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Dynamics in dumbbell domains III. Continuity of attractors

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In this paper we conclude the analysis started in [3] and continued in [4] concerning the behavior of the asymptotic dynamics of a dissipative reactions diffusion equation in a dumbbell domain as the channel shrinks to a line segment. In [3], we have established an appropriate functional analytic framework to address this problem and we have shown the continuity of the set of equilibria. In [4], we have analyzed the behavior of the limitting problem. In this paper we show that the attractors are upper semicontinuous and, moreover, if all equilibria of the limitting problem are hyperbolic, then they are lower semicontinuous and therfore, continuous. The continuity is obtained in L^p and H^1 norms. October, 2008 ICMC-USP

1. INTRODUCTION

We conclude in this paper the analysis started in [3] and continued in [4] concerning the asymptotic dynamics of a dissipative reaction diffusion equation and its dependence on the physical domain when the domain undergoes a singular perturbation given by the

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dumbbell type perturbation of domain. We refer to the introduction in [3] for a broad perspective of the problem.

We consider an equation of the type

$$\begin{cases} u_t - \Delta u + u = f(u) & x \in \Omega_\epsilon \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega_\epsilon \end{cases} \tag{1.1}$$

where $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geqslant 2$ and $\epsilon \in (0,1]$, is a typical dumbbell domain consisting of two disconnected domains, that we will denote by Ω, joined by a thin channel, R_ϵ, which degenerates to a line segment as the parameter ϵ approaches zero, see Figure 1. We refer to [3] Section 2, for a complete and rigorous definition of the dumbbell domain that we are considering. We would like to mention that the channels R_ϵ we consider are rather general and that they are not cylindrical in general.

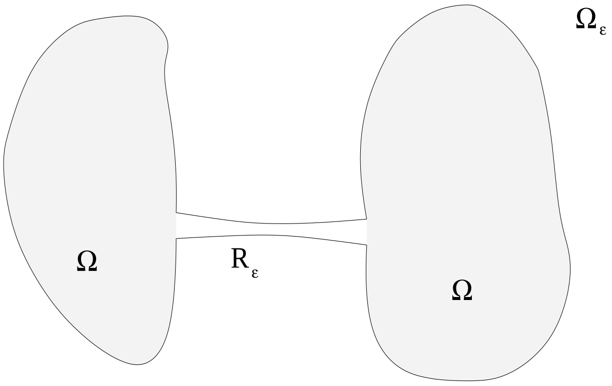

FIG. 1. Dumbbell domain

The limit "domain" will consist of the open set Ω and the line segment R_0, that without loss of generality we may assume that $R_0 = \{(x, 0, \dots, 0) : 0 < x < 1\}$, see Figure 2 of [4].

The limit equation is given by

$$\begin{cases} w_t - \Delta w + w = f(w), \quad x \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \\ \\ v_t - \dfrac{1}{g}(g v_x)_x + v = f(v), \quad x \in (0,1) \\ v(0) = w(P_0),\ v(1) = w(P_1) \end{cases} \tag{1.2}$$

where w is a function that lives in Ω, v lives in the line segment R_0 and P_0, P_1 are the points where the line segment touches the boundary of Ω. Observe that the boundary conditions of v in $(0,1)$ are given in terms of a continuity condition, so that the whole function (w,v) is continuous in the junction of Ω and R_0. The function g is related to the geometry of the channel R_ϵ, more exactly, on the way the channel R_ϵ collapses to the segment line R_0, see [3].

In [3] we developed an appropriate functional setting to treat this singular perturbation problem. We constructed the family of spaces U^p_ϵ, $0<\epsilon\leqslant 1$, in Ω_ϵ, which is the space $L^p(\Omega_\epsilon)$ with the norm

$$\|u_\epsilon\|^p_{U^p_\epsilon} = \int_\Omega |u|^p + \frac{1}{\epsilon^{N-1}}\int_{R_\epsilon} |u_\epsilon|^p.$$

We showed that the appropriate limit space should be $U^p_0 = L^p(\Omega)\oplus L^p_g(0,1)$, that is $(w,v)\in U^p_0$ if $w\in L^p(\Omega)$, $v\in L^p(0,1)$ and the norm is given by

$$\|(w,v)\|^p_{U^p_0} = \int_\Omega |w|^p + \int_0^1 g|v|^p.$$

We studied in [3] the convergence of the set of equilibria in this spaces. In fact, if $A_\epsilon : D(A_\epsilon)\subset U^p_\epsilon \to U^p_\epsilon$ is given by $A_\epsilon(u) = -\Delta u + u$ for $0<\epsilon\leqslant 1$, and $A_0 : D(A_0)\subset U^p_0\to U^p_0$ is given by $A_0(w,v) = (-\Delta u + u, -\frac{1}{g}(gv_x)_x + v)$ and studied the convergence properties of A^{-1}_ϵ to A^{-1}_0, see Proposition 2.7 of [3]. Moreover, considering the equilibria of (1.1) and (1.2) as fixed points of the nonlinear maps $A^{-1}_\epsilon\circ F_\epsilon : U^p_\epsilon\to U^p_\epsilon$ and of $A^{-1}_0\circ F_0 : U^p_0\to U^p_0$ respectively, for the appropriate nonlinearities, we showed the convergence of the equilibria see Theorem 2.3 of [3]. Also, in case the equilibrium of the limit problem (1.2) is hyperbolic, we proved the convergence of the linearizations around the equilibria and the convergence of the linear unstable manifolds. In [4] we studied in detail the properties of the limit problem in terms of generation of nonlinear semigroups, existence of attractors and their characterizations in terms of the unstable manifold of equilibria.

As we mentioned in the introduction of [3], our final objective is to compare the whole dynamics of problems (1.1) and (1.2), that is, to compare the attractors of both problems and we proposed an agenda to prove the continuity of the attractors which was based in a deep and thorough study of the linear part of the problems; that is, on the study of the convergence properties of the resolvent operators. This agenda was established in the introduction of [3] and consisted of six items. The first three were covered in [3].

In this paper we conclude the analysis by completing the last three items of the agenda. We show that the convergence of the resolvent operators A^{-1}_ϵ to A^{-1}_0 we obtain the convergence of the linear semigroups. With the variation of constants formula we show the convergence of the nonlinear semigroups, from which the upper semicontinuity of the attractors follows easily. This is done in a very similar manner as in [2].

Finally, if all the equilibria are hyperbolic, with the convergence of the equilibria and of its linear unstable manifolds, we show the convergence of the local nonlinear unstable manifolds of equilibria. Using the gradient-like structure of the limiting equation we prove lower semicontinuity (and therefore the continuity) of the attractors.

Next, we describe contents of the paper. In Section 2 we recall the general setting of the problem and state the main results of this paper; that is, the upper and lower semicontinuity of the attractors. In Section 3 we obtain the estimates of the linear resolvent operators associated to the evolution problem. We also obtain certain rates of convergence of the equilibria of the system. Based in the resolvent estimates obtained in Section 3, we analyze in Section 4 the convergence of the linear semigroups. In Section 5 we obtain the continuity of the nonlinear semigroups and the upper semicontinuity of the attractors. In Section 6 and under the assumption that all equilibria of the limit problem is hyperbolic we prove the continuity properties of the local unstable manifold which is the key to show the continuity of the attractors. Finally in Section 7 we analyze the continuity properties of the attractors in other norms.

Acknowledgement. We thank Antonio L. Pereira for several helpful comments on the estimates of sections 3 and 4.

Special dedication. The question of the continuity of attractors for reaction-diffusion equations in dumbbell domains, that is addressed in [3, 4] and in the present paper, was proposed initially by Jack K. Hale and a big amount of the ideas explored in the three articles came from him. The three authors are specially indebted for his permanent support and motivation and would like to dedicate this work to him in his 80th anniversary.

2. SETTING OF THE PROBLEM AND MAIN RESULTS

The setting is the same as the one we established initially in [3]. We recall several of the important points.

In order to compare the asymptotic dynamics of (1.1) and (1.2) we need to introduce an adequate functional analytic framework for such comparison.

Consider the spaces U^p_ϵ and U^p_0 defined in the Introduction, see also [3]. Let $0 < \epsilon \leqslant 1$ and let $A_\epsilon : D(A_\epsilon) \subset U^p_\epsilon \to U^p_\epsilon$, $1 \leqslant p < \infty$, be the linear operator defined by

$$\begin{aligned} D(A_\epsilon) &= \left\{u \in W^{2,p}(\Omega_\epsilon) : \Delta u \in U^p_\epsilon,\ \partial u/\partial n = 0 \ \text{ in } \partial\Omega_\epsilon\right\}, \\ A_\epsilon u &= -\Delta u + u, \ \ u \in D(A_\epsilon). \end{aligned} \tag{2.1}$$

Also, for $p > \frac{N}{2}$, let $A_0 : D(A_0) \subset U^p_0 \to U^p_0$ be the operator defined by

$$A_0(w,v) = \Big(-\Delta w + w, -\frac{1}{g}\,(gv')' + v\Big), \quad (w,v) \in D(A_0), \tag{2.2}$$

where

$$\begin{aligned} D(A_0) = \big\{&(w,v) \in U^p_0 : \\ &w \in D(\Delta^\Omega_N), (gv')' \in L^p(0,1), v(0) = w(P_0), v(1) = w(P_1)\big\} \end{aligned}$$

where Δ^Ω_N is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ with $D(\Delta^\Omega_N) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$.

We note that, for $p > \frac{N}{2}$ we have that $D(\Delta^\Omega_N)$ is continuously embedded in $C(\bar{\Omega})$. This tells us that the functions in $D(\Delta^\Omega_N)$ have well defined traces at P_0 and P_1.

Recall that we have defined the operator $M_\epsilon : U^p_\epsilon \to U^p_0$, as follows

$$\psi_\varepsilon \to (M_\epsilon \psi_\varepsilon)(z) = \begin{cases} \psi_\varepsilon(z), & z \in \Omega \\ \\ \dfrac{1}{|\Gamma^z_\varepsilon|} \displaystyle\int_{\Gamma^z_\varepsilon} \psi(z,y)dy, & z \in (0,1), \end{cases} \tag{2.3}$$

where $\Gamma^z_\varepsilon = \{y : (z,y) \in R_\varepsilon\}$. It is easy to see, from Fubini-Tonelli Theorem and Hölder inequality, that M_ϵ is a well defined bounded linear operator with $\|M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, U^p_0)} = 1$.

For the spaces U^p_ϵ and U^p_0 defined as above we consider the extension operators $E_\epsilon : U^p_0 \to U^p_\epsilon$ as

$$E_\epsilon(w,v)(x) = \begin{cases} w(x), \ x \in \Omega \\ v(s), \ (s,y) \in R_\epsilon. \end{cases}$$

It is very easy to see that $\|E_\epsilon(w,v)\|_{U^p_\epsilon} = \|(u,v)\|_{U^p_0}$.

The operator A_ϵ generates an analytic semigroup $\{e^{A_\epsilon t} : t \geqslant 0\}$ on U^p_ϵ whereas, from the results in [4], the operator A_0 generates a *singular semigroup* in U^p_0 that we will denote by $\{e^{-A_0 t} : t \geqslant 0\}$.

We rewrite (1.1) and (1.2) in the abstract form

$$\begin{cases} \dot{u}_\epsilon + A_\epsilon u_\epsilon = f_\epsilon(u_\epsilon) \\ u_\epsilon(0) = u^\epsilon_0 \in U^p_\epsilon \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dot{u} + A_0 u = f_0(u) \\ u(0) = u_0 \in U^p_0 \end{cases} \tag{2.5}$$

With respect to the nonlinearity f, we will assume that

(i) $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function,
(ii) $|f(u)| + |f'(u)| + |f''(u)| \leqslant C_1$ for all $u \in \mathbb{R}$.

See Remark 2.2 of [3] to see how to proceed for more general nonlinearities.

Associated to problems (2.4) and (2.5), there are nonlinear semigroups $\{T_\epsilon(t) : t \geqslant 0\}$ in U^p_ϵ and a nonlinear singular semigroup $\{T_0(t) : t \geqslant 0\}$ in U^p_0, with $p > N/2$, which have compact global attractors $\mathcal{A}_\epsilon \subset U^p_\epsilon$ and $\mathcal{A}_0 \subset U^p_0$ respectively (see [4]). In general, the attractors lie in more regular spaces and in particular, from comparison arguments, they lie in U^∞_ϵ and U^∞_0. We will also denote by $\mathcal{E}_\epsilon$ and $\mathcal{E}_0$ the set of equilibria of (1.1) and (1.2), respectively, that is $e_\epsilon \in \mathcal{E}_\epsilon$ if $A_\epsilon e_\epsilon = f_\epsilon(e_\epsilon)$ and $e_0 \in \mathcal{E}_0$ if $A_0 e_0 = f_0(e_0)$.

The following concept of E-convergence has been proved to be very appropriate when dealing with sequences of functions in different spaces, see [13, 6, 3].

Definition 2.1. We say that a sequence $\{u_\epsilon\}_{\epsilon \in (0,1]}$, $u_\epsilon \in U^p_\epsilon$, E_ϵ−converges to $u_0 \in U^p_0$ if $\|u_\epsilon - E_\epsilon u_0\|_{U^p_\epsilon} \xrightarrow{\epsilon \to 0} 0$. We write this as $u_\epsilon \xrightarrow{E} u_0$.

This notion of convergence can be extended to sets in the following manner (see [6]).

Definition 2.2. Let $\mathcal{A}_\epsilon \subset U^p_\epsilon$, $\epsilon \in [0,1]$ and $\mathcal{A}_0 = \mathcal{A} \subset U^p_0$. Denote by $\mathrm{dist}(\cdot,\cdot)$ the metric induced by the norm in U^p_ϵ, $\epsilon \in [0,1]$, i.e. $\mathrm{dist}(u_\epsilon, v_\epsilon) = \|u_\epsilon - v_\epsilon\|_{U^p_\epsilon}$.

1. We say that the family of sets $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ is E_ϵ-upper semicontinuous at $\epsilon = 0$ if $\sup_{u_\epsilon\in\mathcal{A}_\epsilon} \mathrm{dist}(u_\epsilon, E_\epsilon\mathcal{A}) \xrightarrow{\epsilon\to 0} 0$.
2. We say that the family of sets $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ is E_ϵ-lower semicontinuous at $\epsilon = 0$ if $\sup_{u\in\mathcal{A}} \mathrm{dist}(E_\epsilon u, \mathcal{A}_\epsilon) \xrightarrow{\epsilon\to 0} 0$.

Remark 2.1. In order to show the upper or lower semicontinuity of sets, the following characterizations are usefull

1. If any sequence $\{u_\epsilon\}$ with $u_\epsilon \in \mathcal{A}_\epsilon$ has a E_ϵ-convergent subsequence with limit belonging to $\mathcal{A}$, then $\{\mathcal{A}_\epsilon\}$ is E_ϵ-upper semicontinuous at zero.
2. If $\mathcal{A}$ is compact and for any $u \in \mathcal{A}$ there is a sequence $\{u_\epsilon\}$ with $u_\epsilon \in \mathcal{A}_\epsilon$, which E_ϵ-converges to u, then $\{\mathcal{A}_\epsilon\}$ is E_ϵ-lower semicontinuous at zero.

With all this concepts in mind, our main result is the following,

Theorem 2.1. *The family of attractors $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ is E_ϵ-upper semicontinuous at $\epsilon = 0$ in U^p_ϵ for every $1 \leqslant p < \infty$.*

Moreover, if every equilibria of the limit problem is hyperbolic, then the family of attractors is also E_ϵ-lower semicontinuous at $\epsilon = 0$ in U^p_ϵ for every $1 \leqslant p < \infty$.

Remark 2.2. Observe that once the statement of Theorem 2.1 is shown for a particular $p \geq 1$, then from the boundedness of the attractors in U^∞_ϵ and U^∞_0, it will also be proved for all $1 \leqslant p < \infty$.

Now consider the spaces $U^{1,2}_\epsilon = W^{1,2}(\Omega) \oplus W^{1,2}(R_\epsilon)$ with the norm

$$\|u_\epsilon\|^2_{U^{1,2}_\epsilon} = \|u_\epsilon\|^2_{W^{1,2}(\Omega)} + \frac{1}{\epsilon^{N-1}}\|u_\epsilon\|^2_{W^{1,2}(R_\epsilon)} \tag{2.6}$$

and $U^{1,2}_0 = W^{1,2}(\Omega) \oplus W^{1,2}(0,1)$ with the norm

$$\|(w,v)\|^2_{U^{1,2}_0} = \|w\|^2_{W^{1,2}(\Omega)} + \int_0^1 g(|v_x|^2 + |v|^2).$$

Observe that the spaces $U^{1,2}_\epsilon$ do not coincide algebraically with the spaces $W^{1,2}(\Omega_\epsilon)$ since we are allowing the functions of $U^{1,2}_\epsilon$ to be discontinuous at $\partial\Omega \cap \partial R_\epsilon$.

