CADERNOS DE MATEMÁTICA **09**, 19–44 May (2008)
ARTIGO NÚMERO SMA#289

Semilinear evolution equations with almost sectorial operators

Alexandre N. Carvalho*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: andcarva@icmc.usp.br

Tomasz Dłotko†

Institute of Mathematics, Silesian University, 40-007 Katowice, Poland
E-mail: tdlotko@ux2.math.us.edu.pl

Marcelo J. D. Nascimento‡

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: mardn@icmc.usp.br

Inspired by the theory of semigroups of growth α, we construct *an evolution process of growth* α. The abstract theory is applied to study semilinear singular non-autonomous parabolic problems. We prove that, under natural assumptions, a reasonable concept of solution can be given to such semilinear singularly non-autonomous problems. Applications are considered to non-autonomous parabolic problems in space of Hölder continuous functions and to a parabolic problem in a domain $\Omega \subset \mathbb{R}^n$ with a one dimensional handle.
May, 2008 ICMC-USP

1. INTRODUCTION

In this paper we consider the Cauchy problem for singularly non-autonomous evolution equation of the form

$$\begin{cases} \dfrac{du}{dt} + A(t)u = f(u), \\ u(\tau) = u_0 \in X, \end{cases}$$

* Research partially supported by CNPq grant 305447/2005-0 and by FAPESP grant 03/10042-0, Brazil.
† Research partially supported by FAPESP grant 07/51373-0, Brazil.
‡ Research partially supported by FAPESP grant 02/11855-2, Brazil.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

in a Banach space X. In the usual theory (see [16, 17]) each operator $A(t)$ is sectorial and the family $\{A(t) : t \in \mathbb{R}\}$ is Hölder continuous with respect to time t (which will be made precise later). Our interest is to consider operators $A(t)$ whose resolvent contains a sector, but for which the resolvent operators do not satisfy the required estimate to be a sectorial operator. Instead we assume that for each t the operator $A(t)$ is *almost sectorial*.

To better explain the results we need to introduce some terminology. Recall first the definition of an almost sectorial operator (see [3, 15]). A closed linear operator $A(t) : D(A(t)) \subset X \to X$ is called *almost sectorial* if for some $\theta \in (0, \frac{\pi}{2})$,

$$\rho(-A(t)) \supset \Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| \leq \pi - \theta\} \cup \{0\}$$

and, for some $0 < \alpha < 1$,

$$\|(\lambda + A(t))^{-1}\|_{L(X)} \leq \frac{C}{|\lambda|^\alpha}, \quad \forall \lambda \in \Sigma_\theta\backslash\{0\}. \tag{1}$$

Consequently,

$$\|A(t)(\lambda + A(t))^{-1}\|_{L(X)} \leq 1 + C|\lambda|^{1-\alpha}, \quad \forall \lambda \in \Sigma_\theta. \tag{2}$$

It is impossible to conclude from the above estimates that $A(t)$ generates a strongly continuous semigroup. However we will show that $A(t)$ generates a semigroup with a singularity at $t = 0$ in a sense that will be specified later.

Semigroups of growth $1 - \alpha$ were investigated earlier by Da Prato [6] (see also [10]), when $1 - \alpha$ is a non negative integer. In [12], the author extends the theory to the case when $1 - \alpha$ is a positive real number (see [18]). Next we recall the definition of semigroup of growth $1 - \alpha$.

Definition 1.1. [[12]] Let $1 - \alpha > 0$. A family of operators $\{T(t) : t \geq 0\}$ on X that satisfies

(*i*) $T(0) = I$, (I is the identity operator on X),
(*ii*) $T(t + s) = T(t)T(s)$ for every $t, s > 0$,
(*iii*) If $T(t)u = 0$ for all $t > 0$ then $u = 0$,
(*iv*) $\|t^{1-\alpha}T(t)\|$ is bounded as t tends to zero,
(*v*) $X_0 = \bigcup_{t>0} T(t)[X]$ is dense in X,

is called the *semigroup of growth* $1 - \alpha$.

Note, that the condition (*ii*) of the above definition need not be satisfied for $t = 0$ or $s = 0$. Examples of semigroups of growth $1 - \alpha$ will be found in Kreĭn [9]. Inspired by the theory of semigroups of growth $1 - \alpha$, we define an *evolution process of growth* $1 - \alpha$ in the following manner.

Definition 1.2. Let $1 - \alpha > 0$. A family $\{U(t, \tau) : t > \tau \in \mathbb{R}\} \subset L(X)$ that satisfies

(1) $U(t, \sigma)U(\sigma, \tau) = U(t, \tau)$, for any $t > \sigma > \tau,\ \tau \in \mathbb{R}$,
(2) $\|(t - \tau)^{1-\alpha}U(t, \tau)\| \leq M, \quad \forall t > \tau$,

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

(3) $(t,\tau)\mapsto U(t,\tau)v_0$ is continuous for $t>\tau,\ v_0\in X$,

is called a *linear evolution process of growth* $1-\alpha$.

Let Γ be the boundary of the sector of Σ_θ oriented in such a way that the imaginary part is increasing. If

$$T_{A(\tau)}(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A(\tau))^{-1}d\lambda,$$

by (1) we have that $\{T_{A(\tau)}(t):t>0\}\subset L(X)$. Then, from (1), we obtain that $T_{A(\tau)}(t)$ is a semigroup of growth order $1-\alpha$. Theory of linear evolution process of growth α is presented in Section 2. The notion of *linear evolution process of growth* α, $\{U(t,\tau):t>\tau\in\mathbb{R}\}$, is defined there allowing us to solve linear homogeneous problem

$$\begin{cases}\dfrac{du}{dt}+A(t)u=0,\\ u(0)=u_0\in X,\end{cases}\tag{3}$$

The main result of that section, Theorem 2.1, asserts local solvability of the integral equation

$$U(t,\tau)=T_{A(\tau)}(t-\tau)+\int_\tau^t U(t,s)[A(\tau)-A(s)]T_{A(\tau)}(s-\tau)ds,$$

using Banach contraction principle (following the ideas in [4]) in the complete metric space

$$E=\{U\in C([\tau,t_0),L(X)):\sup_{t\in[\tau,\,t_0)}(t_0-t)^{1-\alpha}\|U(t)\|_{L(X)}<\infty\}.$$

The corresponding semilinear parabolic problem is studied in Section 3. Assuming that the nonlinear term satisfies suitable growth restriction we prove there its local solvability (see Theorem 3.1). Applications of the abstract theory to semilinear parabolic problems in Hölder spaces $C_0^\mu(\bar\Omega)$ and to parabolic problems in n dimensional domains with a one-dimensional handle are described next in Section 4.

The notion of almost sectorial operators was studied by W. von Wahl in [20]. An interesting investigation of functional calculus for such operators was presented recently in [14, 15]. Our studies of the semigroups generated by almost sectorial operators have their origin in the paper [3]. Here we extend the existence results known in case of semilinear autonomous problems to singularly-time dependent (as time varies the changes in $A(t)$ are of the same order as $A(t)$) case obtaining the *evolution processes of growth* α.

2. ABSTRACT THEORY OF EVOLUTION PROCESSES OF GROWTH α

Let X be a Banach space and $A(t):D(A(t))\subset X\to X, t\in\mathbb{R}$, a family of closed operators. In this section we study the problem (3) with the operators $A(t)$ having some special features that we describe below. In particular, the operators $A(t)$ need not generate an analytic semigroup nor a C^0-semigroup.

Remark 2. 1. Throughout this paper the name "analytic semigroup" is reserved for C^0 (strongly continuous) analytic semigroup (as in [13]). The semigroups consider here, called "semigroups of growth α", need not be strongly continuous at $t = 0$, but they are analytic in an open sector of the complex plane $\mathbb{C}$ (compare the definition in [11], p. 34).

Assumption. Assume that there exists another Banach space Z which is continuously embedded in X and such that

$$A_Z(t) : D(A_Z(t)) \subset Z \to Z$$

with $D(A_Z(t)) = \{v \in D(A(t)) \cap Z : A(t)v \in Z\}$, $A(t)v = A_Z(t)v$ for all $v \in D(A_Z(t))$ and $A_Z(t)$ satisfies

$$\|(\lambda + A_Z(t))^{-1}\|_{L(Z)} \leq \frac{C}{|\lambda|}, \quad \forall \lambda \in \Sigma_\theta \backslash \{0\}.$$

Remark 2. 2. While at the first view this assumption looks strange, it is automatically satisfied for the natural and important examples described at the end of the paper.

