x(t)$ of (2.14) are oscillatory.

## 2.2. Case $B$

In this section, we establish an oscillation result for (2.1) under the following hypotheses:

($H_1^*$) $f : [t_0 - \sigma, +\infty] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $uf(t, u, v) > 0$ for all $uv > 0$,

$$\frac{f(t, u, v)}{u} \geq n(t),$$

for all $u \neq 0$, where $n : [t_0 - \sigma, +\infty[ \to \mathbb{R}$ is a positive continuous function.

($H_2^*$) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous, with $I_k(0) = J_k(0) = 0$, and there exists positive number $a_k > 1$ such that

$$I_k(x) = J_k(x) = a_k x,$$

for all $k \in \mathbb{N}$.

We also assume that conditions ($H_3$) and ($H_4$) hold, with $c_k = a_k$ and $b_k = a_k$, for each $k \in \mathbb{N}$.

Now, define the function $\psi_1 : [t_0, +\infty[\to \mathbb{R}$ by

$$\psi_1(t) = \Phi(t)\left\{n(t)[1-p(t)] + r(t)h^2(t)a_k - [r(t)h(t)a_k]'\right\}$$

for each $t_k \le t < t_{k+1}$, $k = 1, 2, 3, \ldots$.

REMARK 2.3. Lemma 2.2 still holds if we replace hypothesis $(H_1)$ by $(H_1^*)$ and $(H_2)$ by $(H_2^*)$.

With the new conditions $(H_1^*)$ and $(H_2)^*$, we can rewrite Proposition 2.1 as follows.

PROPOSITION 2.2. *Suppose* $(H_1^*)$, $(H_2^*)$, $(H_3)$ *and* $(H_4)$ *are fulfilled. If equation* (2.1) *is nonoscillatory, then there exist a number* $k_0 \in \mathbb{N}$ *and a function* $w \in PC^1([t_{k_0}, +\infty[, \mathbb{R})$ *satisfying*

$$w'(t) + \psi_1(t) + \frac{w^2(t)}{r(t)\Phi(t)} \le 0, \quad t_k < t < t_{k+1}, \tag{2.15}$$

*for each* $k = k_0, k_0+1, k_0+2, \ldots$.

**Proof:** This proof follows the main ideas of the proof of Proposition 2.1. Let $x(t)$ be a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $x(t) > 0$ on $[T-\tau-\delta, +\infty[$, for some $T \ge t_0$.

Recall that $z(t) = x(t) + p(t)x(t-\tau)$. By Remark 2.3 and Lemma 2.2, $z(t) > 0$, $z'(t) \ge 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \ge T$ and $k \in \mathbb{N}$ and $z(t)$ is non-decreasing on $[T, +\infty[$.

Let $k_0 = \min\{k : t_k \ge T, k = 1, 2, 3, \ldots\}$. By (2.1) and hypothesis $(H_1^*)$, we obtain

$$[r(t)z'(t)]' = -f(t, x(t), x(t-\delta)) \le -n(t)x(t) < 0,$$

for every $t \ge T$ and $t \ne t_k$, $k \in \mathbb{N}$. Consequently, $r(t)z'(t)$ is a non-increasing function on each interval $[t_k, t_{k+1}[$, $k = k_0, k_0+1, \ldots$ . Since $a_k > 1$, we have

$$r(t)z'(t) \le a_k r(t)z'(t), \tag{2.16}$$

for each $t_k \le t < t_{k+1}$, $k = k_0, k_0+1, \ldots$.

Note that

$$f(t, x(t), x(t-\delta)) \ge n(t)x(t) = n(t)[z(t) - p(t)x(t-\tau)],$$

for $t \ne t_k$, $k \in \mathbb{N}$ and $t \ge T$. Then,

$$[r(t)z'(t)]' + n(t)[z(t) - p(t)x(t-\tau)] \le [r(t)z'(t)]' + f(t, x(t), x(t-\delta)) = 0,$$

for $t \ne t_k$, $k \in \mathbb{N}$ and $t \ge T$, that is

$$[r(t)z'(t)]' + n(t)[z(t) - p(t)x(t-\tau)] \le 0.$$

Since $z(t)$ is non-decreasing from Remark 2.3 and Lemma 2.2, we have

$$x(t-\tau) \leq z(t-\tau) \leq z(t), \qquad t \geq T.$$

Then

$$n(t)z(t)[1-p(t)] \leq n(t)[z(t)-p(t)x(t-\tau)]$$

and, consequently,

$$[r(t)z'(t)]' + n(t)z(t)[1-p(t)] \leq 0,$$

for $t \geq T$, $t \neq t_k$, $k \in \mathbb{N}$.