We also prove that

Theorem 2.2. *The family of attractors* $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ *is* E_ϵ*- upper semicontinuous at* $\epsilon = 0$ *in* $U^{1,2}_\epsilon$*.*

Moreover, if every equilibria of the limit problem is hyperbolic, then the family of attractors is also E_ϵ*- lower semicontinuous at* $\epsilon = 0$ *in* $U^{1,2}_\epsilon$*.*

3. RATE OF CONVERGENCE OF EQUILIBRIA AND OF RESOLVENT OPERATORS

In this section we study the convergence properties of the resolvent operators of the linear elliptic operators A_ϵ, $0 \leqslant \epsilon \leqslant 1$ as $\epsilon \to 0$. We will also need to analyze the convergence properties of the operators $A_\epsilon + V_\epsilon$ where V_ϵ are potentials. Moreover, we will be able to obtain some rates of convergence of the equilibria of the system.

3.1. Rate of convergence of resolvent operators: The case of a fixed potential

Consider a complex potential $V_0 = (V_\Omega, V_{R_0}) \in U^\infty_0$. Often, we write V_0 for $E_\epsilon V_0 \in L^\infty(\Omega_\epsilon)$. Consider also the operator in $\mathcal{L}(L^p(\Omega_\epsilon))$ and in $\mathcal{L}(U^p_0)$ which is the multiplication by the potential V_0. We denote this operator again by V_0, that is, $V_0(u_\epsilon) \equiv (E_\epsilon V_0)u_\epsilon \equiv V_0 u_\epsilon$ and $V_0(w, v) = (V_\Omega w, V_{R_0} v)$.

Let us assume that $\operatorname{Re}\sigma(A_0 + V_0) \geqslant \delta > 0$. It follows from the results in [3] that, for all suitably small ϵ, $\operatorname{Re}\sigma(A_\epsilon + V_0) \geqslant \delta > 0$.

The operator $A_\epsilon + V_0$ is sectorial and the following estimate holds

$$\| (\lambda + A_\epsilon + V_0)^{-1} \|_{\mathcal{L}(L^p(\Omega_\epsilon))} \leqslant \frac{C}{|\lambda| + 1}, \quad \text{for} \quad \lambda \in \Sigma_\theta, \tag{3.1}$$

where $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$ and C is a constant that does not depend on ϵ, although it depends on p and blows up as $p \to \infty$. This estimate follows from the fact that the localization of the numerical range in the complex plane can be done independently of ϵ, see [12].

We know that, for any $0 < \epsilon \leqslant 1$, the operator $A_\epsilon + V_0$ is a sectorial operator in U^p_ϵ and the following result holds

Lemma 3.1. *For any linear bounded operator* $J : L^p(\Omega_\epsilon) \to L^p(\Omega_\epsilon)$ *we have*

$$\|J\|_{\mathcal{L}(U^p_\epsilon)} \leqslant \|J\|_{\mathcal{L}(L^p(\Omega_\epsilon), U^p_\epsilon)} \leqslant \epsilon^{\frac{-N+1}{p}} \|J\|_{\mathcal{L}(L^p(\Omega_\epsilon))} \tag{3.2}$$

Proof: The proof of this result follows immediately from the norm estimate

$$\| \cdot \|_{U^p_\epsilon} \leqslant \epsilon^{\frac{-N+1}{p}} \| \cdot \|_{L^p(\Omega_\epsilon)}. \tag{3.3}$$

which follows directly from the definition of the norm in U^p_ϵ. ■

In particular, from Lemma 3.1 and from estimate (3.1), we have that for all $\lambda \in \Sigma_\theta$

$$\|(\lambda + A_\epsilon + V_0)^{-1}\|_{\mathcal{L}(U^p_\epsilon)} \leqslant \|(\lambda + A_\epsilon + V_0)^{-1}\|_{\mathcal{L}(L^p(\Omega_\epsilon), U^p_\epsilon)} \leqslant C \frac{\epsilon^{\frac{-N+1}{p}}}{|\lambda| + 1}, \tag{3.4}$$

for $\lambda \in \Sigma_\theta$.

As for the limit problem, from [4], we have the following result.

Proposition 3.1. *The operator $A_0 + V_0$ defined by* (2.2) *has the following properties*

i) $D(A_0 + V_0)$ *is dense in* U^p_0,

ii) $A_0 + V_0$ *is a closed operator,*

iii) $A_0 + V_0$ *has compact resolvent and*

iv) $A_0 + V_0 : D(A_0 + V_0) \subset U^p_0 \to U^p_0$ *is such that,* $\rho(A_0 + V_0) \supset \Sigma_\theta$ *where* $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$, *and for* $p \geqslant q > \frac{N}{2}$,

$$\|(\lambda + A_0 + V_0)^{-1}\|_{\mathcal{L}(U^q_0, U^p_0)} \leqslant \frac{C}{|\lambda|^\alpha + 1}, \tag{3.5}$$

$$\|(\lambda + A_0 + V_0)^{-1}\|_{\mathcal{L}(U^\infty_0)} \leqslant \frac{C}{|\lambda| + 1}, \tag{3.6}$$

$$\|(\lambda + A_0 + V_0)^{-1}\|_{\mathcal{L}(U^\infty_0, U^p_0)} \leqslant \frac{C}{|\lambda| + 1}, \tag{3.7}$$

for each $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$ *and* $\lambda \in \Sigma_\theta$.

v) If B_0 *is the realization of* A_0 *in* $C(\bar{\Omega}) \oplus L^p(0,1)$ *we have that* B_0 *is a sectorial operator in* $C(\overline{\Omega}) \oplus L^p_g(0,1)$ *with compact resolvent. Therefore* $-B_0$ *generates an analytic semigroup* $e^{-B_0 t}$ *in* $C(\overline{\Omega}) \oplus L^p_g(0,1)$.

The following result is crucial to the remaining results in this section and to the whole program of the paper.

Proposition 3.2. *There is a constant* C *independent of* ϵ *such that, for* $2 \leqslant q < \infty$ *and* $p > N$,

$$\|A^{-1}_\epsilon f_\epsilon - E_\epsilon A^{-1}_0 M_\epsilon f_\epsilon\|_{H^1(\Omega) \oplus H^1(R_\epsilon)} \leqslant C \epsilon^{N/2} \, \|f_\epsilon\|_{U^p_\epsilon}. \tag{3.8}$$

$$\|A^{-1}_\epsilon f_\epsilon - E_\epsilon A^{-1}_0 M_\epsilon f_\epsilon\|_{L^q(\Omega_\epsilon)} \leqslant C \epsilon^{N/q} \, \|f_\epsilon\|_{U^p_\epsilon}. \tag{3.9}$$

$$\|A^{-1}_\epsilon f_\epsilon - E_\epsilon A^{-1}_0 M_\epsilon f_\epsilon\|_{U^q_\epsilon} \leqslant C \epsilon^{1/q} \|f_\epsilon\|_{U^p_\epsilon}. \tag{3.10}$$

Proof: The first inequality, (3.8) was proved in Proposition A. 8 in [3]. This estimate is the key estimate for [3] and also for the complete analysis we are performing in the dumbbell domains.

Observe that in particular, from (3.8), we obtain that

$$\|A_\epsilon^{-1}f_\epsilon - E_\epsilon A_0^{-1}M_\epsilon f_\epsilon\|_{L^2(\Omega_\epsilon)} \leqslant C\epsilon^{N/2}\,\|f_\epsilon\|_{U_\epsilon^p}. \tag{3.11}$$

From [3, Lemma A.11], for $p > N/2$ we have

$$\|A_\epsilon^{-1}f_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\|f_\epsilon\|_{U_\epsilon^p}. \tag{3.12}$$

Also, if $p > N/2$, $\|A_0^{-1}M_\epsilon f_\epsilon\|_{L^\infty(\Omega)\oplus L^\infty(0,1)} \leqslant C\|M_\epsilon f_\epsilon\|_{L^p(\Omega)\oplus L^p(0,1)}$ then

$$\|E_\epsilon A_0^{-1}M_\epsilon f_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\|f_\epsilon\|_{U_\epsilon^p}. \tag{3.13}$$

which implies that

$$\|A_\epsilon^{-1}f_\epsilon - E_\epsilon A_0^{-1}M_\epsilon f_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\,\|f_\epsilon\|_{U_\epsilon^p}. \tag{3.14}$$

For $q \geqslant 2$, (3.9) follows interpolating between (3.11) and (3.14).

Estimate (3.10) follows from (3.9) and the norm estimate (3.3). ■

To obtain the resolvent convergence of $A_\epsilon + V_0$ we strongly use the previous result and the following uniform (with respect to ϵ) estimate.

Lemma 3.2. *If V_0 is such that (A_0+V_0) is invertible, for $p > \frac{N}{2}$, we have*

$$\|E_\epsilon(A_0+V_0)^{-1}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p)} \leqslant C \tag{3.15}$$

and, for each $p > \frac{N}{2}$, there is a constant C, independent of ϵ, such that

$$\|E_\epsilon(A_0+V_0)^{-1}M_\epsilon\|_{\mathcal{L}(L^p(\Omega_\epsilon))} \leqslant C. \tag{3.16}$$

Proof: Statement (3.15) follows from $\|E_\epsilon\|_{\mathcal{L}(U_0^p,U_\epsilon^p)} = \|M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,U_0^p)} = 1$ (see [3]) and from Proposition 3.1.

For (3.16) we proceed as follows. Let $f_\epsilon \in L^p(\Omega_\epsilon)$ and $u_\epsilon = (w_\epsilon, v_\epsilon) = (A_0+V_0)^{-1}M_\epsilon f_\epsilon$, then

$$\begin{cases} -\Delta w_\epsilon + w_\epsilon + V_\Omega(x)w_\epsilon = f_\epsilon, \quad \Omega, \\ \dfrac{\partial w_\epsilon}{\partial n} = 0, \quad \partial\Omega \\ -\dfrac{1}{g}\left(g(v_\epsilon)_s\right)_s + v_\epsilon + V_{R_0}(s)v_\epsilon = M_\epsilon f_\epsilon, \quad (0,1) \\ v_\epsilon(0) = w_\epsilon(P_0), \quad v_\epsilon(1) = w_\epsilon(P_1). \end{cases}$$

Since $p > \frac{N}{2}$, we have that

$$\|w_\epsilon\|_{L^p(\Omega)} \leqslant C\,\|f_\epsilon\|_{L^p(\Omega)} \quad \text{and } \|w_\epsilon\|_{C(\bar{\Omega})} \leqslant C\,\|f_\epsilon\|_{L^p(\Omega)}.$$

In particular $|w_\epsilon(P_0)| + |w_\epsilon(P_1)| \leqslant C\,\|f_\epsilon\|_{L^p(\Omega)}$. Also

$$\|v_\epsilon\|_{L^p(0,1)} \leqslant |w_\epsilon(P_0)| + |w_\epsilon(P_1)| + \|M_\epsilon f_\epsilon\|_{L^p(0,1)}$$

and

$$\begin{aligned}\|E_\epsilon v_\epsilon\|_{L^p(R_\epsilon)} &= \epsilon^{\frac{N-1}{p}}\,\|v_\epsilon\|_{L^p(0,1)}\\ &\leqslant \epsilon^{\frac{N-1}{p}}(|w_\epsilon(P_0)| + |w_\epsilon(P_1)|) + \epsilon^{\frac{N-1}{p}}\,\|M_\epsilon f_\epsilon\|_{L^p(0,1)}\\ &\leqslant |w_\epsilon(P_0)| + |w_\epsilon(P_1)| + \|f_\epsilon\|_{L^p(R_\epsilon)}\\ &\leqslant C\,\|f_\epsilon\|_{L^p(\Omega_\epsilon)}.\end{aligned}$$

where we have used that $\|M_\epsilon f_\epsilon\|_{L^p(0,1)} \leqslant \epsilon^{-\frac{N-1}{p}}\,\|f_\epsilon\|_{L^p(R_\epsilon)}$. The proof is now complete. ■

The next two lemmas are resolvent identities which allow us (together with the previous lemma) to transfer information from the resolvent convergence of A_ϵ to the resolvent convergence of $A_\epsilon + V_0$.

Lemma 3.3. *If $(A_0 + V_0)$ and $(A_\epsilon + V_0)$ are both invertible the following identity holds*

$$\begin{aligned}&(A_\epsilon + V_0)^{-1} - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon\\ &\qquad = [I - (A_\epsilon + V_0)^{-1}V_0](A_\epsilon^{-1} - E_\epsilon A_0^{-1}M_\epsilon)[I - E_\epsilon V_0(A_0 + V_0)^{-1}M_\epsilon].\end{aligned} \tag{3.17}$$

Proof: Let us first prove the following expression, which is an expresion as operators in $L^p(\Omega_\epsilon)$.

$$\begin{aligned}&(A_\epsilon^{-1} - E_\epsilon A_0^{-1}M_\epsilon)(I - E_\epsilon V_0(A_0 + V_0)^{-1}M_\epsilon) =\\ &\qquad = (I + A_\epsilon^{-1}V_0)((A_\epsilon + V_0)^{-1} - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon).\end{aligned} \tag{3.18}$$

Using that $V_0(A_0 + V_0)^{-1} = I - A_0(A_0 + V_0)^{-1}$ and expanding the left hand side of (3.18) we have

$$\begin{aligned}(A_\epsilon^{-1} - E_\epsilon A_0^{-1}M_\epsilon)(I - E_\epsilon V_0(A_0 + V_0)^{-1}M_\epsilon) &= A_\epsilon^{-1} - A_\epsilon^{-1}E_\epsilon V_0(A_0 + V_0)^{-1}M_\epsilon\\ &\quad - E_\epsilon A_0^{-1}M_\epsilon + E_\epsilon A_0^{-1}(I - A_0(A_0 + V_0)^{-1})M_\epsilon\\ &= A_\epsilon^{-1} - A_\epsilon^{-1}E_\epsilon V_0(A_0 + V_0)^{-1}M_\epsilon - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon\end{aligned}$$

On the other hand, using that $A_\epsilon^{-1} = (I + A_\epsilon^{-1}V_0)(A_\epsilon + V_0)^{-1}$ and expanding the right hand side of (3.18), we have

$$\begin{aligned}&(I + A_\epsilon^{-1}V_0)((A_\epsilon + V_0)^{-1} - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon)\\ &\qquad\qquad = A_\epsilon^{-1} - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon - A_\epsilon^{-1}E_\epsilon V_0(A_0 + V_0)^{-1}M_\epsilon.\end{aligned}$$

which shows that they both coincide.

Since $(I-(A_\epsilon+V_0)^{-1}V_0)(I+A_\epsilon^{-1}V_0)=I$, the lemma is proved. ■

In a very similar way we also have,

LEMMA 3.4. *If* (A_0+V_0) *and* $(A_\epsilon+V_0)$ *are both invertible, the following identity holds*

$$\begin{aligned}(A_\epsilon+V_0)^{-1}-E_\epsilon(A_0+V_0)^{-1}M_\epsilon &=\\ &[I-E_\epsilon(A_0+V_0)^{-1}V_0M_\epsilon](A_\epsilon^{-1}-E_\epsilon A_0^{-1}M_\epsilon)[I-V_0(A_\epsilon+V_0)^{-1}].\end{aligned}\tag{3.19}$$

Proof: The proof is similar to the one provided for the previous lemma. ■

We are now ready to prove the main results of this section

PROPOSITION 3.3. *If* (A_0+V_0) *is invertible, then for* $p,q>N$ *and* $f_\epsilon\in U_\epsilon^p$, *the following estimate,*

$$\|(A_\epsilon+V_0)^{-1}f_\epsilon-E_\epsilon(A_0+V_0)^{-1}M_\epsilon f_\epsilon\|_{L^q(\Omega_\epsilon)}\leqslant C\,\epsilon^{N/q}\|f_\epsilon\|_{U_\epsilon^p},\tag{3.20}$$

where C *depends on* $\|(A_0+V_0)^{-1}\|_{\mathcal{L}(U_0^p,U_0^p)}$ *and on* $\|V_0\|_{L^\infty}$, *but not on* ϵ *or* f_ϵ.

Proof: Let us start pointing out that if (A_0+V_0) is invertible, from [3] we also have that $(A_\epsilon+V_0)$ is invertible for ϵ small enough. Hence (3.20) makes sense.