Observe that

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda,$$

converge in the uniform operator topology of $L(X)$ for all $t > 0$ and using (1) we have that

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda\right\|_{L(X)} &\leq \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}\|(\lambda + A(\tau))^{-1}\|_{L(X)} d|\lambda| \\ &\leq \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}\frac{C}{|\lambda|^\alpha} d|\lambda| \\ &\leq \frac{t^{\alpha-1}}{2\pi}\int_\Gamma e^{-\cos\theta|\mu|}\frac{C}{|\mu|^\alpha} d|\mu|.\end{aligned}$$

Hence, if

$$T_{A(\tau)}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda, \tag{4}$$

$\{T_{A(\tau)}(t) : t > 0\} \subset L(X)$ and there is a constant $C > 0$ such that

$$\|T_{A(\tau)}(t)\|_{L(X)} \leq Ct^{-1+\alpha}. \tag{5}$$

Note that $\{T_{A(\tau)}(t) : t \geq 0\}$ decays exponentially.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

In the same way as when $-A(\tau)$ generates a usual analytic semigroup, one can show that the semigroup property, $T_{A(\tau)}(t+s) = T_{A(\tau)}(t)T_{A(\tau)}(s)$, $t, s > 0$, is satisfied. Also, since the operator $A(\tau)$ is closed and since, from (2), the integral

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda,$$

is convergent, $A(\tau)T_{A(\tau)}(t) \in L(X)$ for all $t > 0$ and from (2) it follows that

$$\begin{aligned}
\|A(\tau)T_{A(\tau)}(t)\|_{L(X)} &\leq \left\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}A(\tau)(\lambda + A(\tau))^{-1}d\lambda\right\|_{L(X)} \\
&\leq \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}\|A(\tau)(\lambda + A(\tau))^{-1}\|_{L(X)}d|\lambda| \\
&\leq \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}C(1+|\lambda|^{1-\alpha})d|\lambda| \\
&= \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\mu|}C(1+t^{\alpha-1}|\mu|^{1-\alpha})t^{-1}d|\mu| \\
&\leq \max\{t^{-1}, t^{-2+\alpha}\}\frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\mu|}C(1+|\mu|^{1-\alpha})d|\mu| \\
&\leq M\max\{t^{-1}, t^{-2+\alpha}\}.
\end{aligned} \tag{6}$$

Note that, for $0 < t < 1$, we have

$$\|A(\tau)T_{A(\tau)}(t)\|_{L(X)} \leq Mt^{-2+\alpha}. \tag{7}$$

In particular, $T_{A(\tau)}(t)u_0 \in D(A)$ for all $t > 0$.

For $0 < \nu < 1$ we can define fractional powers $A_Z(t)^{-\nu} \in L(X)$ of $A_Z(t)$ as

$$A_Z(t)^{-\nu} = \frac{\sin\pi\nu}{\pi}\int_0^\infty s^{-\nu}(s + A_Z(t))^{-1}ds,$$

and observe that, for $\nu > 1-\alpha$, the integral operator

$$A(t)^{-\nu} := \frac{\sin\pi\nu}{\pi}\int_0^\infty s^{-\nu}(s + A(t))^{-1}ds$$

defines an operator in $L(X)$ that coincides with $A_Z(t)^{-\nu}$ on Z. We introduce $A(t)^\nu$ as the inverse of $A(t)^{-\nu}$ and set $X^\nu = D(A(t)^\nu)$ with the graph norm.

Moreover, for $1 > \alpha > \nu > 1-\alpha$ (consequently; $\alpha > \frac{1}{2}$),

$$
\begin{aligned}
&\left\|A(t)^{-1+\nu}u\right\|_X \\
&\le \frac{\sin\pi\nu}{\pi}\Big(\int_0^\tau s^{-1+\nu}\left\|(s+A(t))^{-1}u\right\|_X ds + \int_\tau^\infty s^{-1+\nu}\left\|(s+A(t))^{-1}u\right\|_X ds\Big) \\
&\le \frac{\sin\pi\nu}{\pi}\Big(\int_0^\tau s^{-1+\nu}\left\|A(t)(s+A(t))^{-1}A(t)^{-1}u\right\|_X ds + \int_\tau^\infty s^{-1+\nu}\left\|(s+A(t))^{-1}u\right\|_X ds\Big) \\
&\le \frac{\sin\pi\nu}{\pi}\Big(M_1\int_0^\tau s^{-1+\nu}ds\left\|A(t)^{-1}u\right\|_X + M_2\int_\tau^\infty s^{-1-\alpha+\nu}ds\left\|u\right\|_X\Big) \\
&\le \frac{\sin\pi\nu}{\pi}\Big(\frac{M_1}{\nu}\tau^\nu\left\|A(t)^{-1}u\right\|_X + \frac{M_2}{\alpha-\nu}\tau^{\nu-\alpha}\left\|u\right\|_X\Big) \\
&\le \frac{\sin\pi\nu}{\pi}M_1^{1-\frac{\nu}{\alpha}}M_2^{\frac{\nu}{\alpha}}\Big(\frac{1}{\nu}+\frac{1}{\alpha-\nu}\Big)\left\|A(t)^{-1}u\right\|_X^{1-\frac{\nu}{\alpha}}\left\|u\right\|_X^{\frac{\nu}{\alpha}},
\end{aligned}
$$

where, in the last inequality, we minimize the right hand side for $\tau \in (0,\infty)$. Thus, for $u \in D(A(t))$ we have

$$\left\|A(t)^\nu u\right\|_X \le \frac{\sin\pi\nu}{\pi}M_1^{1-\frac{\nu}{\alpha}}M_2^{\frac{\nu}{\alpha}}\Big(\frac{1}{\nu}+\frac{1}{\alpha-\nu}\Big)\left\|u\right\|_X^{1-\frac{\nu}{\alpha}}\left\|A(t)u\right\|_X^{\frac{\nu}{\alpha}}. \tag{8}$$

Using the fact that $\frac{\pi}{\sin\pi r} = \Gamma(r)\Gamma(1-r)$ and the expression

$$(\lambda + A(t))^{-1} = \int_0^\infty e^{-\lambda s}T_{A(t)}(s)ds, \quad \forall\lambda > 0,$$

we have for $\nu > 1-\alpha$,

$$A(t)^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty s^{\nu-1}T_{A(t)}(s)ds. \tag{9}$$

Let Γ be the boundary of the sector of Σ_θ oriented in such a way that the imaginary part is increasing.

It follows from (5), (7) and (8) that, for $\alpha > \nu$

$$\left\|A^\nu T_{A(\tau)}(t)\right\|_{L(X)} \le Ct^{-1+\alpha-\frac{\nu}{\alpha}}.$$

Note that, if $0 < 1-\alpha < \nu < \alpha^2$ (consequently, $1-\nu > 1-\alpha$ and $1 > \alpha > \frac{\sqrt{5}-1}{2}$), then $-1 < \alpha - 1 - \frac{\nu}{\alpha}$.

The next lemma (for its proof see [3]) says that, in some sense, the semigroup $\{T_{A(\tau)}(t) : t \ge 0\}$ defines the solution operator for linear homogeneous problem

$$\begin{cases} \dfrac{du}{dt} + A(\tau)u = 0, \\ u(0) = u_0 \in X. \end{cases}$$

LEMMA 2.1. *The semigroup $T_{A(\tau)}(t) : (0,\infty) \to L(X)$ is differentiable and*

$$\frac{d}{dt}T_{A(\tau)}(t) = \frac{1}{2\pi i}\int_0^\infty \lambda e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda.$$

In addition, for each $u_0 \in X$, we have that

$$\frac{d}{dt}T_{A(\tau)}(t)u_0 + A(\tau)T_{A(\tau)}(t)u_0 = 0,\ t > 0.$$

Consider the problem

$$\begin{cases} \dfrac{du}{dt} + A(t)u = 0 \\ u(\tau) = I \in L(X),\ \tau \in \mathbb{R}. \end{cases} \tag{10}$$

in a Banach space $L(X)$ and denote by $\{U(t,\tau) : t > \tau \in \mathbb{R}\}$ the solution operator family associated to (10). Note that, for fixed $\tau \in \mathbb{R}$, the family $\{T_{A(\tau)}(t-\tau) : t > \tau\}$ is the solution operator family associated to

$$\begin{cases} \dfrac{du}{dt} = A(\tau)u \\ u(\tau) = I \in L(X),\ \tau \in \mathbb{R}. \end{cases}$$

Therefore, the difference $\{U(t,\tau) - T_{A(\tau)}(t-\tau) : t > \tau\}$ is the solution operator family associated to

$$\begin{cases} \dfrac{du}{dt} + A(t)u = -[A(t) - A(\tau)]T_{A(\tau)}(t-\tau), \\ u(\tau) = 0 \in L(X),\ \tau \in \mathbb{R}. \end{cases}$$

Hence we should have

$$U(t,\tau) = T_{A(\tau)}(t-\tau) + \int_\tau^t U(t,s)[A(\tau) - A(s)]T_{A(\tau)}(s-\tau)ds. \tag{11}$$

To obtain $\{U(t,\tau) \in L(X) : t > \tau \in \mathbb{R}\}$, we will use (11); that is, we show that, for each $\tau \in \mathbb{R}$, equation (11) has a unique continuous solution $\{U(t,\tau) \in L(X) : t > \tau\}$.

Assume that

(a) The operator $A(t) : D \subset X \to X$ is a closed linear operator (its domain D is independent of t) and for some $0 < \alpha < 1$, it satisfies (1). To express this fact we will say that the family $A(t)$ is *uniformly almost sectorial.*

(b) Moreover, there exists $C > 0$ and $\epsilon \in (0,1]$ such that

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{L(X)} \le C(t-\tau)^\epsilon,\ \forall\, t,\tau,s \in \mathbb{R}. \tag{12}$$

To express this fact we will say that the function $A(t)$ is *uniformly Hölder continuous.*

Fix $t_0 > \tau$ and denote by $U(t)$ the operator $U(t_0, t)$. For $\alpha + \epsilon > 1$ consider the complete metric space

$$E = \{U \in C([\tau, t_0), L(X)) : \sup_{t \in [\tau,\, t_0)} (t_0 - t)^{1-\alpha} \|U(t)\|_{L(X)} < \infty\}$$

equipped with the metric

$$\|U\|_E = \sup_{t \in [\tau,\, t_0)} (t_0 - t)^{1-\alpha} \|U(t)\|_{L(X)}.$$

Let S be the map defined by

$$(SU)(t) = T_{A(t)}(t_0 - t) + \int_t^{t_0} U(s)[A(t) - A(s)]T_{A(t)}(s - t)ds. \tag{13}$$

If $U \in E$, then the integral in the above equality is convergent and

$$t \mapsto \int_t^{t_0} U(s)[A(t) - A(s)]T_{A(t)}(s - t)ds$$

is in $C([\tau, t_0), L(X))$.

We are now ready to state our main result in this section.