Now, define

$$w(t) = \Phi(t)\left\{\frac{r(t)z'(t)}{z(t)} + r(t)h(t)a_k\right\},$$

for each $t \in [t_k, t_{k+1}[$, $k = k_0, k_0+1, \ldots$. Note that $w \in PC^1([t_{k_0}, +\infty), \mathbb{R})$.

We also have

$$w'(t) \leq -2h(t)w(t) + \Phi(t)\left\{-n(t)[1-p(t)] - \frac{r(t)z'(t)z'(t)}{z^2(t)} + [r(t)h(t)a_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \ldots$.

Since $r(t)z'(t) \leq a_k r(t)z'(t)$ from (2.16), we have

$$\frac{r(t)z'(t)z'(t)}{z^2(t)} \geq \frac{1}{a_k r(t)}\left(\frac{r(t)z'(t)}{z(t)}\right)^2.$$

Then

$$w'(t) \leq -2h(t)w(t) + \Phi(t)\left\{-n(t)[1-p(t)] + \right.$$

$$\left. -\frac{1}{a_k r(t)}\left(\frac{r(t)z'(t)}{z(t)}\right)^2 + [r(t)h(t)a_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \ldots$.

Since

$$\frac{r(t)z'(t)}{z(t)} = \frac{w(t)}{\Phi(t)} - r(t)h(t)a_k,$$

we have

$$w'(t) \leq -\frac{w^2(t)}{a_k\Phi(t)r(t)} + \Phi(t)\left\{-n(t)[1-p(t)] - r(t)h^2(t)a_k + [r(t)h(t)a_k]'\right\},$$

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Therefore,

$$w'(t) \leq -\psi_1(t) - \frac{w^2(t)}{r(t)\Phi(t)a_k}, \quad t_k < t < t_{k+1},$$

$k = k_0, k_0+1, \ldots$. If $x(t) < 0$ in $[T-\tau-\delta, +\infty[$, for some $T \geq t_0$, the result follows analogously and we complete the proof. □

Lemma 2.4. *If* $\sum_{k=1}^{+\infty} r(t_k)h(t_k)(a_k - a_{k-1}) < +\infty$, *then* $\sum_{k=1}^{+\infty}(w(t_k) - w(t_k^-)) < +\infty$.

**Proof:** Note that

$$\begin{aligned} w(t_k) &= \Phi(t_k)\left\{\frac{r(t_k)z'(t_k)}{z(t_k)} + r(t_k)h(t_k)a_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)a_k z'(t_k^-)}{a_k z(t_k^-)} + r(t_k^-)h(t_k^-)a_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)z'(t_k^-)}{z(t_k^-)} + r(t_k^-)h(t_k^-)a_k\right\}, \end{aligned}$$

$k = 1, 2, 3, \ldots$. Since $w(t_k^-) = \Phi(t_k^-)\left\{\frac{r(t_k^-)z'(t_k^-)}{z(t_k^-)} + r(t_k^-)h(t_k^-)a_{k-1}\right\}$, $k = 1, 2, \ldots$, we have

$$w(t_k) - w(t_k^-) = r(t_k)h(t_k)(a_k - a_{k-1}),$$

$k = 1, 2, 3, \ldots$. Therefore, the result is proved. □

Next, we establish an oscillation criterium for system (2.1) satisfying hypotheses $(H_1^*)$, $(H_2^*)$, $(H_3)$ and $(H_4)$. The proof follows similarly to the proof of Theorem 2.1 by applying Lemma 2.4 instead of Lemma 2.3.

Theorem 2.2. *Suppose* $(H_1^*)$, $(H_2^*)$, $(H_3)$ *and* $(H_4)$ *are fulfilled,*

$$\sum_{k=1}^{+\infty} r(t_k)h(t_k)(a_k - a_{k-1}) < +\infty \text{ and } \sum_{k=n}^{+\infty}\int_{t_k}^{t_{k+1}} \psi(s)ds = +\infty,$$

*for* $n \in \mathbb{N}$. *If there exist sequences* $\{\alpha_n\}_{n\geq 1}$ *and* $\{\xi_n\}_{n\geq 1}$ *of positive real numbers, such that* $\xi_n \in ]t_n, t_{n+1}[$, $n \in \mathbb{N}$, $\limsup_{n\to+\infty}(\xi_n - t_{n+1}) > 0$, $\sum_{n=1}^{+\infty} \frac{1}{\alpha_n} < +\infty$ *and*

$$\int_{t_k}^{\xi_k} \frac{ds}{r(s)\Phi(s)c_k} \geq \alpha_k,$$

$k \in \mathbb{N}$, *then system* (2.1) *is oscillatory.*

Example 2.2. Consider the following second-order neutral delay differential equation