Adding and subtracting the appropriate term in (3.17) we have:

$$\begin{aligned}(A_\epsilon+V_0)^{-1}-E_\epsilon(A_0+V_0)^{-1}M_\epsilon=&(-(A_\epsilon+V_0)^{-1}+E_\epsilon(A_0+V_0)^{-1}M_\epsilon)V_0\\ &\circ(A_\epsilon^{-1}-E_\epsilon A_0^{-1}M_\epsilon)(I-V_0E_\epsilon(A_0+V_0)^{-1}M_\epsilon)\\ +(I-E_\epsilon(A_0+V_0)^{-1}M_\epsilon V_0)&(A_\epsilon^{-1}-E_\epsilon A_0^{-1}M_\epsilon)(I-V_0E_\epsilon(A_0+V_0)^{-1}M_\epsilon).\end{aligned}$$

Let us first estimate

$$\Theta_\epsilon=((A_\epsilon+V_0)^{-1}-E_\epsilon(A_0+V_0)^{-1}M_\epsilon)\circ V_0(A_\epsilon^{-1}-E_\epsilon A_0^{-1}M_\epsilon)(I-V_0E_\epsilon(A_0+V_0)^{-1}M_\epsilon).$$

Note that, from inequality (3.10) and (3.9) we have that

$$\begin{aligned}\|A_\epsilon^{-1}-E_\epsilon A_0^{-1}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^p)}&\leqslant C\epsilon^{1/p}\text{ and}\\ \|A_\epsilon^{-1}-E_\epsilon A_0^{-1}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,L^q(\Omega_\epsilon))}&\leqslant C\epsilon^{N/q}.\end{aligned}$$

Since

$$\|V_0\|_{\mathcal{L}(L^q(\Omega_\epsilon))}\leqslant C\|V_0\|_{L^\infty(\Omega_\epsilon)}\text{ and }\|V_0\|_{\mathcal{L}(U_\epsilon^p)}\leqslant C\|V_0\|_{L^\infty(\Omega_\epsilon)},$$

it follows from (3.15) that

$$\|\Theta_\epsilon\|_{\mathcal{L}(U_\epsilon^p,L^q(\Omega_\epsilon)}\leqslant C\epsilon^{1/p}\|(A_\epsilon+V_0)^{-1}-E_\epsilon(A_0+V_0)^{-1}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,L^q(\Omega_\epsilon))}.$$

where $C = C(\|V_0\|_{L^\infty(\Omega_\epsilon)})$ is independent of ϵ. Choosing ϵ_0 such that $C\epsilon^{1/p} \leqslant \frac{1}{2}$, for all $\epsilon \in [0, \epsilon_0]$, we have that

$$\begin{aligned}\|(A_\epsilon + V_0)^{-1} &- E_\epsilon(A_0 + V_0)^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon)}\\ &\leqslant 2\|(I - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon V_0)\|_{\mathcal{L}(U^p_\epsilon)}\\ &\cdot \|(A_\epsilon^{-1} - E_\epsilon A_0^{-1}M_\epsilon)\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))}\|(I - V_0E_\epsilon(A_0 + V_0)^{-1}M_\epsilon)\|_{\mathcal{L}(L^q(\Omega_\epsilon))}.\end{aligned}$$

Now, from (3.15) and (3.16) there is a constant C, independent of ϵ, such that

$$\begin{aligned}\|(I - V_0E_\epsilon(A_0 + V_0)^{-1}M_\epsilon)\|_{\mathcal{L}(U^p_\epsilon)} &\leqslant 1 + C\|V_0\|_{L^\infty(\Omega_\epsilon)},\\ \|(I - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon V_0)\|_{\mathcal{L}(L^q(\Omega_\epsilon))} &\leqslant 1 + C\|V_0\|_{L^\infty(\Omega_\epsilon)}\end{aligned}$$

Therefore, using (3.9),

$$\|(A_\epsilon + V_0)^{-1} - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \leqslant C\,\epsilon^{N/q},$$

where the constant C does depends on $\|V_0\|_{L^\infty(\Omega_\epsilon)}$. This shows the proposition. ■

3.2. Rate of convergence of resolvent operators: The case of a varying potential

We are going to study now the convergence properties of resolvent operators of the form $(A_\epsilon + W_\epsilon)^{-1}$ to $(A_0 + W_0)^{-1}$, where W_ϵ will converge to W_0 in a sense to be specified. We need to perform this study since we want to compare the resolvent operators of the linearizations around equilibria. Hence, we will have a family of equilibria u^*_ϵ which will converge to an equilibria of the limitting problem u^*_0 and we will need to consider the operators $A_\epsilon - f'(u^*_\epsilon)$ and $A_0 - f'(u^*_0)$ and analyze the convergence properties of their resolvent.

Having this in mind, let us consider the following setting for the potentials,

(H) $V_\epsilon \in L^\infty(\Omega_\epsilon)$, $V_0 = (V_\Omega, V_{R_0}) \in U^\infty_0$ be two potentials which satisfy that $|V_\epsilon|, |V_0| \leqslant a$ for some $a > 0$ and such that for $N < q < \infty$ we have

$$\epsilon^{\frac{-N+1}{q}}\|V_\epsilon - E_\epsilon V_0\|_{L^q(\Omega_\epsilon)} \to 0, \text{ as } \epsilon \to 0 \tag{3.21}$$

Denote by $W_\epsilon = V_\epsilon + a$, $W_0 = V_0 + a = (V_\Omega + a, V_{R_0} + a)$ so that W_ϵ and W_0 are positive and they also satisfy an estimate like (3.21) substituting V_ϵ and V_0 by W_ϵ and W_0 respectively.

As we did in Subsection 3.1, let us identify the potentials W_ϵ, W_0 with their corresponding multiplication operators.

With this notation and writing $\Lambda_\epsilon = A_\epsilon + W_\epsilon$, we have that the operator Λ_ϵ is sectorial and the following estimate holds

$$\|\,(\lambda + \Lambda_\epsilon)^{-1}\,\|_{\mathcal{L}(L^p(\Omega_\epsilon))} \leqslant \frac{C}{|\lambda| + 1}, \quad \text{for} \quad \lambda \in \Sigma_\theta, \tag{3.22}$$

where $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$ and C is a constant that does not depend on ϵ (that follows form the fact that the localization of the numerical range in the complex plane can be done independently of ϵ), however it depends on p and blows up as $p \to \infty$, see [12].

We know that, for any $1 \geqslant \epsilon > 0$, the operator Λ_ϵ is a sectorial operator in U^p_ϵ and the following result holds

Lemma 3.5. *For all* $\lambda \in \Sigma_\theta$ *we have that*

$$\|(\lambda + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(U^p_\epsilon)} \leqslant \|(\lambda + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(L^p(\Omega_\epsilon), U^p_\epsilon)} \leqslant C\frac{\epsilon^{\frac{-N+1}{p}}}{|\lambda| + 1}. \tag{3.23}$$

Proof: It follows immediately from (3.22) and from Lemma 3.1. ■

The following result follows easily from the properties of resolvent operators. It is crucial to obtain convergence properties for resolvent operators from the convergence properties of Λ_ϵ^{-1} to Λ_0^{-1}.

Lemma 3.6. *As an immediate consequence of* (3.5), (3.6) *and* (3.7), *there is a constant* C *such that, for all* $\lambda \in \Sigma_\theta$, $p \geqslant q > \frac{N}{2}$ *and* $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$

$$\|E_\epsilon(\lambda + \Lambda_0)^{-1} M_\epsilon\|_{\mathcal{L}(U^q_\epsilon, U^p_\epsilon)} \leqslant \frac{C}{|\lambda|^\alpha + 1}, \tag{3.24}$$

$$\|E_\epsilon(\lambda + \Lambda_0)^{-1} M_\epsilon\|_{\mathcal{L}(C(\bar{\Omega}_\epsilon), L^\infty(\Omega_\epsilon))} \leqslant \frac{C}{|\lambda| + 1}, \tag{3.25}$$

and

$$\|E_\epsilon \lambda(\lambda + \Lambda_0)^{-1} M_\epsilon\|_{\mathcal{L}(C(\bar{\Omega}_\epsilon), U^p_\epsilon)} \leqslant C \tag{3.26}$$

where C *is a constant that does not depend in* ϵ.

We have now the following key result, which is analogous to Proposition 3.2 and Proposition 3.3

Proposition 3.4. *For* $p, q > N$ *and* $f_\epsilon \in U^p_\epsilon$ *we have*

$$\|\Lambda_\epsilon^{-1} f_\epsilon - E_\epsilon \Lambda_0^{-1} M_\epsilon f_\epsilon\|_{L^q(\Omega_\epsilon)} \leqslant C(\epsilon^{\frac{N}{q}} + \|W_\epsilon - E_\epsilon W_0 M_\epsilon\|_{L^q(\Omega_\epsilon)}) \|f_\epsilon\|_{U^p_\epsilon}. \tag{3.27}$$

with C *independent of* ϵ *and* f_ϵ.

Proof. Let $f_\epsilon \in U^p_\epsilon$ and let $u_\epsilon = \Lambda_\epsilon^{-1} f_\epsilon = (A_\epsilon + W_\epsilon)^{-1} f_\epsilon$. Consider the auxiliary function, $\tilde{u}_\epsilon = (A_\epsilon + E_\epsilon W_0)^{-1} f_\epsilon$, i.e.,

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon + W_\epsilon u_\epsilon & = f_\epsilon, \quad \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} & = 0, \quad \partial\Omega_\epsilon. \end{cases} \tag{3.28}$$

$$\begin{cases} -\Delta\tilde{u}_\epsilon + \tilde{u}_\epsilon + W_0\tilde{u}_\epsilon & = f_\epsilon, \quad \Omega_\epsilon, \\ \frac{\partial\tilde{u}_\epsilon}{\partial n} & = 0, \quad \partial\Omega_\epsilon. \end{cases} \tag{3.29}$$

From comparison results, it is easy to see that $|\tilde{u}_\epsilon| \leqslant \bar{u}_\epsilon$ where

$$\begin{cases} -\Delta\bar{u}_\epsilon + \bar{u}_\epsilon & = |f_\epsilon|, \quad \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} & = 0, \quad \partial\Omega_\epsilon. \end{cases}$$

Applying Lemma A.11 of [3], we have that

$$\|\bar{u}_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\|f_\epsilon\|_{U_\epsilon^p} \quad \text{for}\, p > N/2 \tag{3.30}$$

which implies

$$\|\tilde{u}_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\,\|f_\epsilon\|_{U_\epsilon^p}.$$

Next, observe that

$$u_\epsilon = (A_\epsilon + E_\epsilon W_0)^{-1} f_\epsilon + (A_\epsilon + E_\epsilon W_0)^{-1}(E_\epsilon W_0 - W_\epsilon)u_\epsilon$$

and

$$u_0 = (A_0 + W_0)^{-1} M_\epsilon f_\epsilon.$$

Hence,

$$\begin{aligned} \|u_\epsilon - E_\epsilon u_0\|_{L^q(\Omega_\epsilon)} &\leqslant \|(A_\epsilon + E_\epsilon W_0)^{-1} - E_\epsilon (A_0 + W_0)^{-1} M_\epsilon f_\epsilon\|_{L^q(\Omega_\epsilon)} \\ &\quad + \|(A_\epsilon + E_\epsilon W_0)^{-1}(W_\epsilon - E_\epsilon W_0)u_\epsilon\|_{L^q(\Omega_\epsilon)} \\ &\leqslant \tilde{C}(\epsilon^{\frac{N}{q}} + \|W_\epsilon - E_\epsilon W_0\|_{L^q(\Omega_\epsilon)})\|f_\epsilon\|_{U_\epsilon^p}, \end{aligned}$$

where we have used (3.20) and the fact that that there is a constant C, independent of ϵ and of $q \in [1, \infty]$, such that $\|(A_\epsilon + W_0)^{-1}\|_{\mathcal{L}(L^q(\Omega_\epsilon))} \leqslant C$. This shows the lemma. ■

As an immediate corollary, we have

Corollary 3.1. *For $p, q > N$ we have*

$$\epsilon^{-\frac{N-1}{q}}\,\|\Lambda_\epsilon^{-1} - E_\epsilon \Lambda_0^{-1} M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, L^q(\Omega_\epsilon))} \to 0 \ \textit{as}\ \epsilon \to 0. \tag{3.31}$$

Proof. We just need to apply the previous proposition and hypothesis (**H**). ■

Now consider a compact subset K of the complex plane which is contained in the resolvent set of the operator Λ_0. Let $c(K)$ be a positive constant such that

$$\sup_{\lambda \in K} \|(\lambda + \Lambda_0)^{-1}\|_{\mathcal{L}(U_0^p, U_0^p)} \leqslant c(K).$$

Also, let $\Sigma_\theta := \{z \in C : |\arg(z)| \leqslant \pi - \theta\}$, for $0 < \theta < \pi/2$.

Proposition 3.5. *For $p, q > N$, there exists a constant $C = C(K, \theta)$, a number $\epsilon_0 > 0$ and a function $\eta(\epsilon) \to 0$ as $\epsilon \to 0$ such that for each $\lambda \in K \cup \Sigma_\theta$ and $0 < \epsilon \leqslant \epsilon_0$ we have*

$$\epsilon^{-\frac{N-1}{q}} \|(\lambda+\Lambda_\epsilon)^{-1} - E_\epsilon(\lambda+\Lambda_0)^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \leqslant C\eta(\epsilon)(1+|\lambda|^{1-\alpha}), \tag{3.32}$$

where $0 < \alpha < 1 - \frac{N}{2p} < 1$

Proof: Observe first that the spectrum of the operators Λ_ϵ and Λ_0 are subsets of $[1, +\infty)$. Hence, if $\lambda \in \Sigma_\theta$ both $(\lambda+\Lambda_\epsilon)^{-1}$ and $(\lambda+\Lambda_0)^{-1}$ make perfect sense for $0 < \epsilon \leqslant \epsilon_0$.

Moreover, by the compact convergence of $A_\epsilon^{-1} \to A_0^{-1}$, the convergence of $W_\epsilon \to W_0$ and since $\|(\lambda+\Lambda_0)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} = \|(\lambda+V_0+A_0)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \leqslant c(K)$ for each $\lambda \in K$ which is a compact set in C, we have that $(\lambda + A_\epsilon + V_\epsilon)$ and $(\lambda + A_\epsilon + V_0)$ are invertible for $0 < \epsilon < \epsilon_0$ and $\lambda \in \Lambda_0$ and $\|(\lambda + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \leqslant \tilde{c}(K)$, for some constant $\tilde{c}(K)$ and for all $\lambda \in K$. If this is not the case, then we could get a sequence of $\epsilon_n \to 0$ and $\lambda_n \to \tilde{\lambda} \in K$ such that $\|(\lambda_n + \Lambda_{\epsilon_n})^{-1}\|_{\mathcal{L}(U^p_0, U^p_0)} \to +\infty$. But this is in contradiction with the compact convergence of $(\lambda_n + \Lambda_{\epsilon_n})^{-1}$ to $(\tilde{\lambda} + \Lambda_0)^{-1}$, see Lemma 4.7 of [3].

Hence, with this argument and with (3.22) and (3.24) we obtain

$$\|\lambda\,(\lambda + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(L^q(\Omega_\epsilon))} \leqslant C, \quad \text{for} \quad \lambda \in K \cup \Sigma_\theta, \tag{3.33}$$

$$\|E_\epsilon\lambda(\lambda + \Lambda_0)^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, U^p_\epsilon)} \leqslant C(1 + |\lambda|^{1-\alpha}), \quad \text{for} \quad \lambda \in K \cup \Sigma_\theta. \tag{3.34}$$

with $0 < \alpha < 1 - \frac{N}{2p} < 1$. Applying Lemma 3.3 with Λ_0 and λ instead of A_0 and V_0, applying the appropriate norms, we have

$$\begin{aligned}
&\|(\lambda + \Lambda_\epsilon)^{-1} - E_\epsilon(\lambda + \Lambda_0)^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \\
&\qquad \leqslant \|I + \lambda(\lambda + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(L^q(\Omega_\epsilon))}\|\Lambda_\varepsilon^{-1} - E_\epsilon\Lambda_0^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \\
&\qquad\quad \|[I - E_\epsilon\lambda(\lambda + \Lambda_0)^{-1}M_\epsilon]\|_{\mathcal{L}(U^p_\epsilon)} \\
&\qquad \leqslant C(1 + |\lambda|^{1-\alpha})\|\Lambda_\varepsilon^{-1} - E_\epsilon\Lambda_0^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \\
&\qquad \leqslant C\epsilon^{\frac{N-1}{q}}(1 + |\lambda|^{1-\alpha})\eta(\epsilon)
\end{aligned} \tag{3.35}$$

where $\eta(\epsilon) = \epsilon^{-\frac{N-1}{q}} \|\Lambda_\epsilon^{-1} - E_\epsilon\Lambda_0^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \to 0$ as $\epsilon \to 0$ by Corollary 3.1. This proves the proposition. ■

Remark 3.1. The results of Proposition 3.5 also hold for the operator A_ϵ instead of Λ_ϵ, that is with $W_\epsilon = W_0 = 0$.

Corollary 3.2. *In the conditions of Proposition* 3.5, *we have the following estimates,*

$$\|(\lambda + \Lambda_\epsilon)^{-1} - E_\epsilon(\lambda + \Lambda_0)^{-1}M_\epsilon\|_{\mathcal{L}(U^p_\epsilon, U^q_\epsilon)} \leqslant C\,(1 + |\lambda|^{1-\alpha}) \cdot \eta(\epsilon), \tag{3.36}$$

$$\|(\lambda + \Lambda_\epsilon)^{-1}\|_{\mathcal{L}(U^p_\epsilon, U^q_\epsilon)} \leqslant C(1 + |\lambda|^{1-\alpha}) \tag{3.37}$$

Proof: To prove (3.36) we apply $\epsilon^{-\frac{N-1}{q}}\,\|\cdot\|_{\mathcal{L}(U^p_\epsilon, L^q(\Omega_\epsilon))} \leqslant \|\cdot\|_{\mathcal{L}(U^p_\epsilon, U^q_\epsilon)}$ to (3.32). To prove (3.37) we just use (3.36) and (3.24), to obtain

$$\|(\lambda+\Lambda_\epsilon)^{-1}\|_{\mathcal{L}(U^p_\epsilon, U^q_\epsilon)} \leqslant C(1+|\lambda|^{1-\alpha})\eta(\epsilon) + \frac{C}{|\lambda|^\alpha + 1} \leqslant C(1+|\lambda|^{1-\alpha}),$$

as we wanted to show. ■

These results play a fundamental role on the convergence of the linear semigroups for it will ensure the uniform convergence of the integrals defining them and will allow us to pass to the limit.