THEOREM 2.1. *There exists a unique solution of the integral equation* (13) *in* E.

The proof of this result will be given in a series of results which we present next.

LEMMA 2.2. *For* $t, s, \xi \in \mathbb{R}$, *the following inequalities hold*

$$\|T_{A(t)}(\tau) - T_{A(s)}(\tau)\|_{L(X)} \le c\tau^{-2+2\alpha}(t - s)^{\epsilon}, \quad \textit{for } \tau > 0, \tag{14}$$

$$\|A(\xi)[T_{A(t)}(\tau) - T_{A(s)}(\tau)]\|_{L(X)} \le c\tau^{-3+2\alpha}(t - s)^{\epsilon}, \quad \textit{for } \tau > 0. \tag{15}$$

Proof: Note that

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}[A(s) - A(t)](\lambda + A(s))^{-1}.$$

Thus, due to (4), we get

$$[T_{A(t)}(\tau) - T_{A(s)}(\tau)] = -\frac{1}{2\pi i} \int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1}[A(t) - A(s)](\lambda + A(s))^{-1} d\lambda.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

To prove (14) note that, using (1), (2) and (12),

$$\begin{aligned}
&\|T_{A(t)}(\tau) - T_{A(s)}(\tau)\| \\
&= \| -\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1}[A(t) - A(s)]A(s)^{-1}A(s)(\lambda + A(s))^{-1}d\lambda\| \\
&\leq \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|\tau}\frac{c_1}{|\lambda|^\alpha}|\lambda|^{1-\alpha}d|\lambda|\; C(t-s)^\epsilon \\
&\leq \frac{Cc_1}{2\pi}\int_\Gamma e^{-\cos\theta|u|}\tau^{-1+2\alpha}|u|^{1-2\alpha}\frac{d|u|}{\tau}\;(t-s)^\epsilon \\
&\leq \frac{Cc_1}{2\pi}\tau^{-2+2\alpha}\int_\Gamma e^{-\cos\theta|u|}|u|^{1-2\alpha}d|u|\;(t-s)^\epsilon \leq c\tau^{-2+2\alpha}(t-s)^\epsilon.
\end{aligned}$$

This concludes the proof of (14). To prove (15) we observe that

$$A(t)\Big[T_{A(t)}(\tau) - T_{A(s)}(\tau)\Big] = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}A(t)(\lambda + A(t))^{-1}[A(t) - A(s)](\lambda + A(s))^{-1}d\lambda.$$

Then, it follows from (2) and (12) that

$$\begin{aligned}
\|A(\xi)[T_{A(t)}(\tau) - T_{A(s)}(\tau)]\|_{L(X)} &\leq c_1\|A(t)[T_{A(t)}(\tau) - T_{A(s)}(\tau)]\|_{L(X)} \\
&\leq c_1\| -\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}A(t)(\lambda + A(t))^{-1}[A(t) - A(s)](\lambda + A(s))^{-1}d\lambda\| \\
&\leq \frac{c_2}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|\tau}|\lambda|^{2-2\alpha}d|\lambda|\;(t-s)^\epsilon \\
&\leq \frac{c_2}{2\pi}\int_\Gamma e^{-\cos\theta|u|}\frac{|u|^{2-2\alpha}}{\tau^{2-2\alpha}}\frac{d|u|}{\tau}\;(t-s)^\epsilon \\
&\leq c\tau^{-3+2\alpha}(t-s)^\epsilon.
\end{aligned}$$

This shows (15) and completes the proof. □

LEMMA 2.3. *The functions* $(0,\infty)\times\mathbb{R}\ni(\tau,s)\mapsto T_{A(s)}(\tau)\in L(X)$ *and* $(0,\infty)\times\mathbb{R}\times\mathbb{R}\ni(\tau,t,s)\mapsto A(t)T_{A(s)}(\tau)\in L(X)$ *are continuous in the uniform operator topology of* $L(X)$.

COROLLARY 2.1. *The functions*

$$(t,\tau)\mapsto T_{A(\tau)}(t-\tau),\quad (t,\tau)\mapsto T_{A(t)}(t-\tau),$$

$$(t,\tau)\mapsto[A(\tau)-A(t)]T_{A(\tau)}(t-\tau)\ \textit{and}\ \ (t,\tau)\mapsto[A(\tau)-A(t)]T_{A(t)}(t-\tau)$$

are continuous from $\{(t,\tau)\in\mathbb{R}^2 : t>\tau\}$ *into* $L(X)$ *in the uniform operator topology.*

From the Corollary 2.1, $t \mapsto T_{A(t)}(t_0 - t)$ is in E. In what follows, we prove that S is a contraction from E into itself and therefore has a unique fixed point in E.

Recall that the *beta function* $\mathbf{B}(\cdot,\cdot) : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ is defined by

$$\mathbf{B}(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1} ds.$$

First, note that if $U \in E$, then thanks to (5), (12) and (7),

$$\begin{aligned}
&(t_0 - t)^{1-\alpha}\|(SU)(t)\|_{L(X)} \leq (t_0 - t)^{1-\alpha}\|T_{A(t)}(t_0 - t)\|_{L(X)} \\
&+ (t_0 - t)^{1-\alpha} \int_t^{t_0} \|U(s)\|_{L(X)}\|[A(t) - A(s)]A^{-1}(t)\|_{L(X)}\|A(t)T_{A(t)}(s - t)\|_{L(X)} ds \\
&\leq C + (t_0 - t)^{1-\alpha} \int_t^{t_0} M(t_0 - s)^{\alpha-1} C(s - t)^{\epsilon} M(s - t)^{-2+\alpha} ds \\
&\leq C + C_1\mathbf{B}(\alpha, \alpha + \epsilon - 1)(t_0 - t)^{\alpha+\epsilon-1}.
\end{aligned}$$

Now, given $U, V \in E$, it follows that

$$(SU)(t) - (SV)(t) = \int_t^{t_0} [U(s) - V(s)][A(t) - A(s)]T_{A(t)}(s - t) ds,$$

and thus, from (7) and (12),

$$\begin{aligned}
&(t_0 - t)^{1-\alpha} \|(SU)(t) - (SV)(t)\|_{L(X)} \\
&\leq (t_0 - t)^{1-\alpha} \int_t^{t_0} \|U(s) - V(s)\|_{L(X)} \|A(s) - A(t)\|_{L(X^1,X)} \|T_{A(t)}(s - t)\|_{L(X,X^1)} ds \\
&\leq (t_0 - t)^{1-\alpha} \int_t^{t_0} C(t_0 - s)^{\alpha-1}(s - t)^{\epsilon} C(s - t)^{-2+\alpha} ds \sup_{t \leq s \leq t_0} \left\{\|U(s) - V(s)\|_{L(X)}\right\} \\
&\leq C_1(t_0 - t)^{1-\alpha} \int_t^{t_0} (t_0 - s)^{\alpha-1}(s - t)^{\epsilon+\alpha-2} ds \, \|U - V\|_E \\
&\leq C_1\mathbf{B}(\alpha, \alpha + \epsilon - 1)(t_0 - t)^{\alpha+\epsilon-1} \|U - V\|_E .
\end{aligned}$$

Also, a similar application of (7) and (12) give us,

$$\begin{aligned}
&(t_0 - t)^{1-\alpha}\|(S^2U)(t) - (S^2V)(t)\|_{L(X)} = (t_0 - t)^{1-\alpha}\|S(SU)(t) - S(SV)(t)\|_{L(X)} \\
&\leq (t_0 - t)^{1-\alpha}\int_t^{t_0}\|(SU)(s) - (SV)(s)\|_{L(X)}\|A(s) - A(t)\|_{L(X^1,X)}\|T_{A(t)}(s - t)\|_{L(X,X^1)} ds \\
&\leq C_1\mathbf{B}(\alpha, \alpha + \epsilon - 1)(t_0 - t)^{1-\alpha} \int_t^{t_0} (t_0 - s)^{2\alpha+\epsilon-2}(s - t)^{\epsilon+\alpha-2} ds \, \|U - V\|_E \\
&\leq C_1\mathbf{B}(\alpha, \alpha + \epsilon - 1)\mathbf{B}(2\alpha + \epsilon - 1, \alpha + \epsilon - 1)(t_0 - t)^{2(\alpha+\epsilon-1)} \|U - V\|_E .
\end{aligned}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Then, we have

$$(t_0-t)^{1-\alpha}\left\|(S^3U)(t)-(S^3V)(t)\right\|_{L(X)} = (t_0-t)^{1-\alpha}\left\|S(S^2U)(t)-S(S^2V)(t)\right\|_{L(X)}$$
$$\leq (t_0-t)^{1-\alpha}\int_t^{t_0}\left\|(S^2U)(s)-(S^2V)(s)\right\|_{L(X)}\|A(s)-A(t)\|_{L(X^1,X)}\|T_{A(t)}(s-t)\|_{L(X,X^1)}ds$$
$$\leq C_1\mathbf{B}(\alpha,\alpha+\epsilon-1)\mathbf{B}(2\alpha+\epsilon-1,\alpha+\epsilon-1)(t_0-t)^{1-\alpha}\int_t^{t_0}(t_0-s)^{3\alpha+2\epsilon-3}(s-t)^{\epsilon+\alpha-2}ds\,\|U-V\|_E$$
$$\leq C_1\mathbf{B}(\alpha,\alpha+\epsilon-1)\mathbf{B}(2\alpha+\epsilon-1,\alpha+\epsilon-1)\mathbf{B}(3\alpha+2\epsilon-2,\alpha+\epsilon-1)(t_0-t)^{3(\alpha+\epsilon-1)}\|U-V\|_E .$$