$$\begin{cases} \left(x(t)+\dfrac{1}{t}\,x(t-1)\right)''+x(t)t^2\ln{(t-1)}=0, \quad t\geq 1, \quad t\neq t_k, \\ x(t_k)=\left(\dfrac{k+1}{k}\right)x(t_k^-), \qquad x'(t_k)=\left(\dfrac{k+1}{k}\right)x'(t_k^-), \quad k=1,2,\ldots, \\ x(t)=\phi(t), \qquad -1\leq t\leq 0, \end{cases} \tag{2.17}$$

where $\phi, \phi' : [-1,0] \to \mathbb{R}$ are continuous functions and $t_k = 2k-1$, $k = 2,3,4,....$ Note that $t_{k+1} - t_k = 2 > 1$, for all $k = 2,3,4,....$

We have

$$r(t)=1, \;\; p(t)=\frac{1}{t}, \;\; a_k=\frac{k+1}{k}, \; k=1,2,\ldots.$$

Let us consider $n(t) = t^2 \ln(t-1)$. Then

$$\lim_{t\to+\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)}\prod_{t_0<t_k<s}\frac{c_k}{\max\{b_k,\,c_k\}}\right)ds=$$

$$=\lim_{t\to+\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)}\prod_{t_0<t_k<s}\frac{a_k}{\max\{a_k,\,a_k\}}\right)ds=$$

$$=\lim_{t\to+\infty}\int_{t_0}^{t}\frac{1}{r(s)}\,ds=\lim_{t\to+\infty}\int_{t_0}^{t}1ds=+\infty.$$

Thus hypotheses $(H_1^*)$, $(H_2^*)$, $(H_3)$ and $(H_4)$ are satisfied.

Choose $\Phi(t) = \dfrac{1}{t^2}$. Then $h(t) = \dfrac{1}{t}$ and

$$\psi_1(t)=\frac{\ln(t-1)}{t}(t-1)+\frac{2}{t^4}+\frac{2}{kt^4}, \quad \text{for} \quad t_k\leq t<t_{k+1}, \;\; t>1.$$

Then

$$\sum_{k=n}^{+\infty}\int_{t_k}^{t_{k+1}}\psi_1(s)ds=\int_{t_n}^{+\infty}\left(\frac{\ln(t-1)}{t}(t-1)+\frac{2}{t^4}\right)ds+\sum_{k=n}^{+\infty}\int_{t_k}^{t_{k+1}}\frac{2}{kt^4}\,ds=+\infty,$$

for each $t_n \geq 1$.

As before, let us define the sequences $\{\xi_k\}_{k\geq 2}$ and $\{\alpha_k\}_{k\geq 2}$ by $\xi_k = 2k$ and $\alpha_k = k^2$, for each $k = 2,3,4,....$ Then

$$t_k<\xi_k<t_{k+1}, \qquad k=2,3,4,...,$$

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$$\limsup_{k\to+\infty}[\xi_k - t_{k+1}] = 1,$$

$$\sum_{k=2}^{+\infty}\frac{1}{\alpha_k} = \sum_{k=2}^{+\infty}\frac{1}{k^2} < +\infty,$$

and

$$\int_{t_k}^{\xi_k}\frac{1}{r(s)\Phi(s)a_k}ds = \frac{k}{k+1}\int_{t_k}^{\xi_k} s^2\,ds =$$

$$= \left(\frac{k}{k+1}\right)\left(\frac{12k^2-6k+1}{3}\right) > k^2 = \alpha_k,$$

for each $k = 2, 3, 4, \ldots.$ We also have

$$\sum_{k=2}^{+\infty} r(t_k)h(t_k)(a_k - a_{k-1}) = \sum_{k=2}^{+\infty}\frac{-1}{(2k-1)k(k-1)} < +\infty.$$

Therefore, it follows from Theorem 2.2 that all solutions $x(t)$ of (2.17) are oscillatory.

## 3. FINAL COMMENTS AND AN OPEN PROBLEM

It worths mentioning that in [19], the authors give a counter-example to a result from [24] (namely, Lemma 1) for the non-impulsive case, when the function $p$ in (1.1) takes negative values (in $[\alpha, 0]$, with $\alpha > -1$). As a consequence, counter-examples to results from [1] and [23] appear naturally, since these papers use Lemma 1 from [24]. As a matter of fact, when $-1 < \alpha \leq p(t) \leq 0$, under the conditions of [1], [23] or [24], the solutions of the systems considered in these papers may be non-oscillatory. In view of this, a question arises: is it possible to find adequate impulse operators which, in the case where the function $p$ takes negative values, the system (2.1) is oscillatory?

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