3.3. Rate of convergence of hyperbolic equilibria and of its linearizations

In this Subsection we will obtain certain rates of convergence of hyperbolic equilibria which, besides being interesting by themselves, they show that if we consider the potentials $V_\epsilon = -f'(u^*_\epsilon)$, $V_0 = -f'(u^*_0)$ then hypothesis (**H**) from Subsection 3.2 is satisfied, with $a = \sup\{|f'(s)| : s \in \mathbb{R}\}$. This in turn will imply that if we define $\Lambda_\epsilon = A_\epsilon - f'(u^*_\epsilon) + a$ and $\Lambda_0 = A_0 - f'(u^*_0) + a$, then we can apply all the results from Subsection 3.2 to this case.

PROPOSITION 3.6. *Let u^*_0 be a hyperbolic equilibrium for* (1.2) *and* (*from the results in [3]*) *let u^*_ϵ be the sequence of hyperbolic equilibria for* (1.1) *satisfying that u^*_ϵ E-converges to u^*_0. Then, for $q > N$, we have*

$$\|u^*_\epsilon - E_\epsilon u^*_0\|_{L^q(\Omega)} \leqslant C\epsilon^{\frac{N}{q}} \tag{3.38}$$

and

$$\epsilon^{-\frac{N-1}{q}}\|u^*_\epsilon - E_\epsilon u^*_0\|_{U^p_\epsilon} \to 0, \qquad \textit{as } \epsilon \to 0. \tag{3.39}$$

Proof Let $u^*_0 = (w^*_0, v^*_0)$ be a hyperbolic equilibrium point for (1.2) and u^*_ϵ an equilibrium point for (1.1) with $\|u^*_\epsilon - E_\epsilon u^*_0\|_{U^p_\epsilon} \overset{\epsilon\to 0}{\longrightarrow} 0$. For $V_0(x) = -f'(u^*_0(x))$, we write

$$u^*_\epsilon = (A_\epsilon + V_0)^{-1}(f(u^*_\epsilon) + V_0 u^*_\epsilon) \text{ and } u^*_0 = (A_0 + V_0)^{-1}(f(u^*_0) + V_0 u^*_0).$$

Hence, taking norms in $L^q(\Omega)$, we get

$$\begin{aligned}
\|u^*_\epsilon - E_\epsilon u^*_0\|_{L^q(\Omega)} &= \|(A_\epsilon + V_0)^{-1}(f(u^*_\epsilon) + V_0 u^*_\epsilon) - E_\epsilon(A_0 + V_0)^{-1}(f(u^*_0) + V_0 u^*_0)\|_{L^q(\Omega)} \\
&\leqslant \|(A_\epsilon + V_0)^{-1} - E_\epsilon(A_0 + V_0)^{-1}M_\epsilon)\,(f(u^*_\epsilon) + V_0 u^*_\epsilon)\|_{L^q(\Omega)} \\
&+ \|E_\epsilon(A_0 + V_0)^{-1}M_\epsilon\,[f(u^*_\epsilon) - V_0 u^*_\epsilon - E_\epsilon(f(u^*_0) + V_0 M_\epsilon E_\epsilon u^*_0)]\|_{L^q(\Omega)} \\
&\leqslant C\,\epsilon^{N/q}\|f(u^*_\epsilon) + V_0 u^*_\epsilon\|_{L^q(\Omega_\epsilon)} \\
&+ \|E_\epsilon(A_0 + V_0)^{-1}M_\epsilon[f(u^*_\epsilon) - E_\epsilon f(u^*_0) - V_0(u^*_\epsilon - E_\epsilon u^*_0)]\|_{L^q(\Omega)} \\
&\leqslant C\,\epsilon^{N/q} + \|E_\epsilon(A_0 + V_0)^{-1}M_\epsilon z_\epsilon\|_{L^q(\Omega)}.
\end{aligned}$$

where $z_\epsilon = f(u_\epsilon^*) - f(u_0^*) + V_0(u_\epsilon^* - u_0^*)$ and we have used Proposition 3.3, the boundedness of f' and that u_ϵ^* is also bounded in the sup norm uniformly in ϵ.

We have

$$\begin{aligned}|z_\epsilon(x)| &= |f(u_\epsilon^*(x)) - f(u_0^*(x)) + f'(E_\epsilon u_0^*(x))(u_\epsilon^*(x) - E_\epsilon u_0^*(x))| \\ &\leqslant |[f'(\chi_\epsilon^*(x)) - f'(E_\epsilon u_0^*(x))]\,(u_\epsilon^*(x) - E_\epsilon u_0^*(x))|\end{aligned}$$

where $\chi_\epsilon^*(x) = \theta(x)u_\epsilon^*(x) + (1-\theta(x))E_\epsilon u_0^*(x)$ and $0 \leqslant \theta(x) \leqslant 1$, $x \in \Omega_\epsilon$.

Using that $|f'(\cdot)| \leqslant C$ we have,

$$\|z_\epsilon\|_{L^r(\Omega)} \leqslant C\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^r(\Omega)}, \quad \forall 1 \leqslant r \leqslant +\infty.$$

Also,

$$\|z_\epsilon\|_{L^r(\Omega)} \leqslant \|f'(\chi_\epsilon^*(x)) - f'(E_\epsilon u_0^*(x))\|_{L^s(\Omega)}\, \|u_\epsilon^* - E_\epsilon u_0^*\|_{L^t(\Omega)}, \quad \frac{1}{r} = \frac{1}{s} + \frac{1}{t}$$

But

$$\begin{aligned}\|f'(\chi_\epsilon^*(x)) - f'(E_\epsilon u_0^*(x))\|_{L^\infty(\Omega)} &\leqslant C \\ \|f'(\chi_\epsilon^*(x)) - f'(E_\epsilon u_0^*(x))\|_{L^1(\Omega)} &\leqslant C\|\chi_\epsilon^*(x)) - E_\epsilon u_0^*(x)\|_{L^1(\Omega)} \\ &\leqslant C\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^1(\Omega)}.\end{aligned}$$

Hence, using interpolation

$$\|f'(\chi_\epsilon^*(x)) - f'(E_\epsilon u_0^*(x))\|_{L^s(\Omega)} \leqslant C\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^1(\Omega)}^{1/s}.$$

So

$$\|z_\epsilon\|_{L^r(\Omega)} \leqslant C\,\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^1(\Omega)}^{1/s}\,\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^t(\Omega)} \leqslant C\,\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^t(\Omega)}^{1+\frac{1}{s}}$$

But if we define $w_\epsilon = E_\epsilon(A_0+B)^{-1}M_\epsilon z_\epsilon$, we know from (3.16) that

$$\|w_\epsilon\|_{L^q(\Omega)} \leqslant C\|z_\epsilon\|_{L^r(\Omega)} \text{ for some } r < q.$$

Hence we can choose $\dfrac{1}{r} = \dfrac{1}{s} + \dfrac{1}{q}$ $(t = q, \dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{q} > 0)$. So

$$\|E_\epsilon(A_0+B)^{-1}M_\epsilon z_\epsilon\|_{L^q(\Omega)} \leqslant C\|z_\epsilon\|_{L^r(\Omega)} \leqslant C\|u_\epsilon^* - u_0^*\|_{L^q(\Omega)}^{1+\frac{1}{r}-\frac{1}{q}}.$$

Hence

$$\|u_\epsilon^* - u_0^*\|_{L^q(\Omega)} \leqslant C\epsilon^{N/q} + C\|u_\epsilon^* - u_0^*\|_{L^q(\Omega)}^{1+\frac{1}{r}-\frac{1}{q}}.$$

Since we know that $\|u_\epsilon^* - u_0^*\|_{L^q(\Omega)} \to 0$ (since $\|u_\epsilon^* - u_0^*\|_{U_\epsilon^p} \to 0$ as $\epsilon \to 0$) then $\|u_\epsilon^* - u_0^*\|_{L^q(\Omega)} \leqslant C\,\epsilon^{N/q}$, which shows the first statement of the lemma. For the second one, we just realize that

$$\|u_\epsilon^* - u_0^*\|_{L^q(\Omega)} + \|u_\epsilon^* - u_0^*\|_{L^q(R_\epsilon)} \leqslant C\,\epsilon^{N/q} + C\,\mathbf{o}(\epsilon^{\frac{N-1}{q}}) = \mathbf{o}(\epsilon^{\frac{N-1}{q}}).$$

That is,

$$\epsilon^{-\frac{N-1}{q}}\,\|u_\epsilon^* - u_0^*\|_{L^q(\Omega_\epsilon)} \to 0 \,\text{as}\, \epsilon \to 0.$$

■

Corollary 3.3. *In the conditions of Proposition* 3.6*, if we denote by* $V_\epsilon = -f'(u_\epsilon^*)$*,* $V_0 = -f'(u_0^*)$ *and* $a = \sup\{|f'(s)|;\, s \in \mathbb{R}\}$*, then hypothesis (**H**) from Subsection* 3.2 *is satisfied. Hence, all the results of that Subsection can be applyed to the case where the potentials are given by* $V_\epsilon = -f'(u_\epsilon^*)$ *and* $V_0 = -f'(u_0^*)$*.*

Proof: Just observe that

$$\begin{aligned}\|V_\epsilon - E_\epsilon V_0\|_{L^q(\Omega_\epsilon)} &= \|f'(u_\epsilon^*) - E_\epsilon f'(u_0^*)\|_{L^q(\Omega_\epsilon)} \\ &\leqslant \|f''\|_{L^\infty(\mathbb{R})}\|u_\epsilon^* - E_\epsilon u_0^*\|_{L^q(\Omega_\epsilon)} = \mathbf{o}(\epsilon^{\frac{N-1}{q}})\end{aligned}$$

which shows the result. ■

4. CONVERGENCE OF LINEAR SEMIGROUPS

In this section we analyze the convergence properties of the linear semigroups generated by the operators $A_\epsilon + V_\epsilon$, $A_0 + V_0$ where the potentials V_ϵ, V_0 satisfy hypothesis (H) from Subsection 3.2. Later on we will be interested in applying the results from this section to the semigroups generated by A_ϵ, A_0 and also by $A_\epsilon - f'(u_\epsilon^*)$ and $A_0 - f'(u_0^*)$, where u_ϵ^*, u_ϵ^* are hyperbolic equilibria of the perturbed and limit problem respectively.

We will keep the notation from previous sections and we will denote by $W_\epsilon = V_\epsilon + a \geqslant 0$, $W_0 = V_0 + a \geqslant 0$, (see hypothesis (**H**)) and the operators $\Lambda_\epsilon = A_\epsilon + W_\epsilon$, $\Lambda_0 = A_0 + W_0$.

As we have already seen in [4], the operators $-A_0$, $-(A_0 + V_0)$ and $-\Lambda_0$ do not generate strongly continuous semigroups in U_0^p. Nonetheless they generate certain singular semigroups as we briefly recall.

Let $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$, $0 < \theta < \frac{\pi}{2}$ and let Γ be the boundary of Σ_θ oriented such that the imaginary part grows as λ runs in Γ. Notice that the semigroups generated by $-\Lambda_0$ and by $-(A_0 + V_0)$ are related by a multiplicative factor of the form e^{at}.

Proceeding as in [4] we define

$$e^{-\Lambda_0 t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(\lambda + \Lambda_0)^{-1}\,d\lambda, \quad t > 0. \tag{4.1}$$

Then, $e^{-\Lambda_0 t}$ satisfies the semigroup properties but strong continuity fails at $t = 0$ for data which are not sufficiently smooth. Nonetheless, several of the properties of analytic semigroup will still hold for sufficiently regular data. We say that $\{e^{-\Lambda_0 t} : t \geqslant 0\}$ is the semigroup generated by $-\Lambda_0$ and do not make any allusion to continuity. We refer to [4] for a detailed study of the semigroup generated by $-\Lambda_0$.

In what follows we recall some simple properties of the semigroup $\{e^{-\Lambda_0 t} : t \geqslant 0\}$ that we will employed later in this paper.

The next result investigates the singularity of $\{E_\epsilon e^{-\Lambda_0 t} M_\epsilon : t > 0\}$ at $t = 0$ in $\mathcal{L}(U_\epsilon^p)$. Its proof is a consequence of Proposition 3.5 and (4.1).

Lemma 4.1. *For any* $p \geqslant q > \frac{N}{2}$ *and for* $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$, *there is a constant* C, *independent of* ϵ, *such that*

$$\|E_\epsilon e^{-\Lambda_0 t} M_\epsilon u\|_{U_\epsilon^p} \leqslant C t^{\alpha-1} \|u\|_{U_\epsilon^q}, \quad t > 0, \quad u \in U_\epsilon^q, \tag{4.2}$$

and

$$\|E_\epsilon e^{-\Lambda_0 t} M_\epsilon u\|_{U_\epsilon^p} \leqslant C \|u\|_{U_\epsilon^\infty}, \quad t > 0, \quad u \in U_\epsilon^\infty. \tag{4.3}$$

From Lemma 3.5 it follows that, $-\Lambda_\epsilon$ generates an analytic semigroup $\left\{e^{-\Lambda_\epsilon t} : t \geqslant 0\right\}$ in U_ϵ^p given by

$$e^{-\Lambda_\epsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left(\lambda + \Lambda_\epsilon\right)^{-1} d\lambda, \quad t > 0, \tag{4.4}$$

where $\Gamma \subset \rho(-\Lambda_\epsilon)$ is the boundary of $\Sigma_\theta\}$ oriented such that the imaginary part grows as λ runs in Γ. Note that Γ is independent of ϵ. It follows from (3.22), (3.23) and (4.4) that the following estimates hold

$$\|e^{-\Lambda_\epsilon t} w\|_{U_\epsilon^p} \leqslant C \epsilon^{\frac{-N+1}{p}} \|w\|_{U_\epsilon^p}, \quad t \geqslant 0,\ w \in U_\epsilon^p, \tag{4.5}$$

$$\|e^{-\Lambda_\epsilon t} w\|_{L^p(\Omega_\epsilon)} \leqslant C \|w\|_{L^p(\Omega_\epsilon)}, \quad t \geqslant 0,\ w \in L^p(\Omega_\epsilon), \tag{4.6}$$

and

$$\|e^{-\Lambda_\epsilon t} w\|_{U_\epsilon^p} \leqslant C \|w\|_{U_\epsilon^\infty}, \quad t \geqslant 0,\ w \in U_\epsilon^\infty, \tag{4.7}$$

for some constant $C > 0$ that does not depend on ϵ, that is, the linear semigroup $e^{\Lambda_\epsilon t}$ is bounded, uniformly in ϵ, in $\mathcal{L}(L^p(\Omega_\epsilon))$ into itself.

We analyze now the convergence properties of the semigroups. To accomplish this task we will use extensively the resolvent estimates of the previous section applied to the integral expression of the semigroup.

Proposition 4.1. *For* $p, q > N$ *large enough, we have*

$$\|e^{(A_\epsilon + V_\epsilon)t} - E_\epsilon e^{(A_0 + V_0)t} M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant C\, e^{\beta t} t^{-\gamma} \cdot \rho(\epsilon), \quad t > 0. \tag{4.8}$$

Where $\beta \in \mathbb{R}$, $\gamma < 1$ *and the function* $\rho(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof: Observe first that $e^{-(A_\epsilon+V_\epsilon)t} - E_\epsilon e^{-(A_0+V_0)t}M_\epsilon = e^{at}(e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon)$, so that it is sufficient to prove an estimate of the type (4.8) for the difference $e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon$.

Hence,

$$e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon = \frac{1}{2\pi i}\int_\Gamma ((\lambda+\Lambda_\epsilon)^{-1} - E_\epsilon(\lambda+\Lambda_0)^{-1}M_\epsilon)e^{\lambda t}d\lambda. \tag{4.9}$$

Applying Proposition 3.5, we have

$$\begin{aligned} \epsilon^{-\frac{N-1}{q}}\|e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, L^q(\Omega_\epsilon))} &\leqslant \frac{C}{2\pi}\left|\int_\Gamma (1+|\lambda|^{1-\alpha})|e^{\lambda t}|d\lambda\right| \cdot \eta(\epsilon) \\ &\leqslant C\,t^{-(2-\alpha)} \cdot \eta(\epsilon). \end{aligned}$$

which implies

$$\|e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant C\,t^{-(2-\alpha)} \cdot \eta(\epsilon).$$

On the other hand, by comparison (maximum principle) we have

$$\|e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^\infty)} \leqslant \|e^{-\Lambda_\epsilon t}\|_{\mathcal{L}(U_\epsilon^\infty)} + \|E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^\infty)} \leqslant C.$$

which implies, using that $\|\cdot\|_{U_\epsilon^q} \leqslant \|\cdot\|_{U_\epsilon^\infty}$,

$$\|e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^\infty, U_\epsilon^q)} \leqslant C.$$

Interpolation shows that (see [7, Theorem 6.27])

$$\|e^{-\Lambda_\epsilon} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^{\bar p}, U_\epsilon^q)} \leqslant C\,t^{-\theta(2-\alpha)} \cdot \eta^\theta(\epsilon).$$

where $p \leqslant \bar p < \infty$ and $0 \leqslant \theta \leqslant 1$. Taking θ small we can get $\theta(2-\alpha) < 1$.