Following with the iterations, we get

$$\begin{aligned}&\|(S^nU)(t)-(S^nV)(t)\|_{L(X)}\\ &\leq C_1\mathbf{B}(\alpha,\alpha+\epsilon-1)\mathbf{B}(2\alpha+\epsilon-1,\alpha+\epsilon-1)\ldots\mathbf{B}(n\alpha+(n-1)(\epsilon-1),\alpha+\epsilon-1)\\ &\times(t_0-t)^{n(\alpha+\epsilon-1)}\|U-V\|_E .\end{aligned}$$

Recalling that $\mathbf{B}(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$, where Γ is the *Gamma function*, we have

$$\mathbf{B}(\alpha,\alpha+\epsilon-1)\ldots\mathbf{B}(n\alpha+(n-1)(\epsilon-1),\alpha+\epsilon-1)=\frac{\Gamma(\alpha)[\Gamma(\alpha+\epsilon-1)]^n}{\Gamma((n+1)\alpha+n(\epsilon-1))}.$$

That gives,

$$\|(S^nU)(t)-(S^nV)(t)\|_{L(X)}\leq\frac{C\Gamma(\alpha)[\Gamma(\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}]^n}{\Gamma((n+1)\alpha+n(\epsilon-1))}\|U-V\|_E .$$

Consider the power series

$$\sum_{n=1}^{\infty}\frac{C\Gamma(\alpha)}{\Gamma((n+1)\alpha+n(\epsilon-1))}s^n.$$

If the radius of convergence of this power series is infinite, there is $n\in\mathbb{N}$, such that

$$\frac{C\Gamma(\alpha)[\Gamma(\alpha+\epsilon-1)(t_0-\tau)^{\alpha+\epsilon-1}]^n}{\Gamma((n+1)\alpha+n(\epsilon-1))}<1,$$

and then by the Banach contraction principle S^n has a unique fixed point in E and, consequently, S has a unique fixed point in E. To verify that the radius of convergence is infinite, we apply the ratio test. Dividing the $(n+1)^{\text{th}}$ coefficient of the series by its n^{th} coefficient we obtain that

$$\frac{1}{\Gamma(\alpha+\epsilon-1)}\mathbf{B}((n+1)\alpha+n(\epsilon-1),\alpha+\epsilon-1)=\frac{1}{\Gamma(\alpha+\epsilon-1)}\int_0^1\theta^{(n+1)\alpha+n(\epsilon-1)-1}(1-\theta)^{\alpha+\epsilon-2}d\theta.$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Now, for each $0 < \theta \leq 1$, the integrand in the above expression is bounded by $(1-\theta)^{\alpha+\epsilon-1}$ and converges pointwise to zero as $n \to \infty$. It follows from the Lebesgue Dominated Convergence Theorem that $\mathbf{B}((n+1)\alpha + n(\epsilon-1), \alpha+\epsilon-1) \overset{n\to\infty}{\longrightarrow} 0$. The proof of existence of a unique fixed point of S in E is completed.

Since t_0 is arbitrarily chosen, (11) has a unique solution $U(t,\tau)$ which is continuous in the uniform operator topology for all $t > \tau$.

PROPOSITION 2.1. *Let $0 < \alpha < 1$ and $\beta < \alpha$, then*

$$\|A(t)^\beta T_{A(t)}(\tau)\|_{L(X)} \leq C \max\{\tau^{-1+\alpha-\frac{\beta}{\alpha}}, \tau^{-1+\alpha-\beta}\}.$$

Proof: It follows from (5), (6) and (8), that

$$\begin{aligned}
\|A(t)^\beta T_{A(t)}(\tau)\| &\leq C(\alpha,\beta)\|T_{A(t)}(\tau)\|^{1-\frac{\beta}{\alpha}}\|A(t)T_{A(t)}(\tau)\|^{\frac{\beta}{\alpha}} \\
&\leq C(\alpha,\beta)\Big(\tau^{-1+\alpha}\Big)^{1-\frac{\beta}{\alpha}}\Big(\max\{\tau^{-1}, \tau^{-2+\alpha}\}\Big)^{\frac{\beta}{\alpha}} \\
&\leq C(\alpha,\beta)\max\{\tau^{-1+\alpha-\frac{\beta}{\alpha}}, \tau^{-1+\alpha-\beta}\}.
\end{aligned}$$

The theorem is proved. □

The proof of the two next results is due to Sobolevskiĭ [16], we only point out the needed adjustments.

PROPOSITION 2.2. *Let $0 < \alpha < 1$, $\gamma - \beta < \alpha$, and $\beta < -2\alpha + \frac{\epsilon\alpha}{1-\alpha}$, then*

$$\|A(t)^\gamma T_{A(t)}(t-\tau)A(\tau)^{-\beta}\|_{L(X)} \leq C(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}. \tag{16}$$

Proof: Note that

$$A(t)^\gamma T_{A(t)}(t-\tau)A(\tau)^{-\beta} = A^{\gamma-\beta}(t)T_{A(t)}(t-\tau) + A(t)^\gamma T_{A(t)}(t-\tau)[A(\tau)^{-\beta} - A(t)^{-\beta}].$$

The norm of the first term does not exceed

$$C(\gamma,\beta)(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}.$$

For the second term, it follows from (9) that

$$A(\tau)^{-\beta} - A(t)^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty [T_{A(\tau)}(s) - T_{A(t)}(s)]s^{\beta-1}ds. \tag{17}$$

Now, for any $\eta \in [0,1]$, using (14) and (15) in (8) we have that for $\eta < \alpha$

$$\|A(t)^\eta[T_{A(\tau)}(s) - T_{A(t)}(s)]\| \leq Cs^{-2+2\alpha-\frac{\eta}{\alpha}}(t-\tau)^\epsilon. \tag{18}$$

Let $0 < \beta < 1$. From (17), (18) and closedness of the operators it follows that for any η, with $2\alpha + \beta > 2 + \frac{\eta}{\alpha}$,

$$\begin{aligned} \|A(t)^{\eta}[A(t)^{-\beta} - A(\tau)^{-\beta}]\| &\leq \frac{1}{\Gamma(\beta)} \int_0^{\infty} Ce^{-\delta s} s^{-2+2\alpha-\frac{\eta}{\alpha}} (t-\tau)^{\epsilon} s^{\beta-1} ds \\ &\leq \frac{C(t-\tau)^{\epsilon}}{\Gamma(\beta)} \int_0^{\infty} e^{-\delta s} s^{-2+2\alpha-\frac{\eta}{\alpha}} s^{\beta-1} ds \\ &\leq C(\alpha,\beta,\eta)(t-\tau)^{\epsilon}. \end{aligned} \tag{19}$$

If $2 > \beta \geq 1$, then (19) holds for $\eta = 1$. In fact,

$$A(t)[A(t)^{-\beta} - A(\tau)^{-\beta}] = [A(\tau) - A(t)]A^{-1}(\tau)A^{-(\beta-1)}(\tau) + [A^{-(\beta-1)}(t) - A^{-(\beta-1)}(\tau)],$$

and the result follows by (12) and from the bound already obtained for the case $\beta < 1$.

Using (19), we have

$$\begin{aligned} \|A(t)^{\gamma}T_{A(t)}(t-\tau)[A(\tau)^{-\beta} - A(t)^{-\beta}]\| &\leq \|A^{\gamma-\eta}(t)T_{A(t)}(t-\tau)\| \, \|A(t)^{\eta}[A(t)^{-\beta} - A(\tau)^{-\beta}]\| \\ &\leq C(\alpha,\gamma)e^{-\delta(t-\tau)}(t-\tau)^{-1+\alpha-\frac{\gamma-\eta}{\alpha}} C(\gamma,\beta)(t-\tau)^{\epsilon} \\ &\leq C(\alpha,\beta,\gamma)e^{-\delta(t-\tau)}(t-\tau)^{-1+\alpha-\frac{\gamma-\eta}{\alpha}+\epsilon}, \end{aligned}$$

where $\frac{\beta}{\alpha} - \epsilon < \frac{\eta}{\alpha} < -2 + 2\alpha + \beta$ and, consequently $\beta < -2\alpha + \frac{\epsilon\alpha}{1-\alpha}$. This completes the proof. □

Proposition 2.3. *Let* $0 < \alpha < 1$, $\alpha + \frac{\epsilon}{2} > \max\{1, \frac{\gamma}{\alpha}\}$, $\gamma - \beta < \alpha$ *and* $\beta < -2\alpha + \frac{\epsilon\alpha}{1-\alpha}$, *then*

$$\|A(t)^{\gamma}U(t,\tau)A(\tau)^{-\beta}\|_{L(X)} \leq C(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}. \tag{20}$$

Proof: For any $u \in D(A^{1-\beta}(\tau))$ $(0 \leq \beta \leq 1)$, it follows as in [16, page 20], we know that

$$\begin{aligned} U(t,\tau)A(\tau)^{-\beta}u &= T_{A(t)}(t-\tau)A(\tau)^{-\beta}u \\ &\quad + \int_{\tau}^{t} T_{A(t)}(t-s)[A(s) - A(t)]U(s,\tau)A(\tau)^{-\beta}u ds. \end{aligned}$$

Using (16) with $\gamma = 1$ for $1 - \beta < \alpha$, we have

$$\begin{aligned} \|A(t)U(t,\tau)A^{-\beta}(\tau)u\| &\leq \|A(t)T_{A(t)}(t-\tau)A(\tau)^{-\beta}u\| \\ &\quad + \int_{\tau}^{t} \|A(t)T_{A(t)}(t-s)\| \, \|[A(s) - A(t)]A^{-1}(s)\| \, \|A(s)U(s,\tau)A(\tau)^{-\beta}u\| ds \\ &\leq C(1,\beta) \; (t-\tau)^{-1+\alpha-\frac{\beta-1}{\alpha}} \|u\| \\ &\quad + C^2 \int_{\tau}^{t} (t-s)^{-2+\alpha+\epsilon} \|A(s)U(s,\tau)A(\tau)^{-\beta}u\| ds. \end{aligned}$$

From this and the Generalized Gronwall Lemma ([8], Exercise 4, p. 190) it follows that

$$\|A(t)U(t,\tau)A^{-\beta}(\tau)u\| \le C_1(1,\beta)(t-\tau)^{-1+\alpha-\frac{1-\beta}{\alpha}}\|u\|.$$

Since $D(A^{1-\beta}(\tau))$ is dense in X, the inequality (20) holds for $1-\beta<\alpha^2$ and $\gamma=1$.