That is

$$\|e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^{\bar p}, U_\epsilon^q)} \leqslant C\,t^{-\gamma} \cdot \eta(\epsilon)^\theta,\ \gamma < 1.$$

Hence, if we define $\rho(\epsilon) = \eta(\epsilon)^\theta$, we have

$$\begin{aligned} \|e^{(A_\epsilon+V_\epsilon)t} - E_\epsilon e^{(A_0+V_0)t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^{\bar p}, U_\epsilon^q)} &= e^{at}\|e^{-\Lambda_\epsilon t} - E_\epsilon e^{-\Lambda_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^{\bar p}, U_\epsilon^q)} \\ &\leqslant C\,e^{at}t^{-\gamma} \cdot \rho(\epsilon) \end{aligned}$$

which shows the result with $\rho(\epsilon) = \eta(\epsilon)^\theta$ and $\beta = a$. ■

Let us consider now a real number b with the property that there exists a $\delta > 0$, small, such that $[b-\delta, b+\delta] \cap \sigma(-(A_0+V_0)) = \emptyset$. That is, the spectrum of the operator

$-(A_0+V_0)$, which is all real, is divided in two parts, σ_0^+ which is above $b+\delta$ and it is a finite set and σ_0^- which is below $b-\delta$ and it is an infinite set (a sequence that goes to $-\infty$). From the continuity properties of the spectrum, (see [3]) we have that for ϵ small enough $[b-\delta, b+\delta] \cap \sigma(-(A_\epsilon+V_\epsilon)) = \emptyset$ and the spectra of $-(A_\epsilon+V_\epsilon)$, which is also real, is divided in two parts σ_ϵ^+, above $b+\delta$ and σ_ϵ^-, below $b-\delta$. Moreover, we can choose a fixed closed curve $\Gamma_b^+ \subset \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geqslant b+\delta\}$ which encloses σ_ϵ^+ for all $0 \leqslant \epsilon \leqslant \epsilon_0$ for some ϵ_0 small. Moreover, we denote by $\Gamma_b^- = \{z \in \mathbb{C} : \arg(z-(b-\delta)) = \pi - \theta\}$ for some $0 < \theta < \pi/2$.

We decompose U_ϵ^p using the projection

$$Q_\epsilon^+ = Q(\sigma_\epsilon^+) = \frac{1}{2\pi i}\int_{\Gamma_b^+} (\lambda + A_\epsilon + V_\epsilon)^{-1} d\lambda. \tag{4.10}$$

Proposition 4.2. *For $p, q > N$ large enough, we have that there are constants $C > 0$, $\gamma < 1$, independent of ϵ and a function $\rho(\epsilon)$, with $\rho(\epsilon) \to 0$ as $\epsilon \to 0$, such that for $t > 0$*

$$\|e^{-(A_\epsilon+V_\epsilon)t}(I - Q(\sigma_\epsilon^+)) - E_\epsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant C\, e^{bt} t^{-\gamma} \cdot \rho(\epsilon) \tag{4.11}$$

$$\|E_\epsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant C\, e^{bt} t^{-\gamma} \tag{4.12}$$

$$\|e^{-(A_\epsilon+V_\epsilon)t}(I - Q(\sigma_\epsilon^+))\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant C\, e^{bt} t^{-\gamma}. \tag{4.13}$$

Proof: We have

$$e^{-(A_0+V_0)t}(I - Q(\sigma_0^+)) = \frac{1}{2\pi i}\int\limits_{\Gamma_b^-} (\lambda + A_0 + V_0(x))^{-1} e^{\lambda t} d\lambda.$$

Plugging norms and using estimate (3.5) we get

$$\|e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))\|_{\mathcal{L}(U_0^p, U_0^q)} \leqslant \left| \frac{1}{2\pi}\int\limits_{\Gamma_b^-} \frac{|e^{\lambda t}|}{|1+|\lambda|^{1-\alpha}} d\lambda \right|$$

and elementary integration shows

$$\|e^{(A_0+V_0)t}(I - Q(\sigma_0^+))\|_{\mathcal{L}(U_0^p, U_0^q)} \leqslant C e^{bt} t^{-\alpha} \tag{4.14}$$

which shows (4.12) with $\gamma = \alpha$.

In a similar way,

$$e^{-(A_\epsilon+V_\epsilon)t}(I - Q(\sigma_\epsilon^+)) - E_\epsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\epsilon =$$
$$\frac{1}{2\pi i}\int\limits_{\Gamma_b^-} ((\lambda + A_\epsilon + V_\epsilon(x))^{-1} - E_\epsilon(\lambda + A_0 + V_0(x))^{-1} M_\epsilon) e^{\lambda t} d\lambda.$$

So

$$
\begin{aligned}
&||e^{-(A_\epsilon+V_\epsilon)t}(I-Q(\sigma_\epsilon^+))-E_\epsilon e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))M_\epsilon||_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)} \leqslant \\
&\qquad\qquad \frac{1}{2\pi}\left|\int_{\Gamma_b^-} |e^{\lambda t}| \|(\lambda+A_\epsilon+V_\epsilon(x))^{-1}-E_\epsilon(\lambda+A_0+V_0(x))^{-1}M_\epsilon)\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)} d\lambda\right| \\
&\qquad\qquad \leqslant \frac{1}{2\pi}\int_{\Gamma_b^-} |e^{\lambda t}|(1+|\lambda|^{1-\alpha})d\lambda\cdot\eta(\epsilon)d\lambda \\
&\qquad\qquad \leqslant \frac{C}{2\pi}e^{bt}t^{-(2-\alpha)}\cdot\eta(\epsilon),
\end{aligned}
$$

where we have applied Proposition 3.5. Therefore,

$$
\|e^{-(A_\epsilon+V_\epsilon)t}(I-Q(\sigma_\epsilon^+))-E_\epsilon e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)} \leqslant C\,e^{bt}t^{-(2-\alpha)}\cdot\eta(\epsilon). \quad (4.15)
$$

This estimate does not show yet the proposition since the exponent $2-\alpha>1$. We will do an interpolation argument to conclude with the correct estimate. For this, let us see now that $Q(\sigma_\epsilon^+): U_\epsilon^p \to U_\epsilon^p$ satisfies $\|Q(\sigma_\epsilon^+)\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^p)} \leqslant C$ independent of ϵ. To see this, just observe that

$$
Q(\sigma_\epsilon^+) = \frac{1}{2\pi i}\int_{\Gamma_b^+}(\lambda+A_\epsilon+V_\epsilon)^{-1}d\lambda.
$$

Applying now the estimate of Propostion 3.5, we obtain that

$$
\|(\lambda+A_\epsilon+V_\epsilon)^{-1}\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^p)} \leqslant C
$$

for $\lambda\in\Gamma_b^-$ and with C independent of ϵ. From this last expresion and using the boundedness of Γ_b^- we get $\|Q(\sigma_\epsilon^+)\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^p)} \leqslant C$, for all $0\leqslant\epsilon\leqslant 1$.

Moreover, for the limit semigroup and for $0<t\leqslant 1$, we obtain from (4.14)

$$
\|E_\epsilon e^{-(A_0+V_0)t}(I-Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)} \leqslant Ct^{-\alpha}.
$$

Hence for $0<t\leqslant 1$, we get that

$$
\|e^{-(A_\epsilon+V_\epsilon)t}(I-Q(\sigma_\epsilon^+))\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)} \leqslant \|e^{-(A_\epsilon+V_\epsilon)t}\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)}(1+\|Q(\sigma_\epsilon^+)\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^p)})
$$

and

$$
\|e^{-(A_\epsilon+V_\epsilon)t}-E_\epsilon e^{-(A_0+V_0)t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)}+\|E_\epsilon e^{-(A_0+V_0)t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p,U_\epsilon^q)} \leqslant C(t^{-\gamma}+t^{-\alpha+1}),
$$

where we are using the bounds given by Proposition 4.1.

Hence, for $0 < t \leqslant 1$

$$\|e^{-(A_\epsilon+V_\epsilon)t}(I - Q(\sigma_\epsilon^+)) - E_\epsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant Ct^{-\bar{\gamma}}, \tag{4.16}$$

where $\bar{\gamma} = \max\{\gamma, 1-\alpha\}$.

Interpolating (4.15) and (4.16) we obtain, for $0 < t \leqslant 1$,

$$\begin{aligned}\|e^{-(A_\epsilon+V_\epsilon)t}(I - Q(\sigma_\epsilon^+)) - E_\epsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \\ \leqslant (Ct^{-(2-\alpha)} \cdot \eta(\epsilon))^\theta (Ct^{-\bar{\gamma}})^{1-\theta} \leqslant Ct^{-(2-\alpha)\theta-(1-\theta)\bar{\gamma}}\eta(\epsilon)^\theta,\end{aligned} \tag{4.17}$$

where we have used that $e^{bt} \leqslant C$ for $0 \leqslant t \leqslant 1$. Choosing $\theta > 0$ small enough so that $(2-\alpha)\theta + (1-\theta)\bar{\gamma} < 1$, we obtain the estimate for $0 < t \leqslant 1$.

Now for $t \geqslant 1$, from (4.15) we get

$$\|e^{-(A_\epsilon+V_\epsilon)t}(I - Q(\sigma_\epsilon^+)) - E_\epsilon e^{-(A_0+V_0)t}(I - Q(\sigma_0^+))M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant Ce^{bt}\eta(\epsilon).$$

Putting together both estimates, we prove (4.11). To prove (4.13) we just use (4.11) and (4.12). This concludes the proof of the proposition. ■

We also have

Corollary 4.1. *For the case* $V_\epsilon = V_0 \equiv 0$ *and with* $b \in (-1, 0)$ *a fixed number, we have that* $Q(\sigma_\epsilon^+) \equiv 0$ *for* ϵ *small enough and we have*

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t}M_\epsilon\|_{\mathcal{L}(U_\epsilon^p, U_\epsilon^q)} \leqslant C\, e^{bt}t^{-\gamma} \cdot \rho(\epsilon).$$

Remark 4.1. Observe that we can consider the case where $V_0 = -f'(u_0^*)$, $V_\epsilon = -f'(u_\epsilon^*)$ with u_0^* and u_ϵ^* hyperbolic equilibria satisfying u_ϵ^* converging to u_0^* (see [3]). In this case, we can always apply Proposition 4.2 with $b < 0$, a number dividing the spectrum among the stable part, that is with negative real part, and the unstable spectrum, that is with positive real part.

Let us conclude the section with the following useful unifom estimates of the semigroup on the linear unstable manifold

Proposition 4.3. *There are constants* $C \geqslant 1$ *and* $\beta > 0$ *such that*

$$\|e^{-(A_\epsilon+V_\epsilon)t}Q_\epsilon^+\|_{\mathcal{L}(U_\epsilon^q, U_\epsilon^p)} \leqslant Ce^{\beta t},\ t \leqslant 0 \tag{4.18}$$

Proof: Observe that

$$e^{-(A_\epsilon+V_\epsilon)t}Q_\epsilon^+ = \int_{\Gamma^+} e^{\lambda t}(\lambda + A_\epsilon + V_\epsilon)^{-1} d\lambda.$$

Using (3.37) and noticing that the curve Γ^+ is bounded, we have

$$\|e^{-(A_\epsilon+V_\epsilon)t}Q_\epsilon^+\|_{\mathcal{L}(U_\epsilon^q,U_\epsilon^p)} \leqslant C\left|\int_{\Gamma^+}|e^{\lambda t}|d\lambda\right| \leqslant Ce^{\beta t}$$

which shows the result. ∎

5. CONTINITY OF NONLINEAR SEMIGROUPS AND UPPER SEMICONTINUITY OF ATTRACTORS

Now that we have obtained in the previous section the continuity of linear semigroups we proceed to obtain the continuity of nonlinear semigroups using the Variation of Constants Formula. After we obtain the continuity of nonlinear semigroups we will proceed to obtain the upper semicontinuity of the family of attractors $\{\mathcal{A}_\epsilon : \epsilon \in [0,1]\}$.

To this end we will follow the ideas in [1] that relate the continuity of the linear semigroups with the continuity of the nonlinear semigroups for dissipative parabolic equations by using the variation of constants formula. This in turn will imply the upper semicontinuity of the attractors and the stationary states.

For $\epsilon \in [0,1]$, let $\{T_\epsilon(t) : t \geqslant 0\}$ be the semigroups defined in U_ϵ^p by the variation of constants formula

$$T_\epsilon(t,u_\epsilon) = e^{-A_\epsilon t}u_\epsilon + \int_0^t e^{-A_\epsilon(t-s)}f_\epsilon(T_\epsilon(s,u_\epsilon))ds. \tag{1}$$

We will show the following result

Proposition 5.1. *There exists a* $0 \leqslant \gamma < 1$ *and a function* $c(\epsilon)$ *with* $c(\epsilon) \overset{\epsilon\to 0}{\longrightarrow} 0$ *such that, for each* $\tau > 0$ *we have*

$$\|T_\epsilon(t,u_\epsilon) - E_\epsilon T_0(t,M_\epsilon u_\epsilon)\|_{U_\epsilon^p} \leqslant M(\tau)c(\epsilon)t^{-\gamma},\ t \in (0,\tau],\ u_\epsilon \in \mathcal{A}_\epsilon, \tag{2}$$

$\epsilon \in (0,\epsilon_0]$. *Moreover, the family of attractors* $\{\mathcal{A}_\epsilon : \epsilon \in [0,\epsilon_0]\}$ *is upper semicontinuous at* $\epsilon = 0$ *in* U_ϵ^p*, in the sense that*

$$\sup_{u_\epsilon\in\mathcal{A}_\epsilon}\left[\inf_{u_0\in\mathcal{A}_0}\{\|u_\epsilon - E_\epsilon u_0\|_{U_\epsilon^p}\}\right] \to 0, \quad \textit{as } \epsilon \to 0 \tag{3}$$

Also, if $\mathcal{E}_\epsilon$ *denotes the set of stationary states* (2.4), $\epsilon \in [0,\epsilon_0]$, *then* $\{\mathcal{E}_\epsilon : \epsilon \in [0,\epsilon_0]\}$ *is upper semicontinuous at* $\epsilon = 0$ *in* U_ϵ^p*; that is,*

$$\sup_{u_\epsilon^*\in\mathcal{E}_\epsilon}\left[\inf_{u_0^*\in\mathcal{E}_0}\{\|u_\epsilon^* - E_\epsilon u_0^*\|_{U_\epsilon^p}\}\right] \to 0, \quad \textit{as } \epsilon \to 0 \tag{4}$$

Proof: To prove this result we follow [1, 6]. Notice that the nonlinear semigroups $T_\epsilon(t)$ are given by (1). Hence, estimating $T_\epsilon(t,u_\epsilon) - E_\epsilon T_0(t,M_\epsilon u_\epsilon)$ and with some elementary computations we obtain

$$\begin{aligned}\|T_\epsilon(t,u_\epsilon)-E_\epsilon T_0(t,M_\epsilon u_\epsilon)\|_{U_\epsilon^p} &\leqslant \|e^{-A_\epsilon t}u_\epsilon - E_\epsilon e^{-A_0 t}M_\epsilon u_\epsilon\|_{U_\epsilon^p}\\ &+\int_0^t \|\left(e^{-A_\epsilon t}-E_\epsilon e^{-A_0 t}M_\epsilon\right) f_\epsilon(T_\epsilon(s,u_\epsilon))\|_{U_\epsilon^p}\,ds\\ &+\int_0^t \|E_\epsilon e^{-A_0 t}(M_\epsilon f_\epsilon(T_\epsilon(s,u_\epsilon))-f_0(T_0(s,M_\epsilon u_\epsilon)))\|_{U_\epsilon^p}\,ds,\end{aligned}$$

$\epsilon\in[0,\epsilon_0]$. Note that

$$\begin{aligned}&\int_0^t \|E_\epsilon e^{-A_0 t}(M_\epsilon f_\epsilon(T_\epsilon(s,u_\epsilon))-f_0(T_0(s,M_\epsilon u_\epsilon)))\|_{U_\epsilon^p}\,ds\\ &\quad=\int_0^t \|E_\epsilon e^{-A_0 t}(M_\epsilon f_\epsilon(T_\epsilon(s,u_\epsilon))-M_\epsilon E_\epsilon f_0(T_0(s,M_\epsilon u_\epsilon)))\|_{U_\epsilon^p}\,ds\\ &\quad=\int_0^t \|E_\epsilon e^{-A_0 t}M_\epsilon(f_\epsilon(T_\epsilon(s,u_\epsilon))-f_\epsilon(E_\epsilon T_0(s,M_\epsilon u_\epsilon)))\|_{U_\epsilon^p}\,ds\end{aligned}$$

where we have used that $M_\epsilon E_\epsilon = I$ and that $f_\epsilon(E_\epsilon u)=E_\epsilon f_0(u)$. Applying now Corollary 4.1 and Lemma 4.1 we have, for $0<t\leqslant\tau$,

$$\begin{aligned}\|T_\epsilon(t,u_\epsilon)-E_\epsilon T_0(t,M_\epsilon u_\epsilon)\|_{U_\epsilon^p} &\leqslant Ce^{bt}t^{-\gamma}\rho(\epsilon)\|u_\epsilon\|_{U_\epsilon^p}\\ &+C\rho(\epsilon)\int_0^t (t-s)^{-\gamma}e^{b(t-s)}\|f_\epsilon(T_\epsilon(s,u_\epsilon))\|_{U_\epsilon^p}\\ &+C\int_0^t (t-s)^{\alpha-1}\|T_\epsilon(s,u_\epsilon)-E_\epsilon T_0(s,M_\epsilon u_\epsilon)\|_{U_\epsilon^p}\end{aligned}$$

But since we have uniform bounds in $L^\infty(\Omega_\epsilon)$ of all the attractors, the first two terms in the above inequality can be bounded by $C\rho(\epsilon)t^{-\gamma}$. The result now follows applying the singular Gronwall's lemma (see [10]).