For $\gamma<\alpha$, and $\beta<-2\alpha+\frac{\epsilon\alpha}{1-\alpha}$, it follows from (12) and (16) that

$$\begin{aligned}\|A(t)^{\gamma}U(t,\tau)A(\tau)^{-\beta}u\| &\le \|A(t)^{\gamma}T_{A(t)}(t-\tau)A(\tau)^{-\beta}u\|\\ &+\int_\tau^t \|A(t)^{\gamma}T_{A(t)}(t-s)\|\,\|[A(s)-A(t)]A^{-1}(s)\|\,\|A(s)U(s,\tau)A(\tau)^{-\beta}u\|ds\\ &\le C(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}\|u\|+C\int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma}{\alpha}+\epsilon}(s-\tau)^{\alpha-\frac{1-\beta}{\alpha}-1}ds\|u\|\end{aligned}$$

for $u\in D(A^{1-\beta}(\tau))$. This shows (20). □

Let

$$\varphi_1(t,\tau)=[A(\tau)-A(t)]T_{A(\tau)}(t-\tau),$$

and assume that the integral equation

$$\Phi(t,\tau)=\varphi_1(t,\tau)+\int_\tau^t \Phi(t,s)\varphi_1(s,\tau)ds \tag{21}$$

has a unique solution $\Phi(t,\tau)$, $t>\tau$. Setting

$$\bar U(t,\tau)=T_{A(\tau)}(t-\tau)+\int_\tau^t T_{A(s)}(t-s)\Phi(s,\tau)ds,$$

and noting that

$$\begin{aligned}\int_\tau^t \bar U(t,s)\varphi_1(s,\tau)ds &= \int_\tau^t [T_{A(s)}(t-s)+\int_s^t T_{A(\theta)}(t-\theta)\Phi(\theta,s)d\theta]\varphi_1(s,\tau)ds\\ &=\int_\tau^t T_{A(s)}(t-s)\varphi_1(s,\tau)ds+\int_\tau^t T_{A(s)}(t-s)\int_\tau^s \Phi(s,\theta)\varphi_1(\theta,\tau)d\theta ds\\ &=\int_\tau^t T_{A(s)}(t-s)\Phi(s,\tau)ds,\end{aligned}$$

we have that $\bar U(t,\tau)=U(t,\tau)$ for all $t\ge\tau$, and

$$U(t,\tau)=T_{A(\tau)}(t-\tau)+\int_\tau^t T_{A(s)}(t-s)\Phi(s,\tau)ds. \tag{22}$$

It remains to show that (21) has a unique solution. The expression (22) will be used to study regularity properties of $t\mapsto U(t,\tau)u_0$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Note that $\varphi_1(s,\tau) = [A(\tau) - A(s)]T_{A(\tau)}(s-\tau)$ is continuous in the uniform operator topology in all variables for $s > \tau$ (see Corollary 2.1). From (12) and (7) we have that

$$\|\varphi_1(s,\tau)\|_{L(X)} \leq C(s-\tau)^{\alpha+\epsilon-2}.$$

With this in mind we show that (21) has a unique solution $\Phi(t,\tau)$ which is continuous in the uniform operator topology for all (t,τ) with $t > \tau$ and which satisfies

$$\|\Phi(t,\tau)\|_{L(X)} \leq C(t-\tau)^{\alpha+\epsilon-2}.$$

The procedure is analogous to that used to obtain $U(t,\tau)$. For $\alpha + \epsilon > 1$ and fixed $t_0 \in \mathbb{R}$, with $t_0 > \tau$ consider the complete metric space

$$Y = \{\Phi \in C([\tau,t_0), L(X)) : \sup_{t\in[\tau,\, t_0)} (t_0-t)^{2-\alpha-\epsilon} \|\Phi(t)\|_{L(X)} < \infty\},$$

with the metric $\|\Phi\|_Y = \sup_{t\in[\tau,\, t_0)}(t_0-t)^{2-\alpha-\epsilon} \|\Phi(t)\|_{L(X)}$. Define the map G by the formula

$$(G\Phi)(t) = \varphi_1(t_0,t) + \int_t^{t_0} \Phi(s)\varphi_1(s,t)ds.$$

Clearly G takes Y into $C([\tau,t_0), L(X))$. Next, we prove that G takes Y into itself and that, for some $n \in \mathbb{N}$, G^n is a contraction. Indeed,

$$(t_0-t)^{2-\alpha-\epsilon}\|(G\Phi)(t)\|_{L(X)} \leq C\left[1 + (t_0-t)^{\alpha+\epsilon-1}\|\Phi\|_Y \int_0^1 (1-s)^{\alpha+\epsilon-2}s^{\alpha+\epsilon-2}ds\right]$$

which proves that $G : Y \longrightarrow Y$. Now, for $\Phi, \Psi \in Y$,

$$(G\Phi)(t) - (G\Psi)(t) = \int_t^{t_0} [\Phi(s) - \Psi(s)]\varphi_1(s,t)ds$$

and

$$(t_0-t)^{2-\alpha-\epsilon}\|(G\Phi)(t) - (G\Psi)(t)\|_{L(X)} \leq C\mathbf{B}(\alpha+\epsilon-1, \alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1} \|\Phi-\Psi\|_Y .$$

Using this estimate and iterating we obtain

$$\begin{aligned}&(t_0-t)^{-n(\alpha+\epsilon-1)} \|(G^n\Phi)(t) - (G^n\Psi)(t)\|_Y \\ &\leq C^n\mathbf{B}(\alpha+\epsilon-1, \alpha+\epsilon-1)\mathbf{B}(2(\alpha+\epsilon-1), \alpha+\epsilon-1)\cdots\mathbf{B}(n(\alpha+\epsilon-1), \alpha+\epsilon-1) \|\Phi-\Psi\|_Y ,\end{aligned}$$

which means

$$\|(G^n\Phi)(t) - (G^n\Psi)(t)\|_{L(X)} \leq \frac{C^n\Gamma(\alpha+\epsilon-1)[\Gamma(\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}]^n \|\Phi-\Psi\|_Y}{\Gamma((n+1)(\alpha+\epsilon-1))}.$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

From Banach contraction principle G has a unique fixed point in $C([\tau,\ t_0), L(X))$, which concludes the proof that (21) has a unique solution. Also the proof of Theorem 2.1 is thus finished.

3. LOCAL SOLVABILITY OF SEMILINEAR PARABOLIC PROBLEMS

For $\gamma-\beta<1$ and $\alpha-\frac{\gamma-\beta}{\alpha}>1-\alpha$, we consider an abstract semilinear parabolic problem of the form

$$\begin{aligned} &\frac{du}{dt}+A(t)u=f(u),\ t>\tau,\\ &u(\tau)=u_0\in X^{\gamma}, \end{aligned} \tag{23}$$

where $\{A(t): t\in\mathbb{R}\}$ is a family of linear operators satisfying conditions (a) and (b) of Section 2. Assume that $f: X^{\gamma}\to X^{\beta}$ and that there are constants $c>0$ and $q>1$ such that

$$\|f(v)-f(w)\|_{X^{\beta}}\le c\|v-w\|_{X^{\gamma}}\left(1+\|v\|_{X^{\gamma}}^{q-1}+\|w\|_{X^{\gamma}}^{q-1}\right),\ \forall\, v,w\in X^{\gamma} \tag{24}$$

with

$$q<\frac{\alpha-\frac{\gamma-\beta}{\alpha}}{1-\alpha}. \tag{25}$$

We will show that for each $\tau\in\mathbb{R}$ and arbitrary $u_0\in X^{\gamma}$ the problem (23) has a unique local mild solution defined on certain interval $(\tau,\tau+\tau_0)$ with τ_0 dependent on u_0.

Definition 3.1. A function $u:(\tau,\tau+\tau_0)\to X^{\gamma}$ is said to be local mild solution to (23), if $(\tau,\tau+\tau_0)\ni t\mapsto w(t):=u(t)-U(t,\tau)u_0\in X^{\gamma}$ is continuous, $\lim_{t\to\tau^+}\|w(t)\|_{X^{\gamma}}=0$ and

$$u(t)=U(t,\tau)u_0+\int_{\tau}^{t}U(t,s)f(u(s))ds,$$

holds for all $t\in(\tau,\tau+\tau_0)$, where $\{U(t,\tau): t>\tau\in\mathbb{R}\}$ is the process of growth α corresponding to the linear part of (23).

According to the above definition, since the process $\{U(t,\tau): t>\tau\in\mathbb{R}\}$ is singular at $t=\tau$, solutions to (23) are assumed to have the same kind of singularity at $t=\tau$ as the process U.

For τ, u_0 as in (23), $\mu>0$ (to be chosen later) introduce the metric space

$$K(\tau_0,u_0):=\{v\in C((\tau,\tau+\tau_0),X^{\gamma}): \sup_{t\in(\tau,\tau+\tau_0)}\|v(t)-U(t,\tau)u_0\|_{X^{\gamma}}\le\mu\},$$

with metric in $K(\tau_0,u_0)$ given by

$$\|\phi-\psi\|_{K(\tau_0,u_0)}=\sup_{t\in(\tau,\tau+\tau_0)}\|\phi(t)-\psi(t)\|_{X^{\gamma}},\quad \phi,\psi\in K(\tau_0,u_0).$$

The following result holds

Theorem 3.1. *Under the condition* (24) *with parameters satisfying* (25) *and for suitably small* $\tau_0 > 0$*, there exists a unique local mild solution to* (23) *in* $K(\tau_0, u_0)$.

The Banach Fixed Point Theorem in $K(\tau_0, u_0)$ is used to prove the theorem. Consider the operator $T : K(\tau_0, u_0) \to C((\tau, \tau + \tau_0), X^\gamma)$ defined by

$$Tv(t) := U(t, \tau)u_0 + \int_\tau^t U(t, s)f(v(s))ds.$$

We show that, for all suitably small τ_0,

(i) T is well defined and takes $K(\tau_0, u_0)$ into itself,
(ii) T is a contraction in $K(\tau_0, u_0)$.

To prove the first assertion let $v \in K(\tau_0, u_0)$. Note that

$$\begin{aligned}(t-\tau)^{1-\alpha}\|v(t)\|_{X^\gamma} &\le (t-\tau)^{1-\alpha}\|v(t) - U(t,\tau)u_0\|_{X^\gamma} + (t-\tau)^{1-\alpha}\|U(t,\tau)u_0\|_{X^\gamma} \\ &\le (t-\tau)^{1-\alpha}\mu + c\|u_0\|_{X^\gamma}.\end{aligned}$$

Hence, if B is a bounded subset of X^γ and

$$k = \tau_0^{1-\alpha}\mu + c \sup_{u_0 \in B} \|u_0\|_{X^\gamma},$$

it follows from (20) and (24) that

$$\begin{aligned}\|Tv(t) - U(t,\tau)u_0\|_{X^\gamma} &\le \int_\tau^t \|U(t,s)f(v(s))\|_{X^\gamma} ds \\
&\le c \int_\tau^t \|U(t,s)\|_{L(X^\beta, X^\gamma)} \|f(v(s))\|_{X^\beta} ds \\
&\le c \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \left(1 + \|v(s)\|^q_{X^\gamma}\right) ds \le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}} (t-\tau)^{\alpha - \frac{\gamma-\beta}{\alpha}} \\
&+ c \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} (s-\tau)^{-(1-\alpha)q} ((s-\tau)^{(1-\alpha)q} \|v(s)\|_{X^\gamma})^q ds \\
&\le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}} (t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}} + c\, k^q \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} (s-\tau)^{-(1-\alpha)q} ds \\
&\le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}} (t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}} + c\, k^q (t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}-(1-\alpha)q} \mathbf{B}\Big(1-(1-\alpha)q, \alpha - \frac{\gamma-\beta}{\alpha}\Big) \\
&\le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}} \tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}} + c\, k^q \tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}-(1-\alpha)q} \mathbf{B}\Big(1-(1-\alpha)q, \alpha - \frac{\gamma-\beta}{\alpha}\Big) \\
&\le \mu,\end{aligned}$$

for all suitably small τ_0 and for all $u_0 \in B$. Note that condition $\alpha - \frac{\gamma-\beta}{\alpha} - (1-\alpha)q > 0$ is equivalent to assumption (25).

In particular, for each $v \in K(\tau_0, u_0)$ we have that

$$\|Tv(t) - U(t,\tau)u_0\|_{X^\gamma} \to 0, \quad \text{as } t \to 0.$$

Hence, with this choice of τ_0, we have that $T : K(\tau_0, u_0) \to K(\tau_0, u_0)$. Note that this procedure can be carried out uniformly for u_0 in bounded subsets of X^γ.

The proof of the second assertion is similar. Using (20) and (24), we have that

$$\|Tv(t) - Tw(t)\|_{X^\gamma} \leq \int_\tau^t \|U(t,s)(f(v(s) - f(w(s)))\|_{X^\gamma} ds$$
$$\leq c \int_\tau^t \|U(t,s)\|_{L(X^\beta, X^\gamma)} \|f(v(s)) - f(w(s))\|_{X^\beta} ds$$
$$\leq c \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \|v(s) - w(s)\|_{X^\gamma} \left(1 + \|v(s)\|_{X^\gamma}^{q-1} + \|w(s)\|_{X^\gamma}^{q-1}\right) ds,$$

for all $v, w \in K(\tau_0, u_0)$. From this we obtain that

$$\|Tv - Tw\|_{X^\gamma} \leq c \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \|v-w\|_{K(\tau_0,u_0)} ds$$
$$+ 2c\, k^{q-1} \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \|v-w\|_{K(\tau_0,u_0)} (s-\tau)^{-(1-\alpha)(q-1)} ds$$
$$\leq c\|v-w\|_{K(\tau_0,u_0)} \Big[\frac{\tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}}}{\alpha - \frac{\gamma-\beta}{\alpha}} + 2k^{q-1} \tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}-(1-\alpha)(q-1)} \mathbf{B}\Big(1-(1-\alpha)(q-1), \alpha - \frac{\gamma-\beta}{\alpha}\Big)\Big].$$

Note that the expression within parenthesis tends to zero as τ_0 tends to zero provided that $\alpha - \frac{\gamma-\beta}{\alpha} - (1-\alpha)(q-1) > 0$, or equivalently that

$$q < \frac{1 - \frac{\gamma-\beta}{\alpha}}{1-\alpha}. \tag{26}$$

Clearly, if the exponent q in (24) satisfies (25) then it also satisfies (26). Hence, we obtain that, for all suitably small τ_0, T is a contraction in $K(\tau_0, u_0)$ and this concludes the proof of Theorem 3.1.

4. APPLICATIONS

In this section we apply the abstract theory developed in the previous sections to two examples. One for parabolic equations in spaces of Hölder continuous functions and another for parabolic equations in n dimensional domains with a one-dimensional handle.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

4.1. Semilinear parabolic problems in Hölder spaces

Recall that (see e.g. [7, Chapter 3]) the Hölder space $C^{\mu}(\bar{\Omega}), \mu \in (0,1)$, is a Banach space of uniformly continuous functions $v : \bar{\Omega} \to \mathbb{R}$, equipped with the norm

$$\|v\|_{\mu} = \sup_{x\in\bar{\Omega}} |v(x)| + \sup_{x,y\in\bar{\Omega},0<|x-y|\leq 1} \frac{|v(x)-v(y)|}{|x-y|^{\mu}} =: \|v\|_{C(\bar{\Omega})} + H^{\mu}(v).$$

Examples of almost sectorial operators which are not sectorial were first introduced by W. von Wahl in [20] (see also [15], and [11, Example 3.1.33]). We recall it here for completeness of the presentation.

Example 4.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^{4m} regular boundary. In the space $C^{\mu}(\bar{\Omega})$ of Hölder continuous functions ($\mu \in (0,1)$) consider elliptic differential operator of order $2m$

$$Av(x) = \sum_{|\delta|\leq 2m} a_{\delta}(x) D^{\delta} v(x),$$

with the domain $D(A) = \{v \in C^{2m+\mu}(\bar{\Omega}) : D^{\delta}u = 0 \text{ on } \partial\Omega, |\delta| \leq m-1\}$, where $D^{\delta} = \prod_{j=1}^{n}(-i\frac{\partial}{\partial x_j})^{\delta_j}$, δ is a multi-index. The coefficients $a_{\delta} : \bar{\Omega} \to \mathbb{C}$ are assumed to satisfy the following conditions

(i) $a_{\delta} \in C^{\mu}(\bar{\Omega})$ for $|\delta| \leq 2m$,

(ii) $a_{\delta}(x) \in \mathbb{R}$ for $x \in \bar{\Omega}$ and $|\delta| = 2m$,

(iii) There exists $M > 0$ such that $M^{-1}|\xi|^{2m} \leq \sum_{|\delta|=2m} a_{\delta}(x)\xi^{\delta} \leq M|\xi|^{2m}, \forall\xi\in\mathbb{R}^n, \ \forall x\in\bar{\Omega}.$

(27)

For certain $\nu > 0$ the operator $A + \nu$ will be almost sectorial with parameter $\alpha = 1 - \frac{\mu}{2m}$. Note that in particular $-\Delta$ satisfies the above conditions. Also time dependent families of almost sectorial operators were studied in [20].

In case of time dependent processes consider the second order parabolic problem

$$\begin{cases} u_t = L(t,x)u + g(u), \quad t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{28}$$

where

$$L(t,x)u := \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Assume that there is a constant $L > 0$ such that the coefficients $a_{ij} : \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ are uniformly continuous together with its derivatives and satisfy the conditions:

$$(i)\, a_{ij}(t,\cdot) \text{ are uniformly continuous in } \bar{\Omega} \text{ and equicontinuous in } t,$$

$$(ii)\, \sup_{x\in\Omega} |a_{ij}(t,x) - a_{ij}(s,x)| \le L|t-\tau|, \quad \text{for all } t,\tau \in \mathbb{R},\ |t-\tau| < 1, \tag{29}$$
$$(iii)\, \sup_{1\le k\le n} \sup_{x\in\Omega} |\frac{\partial a_{ij}}{\partial x_k}(t,x) - \frac{\partial a_{ij}}{\partial x_k}(s,x)| \le L|t-\tau|\ \forall\, t,\tau \in \mathbb{R},\ |t-\tau| < 1,$$

and, uniformly with respect to t, ellipticity condition (27) (iii).