To show the uppersemicontinuity of the attractors $\mathcal{A}_\epsilon$, we notice first that by the uniform $L^\infty(\Omega_\epsilon)$ bounds of the attractors we have

$$\bigcup_{0\leqslant\epsilon\leqslant\epsilon_0} M_\epsilon\mathcal{A}_\epsilon$$

is a bounded set in U_0^∞. Hence, by the attractivity properties of $\mathcal{A}_0$, for a fixed $\eta>0$ there exists a time $\tau>0$ such that

$$\operatorname{dist}_{U_0^p}\Big(T_0(\tau)(M_\epsilon\varphi_\epsilon),\mathcal{A}_0\Big)\equiv\inf_{\varphi\in\mathcal{A}_0}\|T_0(\tau)(M_\epsilon\varphi_\epsilon)-\varphi\|_{U_0^p}\leqslant\eta,\forall\varphi_\epsilon\in\mathcal{A}_\epsilon,\ \epsilon\in[0,\epsilon_0]$$

which implies that

$$\operatorname{dist}_{U_\epsilon^p}\Big(E_\epsilon T_0(\tau)(M_\epsilon\varphi_\epsilon),E_\epsilon\mathcal{A}_0\Big)\leqslant\eta,\ \forall\varphi_\epsilon\in\mathcal{A}_\epsilon,\ \epsilon\in[0,\epsilon_0].$$

Applying now the convergence of the nonlinear semigroups (2) with $t = \tau$, we have that there exists a ϵ_1 such that for $0 < \epsilon \leqslant \epsilon_1$, we have

$$\|T_\epsilon(\tau, \varphi_\epsilon) - E_\epsilon T_0(\tau, M_\epsilon \varphi_\epsilon)\|_{U^p_\epsilon} \leqslant \eta, \quad \forall \varphi_\epsilon \in \mathcal{A}_\epsilon, \quad 0 \leqslant \epsilon \leqslant \epsilon_1$$

which implies

$$\mathrm{dist}_{U^p_\epsilon}\Big(T_\epsilon(\tau, \varphi_\epsilon), E_\epsilon \mathcal{A}_0\Big) \leqslant \eta, \quad \forall \varphi_\epsilon \in \mathcal{A}_\epsilon, \quad 0 \leqslant \epsilon \leqslant \epsilon_0$$

The invariance of the attractor under the flow given by T_ϵ, implies that

$$\mathrm{dist}_{U^p_\epsilon}\Big(\varphi_\epsilon, E_\epsilon \mathcal{A}_0\Big) \leqslant \eta, \quad \forall \varphi_\epsilon \in \mathcal{A}_\epsilon, \quad 0 \leqslant \epsilon \leqslant \epsilon_0$$

which implies (3).

The upper semicontinuity of the equilibria was already obtained in [3]. ■

Remark 5.1. Observe that Proposition 5.1 proves the upper semicontinuity part of Theorem 2.1.

6. CONTINUITY OF LOCAL UNSTABLE MANIFOLDS AND OF ATTRACTORS

We already know that, if all equilibrium points of (2.5), which is the abstract version of (1.2) are hyperbolic then they are all isolated and there is only a finite number of them, say $\mathcal{E}_0 = \{e_0^1, \cdots, e_0^m\}$. In this case, we also know that there is an $\epsilon_0 > 0$ such that the set of equilibria of (2.4), which is the abstract version of (1.1), $\mathcal{E}_\epsilon = \{e_\epsilon^1, \cdots, e_\epsilon^m\}$ for all $0 < \epsilon \leqslant \epsilon_0$ and $e_\epsilon^i \xrightarrow{E} e_0^i$ for $1 \leqslant i \leqslant m$ (see Theorem 2.3 of [3]). Moreover, we also know that the linear unstable manifolds associated to e_j^ϵ converge to the linear unstable manifold of e_j^ϵ, see Theorem 2.5 of [3]. For each $e_\epsilon^j \in \mathcal{E}_\epsilon$, $\epsilon \in [0, 1]$, we define its unstable manifold

$$\begin{aligned} W^u(e_\epsilon^j) = \{\eta_\epsilon \in U^p_\epsilon{:}\ & \text{there is a backward solution } u_\epsilon(t, \eta_\epsilon) \\ & \text{through } \eta_\epsilon \text{ such that } u_\epsilon(t, \eta_\epsilon) \to e_\epsilon^j \text{ as } t \to -\infty\}. \end{aligned}$$

and its δ-local unstable manifold as

$$W^u_\delta(e_\epsilon^j) = W^u(e_\epsilon^j) \cap B(e_\epsilon^j, \delta) = \{u \in W^u(e_\epsilon^j); \|u - e_\epsilon^j\|_{U^p_\epsilon} < \delta\}.$$

These definitions are standard and we refer to [8] for further properties of local unstable manifolds.

In this section we show that the local unstable manifolds of e_ϵ^j, for $j = 1, \ldots, m$ fixed, behaves continuously with ϵ in U^p_ϵ.

Proposition 6.1. *Assume that $e_0 \in \mathcal{E}_0$ is hyperbolic; that is, $0 \notin \sigma(A_0 + f'(e_0)I)$. By [3, Theorem 5.8 and Example 5.9], there are $\delta > 0$ and ϵ_0 such that, there is a unique*

$e_\epsilon \in \mathcal{E}_\epsilon$ with $\|e_\epsilon - E_\epsilon e_0\|_{U^p_\epsilon} < \delta$, for all $0 \leqslant \epsilon \leqslant \epsilon_0$. Then, there is $\delta > 0$ such that

$$\mathrm{dist}_{U^p_\epsilon}(W^u_\delta(e_\epsilon), E_\epsilon W^u_\delta(e_0)) + \mathrm{dist}_{U^p_\epsilon}(E_\epsilon W^u_\delta(e_0), W^u_\delta(e_\epsilon)) \xrightarrow{\epsilon \to 0} 0$$

that is,

$$\sup_{u_\epsilon \in W^u_\delta(e_\epsilon)} \inf_{u_0 \in W^u_\delta(u_0)} \|u_\epsilon - E_\epsilon u_0\|_{U^p_\epsilon} + \sup_{u_0 \in W^u_\delta(u_0)} \inf_{u_\epsilon \in W^u_\delta(e_\epsilon)} \|u_\epsilon - E_\epsilon u_0\|_{U^p_\epsilon} \xrightarrow{\epsilon \to 0} 0$$

Before proving this result, let us see how we can proceed to give a proof of our main result, Theorem 2.1.

Proof of Theorem 2.1: The upper-semicontinuity has already been proved in Proposition 5.1 from Section 5. Observe that to obtain the upper-semicontinuity of the attractors, we have used the continuity of the nonlinear semigroups, but no gradient structure of the flows have been used.

To obtain the lower-semicontinuity, we need to show that for each $\varphi_0 \in \mathcal{A}_0$ we have a sequence of $\varphi_\epsilon \in \mathcal{A}_\epsilon$, with the property that $\|\varphi_\epsilon - E_\epsilon \varphi_0\|_{U^p_\epsilon} \to 0$ as $\epsilon \to 0$. To accomplish this, we follow similar arguments as the one developed in [8], [9] or [2].

We are assuming that each equilibrium of the limiting problem $\mathcal{E}_0$ is hperbolic. This implies that we have a finite number of them and that the flow $T_0(t)$ has a gradient structure, see [4] and in particular, given $\varphi_0 \in \mathcal{A}_0$ it will lie in the unstable manifold of some $e_0 \in \mathcal{E}_0$. This implies that there exist an element $\phi_0 \in W^u_\delta(e_0)$ and a $\tau > 0$ such that $T_0(\tau, \phi_0) = \varphi_0$, where $\delta > 0$ is the one from Proposition 6.1. Using the continuity of the local unstable manifolds obtained in Proposition 6.1, we have that there exists a sequence of elements $\phi_\epsilon \in W^u_\delta(e_\epsilon)$ such that $\|\phi_\epsilon - E_\epsilon \phi_0\|_{U^p_\epsilon} \to 0$. But, from the invariance of the attractor $\mathcal{A}_\epsilon$ under the flow T_ϵ, we have $\varphi_\epsilon = T_\epsilon(\tau, \phi_\epsilon) \in \mathcal{A}_\epsilon$. Moreover,

$$\begin{aligned} \|\varphi_\epsilon - E_\epsilon \varphi_0\|_{U^p_\epsilon} &= \|T_\epsilon(\tau, \phi_\epsilon) - E_\epsilon T_0(\tau, \phi_0)\|_{U^p_\epsilon} \\ &\leqslant \|T_\epsilon(\tau, \phi_\epsilon) - E_\epsilon T_0(\tau, M_\epsilon \phi_\epsilon)\|_{U^p_\epsilon} + \|E_\epsilon T_0(\tau, M_\epsilon \phi_\epsilon) - E_\epsilon T_0(\tau, \phi_0)\|_{U^p_\epsilon} \\ &\leqslant M(\tau)\tau^{-\gamma} c(\epsilon) + \|T_0(\tau, M_\epsilon \phi_\epsilon) - T_0(\tau, \phi_0)\|_{U^p_0}, \end{aligned}$$

where we are using (2) and the fact that $\|E_\epsilon\|_{\mathcal{L}(U^p_0, U^p_\epsilon)} = 1$.

The continuity of the map $T(\tau, \cdot) : U^p_0 \to U^p_0$, the fact that $\|\phi_0 - M_\epsilon \phi_\epsilon\|_{U^p_0} \to 0$ as $\epsilon \to 0$ and that $c(\epsilon) \to 0$, shows that $\|\varphi_\epsilon - E_\epsilon \varphi_0\|_{U^p_\epsilon} \to 0$ as $\epsilon \to 0$. This concludes the proof of Theorem 2.1. ■

Proof of Proposition 6.1: Let $\{e_\epsilon\}$ with $e_\epsilon \in \mathcal{E}_\epsilon$, $\epsilon \in [0, 1]$, such $\|e_\epsilon - E_\epsilon e_0\|_{U^p_\epsilon} \overset{\epsilon \to 0}{\to} 0$. Rewriting (2.4) for $w_\epsilon = u_\epsilon - e_\epsilon$ to deal with the neighborhood of e_ϵ we arrive at

$$w_t + A_\epsilon w - f'_\epsilon(e_\epsilon) w = f(w + e_\epsilon) - f(e_\epsilon) - f'_\epsilon(e_\epsilon) w. \tag{6.1}$$

Let us denote by $V_0 = -f'(e_0)$, $V_\epsilon = -f'(e_\epsilon)$. Using the hiperbolicity of e_0, e_ϵ we consider $b < 0$ and define σ^+_ϵ, $Q(\sigma^*_\epsilon)$ as in (4.10), see Remark 4.1.

Decomposing (6.1) with the aid of projection $Q(\sigma_\epsilon^+)$ and denoting by $\tilde{A}_\epsilon$ the restriction of $A_\epsilon + V_\epsilon$ to the kernel of $Q(\sigma_\epsilon^+)$, by B_ϵ the restriction of $A_\epsilon + V_\epsilon$ to the range of $Q(\sigma_\epsilon^+)$ and making $S_\epsilon^{-1}v = Q(\sigma_\epsilon^+)w$, $z = (I - Q(\sigma_\epsilon^+))w$ we rewrite (6.1) as

$$\begin{aligned} &\dot{v} + B_\epsilon v = Q(\sigma_\epsilon^+)F_\epsilon(S_\epsilon v, z) := G_\epsilon(S_\epsilon v, z) \\ &\dot{z} + \tilde{A}_\epsilon z = (I - Q(\sigma_\epsilon^+))F_\epsilon(S_\epsilon v, z) := H_\epsilon(S_\epsilon v, z), \end{aligned} \tag{6.2}$$

where $F_\epsilon(0,0) = 0$ and $F'_\epsilon(0,0) = 0$. Proceeding as in Example 5.9 in [3] we have that, given $\rho > 0$ there is a $\delta > 0$ such that

$$\begin{aligned} &\|F_\epsilon(S_\epsilon v, z)\|_{U_\epsilon^q} < \rho, \\ &\|F_\epsilon(S_\epsilon v, z) - F_\epsilon(S_\epsilon \tilde{v}, \tilde{z})\|_{U_\epsilon^q} < \rho(\|v - \tilde{v}\|_{\mathbb{R}^n} + \|z - \tilde{z}\|_{U_\epsilon^p}). \end{aligned} \tag{6.3}$$

for all $(v, z) \in B_\delta(0,0)$ and for all $\epsilon \in (0,1]$. Since we are interested on the behavior of the solutions near $(0,0)$ we cut F_ϵ outside $B_\delta(0,0)$ in such a way that it satisfies (6.3) globally.

Proceeding as in [2, 6] we can show that for a suitably small $\rho > 0$, there is an unstable manifold for e_ϵ

$$S^\epsilon = \{(v, z) : z = \Sigma_\epsilon^*(v),\ v \in \mathbb{R}^n\}$$

where $\Sigma_\epsilon^* : \mathbb{R}^n \to \mathrm{Ker}(Q_\epsilon)$ is bounded and Lipschitz continuous. Furthermore

$$\sup_{v \in \mathbb{R}^n} \|\Sigma_\epsilon^*(v) - E_\epsilon \Sigma_0^*(v)\|_{U_\epsilon^p} \overset{\epsilon \to 0}{\longrightarrow} 0.$$

Let sketch the proof of existence of the unstable manifold as a graph and prove its continuity. Let $\Sigma_\epsilon : \mathbb{R}^n \to \mathrm{Ker}(Q_\epsilon)$ such that

$$|\!|\!|\Sigma_\epsilon|\!|\!| := \sup_{v \in \mathbb{R}^n} \|\Sigma_\epsilon(v)\|_{U_\epsilon^p} \leqslant D,\ \|\Sigma_\epsilon(v) - \Sigma_\epsilon(\tilde{v})\|_{U_\epsilon^p} \leqslant L\|v - \tilde{v}\|_{\mathbb{R}^n}. \tag{6.4}$$

If $v_\epsilon(t) = \psi(t, \tau, \eta, \Sigma_\epsilon)$ denotes the solution of

$$\frac{dv_\epsilon}{dt} + B_\epsilon v_\epsilon = F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon(v_\epsilon)),\quad \text{for } t < \tau,\ v_\epsilon(\tau) = \eta,$$

We seek for a fixed point Σ_ϵ^* of

$$\Phi(\Sigma_\epsilon)(\eta) = \int_{-\infty}^{\tau} e^{-\tilde{A}_\epsilon(\tau - s)}(I - Q(\sigma_\epsilon^+))F_\epsilon(S_\epsilon v_\epsilon(s), \Sigma_\epsilon(v_\epsilon(s)))ds,\ \epsilon \in [0,1]. \tag{6.5}$$

in the class of Lipschitz maps $\Sigma_\epsilon : \mathbb{R}^n \to \mathrm{Ker}(Q_\epsilon)$ which are globally bounded with bound D and globally Lipschitz with Lipschitz constant L.

Note that, from (4.13),

$$\|\Phi(\Sigma_\epsilon)(\eta)\|_{U_\epsilon^p} = \int_{-\infty}^{\tau} \rho C(\tau - s)^{-\gamma} e^{-b(\tau - s)} ds. \tag{6.6}$$

and for suitably chosen ρ we have that $|||\Phi(\Sigma_\epsilon)||| \leqslant D$.