We need to check that assumptions (a) and (b) of Section 2 are satisfied for this example. Let $A(t) : D(A(t)) \subset C_0^{\mu}(\bar{\Omega}) \to C_0^{\mu}(\bar{\Omega})$ (subscript 0 means the subspace consisting of all functions vanishing at $\partial\Omega$) be an abstract operator defined by $L(t,x)$

$$A(t)v = L(t,\cdot)v, \quad v \in D(A(t)).$$

First we will show that condition (a) is satisfied, which means that there are $\alpha \in (0,1)$ and $M > 0$ such that

$$\|(\lambda + A(t))^{-1}\|_{L(C_0^{\mu}(\Omega))} \le \frac{M}{|\lambda|^{\alpha}}, \quad \text{for all } \lambda \in \Sigma_{\theta}\backslash\{0\} \text{ and } \forall\, t \in \mathbb{R}. \tag{30}$$

For clarity of further calculations we will discuss now the connections between various constants appearing in the considerations below. Fix $p > n$, $\mu \in (0,1)$ and choose s such that

$$s \in (\mu + \frac{n}{p}, 1 + \frac{n}{p}). \tag{31}$$

Next, let $\epsilon, \epsilon' \in (0,1)$ be such that

$$0 < s - 2\epsilon < s - 2\epsilon' < \mu. \tag{32}$$

Note that the last condition together with (31) require that

$$\frac{n}{2p} \le \epsilon' < \epsilon < \frac{\mu}{2} + \frac{n}{2p}. \tag{33}$$

The notation of the function spaces borrowed from the monograph [19] will be used below for the reader's convenience. In the proof of (30) we will use known properties of the operators $A(t)$ in the scale $\tilde{X}^r := H^{2r+s}_{p,\{B_j\}}(\Omega)$ of Bessel potentials spaces (see [19, p. 321]), that is Calderon-Zygmund type estimate

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\tilde{X}^0,\tilde{X}^1)} \le c_1, \tag{34}$$

and sectoriality condition for these operators

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\tilde{X}^0)} \leq \frac{c_2}{|\lambda|}. \tag{35}$$

Thanks to the Sobolev type embedding (see [19, p. 328]) $H^s_{p,\{B_j\}}(\Omega) \subset C^\mu_0(\bar{\Omega})$, valid for μ, s satisfying (31), we have

$$\begin{aligned} \|(\lambda{+}A(t))^{-1}\phi\|_{C^\mu_0(\bar{\Omega})} &\leq c\|(\lambda + A(t))^{-1}\phi\|_{\tilde{X}^0} \\ &\leq \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(H^{s-2\epsilon}_{p,\{B_j\}}(\Omega), H^s_{p,\{B_j\}}(\Omega))}\|\phi\|_{H^{s-2\epsilon}_{p,\{B_j\}}(\Omega)} \leq \frac{c}{|\lambda|^{1-\epsilon}}\|\phi\|_{H^{s-2\epsilon}_{p,\{B_j\}}(\Omega)}, \end{aligned} \tag{36}$$

where we have used that

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\tilde{X}^{-\epsilon}, \tilde{X}^0)} \leq \frac{c}{|\lambda|^{1-\epsilon}},$$

which follows from (34), (35) and the *moment inequality*.

Remark 4. 1. To assure that the operators $A(t)$ in our example satisfy (1) with C, α independent of t (are 'uniformly almost sectorial') we need to check that the constants c_1, c_2 in (34), (35) are independent of t. As well known ([7, Theorem 18.2]), these constants depend only on p, n, Ω, uniform ellipticity constant M and modulus of continuity of the coefficients a_{ij}. Thanks to our assumption (29)(i) they will be thus chosen independent of t.

We will extend next (36) using the following result:

LEMMA 4.1. *Assuming* (31), (32), *the following chain of continuous embedding hold*

$$C^\mu_0(\bar{\Omega}) \subset B^{s-2\epsilon'}_{p,p,\{B_j\}}(\Omega) \subset H^{s-2\epsilon}_{p,\{B_j\}}(\Omega).$$

Proof: Recall that for $H^{s-2\epsilon}_{p,\{B_j\}}(\Omega)$ and $B^{s-2\epsilon}_{p,p,\{B_j\}}(\Omega)$ defined in [19, §4.3.3] we have

$$B^{s-2\epsilon}_{p,p,\{B_j\}}(\Omega) = (L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega))_{\frac{s-2\epsilon}{2m},p}$$

and

$$H^{s-2\epsilon}_{p,\{B_j\}}(\Omega) = [L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega)]_{\frac{s-2\epsilon}{2m},p}.$$

Consider next any auxiliary operator $\tilde{A}$ letting e.g. $\tilde{A}$ be the m-th power of the negative Laplacian in $L^p(\Omega)$ with the domain $H^{2m}_{p,\{B_j\}}(\Omega)$ ($\{B_j\}$ being the homogeneous Dirichlet boundary conditions). Then $\tilde{A}$ is a positive sectorial operator with bounded (purely) imaginary powers and we have

$$[L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega)]_{\frac{s-2\epsilon}{2m},p} = D(\tilde{A}^{\frac{s-2\epsilon}{2m}}).$$

The main idea that supports the argument is that we can now suitably change the interpolation functors. Namely, from the above and [19, §1.15.2 (3)], we get

$$B^{s-2\epsilon}_{p,1,\{B_j\}}(\Omega) = (L^p(\Omega), D(\tilde{A}))_{\frac{s-2\epsilon}{2m},1} \subset D(\tilde{A}^{\frac{s-2\epsilon}{2m}}) = H^{s-2\epsilon}_{p,\{B_j\}}(\Omega),$$

whereas from [19, §1.15.2 Step 7] (or [2, I.2.5])

$$(L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega))_{\frac{s-2\epsilon'}{2m},\infty} \subset (L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega))_{\frac{s-2\epsilon}{2m},1}.$$

Since evidently (see [19, §1.15.2 (3)])

$$(L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega))_{\frac{s-2\epsilon'}{2m},p} \subset (L^p(\Omega), H^{2m}_{p,\{B_j\}}(\Omega))_{\frac{s-2\epsilon'}{2m},\infty},$$

we conclude that

$$B^{s-2\epsilon'}_{p,p,\{B_j\}}(\Omega) \subset H^{s-2\epsilon}_{p,\{B_j\}}(\Omega).$$

Coming back to the definition of $B^{s-2\epsilon'}_{p,p,\{B_j\}}(\Omega)$ in [19, §4.3.3] we interpret it as follows; for the prescribed parameters and p being large enough

$$B^{s-2\epsilon'}_{p,p,\{B_j\}}(\Omega) = \{\phi \in B^{s-2\epsilon'}_{p,p}(\Omega) : \phi|_{\partial\Omega} = 0\}.$$

It is well known (see [19, 4.4.1(8)], [1, p. 208]), that for non-integer r an equivalent norm of the space $B^r_{p,p}(\Omega) = W^r_p(\Omega)$, called *intrinsic norm*, is given by the integral expression

$$\|\phi\|_{W^r_p(\Omega)} = \left(\|\phi\|^p_{W^m_p(\Omega)} + \sum_{|\delta|=m} \int_\Omega \int_\Omega \frac{|D^\delta\phi(x) - D^\delta\phi(y)|^p}{|x-y|^{n+\sigma p}} dxdy \right)^{\frac{1}{p}},$$

where $r = m + \sigma$, with $\sigma \in (0,1), m \in \mathbb{N}$.

Analyzing the above norm with $r = s - 2\epsilon'$, $m = 0$ and p so large that $\mu > s - 2\epsilon' + \frac{n}{p}$ (compare (32)), we observe that

$$\|\phi\|_{B^{s-2\epsilon'}_{p,p}(\Omega)} \le c\|\phi\|_\mu, \ \phi \in C^\mu(\bar{\Omega}).$$

Therefore, for the space of Hölder functions equal zero at the boundary of Ω we have

$$C^\mu_0(\bar{\Omega}) \subset \{\phi \in B^{s-2\varepsilon'}_{p,p}(\Omega) : \phi|_{\partial\Omega} = 0\} = B^{s-2\epsilon'}_{p,p,\{B_j\}}(\Omega),$$

which completes the proof. □

Extending (36) with the use of the above Lemma we obtain an estimate

$$\|(\lambda + A(t))^{-1}\phi\|_{C^\mu_0(\bar{\Omega})} \le \frac{c}{|\lambda|^{1-\epsilon}} \|\phi\|_{C^\mu_0(\bar{\Omega})}, \ Re\lambda > 0,$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

verifying condition (30) with $\alpha = 1 - \epsilon$ (the value of ϵ was evaluated in (33)). Note that for large p we agree with von Wahl's estimate

$$\alpha = 1 - \epsilon > 1 - \frac{\mu}{2}.$$

Next, we need to check condition (12); that is, there exists $\epsilon \in (0,1)$ such that

$$\|[A(t) - A(\tau)]A^{-1}(s)]\phi\|_\mu \leq c(t-\tau)^\epsilon \|\phi\|_\mu, \ \forall\, t, \tau \in \mathbb{R}, \ |t-\tau| < 1.$$

Since the operator $A^{-1}(s)$ is bounded linear from $C_0^\mu(\bar{\Omega})$ into $D(A(t))$, it suffices to show that for $\psi := A^{-1}(s)\phi$,

$$\|[A(t) - A(\tau)]\psi\|_\mu \leq c(t-\tau)^\epsilon \|\phi\|_\mu, \quad \tau \leq t \in \mathbb{R},$$

which, in an explicit form, means the estimate

$$\left\| \sum_{i,j=1}^n [a_{ij}(t,\cdot) - a_{ij}(\tau,\cdot)] \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right\|_\mu \leq c(t-\tau)^\epsilon \|\phi\|_\mu.$$