Next, suppose that Σ_ϵ and $\tilde{\Sigma}_\epsilon$ are functions satisfying (6.4), $\eta, \tilde{\eta} \in \mathbb{R}^n$ and denote $v_\epsilon(t) = \psi(t, \tau, \eta, \Sigma_\epsilon)$, $\tilde{v}_\epsilon(t) = \psi(t, \tau, \tilde{\eta}, \tilde{\Sigma}_\epsilon)$. Then,

$$\begin{aligned} v_\epsilon(t) - \tilde{v}_\epsilon(t) = {} & e^{-B_\epsilon(t-\tau)}(\eta - \tilde{\eta}) \\ & + \int_\tau^t e^{-B_\epsilon(t-s)} Q_\epsilon[F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon(v_\epsilon)) - F_\epsilon(S_\epsilon \tilde{v}_\epsilon, \tilde{\Sigma}_\epsilon(\tilde{v}_\epsilon))]ds \end{aligned}$$

and

$$\begin{aligned} \|v_\epsilon(t) - \tilde{v}_\epsilon(t)\|_{\mathbb{R}^n} \leqslant {} & Ce^{b(t-\tau)}\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} \\ & + C\int_t^\tau e^{b(t-s)}\|Q_\epsilon F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon(v_\epsilon)) - Q_\epsilon F_\epsilon(S_\epsilon \tilde{v}_\epsilon, \tilde{\Sigma}_\epsilon(\tilde{v}_\epsilon))\|_{\mathbb{R}^n} ds \\ \leqslant {} & Ce^{b(t-\tau)}\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} \\ & + \rho C\int_t^\tau e^{-b(t-s)}\left(\|\Sigma_\epsilon(v_\epsilon) - \tilde{\Sigma}_\epsilon(\tilde{v}_\epsilon)\|_{U_\epsilon^p} + \|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n}\right) ds \\ \leqslant {} & Ce^{b(t-\tau)}\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} \\ & + \rho C\int_t^\tau e^{b(t-s)}\left(\|\Sigma_\epsilon(\tilde{v}_\epsilon) - \tilde{\Sigma}_\epsilon(\tilde{v}_\epsilon)\|_{U_\epsilon^p} + (1+L)\|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n}\right) ds \\ \leqslant {} & Ce^{b(t-\tau)}\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} \\ & + \rho C\int_t^\tau e^{b(t-s)}\left((1+L)\|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n} + |||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||_{U_\epsilon^p}\right) ds \\ \leqslant {} & Ce^{b(t-\tau)}\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} \\ & + \rho C(1+L)\int_t^\tau e^{b(t-s)}\|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n} ds + \rho C|||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||_{U_\epsilon^p}\int_t^\tau e^{b(t-s)}ds. \end{aligned}$$

Let $\phi(t) = e^{-b(t-\tau)}\|v_\epsilon(t) - \tilde{v}_\epsilon(t)\|_{\mathbb{R}^n}$. Then,

$$\phi(t) \leqslant C\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} + \rho C\int_t^\tau e^{b(\tau-s)}ds|||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||_{U_\epsilon^p} + C\,\rho\,(1+L)\int_t^\tau \phi(s)ds.$$

By Gronwall's inequality

$$\begin{aligned} \|v_\epsilon(t) - \tilde{v}_\epsilon(t)\|_{\mathbb{R}^n} \leqslant {} & [C\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} e^{b(t-\tau)} \\ & + \rho C\int_t^\tau e^{b(t-s)}ds|||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||_{U_\epsilon^p}]e^{-\rho C(1+L)(t-\tau)} \\ \leqslant {} & [C\|\eta - \tilde{\eta}\|_{\mathbb{R}^n} + \rho C b^{-1}|||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||_{U_\epsilon^p}]e^{-\rho C(1+L)(t-\tau)}. \end{aligned}$$

Thus,

$$\begin{aligned}
&\|\Phi(\Sigma_\epsilon)(\eta) - \Phi(\tilde{\Sigma}_\epsilon)(\tilde{\eta})\|_{U_\epsilon^p} \\
&\qquad \leqslant C\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-b(\tau-s)}\|F_\epsilon(S_\epsilon v_\epsilon,\Sigma_\epsilon(v_\epsilon)) - F_\epsilon(S_\epsilon\tilde{v}_\epsilon,\tilde{\Sigma}_\epsilon(\tilde{v}_\epsilon))\|_{L^2(\Omega_\epsilon)}ds \\
&\qquad \leqslant \rho C\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-b(\tau-s)}\left(\|\Sigma_\epsilon(v_\epsilon) - \tilde{\Sigma}_\epsilon(\tilde{v}_\epsilon)\|_{U_\epsilon^p} + \|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n}\right)ds \\
&\qquad \leqslant \rho C\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-b(\tau-s)}\left[(1+L)\|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n} + |||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||\right]ds.
\end{aligned}$$

Using the estimates for $\|v_\epsilon - \tilde{v}_\epsilon\|_{\mathbb{R}^n}$ we obtain

$$\begin{aligned}
&\|\Phi(\Sigma_\epsilon)(\eta) - \Phi(\tilde{\Sigma}_\epsilon)(\tilde{\eta})\| \\
&\qquad \leqslant \rho C\Gamma(1-\gamma)\left[b^{-1+\gamma}+\frac{\rho C(1+L)}{b(b-\rho C(1+L))^{1-\gamma}}\right]|||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon||| \\
&\qquad + \frac{\rho C^2(1+L)\Gamma(1-\gamma)}{(b-\rho C(1+L))^{-1+\gamma}}\|\eta - \tilde{\eta}\|_{\mathbb{R}^n}.
\end{aligned}$$

Let

$$I_\Sigma(\rho) = \rho C\Gamma(1-\gamma)\left[b^{-1+\gamma}+\frac{\rho C(1+L)}{b(b-\rho C(1+L))^{1-\gamma}}\right]$$

and

$$I_\eta(\rho) = \frac{\rho C^2(1+L)\Gamma(1-\gamma)}{(b-\rho C(1+L))^{1-\gamma}}.$$

It is easy to see that, given $\theta < 1$, there exists a ρ_0 such that, for $\rho \leqslant \rho_0$, $I_\Sigma(\rho) \leqslant \theta$ and $I_\eta(\rho) \leqslant L$ and

$$\|\Phi(\Sigma_\epsilon)(\eta) - \Phi(\tilde{\Sigma}_\epsilon)(\tilde{\eta})\|_{U_\epsilon^p} \leqslant L\|\eta - \eta'\|_{\mathbb{R}^n} + \theta|||\Sigma_\epsilon - \tilde{\Sigma}_\epsilon|||. \tag{6.7}$$

The inequalities (6.6) and (6.7) imply that G is a contraction map from the class of functions that satisfy (6.4) into itself. Therefore, it has a unique fixed point $\Sigma_\epsilon^* = \Phi(\Sigma_\epsilon^*)$ in this class. The invariance and the fact that the graph is the whole unstable manifold follows in a usual manner.

It remains to prove the continuity of the unstable manifolds. This is accomplished in the following manner. If $0 \leqslant \epsilon \leqslant \epsilon_0$ is such that the unstable manifold is given by the graph of Σ_ϵ^*, $0 \leqslant \epsilon \leqslant \epsilon_0$, we want to show that

$$\sup_{\eta\in\mathbb{R}^n}\|\Sigma_\epsilon^*(\eta) - E_\epsilon\Sigma_0^*(\eta)\|_{U_\epsilon^p} = |||\Sigma_\epsilon^* - E_\epsilon\Sigma_0^*|||.$$

It follows from Proposition 4.2 that

$$\begin{aligned}
&\|\Sigma_\epsilon^*(\eta_\epsilon) - E_\epsilon\Sigma_0^*(\eta)\|_{U_\epsilon^p}\\
&\qquad\leqslant \int_{-\infty}^{\tau} \|e^{-\tilde{A}_\epsilon(\tau-s)}H_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon^*(v_\epsilon)) - E_\epsilon e^{-\tilde{A}_0(\tau-s)}H_0(S_0v_0, \Sigma_0^*(v_0))\|_{U_\epsilon^p} ds\\
&\qquad\leqslant C\int_{-\infty}^{\tau} e^{b(\tau-s)}(\tau-s)^{-\gamma}\|F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon^*(v_\epsilon)) - F_\epsilon(E_\epsilon(S_0v_0, \Sigma_0^*(v_0)))\|_{U_\epsilon^q} ds\\
&\qquad+ C\rho(\epsilon)\int_{-\infty}^{\tau} e^{b(\tau-s)}\|F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon^*(v_\epsilon))\|_{C(\bar{\Omega}_\epsilon)} ds\\
&\qquad\leqslant \rho\, Cb^{-1}\rho(\epsilon) + \rho C b^{\gamma-1}\Gamma(1-\gamma)\,|||\Sigma_\epsilon^* - E_\epsilon\Sigma_0^*|||\\
&\qquad+ \rho\, C(1+L)\int_{-\infty}^{\tau} e^{-b(\tau-s)}(\tau-s)^{-\gamma}\|v_\epsilon - v_0\|_{\mathbb{R}^n} ds.
\end{aligned} \tag{6.8}$$

Thus, it is enough to estimate $\|v_\epsilon - v_0\|_{\mathbb{R}^n}$. Note that

$$\begin{aligned}
\|v_\epsilon - v_0\|_{\mathbb{R}^n} &\leqslant \int_t^{\tau} \|e^{-B_\epsilon(t-s)} - e^{-B_0(t-s)}\|\,\|F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon^*(v_\epsilon))\|_{\mathbb{R}^n} ds\\
&+ \int_t^{\tau} \|e^{-B_0(t-s)}\|\,\|F_\epsilon(S_\epsilon v_\epsilon, \Sigma_\epsilon^*(v_\epsilon)) - F_0(S_0v_0, \Sigma_0^*(v_0))\|_{\mathbb{R}^n} ds\\
&\leqslant \rho M b^{-1}[o(1) + |||\Sigma_\epsilon^* - \Sigma_0^*|||] + \rho C(1+L)\int_t^{\tau} e^{b(t-s)}\|v_\epsilon - v_0\|_{\mathbb{R}^n} ds.
\end{aligned}$$

Therefore

$$\|v_\epsilon - v_0\|_{\mathbb{R}^n} \leqslant \rho C b^{-1}\,[o(1) + |||\Sigma_\epsilon^* - \Sigma_0^*|||]\, e^{-\rho C(1+L)(\tau-t)}$$

which shows that

$$\sup_{\eta\in\mathbb{R}^n} \|\Sigma_\epsilon^*(\eta) - \Sigma_0^*(\eta)\|_{U_\epsilon^p} \overset{\epsilon\to 0}{\to} 0.$$

This proves the result. ■

7. CONTINUITY OF ATTRACTORS IN OTHER NORMS

In this section we study the continuity of attractors in other norms and very specially in the norm of the space $U_\epsilon^{1,2}$, see (2.6). This continuity is obtained as a consequence of the regularization properties of the nonlinear semigroups. As a matter of fact, in many instances the attractors $\mathcal{A}_\epsilon$, $\mathcal{A}_0$ live in better spaces X_ϵ and X_0 respectively for which the linear map $E_\epsilon : X_0 \to X_\epsilon$ is well defined as well. We would like to give conditions that, once the continuity of the attractors in U_ϵ^p is obtained, will guarantee the continuity results for the attractors in these better spaces. In fact, the following result holds

Proposition 7.1. *If there exists a $\tau > 0$ fixed such that for each sequence of $\epsilon_n \to 0$, $\phi_{\epsilon_n} \in \mathcal{A}_{\epsilon_n}$ and $\phi_0 \in \mathcal{A}_0$ with $\|\phi_{\epsilon_n} - E_{\epsilon_n}\phi_0\|_{U_{\epsilon_n}^p} \to 0$ implies that*

$$\|T_{\epsilon_n}(\tau, \phi_{\epsilon_n}) - E_{\epsilon_n}T_0(\tau, \phi_0)\|_{X_{\epsilon_n}} \to 0 \tag{7.1}$$

then, the upper semicontinuity of the attractors in U^p_ϵ *implies the upper semicontinuity in* X_ϵ *and the lower semicontinuity of the attractors in* U^p_ϵ *implies the lower semicontinuity of the attractors in* X_ϵ*.*

Proof: Assume we have a family of $\varphi_\epsilon \in \mathcal{A}_\epsilon$. From the invariance of the attractors under the semigroup T_ϵ, we have that there exist $\phi_\epsilon \in \mathcal{A}_\epsilon$ with $T_\epsilon(\tau, \phi_\epsilon) = \varphi_\epsilon$.

If the attractors are E_ϵ-upper semicontinuous in U^p_ϵ, we have that for each sequence $\epsilon_n \to 0$, there will exist a subsequence, that we still denote by ϵ_n and an element $\phi_0 \in \mathcal{A}_0$ such that $\|\phi_{\epsilon_n} - E_{\epsilon_n}\phi_0\|_{U^p_{\epsilon_n}} \to 0$ as $\epsilon_n \to 0$. With (7.1) we get that if we define $\varphi_0 = T_0(\tau, \phi_0)$, we have $\|\varphi_{\epsilon_n} - E_{\epsilon n}\varphi_0\|_{X_{\epsilon_n}} \to 0$, which shows the E_ϵ-upper semicontinuity in X_ϵ.

Assume now that the attractors are E_ϵ-lower semicontinuous in U^p_ϵ. If $\varphi_0 \in \mathcal{A}_0$ and if we define $\phi_0 \in \mathcal{A}_0$ with $T_0(\tau, \phi_0) = \varphi_0$, then there will exist a sequence of $\phi_\epsilon \in \mathcal{A}_\epsilon$ with $\|\phi_\epsilon - E_\epsilon\phi_0\|_{U^p_\epsilon} \to 0$ as $\epsilon \to 0$. Using (7.1) again, we get that $\|\varphi_\epsilon - E_\epsilon\varphi_0\|_{X_\epsilon} \to 0$ which shows the E_ϵ-lower semicontinuty in X_ϵ. ■

With this result we can provide now a proof of Theorem 2.2.

Proof of Theorem 2.2: We will apply Proposition 7.1, proving first that

$$\|T_{\epsilon_n}(\tau, \phi_{\epsilon_n}) - E_{\epsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\epsilon_n}} \to 0$$

for some $\tau > 0$ fixed, sequences $\epsilon_n \to 0$, $\phi_{\epsilon_n} \in \mathcal{A}_{\epsilon_n}$ and $\phi_0 \in \mathcal{A}_0$ with $\|\phi_{\epsilon_n} - E_{\epsilon_n}\phi_0\|_{U^p_{\epsilon_n}} \to 0$.

Observe first that

$$\begin{aligned}\|T_{\epsilon_n}(\tau, \phi_{\epsilon_n}) - E_{\epsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\epsilon_n}} \leq{}& \|T_{\epsilon_n}(\tau, \phi_{\epsilon_n}) - E_{\epsilon_n}T_0(\tau, M_\epsilon\phi_{\epsilon_n})\|_{U^{1,2}_{\epsilon_n}} \\ &+ \|E_\epsilon T_0(\tau, M_\epsilon\phi_{\epsilon_n}) - E_{\epsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\epsilon_n}}\end{aligned} \tag{7.2}$$

and for a fixed $\tau > 0$,

$$\|E_\epsilon T_0(\tau, M_\epsilon\phi_{\epsilon_n}) - E_{\epsilon_n}T_0(\tau, \phi_0)\|_{U^{1,2}_{\epsilon_n}} \leq \|T_0(\tau, M_\epsilon\phi_{\epsilon_n}) - T_0(\tau, \phi_0)\|_{U^{1,2}_0} \to 0$$

since $T_0(\tau, \cdot) : U^p_0 \to U^{1,2}_0$ is continuous, see [4] .

To estimate the first term of the second line of (7.2) we use the Variation of Constants Formula (1) for $\epsilon \in [0, 1]$ and with simple computations we obtain

$$\begin{aligned}\|T_\epsilon(t, \phi_\epsilon) - E_\epsilon T_0(t, M_\epsilon\phi_\epsilon)\|_{U^{1,2}_\epsilon} \leq{}& \|e^{-A_\epsilon t}\phi_\epsilon - E_\epsilon e^{-A_0 t}M_\epsilon\phi_\epsilon\|_{U^{1,2}_\epsilon} \\ &+ \int_0^t \| \left(e^{-A_\epsilon(t-s)} - E_\epsilon e^{-A_0(t-s)}M_\epsilon\right) f_\epsilon(T_\epsilon(s, \phi_\epsilon))\|_{U^{1,2}_\epsilon}\, ds \\ &+ \int_0^t \|E_\epsilon e^{-A_0(t-s)}M_\epsilon(f_\epsilon(T_\epsilon(s, \phi_\epsilon)) - f_\epsilon(E_\epsilon T_0(s, M_\epsilon\phi_\epsilon)))\|_{U^{1,2}_\epsilon}\, ds,\end{aligned} \tag{7.3}$$

$\epsilon \in [0, \epsilon_0]$. But note that $\mathcal{A}_\epsilon \subset C(\bar{\Omega}_\epsilon)$ for $0 < \epsilon \leq \epsilon_0$, $\mathcal{A}_0 \subset C(\bar{\Omega}) \oplus C([0, 1])$ and that we have uniform bounds in these spaces.

If we are able to obtain the following two estimates:

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(C(\bar{\Omega})\oplus C(\bar{\mathrm{R}}_\epsilon), U_\epsilon^{1,2})} \leq C t^{-\gamma}\nu(\epsilon),\ t > 0. \tag{7.4}$$

for some $0 \leq \gamma < 1$ and with $\nu(\epsilon) \to 0$ as $\epsilon \to 0$, and

$$\|e^{-A_0 t}\|_{\mathcal{L}(U_0^p, U_0^{1,2})} \leq C t^{-\beta}, \quad t > 0. \tag{7.5}$$

for some $0 \leq \beta < 1$, then using (7.4) and (7.5) in (7.3) and using the convergence of the nonlinear semigroup in U_0^p we obtain that $\|T_\epsilon(t,\phi_\epsilon) - E_\epsilon T_0(t, M_\epsilon \phi_\epsilon)\|_{U_\epsilon^{1,2}} \to 0$ as $\epsilon \to 0$.

The proof of (7.5) is in [4] Remark 3.2.

Hence we just need to show (7.4). To obtain this estimate we need some extra resolvent estimates, similar to the ones obtained in Section 3.1. To that end we introduce the continuous extension operator

$$\begin{array}{rcl} E_\epsilon^{\mathcal{C}} : \ \mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(0,1) & \to & \mathcal{C}(\overline{\Omega_\varepsilon}) \\ (w_\varepsilon, v_\varepsilon) & \to & E_\epsilon^{\mathcal{C}}(w_\varepsilon, v_\varepsilon) = \begin{cases} w_\varepsilon, & x \in \Omega \\ \tilde{v}_\varepsilon, & x \in \mathrm{R}_\varepsilon, \end{cases} \end{array} \tag{7.6}$$

where, for $x = (s,y) \in \mathrm{R}_\varepsilon$,

$$\tilde{v}_\varepsilon(x) = v_\varepsilon(s) + h_\varepsilon(s)\left(w_\varepsilon(0,y) - v_\varepsilon(0)\right) + h_\varepsilon(1-s)\left(w_\varepsilon(1,y) - v_\varepsilon(1)\right), \tag{7.7}$$

and the function $h_\delta(s) = h(\frac{s}{\delta})$, where $h : \mathbb{R}^+ \to [0,1]$ is a $\mathcal{C}^\infty$ function such that

$$h(s) = \begin{cases} 1, & \text{for } s \in [0, 1/4], \\ 0, & \text{for } s \geqslant 3/4 \end{cases}$$

and $|h'(s)| \leqslant C$.

Observe that with this definition $E_\epsilon^{\mathcal{C}}(w_\epsilon, v_\epsilon)$ is always a continuous function in $\bar{\Omega}_\epsilon$ if $(w_\epsilon, v_\epsilon) \in \mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(0,1)$. Moreover, if $(w_\epsilon, v_\epsilon) \in U_0^{1,2}$ then, $E_\epsilon^{\mathcal{C}}(w_\epsilon, v_\epsilon) \in H^1(\Omega_\epsilon)$.

We also need the following lemmas whose proofs will be provided later.

Lemma 7.1. *Let $\lambda \in \rho(A_\varepsilon) \cap \rho(A_0)$, then the following holds*

$$\begin{aligned} (\lambda + A_\varepsilon)^{-1} &- E_\epsilon(\lambda + A_0)^{-1} M \\ &= (I - \lambda(A_\varepsilon + \lambda)^{-1})(A_\varepsilon^{-1} - E_\epsilon^{\mathcal{C}} A_0^{-1} M_\epsilon)(I - \lambda E_\epsilon (A_0 + \lambda)^{-1} M_\epsilon) \\ &+ (I - \lambda(A_\varepsilon + \lambda)^{-1})\left(E_\epsilon^{\mathcal{C}} - E_\epsilon\right)(A_0 + \lambda)^{-1} M_\epsilon \end{aligned}$$

Lemma 7.2. *There is a constant $C > 0$ such that for each $\lambda \in \Sigma_\theta$ we have*

$$\begin{aligned} \|(E_\epsilon^{\mathcal{C}} - E_\epsilon)(A_0 + \lambda)^{-1} M_\epsilon\|_{\mathcal{L}(\mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(\bar{\mathrm{R}}_\varepsilon), H^1(\Omega)\oplus H^1(\mathrm{R}_\varepsilon))} &\leqslant C \frac{\varepsilon^{\frac{N}{2}}}{1 + |\lambda|^{\frac{1}{2}}}, \\ \|A_\epsilon (A_\varepsilon + \lambda)^{-1}(E_\epsilon^{\mathcal{C}} - E_\epsilon)(A_0 + \lambda)^{-1} M_\epsilon\|_{\mathcal{L}(\mathcal{C}(\overline{\Omega})\oplus\mathcal{C}(\bar{\mathrm{R}}_\varepsilon), H^1(\Omega)\oplus H^1(\mathrm{R}_\varepsilon))} &\leqslant C \varepsilon^{\frac{N}{2}}. \end{aligned} \tag{7.8}$$

Lemma 7.3. *There is a constant $C > 0$, independent of ϵ, such that*

(i) $\left\| E_\epsilon (I - \lambda(A_0 + \lambda)^{-1}) M_\epsilon f_\varepsilon \right\|_{C(\bar{\Omega}) \oplus C(\bar{R}_\epsilon)} \leqslant C \left\| f_\varepsilon \right\|_{C(\bar{\Omega}) \oplus C(\bar{R}_\epsilon)}$,
(ii) $\left\| (I - \lambda(A_\varepsilon + \lambda)^{-1}) g_\varepsilon \right\|_{H^1(\Omega_\varepsilon)} \leqslant C \left\| g_\varepsilon \right\|_{H^1(\Omega_\varepsilon)}$.

Lemma 7.4. *There exists a constant $C > 0$ such that for all $\lambda \in \Sigma_\theta$ and all $f_\varepsilon \in \mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(\mathrm{R}_\varepsilon)$, then*

$$\left\| (A_\varepsilon + \lambda)^{-1} - E_\epsilon (A_0 + \lambda)^{-1} M_\epsilon \right\|_{\mathcal{L}(\mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(\bar{\mathrm{R}}_\varepsilon), H^1(\Omega) \oplus H^1(\mathrm{R}_\varepsilon))} \leqslant C \varepsilon^{\frac{N}{2}}. \tag{7.9}$$

Clearly, from Lemma 7.4 and the expression of the differences of the semigroups in terms of the integral of the difference of the resolvents as in (4.9), we have that there is a constant $C > 0$ such that

$$\left\| e^{-A_\varepsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon \right\|_{\mathcal{L}(C(\bar{\Omega}) \oplus C(\bar{R}_\epsilon), H^1(\Omega) \oplus H^1(\mathrm{R}_\varepsilon))} \leqslant C \varepsilon^{N/2} t^{-1}. \tag{7.10}$$

On the other hand,

$$\begin{aligned} &\left\| e^{-A_\varepsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon \right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon), H^1(\Omega) \oplus H^1(\mathrm{R}_\varepsilon))} \\ &\qquad \leqslant \left\| e^{-A_\varepsilon t} \right\|_{\mathcal{L}(L^2(\Omega_\epsilon), H^1(\Omega_\epsilon))} + \left\| e^{-A_0 t} \right\|_{\mathcal{L}(U_0^p, H^1(\Omega) \oplus H^1(0,1))} \leq C t^{-\beta} \end{aligned} \tag{7.11}$$

for some β with $1/2 < \beta < 1$, see [4], Remark 3.2. . Interpolating (7.10) and (7.11), we have that that, for any $\eta < 1$,

$$\left\| e^{-A_\varepsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon \right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon), H^1(\Omega) \oplus H^1(\mathrm{R}_\varepsilon))} \leqslant C \varepsilon^{\eta N/2} t^{-(\eta + (1-\eta)\beta)}. \tag{7.12}$$

Choosing $\frac{N-1}{N} < \eta < 1$ so that $\eta N/2 > (N-1)/2$, the result follows with $\gamma = \eta + (1-\eta)\beta < 1$. This shows estimate (7.4) and the theorem is proved. ■

Remark 7.1. We may also obtain the convergence of the attractors in some other norms. As a matter of fact if K is a compact subset of $\bar{\Omega} \backslash P_0, P_1$ we can easily obtain uniform bounds of all the attractors for instance in $C^{1,\eta}(K)$. This estimates may be obtained with an appropriate cut-off function and using standard regularity properties of the nonlinear semigroups (we are far away from the channel R_ϵ). Hence, since we have obtained already the continuity (lower or upper) of the attractor in $L^p(K)$, with the compact embedding of $C^{1,\eta}(K)$ in $C^{1,\eta^-}(K)$ we also get the continuity (lower or upper) in $C^{1,\eta^-}(K)$.

We provide now the proofs of the different lemmas we have stated above.

Proof of Lemma 7.1: This proof is similar as Lemma 3.3. ■

Proof of Lemma 7.2: Let $f_\varepsilon \in \mathcal{C}(\overline{\Omega}) \oplus \mathcal{C}(\bar{\mathrm{R}}_\varepsilon)$ and define

$$K_\varepsilon := (E_\epsilon^{\mathcal{C}} - E_\epsilon)(A_0+\lambda)^{-1}Mf_\varepsilon = \tilde{z}_\varepsilon - z_\varepsilon,$$

where $\tilde{z}_\varepsilon = E_{\mathcal{C}}(A_0+\lambda)^{-1}Mf_\varepsilon$ e $z_\varepsilon = E_\epsilon(A_0+\lambda)^{-1}Mf_\varepsilon$.

Observe that $(A_0+\lambda)^{-1}Mf_\varepsilon = (w_\varepsilon, v_\varepsilon)$ where

$$\begin{cases} \quad -\Delta w_\varepsilon + \lambda w_\varepsilon = f_\varepsilon, & x \in \Omega \\ \qquad\qquad \frac{\partial w_\varepsilon}{\partial n} = 0, & x \in \partial\Omega \\ -\dfrac{1}{g}(g v_{\varepsilon s})_s + \lambda v_\varepsilon = Mf_\varepsilon, & s \in (0,1) \\ v_\varepsilon(0) = w_\varepsilon(0), v_\varepsilon(1) = w_\varepsilon(1), \end{cases} \tag{7.13}$$

$\tilde{v}_\varepsilon(s,y) = v_\varepsilon(s) + h_\varepsilon(s)\,(w_\varepsilon(0,y) - v_\varepsilon(0)) + h_\varepsilon(1-s)\,(w_\varepsilon(1,y) - v_\varepsilon(1))$, for all $(s,y) \in \mathrm{R}_\varepsilon$ and $z_\varepsilon(s,y) = v_\varepsilon(s)$, $\forall (s,y) \in \mathrm{R}_\varepsilon$.

Also, since $K_\epsilon \equiv 0$ in Ω, we have $\|K_\varepsilon\|_{H^1(\Omega)\oplus H^1(\mathrm{R}_\epsilon)} = \|K_\varepsilon\|^2_{H^1(\mathrm{R}_\varepsilon)}$. Moreover,

$$\begin{aligned} \|K_\varepsilon\|^2_{L^2(\mathrm{R}_\varepsilon)} \le\; & 2\int_0^\varepsilon \int_{\Gamma_\varepsilon^s} |h_\varepsilon(s)|^2 \, |w_\varepsilon(0,y) - v_\varepsilon(0)|^2 \, dsdy \\ & + 2\int_{1-\varepsilon}^1 \int_{\Gamma_\varepsilon^s} |h_\varepsilon(1-s)|^2 \, |w_\varepsilon(1,y) - v_\varepsilon(1)|^2 \, dsdy \\ & \leqslant C_2\, \varepsilon^N \, \|w_\varepsilon\|^2_{\mathcal{C}(\overline{\Omega})} . \end{aligned}$$

Now note that $h'_\varepsilon(s) = \varepsilon^{-1}h'(x/\varepsilon)$, $h'_\varepsilon(1-s) = -\varepsilon^{-1}h'((1-s)/\varepsilon)$. Hence, with similar estimates as above,

$$\begin{aligned} \|\nabla K_\varepsilon\|^2_{L^2(\mathrm{R}_\varepsilon)} \leqslant\; & 2\int_0^\varepsilon |h'_\varepsilon(s)|^2 \int_{\Gamma_\varepsilon^s} |w_\varepsilon(0,y) - v_\varepsilon(0)|^2 \, dsdy \\ & + 2\int_{1-\varepsilon}^1 |h'_\varepsilon(1-s)|^2 \int_{\Gamma_\varepsilon^s} |w_\varepsilon(1,y) - v_\varepsilon(1)|^2 \, dsdy \\ & + 2\int_0^\varepsilon |h_\varepsilon(s)|^2 \int_{\Gamma_\varepsilon^s} |\nabla_y w_\varepsilon(0,y)|^2 \, dsdy \\ & + 2\int_{1-\varepsilon}^1 |h_\varepsilon(1-s)|^2 \int_{\Gamma_\varepsilon^s} |\nabla_y w_\varepsilon(1,y)|^2 \, dsdy \\ & \leqslant \mathrm{C}\, \varepsilon^N \, \|w_\varepsilon\|^2_{\mathcal{C}^1(\overline{\Omega})} , \end{aligned}$$

where we have used that $\displaystyle\int_0^\varepsilon \int_{\Gamma_\varepsilon^s} r dsdy = O(\varepsilon^N)$.

The following estimates hold (see [11]), for some $C > 0$,

$$\|w_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \;\leqslant\; \frac{C}{|\lambda|+1} \, \|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \tag{7.14}$$

$$\|w_\varepsilon\|_{\mathcal{C}^1(\overline{\Omega})} \leqslant \frac{C}{|\lambda|^{1/2}+1}\,\|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})}\,. \tag{7.15}$$

Using (7.15) we have that

$$\|K_\varepsilon\|_{H^1(\mathrm{R}_\varepsilon)} \leqslant C\,\frac{\varepsilon^{N/2}}{|\lambda|^{1/2}+1}\,\|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})}\,. \tag{7.16}$$

which shows the first inequality of (7.8).

On the other hand we also have that

$$\left\|\lambda(A_\varepsilon+\lambda)^{-1}K_\varepsilon\right\|_{H^1(\Omega_\varepsilon)} \leqslant |\lambda|\frac{1}{|\lambda|^{1/2}+1}\,\|K_\varepsilon\|_{L^2(\mathrm{R}_\varepsilon)} \leqslant C\,\frac{\varepsilon^{N/2}}{|\lambda|^{1/2}+1}\,\|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} \tag{7.17}$$

and

$$\begin{aligned}\|(I-\lambda(A_\varepsilon+\lambda)^{-1})(E_\epsilon^{\mathcal{C}}-E_\epsilon)(A_0+\lambda)^{-1}M\,f_\varepsilon\|_{H^1(\Omega)\oplus H^1(\mathrm{R}_\varepsilon)}&\\ \leqslant \|K_\varepsilon\|_{H^1(\Omega)\oplus H^1(\mathrm{R}_\varepsilon)} + \left\|\lambda(A_\varepsilon+\lambda)^{-1}K_\varepsilon\right\|_{H^1(\Omega_\varepsilon)}&\\ \leqslant C\,\frac{\varepsilon^{N/2}}{|\lambda|^{1/2}+1}\,\|f_\varepsilon\|_{\mathcal{C}(\overline{\Omega})}\,.&\end{aligned}$$

The proof is complete. ■

Proof of Lemma 7.3: It follows from the definition of E_ϵ and M_ϵ that

$$\|E_\epsilon\|_{\mathcal{L}(L^\infty(\Omega)\oplus L^\infty(0,1),L^\infty(\Omega_\varepsilon))} = 1$$

and $\|M_\varepsilon\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon),L^\infty(\Omega)\oplus L^\infty(0,1))} \leqslant 1$. Hence,

$$\|E_\epsilon A_0(A_0+\lambda)^{-1}M\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon),L^\infty(\Omega_\varepsilon))} \leqslant C\|A_0(A_0+\lambda)^{-1}\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon)\oplus L^\infty(0,1))} \tag{7.18}$$

Let $g \in \mathcal{C}(\overline{\Omega}) \oplus L^\infty(0,1)$, be such that

$$(A_0+\lambda)^{-1}g = (w,v). \tag{7.19}$$

or equivalently

$$\begin{cases} -\Delta w_\varepsilon + \lambda w_\varepsilon = g, & \text{em} \quad \Omega \\ \quad \frac{\partial w_\varepsilon}{\partial n} = 0, & \text{em} \quad \partial\Omega \\ -\dfrac{1}{g}(g v_{\varepsilon s})_s + \lambda v_\varepsilon = Mg, & \text{em} \quad (0,1) \\ v_\varepsilon(0) = w_\varepsilon(0), \quad v_\varepsilon(1) = w_\varepsilon(1) & \end{cases} \tag{7.20}$$

proceeding as in the proof of Proposition 3.1 (*iv*), we have that

$$\begin{aligned}\|w\|_{\mathcal{C}(\overline{\Omega})} &\leqslant \frac{C}{|\lambda|+1}\,\|g\|_{\mathcal{C}(\overline{\Omega})}\,,\\ \|v\|_{\mathcal{C}(\overline{\Omega})} &\leqslant \frac{C}{|\lambda|+1}\Big(\|g\|_{\mathcal{C}(\overline{\Omega})}+\|g\|_{\mathcal{C}(0,1)}\Big)\,.\end{aligned}$$

Since $A_0(A_0+\lambda)^{-1}=I-\lambda(A_0+\lambda)^{-1}$, then

$$\begin{aligned}\left\|A_0(A_0+\lambda)^{-1}g\right\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)} &= \left\|g-\lambda(A_0+\lambda)^{-1}g\right\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)}\\ &\leqslant \|g\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)}+C\,\|g\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)}\\ &\leqslant \widetilde{C}\,\|g\|_{\mathcal{C}(\overline{\Omega})\oplus L^\infty(0,1)}\,.\end{aligned}$$

Applying this to (7.18), we have that

$$\left\|E_\epsilon A_0(A_0+\lambda)^{-1}M\right\|_{\mathcal{L}(L^\infty(\Omega_\varepsilon),L^\infty(\Omega_\varepsilon))}\ \leqslant\ \mathrm{C}, \tag{7.21}$$

where C is independent of λ and ε.

Part (*ii*) is immediate from the fact that A_ϵ is positive and self-adjoint. ■

Proof of Lemma 7.4: The proof follows from Lemma 7.1, Lemma 7.2, Lemma 7.3 and statement (3.8) from Proposition 3.2. ■

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