It follows from our assumption (29) on the coefficients that

$$\|a_{ij}(t,\cdot) - a_{ij}(\tau,\cdot)\|_\mu \leq c\|a_{ij}(t,\cdot) - a_{ij}(\tau,\cdot)\|_{C^1(\Omega)} \leq L|t-\tau| \leq L|t-\tau|^\epsilon,$$

for $|t-\tau| < 1, \epsilon \in (0,1)$, which justifies condition (12) since $C^\mu(\bar{\Omega})$ is a Banach algebra and

$$\left\| \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right\|_\mu \leq c\|\phi\|_\mu.$$

To apply the results of Section 3, we fix $X^\gamma = C_0^\mu(\bar{\Omega})$ and impose conditions on the nonlinear term g in (28) to ensure that (24) holds and, consequently, the problem is locally solvable. Since in our example, for $1 > \mu^+ > \mu$, for $z > \mu^+ - \mu$ and $p > n$, $L^{p_0}(\Omega) \subset H^{-z}_{p,\{B_j\}}(\Omega) \subset X^{\gamma - \frac{\mu^+}{2}}$ with $p_0 \geq \frac{np}{n+zp}$ and $X^\gamma \subset L^\infty(\Omega)$, then it suffices to assume that $g: \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function. Indeed, when $B \subset C_0^\mu(\bar{\Omega})$ is bounded and $v, w \in B$, we have

$$\|g(v) - g(w)\|_{X^{\gamma - \frac{\mu^+}{2}}} \leq c\|g(v) - g(w)\|_{L^{p_0}(\Omega)} \leq cL_B\|v-w\|_{L^{p_0}(\Omega)} \leq cL_B\|v-w\|_{X^\gamma},$$

where L_B is the Lipschitz constant for g in a large interval containing the image of every function in B. This proves the required condition and concludes the example.

4.2. Domains with a handle

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Consider the problem

$$\begin{cases} u_t = a(t,x)\Delta u - u + f(u), \quad x \in \Omega,\ t > 0,\\ u(t,x) = 0, \quad x \in \partial\Omega,\ t \geq 0,\\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases}$$

This problem can be written abstractly as

$$\frac{du}{dt} + A(t)u = f(u), \quad u(0) = u_0,$$

where $X = L^p(\Omega)$, $A(t) : D(A(t)) \subset X \to X$, $D(A(t)) = H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$, and

$$A(t)u = -a(t,x)\Delta u + u.$$

Assume that $a(t,x)$ is continuously differentiable in x and that there are positive constants c_0 and c_1 such that $0 < c_0 \leq a(t,x) \leq c_1$.

Assume further that $b(t,x) := \nabla_x a(t,x) \in L^\infty(\Omega^n)$ for all $t \in \mathbb{R}$ and that $t \mapsto a(t,\cdot) \in L^\infty(\Omega)$, $t \mapsto b(t,\cdot) \in L^\infty(\Omega^n)$ are Hölder continuous functions with exponent $\epsilon > 0$ and constant C.

As for the nonlinearities f, we assume that $f \in C^1(\mathbb{R},\mathbb{R})$ and satisfies the growth condition

$$|f'(u)| \leq c.$$

Operator $-A(t)$ is the infinitesimal generator of an analytic semigroup which satisfies

$$\|(A(t) + \lambda I)^{-1}\|_{L(X)} \leq \frac{M}{|\lambda|}, \text{ for } t \in \mathbb{R} \text{ and } \lambda \in \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\},$$

and

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{L(X)} \leq C(t-\tau)^\epsilon, \text{ for some } \epsilon \in (0,1] \text{ and for all } t, \tau, s \in \mathbb{R}.$$

LEMMA 4.2 ([5]). *The operator $A(t) : D(A(t)) \subset X \to X$ is the infinitesimal generator of an analytic semigroup and satisfies conditions* (1) (*with* $\alpha = 1$) *and* (12).

Inspired by [3] we consider the following example. For $g \in C^1([0,1],(0,\infty))$, let $U^p = L^p(\Omega) \oplus L^p_g(0,1)$, where $L^p_g(0,1)$ is the space $L^p(0,1)$ with the norm

$$\|\phi\|_{L^p_g(0,1)} = \left(\int_0^1 g(s)|\phi(s)|^p ds\right)^{\frac{1}{p}}.$$

For $P_0, P_1 \in \partial\Omega$ and $p > n/2$ let $\mathcal{A}(t) : \mathcal{D}(\mathcal{A}(t)) \subset U^p \to U^p$ be the operator defined by

$$\begin{aligned} &D(\mathcal{A}(t)) = \big\{(w,v) \in U^p : w \in D(\Delta_N^\Omega),\ (gv_x)_x \in L^p(0,1),\ v(0) = w(P_0),\ v(1) = w(P_1)\big\}\\ &\mathcal{A}(t)(w,v) = \Big(-a(t,x)\Delta w + w, -\frac{1}{g}(gv_x)_x + v\Big), \quad (w,v) \in D(\mathcal{A}(t)), \end{aligned} \tag{37}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

where Δ_N^Ω is the Laplacian with homogeneous Neumann boundary conditions in $L^p(\Omega)$ and the domain $D(\Delta_N^\Omega) = \{u \in H^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$.

For $p > n/2$ we have that $D(\Delta_N^\Omega)$ is continuously embedded in $C(\bar{\Omega})$. This tells us that the functions in $D(\Delta_N^\Omega)$ have trace at P_0 and P_1.

Proposition 4.1 ([3]). *For each $t \in \mathbb{R}$, the operator $\mathcal{A}(t)$ defined by* (37) *has the following properties*

(i) $D(\mathcal{A}(t))$ is dense in U^p,

(ii) $\mathcal{A}(t)$ is a closed operator,

(iii) $\mathcal{A}(t)$ has compact resolvent, and

(iv) $\rho(\mathcal{A}(t)) \supset \Sigma_\theta$ and the estimates hold

$$\|(\mathcal{A}(t) + \lambda)^{-1}\|_{L(U^p)} \leqslant \frac{C}{|\lambda|^\alpha}, \ \lambda \in \Sigma_\theta \backslash \{0\}$$

$$\|\mathcal{A}(t)(\mathcal{A}(t) + \lambda)^{-1}\|_{L(U^p)} \leqslant C(1 + |\lambda|^{1-\alpha}), \ \lambda \in \Sigma_\theta,$$

for some $0 < \alpha < 1 - \frac{n}{2p} < 1$ and there are constants $C > 0$ and $\epsilon \in (0,1]$ such that

$$\|[\mathcal{A}(t) - \mathcal{A}(\tau)]\mathcal{A}^{-1}(s)\|_{L(U^p)} \leq C(t-\tau)^\epsilon, \ \forall t, \tau, s \in \mathbb{R}.$$

If $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, it defines a Nemitskiĭ operator from U^p into itself by $f^e(u, v) = (f_\Omega(u), f_I(v))$, with $f_\Omega(u)(x) = f(u(x))$, $x \in \Omega$, and $f_I(v)(s) = f(v(s))$, $s \in (0, 1)$.

Consider the semilinear evolution equation

$$\begin{cases} \dfrac{du}{dt} = \mathcal{A}(t)u + f^e(u), \\ u(0) = u_0 \in U^p, \end{cases} \tag{38}$$

where u vary in the Banach space U^p and $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset U^p \to U^p$ is the linear operator defined by (37) and where $f^e : U^p \to U^p$ is such that

$$\|f^e(u) - f^e(v)\|_{U^p} \leq c\|u - v\|_{U^p}.$$

It follows from the result in Section 3 with $\gamma = \beta = 0$ that (38) has a unique solution for each $u_0 \in U^p$.

Acknowledgements. The second author is very grateful to the Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Brasil and to FAPESP for supporting his scientific visit in São Carlos. All the authors wish to thank Professor Jan Cholewa for many helpful discussions and comments throughout the paper. Finally, all authors wish to express their gratitude to an anonymous referee whose comments and suggestions greatly improved the original manuscript.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

REFERENCES

1. R. A. Adams, *Sobolev Spaces*, Academic Press Inc., New York, 1975.
2. H. Amann, *Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory*, Birkhäuser Verlag, Basel, 1995.
3. J. Arrieta, A. N. Carvalho and G. Lozada-Cruz, Dynamics in dumbbell domains II: The limiting problem, *submitted*.
4. J. Arrieta and A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Trans. Amer. Math. Soc.*, **352**, (2000), 285–310.
5. A. N. Carvalho and M. J. D. Nascimento, Singularly non-autonomous semilinear parabolic problems with critical exponents and applications, *submitted*.
6. G. Da Prato, Semigruppi di crescenza n, *Ann. Scuola Norm. Sup. Pisa*, **20**, (1966), 753–782.
7. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1964.
8. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.
9. S. G. Kreĭn, *Linear Differential Equations in Banach Space*, AMS, Providence, 1971.
10. R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Springer-Verlag, Berlin, 1994.
11. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.
12. N. Okazawa, A generation theorem for semigroups of growth order α, *Tôhoku Math. J.*, **26**, (1974), 39–51.
13. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
14. F. Periago, Global existence, uniqueness, and continuous dependence for a semilinear initial value problem, *J. Math. Anal. Appl.*, **280**, (2003), 413–423.
15. F. Periago and B. Straub, A functional calculus for almost sectorial operators and applications to abstract evolution equations, *J. Evol. Equations*, **2**, (2002), 41–68.
16. P. E. Sobolevskiĭ, Equations of parabolic type in a Banach space, *Amer. Math. Soc. Transl.*, **49**, (1965), 1–62.
17. H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.
18. S. P. Toropova, Relation of semigroups with singularity to integrated semigroups, *Russian Mathematics*, **47**, (2003), 66–74.
19. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.
20. W. von Wahl, Gebrochene Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen hölderstetinger Funktionen, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.*, **11**, (1972), 231–258.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC