

## On nonsymmetric theorems for $(H, G)$ -coincidences

Denise de Mattos\*

*Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação,  
Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668,  
13560-970 São Carlos SP, Brazil  
E-mail: deniseml@icmc.usp.br*

Edivaldo L. dos Santos

*Departamento de Matemática, Universidade Federal de São Carlos-UFSCar, Caixa Postal 676,  
13565-905, São Carlos-SP, Brazil  
E-mail: edivaldo@dm.ufscar.br*

Let  $X$  be a compact Hausdorff space,  $\varphi : X \rightarrow S^n$  a continuous map into the  $n$ -sphere  $S^n$  that induces a nonzero homomorphism  $\varphi^* : H^n(S^n; \mathbb{Z}_p) \rightarrow H^n(X; \mathbb{Z}_p)$ ,  $Y$  a  $k$ -dimensional CW-complex and  $f : X \rightarrow Y$  a continuous map. Let  $G$  a finite group which acts freely on  $S^n$ . Suppose that  $H \subset G$  is a normal cyclic subgroup of a prime order. In this paper, we define and we estimate the cohomological dimension of the set  $A_\varphi(f, H, G)$  of  $(H, G)$ -coincidence points of  $f$  relative to  $\varphi$ . May, 2008 ICMC-USP

### 1. INTRODUCTION

K.D. Joshi [10] has proved a nonsymmetric generalization of the Borsuk-Ulam theorem [1], in which the  $n$ -sphere  $S^n$  is replaced by a certain compact subset  $X$  of the  $(n + 1)$ -dimensional Euclidean space  $\mathbb{R}^{n+1}$ . In this context, a pair of points  $x, y \in X$  are said to be antipodal if  $y = -\lambda x$ , for some  $\lambda > 0$ . The Joshi's theorem shows that for every continuous map  $f : X \rightarrow \mathbb{R}^{n+1}$  there exist antipodal points  $x, y \in X$  such that  $f(x) = f(y)$ .

K. Borsuk has suggested to define antipodal points in an arbitrary space of the following way:  $x_1, x_2 \in X$  are said to be antipodal points relative to an essential map<sup>1</sup>  $\varphi : X \rightarrow S^n$  if  $\varphi(x_1) = -\varphi(x_2)$ . Using the Borsuk's suggestion, Spież [11] has proved that if  $X$  is a compact Hausdorff space and if  $\varphi : X \rightarrow S^n$  is an essential map, then for every continuous map  $f : X \rightarrow \mathbb{R}^k$ , the covering dimension of the set

$$A_\varphi(f) = \{x \in X; \exists y \in X, \text{ such that } \varphi(x) = -\varphi(y) \text{ and } f(x) = f(y)\}$$

\*The author was supported in part by FAPESP of Brazil Grant 06/03932-8.

<sup>1</sup>A map  $\varphi : X \rightarrow S^n$  is called to be an essential map if  $\varphi$  induces nonzero homomorphism  $\varphi^* : H^n(S^n; \mathbb{Z}_p) \rightarrow H^n(X; \mathbb{Z}_p)$ .

is not less than  $n - k$ , obtaining thus a generalization of the Joshi's theorem.

M. Izydorek [7] has extended the proposition of Borsuk for a cyclic group  $G$  of order prime which acts freely on a  $n$ -dimensional sphere  $S^n$  and has proved the following generalization of the Spieź's theorem: if  $X$  is a compact Hausdorff space and if  $\varphi : X \rightarrow S^n$  is an essential map, then for every continuous map  $f : X \rightarrow \mathbb{R}^k$ , the covering dimension of the set

$$A_\varphi(f) = \{x \in X; \exists x_2, \dots, x_p \mid \varphi(x) = g^{-1}\varphi(x_2) = \dots = g^{1-p}\varphi(x_p) \\ \text{and } f(x) = f(x_2) = \dots = f(x_p)\}$$

is not less than  $n - (p-1)k$ , where  $g$  is a fixed generator of  $G$ . Moreover, if  $\mathbb{R}^k$  is replaced by a generalized  $k$ -dimensional manifold  $M^k$  over  $\mathbb{Z}_p$ , then an analogous theorem has proved (see [7, Theorem 4]).

Gonçalves, Jaworowski and Pergher [3] has defined  $(H, G)$ -coincidence for a continuous map  $f$  from a  $n$ -sphere  $S^n$  into a  $k$ -dimensional CW-complex  $Y$ , where  $G$  is a finite group which acts freely on  $S^n$  and has proved that if  $H$  is a nontrivial normal cyclic subgroup of a prime order, then

$$A(f, H, G) \geq n - |G|k,$$

where  $A(f, H, G)$  is the set of  $(H, G)$ -coincidence points of  $f$  and  $\text{cohom.dim}$  denotes the cohomological dimension. Others papers closely to the paper [3] are the papers [4], [5], [6], [8], [9] and [13].

The purpose of this paper is to define the set  $A_\varphi(f, H, G)$  of  $(H, G)$ -coincidence points of a continuous map  $f : X \rightarrow Y$  relative to an essential map  $\varphi : X \rightarrow S^n$ , where  $X$  is a compact Hausdorff space,  $Y$  is a topological space,  $G$  is a finite group which acts freely on the  $n$ -dimensional sphere  $S^n$  and  $H$  is a subgroup of  $G$ . Using this definition, under certain conditions, we estimate the cohomological dimension of the set  $A_\varphi(f, H, G)$ . Specifically, we will prove the following nonsymmetric version of the main Theorem of [3]:

**THEOREM 1.1.** *Let  $X$  be a compact Hausdorff space,  $Y$  a  $k$ -dimensional CW-complex and  $\varphi : X \rightarrow S^n$  an essential map. Given a finite group  $G$  which acts freely on  $S^n$  and  $H$  a normal cyclic subgroup of prime order, then for every continuous map  $f : X \rightarrow Y$  such that  $f^* : H^i(Y; \mathbb{Z}_p) \rightarrow H^i(X; \mathbb{Z}_p)$  is trivial for  $i \geq 1$ ,  $\text{cohom.dim } A_\varphi(f, H, G) \geq n - |G|k$ .*

For the proof of Theorem 1.1, it was fundamental to prove the following version of the main Theorem of [3]:

**THEOREM 1.2.** *Let  $X$  be a paracompact Hausdorff space,  $Y$  a  $k$ -dimensional CW-complex,  $G$  a finite group which acts freely on  $X$  and  $H \subset G$  a normal cyclic subgroup of prime order. Let  $f : X \rightarrow Y$  be a continuous map such that  $f^* : H^i(Y; \mathbb{Z}_p) \rightarrow H^i(X; \mathbb{Z}_p)$  is trivial, for  $i \geq 1$ . Suppose that the  $\mathbb{Z}_p$ -index of  $X$  is greater than or equal to  $n$ , then the  $\mathbb{Z}_p$ -index of the set  $A(f, H, G)$  is greater than or equal to  $n - |G|k$ . Consequently,  $\text{cohom.dim } A(f, H, G) \geq n - |G|k$ .*

## 2. PRELIMINARIES

Throughout this paper the symbols  $H_*$  and  $H^*$  will denote Čech homology and cohomology groups with coefficients in  $\mathbb{Z}_p$ , unless otherwise indicated. If  $G$  is a group which acts on topological space  $X$ , we will denote by  $X^*$  the orbit space  $X/G$ .

We start by introducing some basic notions and definitions as follows.

### 2.1. $(H, G)$ -coincidence

Suppose that  $X, Y$  are topological spaces,  $G$  is a group acting freely on  $X$  and  $f : X \rightarrow Y$  is a map. If  $H$  is a subgroup of  $G$ , then  $H$  acts on the right on each orbit  $Gx$  of  $G$  as follows: if  $y \in Gx$  and  $y = gx$ ,  $g \in G$ , then  $hy = ghx$ . Following [4, 6, 9] the concept of  $G$ -coincidence is generalized as follows: a point  $x \in X$  is said to be a  $(H, G)$ -coincidence point of  $f$  if  $f$  sends every orbit of the action of  $H$  on the  $G$ -orbit of  $x$  to a single point (see [5]). We will denote by  $A(f, H, G)$  the set of all  $(H, G)$ -coincidence points of  $f$ . If  $H$  is the trivial subgroup, then every point of  $X$  is a  $(H, G)$ -coincidence. If  $H = G$ , this is the usual definition of coincidence. If  $G = Z_p$  with  $p$  prime, then a nontrivial  $(H, G)$ -coincidence point is a  $G$ -coincidence point.

### 2.2. The space $X_\varphi$ and the set $A_\varphi(f, H, G)$

Let us consider  $X$  a compact Hausdorff space and an essential map  $\varphi : X \rightarrow S^n$ . Suppose  $G$  be a finite group of order  $r$  which acts freely on  $S^n$  and  $H$  be a subgroup of order  $p$  of  $G$ . Let  $G = \{g_1, g_2, \dots, g_r\}$  be a fixed enumeration of elements of  $G$ , where  $g_1$  is the identity of  $G$ . A nonempty space  $X_\varphi$  can be associated with the essential map  $\varphi : X \rightarrow S^n$  as follows:

$$\begin{aligned} X_\varphi &= \{(x_1, x_2, \dots, x_r) \in X^r \mid \varphi(x_1) = (g_2)^{-1}\varphi(x_2) = \dots = (g_r)^{-1}\varphi(x_r)\} \\ &= \{(x_1, x_2, \dots, x_r) \in X^r \mid g_i\varphi(x_1) = \varphi(x_i), i = 1, \dots, r\}, \end{aligned}$$

where  $X^r$  denotes the  $r$ -fold cartesian product of  $X$ . The set  $X_\varphi$  is a closed subset of  $X^r$  and so it is compact. We define a  $G$ -action on  $X_\varphi$  as follows: for each  $g_i \in G$  and for each  $(x_1, \dots, x_r) \in X_\varphi$ ,

$$g_i(x_1, \dots, x_r) = (x_{\sigma_{g_i}(1)}, \dots, x_{\sigma_{g_i}(r)}), \quad (1)$$

where the permutation  $\sigma_{g_i}$  is defined by  $\sigma_{g_i}(k) = j$ , se  $g_k g_i = g_j$ . We observe that if  $x = (x_1, \dots, x_r) \in X_\varphi$  then  $x_i \neq x_j$ , for any  $i \neq j$  and therefore  $G$  acts freely on  $X_\varphi$ .

Now, let us consider a continuous map  $f : X \rightarrow Y$ , where  $Y$  is a topological space and  $\tilde{f} : X_\varphi \rightarrow Y$  given by  $\tilde{f}(x_1, \dots, x_r) = f(x_1)$ . Let  $y = (x_1, \dots, x_r) \in A(\tilde{f}, H, G)$  and consider the orbit  $Gy = \{g_1 y, g_2 y, \dots, g_r y\}$ . Note that

- (i) From (1), we have that for each  $i$ , the 1-th coordinate of  $g_i y$  is  $x_i$ .
- (ii) The action of  $H$  on  $Gy$  determines a partition of the orbit  $Gy$  in  $s = (r/p)$  disjoint suborbits, and we can be rewrite

$$Gy = \{g_{1_p} y, \dots, g_{1_p} y; \dots; g_{j_1} y, \dots, g_{j_p} y; \dots; g_{s_1} y, \dots, g_{s_p} y\},$$

where  $\{1, 2, \dots, r\} \longleftrightarrow \{1_1, \dots, 1_p; \dots; j_1, \dots, j_p; \dots; s_1, \dots, s_p\}$  is a bijection.

Since  $y$  is a  $(H, G)$  - coincidence point of  $\tilde{f}$ , it follows from (i) and (ii) that,

$$f(x_{1_1}) = \cdots = f(x_{1_p}); \dots; f(x_{j_1}) = \cdots = f(x_{j_p}); \dots; f(x_{s_1}) = \cdots = f(x_{s_p}).$$

In these conditions, we have the following

DEFINITION 2.1. The set  $A_\varphi(f, H, G)$  of  $(H, G)$ -coincidence points of  $f$  relative to  $\varphi$  is defined by

$$A_\varphi(f, H, G) = A(\tilde{f}, H, G) = \{(x_1, \dots, x_r) \in X^r \mid g_i \varphi(x_1) = \varphi(x_i), i = 1, \dots, r \\ \text{and } f(x_{j_1}) = \cdots = f(x_{j_p}), j = 1, \dots, s\}.$$

*Remark 2. 1.* Let us observe that if  $G = H = \mathbb{Z}_p$ ,

$$A_\varphi(f, H, G) = A_\varphi(f) = \{(x_1, \dots, x_p) \in X^p \mid g_i \varphi(x_1) = \varphi(x_i), i = 1, \dots, p \\ \text{and } f(x_1) = \cdots = f(x_p)\}.$$

### 2.3. The $\mathbb{Z}_p$ -index

Suppose that the cyclic group  $G = \mathbb{Z}_p$  of order prime  $p$  acts freely on a Hausdorff and paracompact space  $X$ . Then  $X \rightarrow X^*$  is a principal  $\mathbb{Z}_p$ -bundle and one can take  $h : X^* \rightarrow B\mathbb{Z}_p$  a classifying map for the  $G$ -bundle  $X \rightarrow X^*$ .

*Remark 2. 2.* Let us observe that if  $\hat{h}$  is another classifying map for the principal  $\mathbb{Z}_p$ -bundle  $X \rightarrow X^*$ , then there is a homotopy between  $h$  and  $\hat{h}$ .

We will be considering the following definition for the  $\mathbb{Z}_p$ -index of  $X$  (see [7]).

DEFINITION 2.2. We say that the  $\mathbb{Z}_p$ -index of  $X$  is greater than or equal to  $k$  if the homomorphism  $h^* : H^k(B\mathbb{Z}_p) \rightarrow H^k(X^*)$  is nontrivial.

We say that the  $\mathbb{Z}_p$ -index of  $X$  is equal to  $k$  if it is greater than or equal to  $k$  and moreover  $h^* : H^i(B\mathbb{Z}_p) \rightarrow H^i(X^*)$  is zero, for any  $i \geq k + 1$ .

*Remark 2. 3.* A model for  $B\mathbb{Z}_2$  the classifying space for  $\mathbb{Z}_2$  is the infinite real projective space  $P^\infty$ . Then  $H^*(B\mathbb{Z}_2) \cong H^*(P^\infty)$  is isomorphic to  $\mathbb{Z}_2[a]$ , where  $a \in H^1(P^\infty)$  is the generator. The generator of  $H^i(B\mathbb{Z}_2)$  is  $a^i$  for any  $i \geq 0$ . If  $p > 2$  a model for  $B\mathbb{Z}_p$  the classifying space for  $\mathbb{Z}_p$  is the infinite lens space  $L_p^\infty = S^\infty/\mathbb{Z}_p$ . Thus,  $H^i(B\mathbb{Z}_p) = H^i(L_p^\infty) \cong \mathbb{Z}_p$  for any  $i \geq 0$  and given any nonzero element  $a \in H^1(L_p^\infty)$ , one has that  $b = \beta(a)$  is a nonzero element of  $H^2(L_p^\infty)$ , where  $\beta : H^1(L_p^\infty) \rightarrow H^2(L_p^\infty)$  is the Bockstein

homomorphism. More generally, a generator  $\mu \in H^i(B\mathbb{Z}_p)$  is given by

$$\mu = \begin{cases} a \smile b^{(i-1)/2}, & \text{if } i \text{ is odd} \\ b^{i/2}, & \text{if } i \text{ is even.} \end{cases}$$

#### 2.4. The Smith special cohomology groups with coefficients in $\mathbb{Z}_p$

In this work, we will be considering the definition of the Smith special cohomology groups with coefficients in  $\mathbb{Z}_p$  in the sense of [2]. Smith homology and cohomology were originally defined in [12] and in a series of subsequent papers. A systematic exposition of the Smith theory can be found in [2]. Let  $X$  be a topological space. Given a finite group  $G$  of prime order  $p$  which acts freely on  $X$ , let  $g$  be a fixed generator of  $G$  and put

$$\sigma = 1 + g + g^2 + \cdots + g^{p-1} \quad \text{and} \quad \tau = 1 - g,$$

in the group ring  $\mathbb{Z}_p(G)$ . We have that  $\sigma = \tau^{p-1}$ . If  $\rho = \tau^i$ , we put  $\bar{\rho} = \tau^{p-i}$ , then  $\tau = \bar{\sigma}$  and  $\sigma = \bar{\tau}$ . There exists an exact sequence with coefficients in  $\mathbb{Z}_p$  [2, (3.9), p.125],

$$\longrightarrow H_\rho^n(X) \xrightarrow{\rho^*} H^n(X) \xrightarrow{i^*} H_\rho^n(X) \oplus H^n(X^G) \xrightarrow{\delta} H_\rho^{n+1}(X) \xrightarrow{\rho^*} \longrightarrow$$

called Smith exact sequence, where  $H_\rho^*(X)$  and  $H_\rho^*(X)$  denotes the Smith special cohomology groups and  $X^G$  denotes the fixed set points of  $X$  under the action of  $G$ . If this action is free  $X^G$  is empty. Moreover, the group  $H_\sigma^*(X)$  is naturally isomorphic to the ordinary cohomology group  $H^*(X^*)$  of the orbit space [2, (3.10), p.125]. Thus, for free actions the Smith sequence is given by

$$\longrightarrow H_\rho^n(X) \xrightarrow{\rho^*} H^n(X) \xrightarrow{\mathcal{T}} H^n(X^*) \xrightarrow{\delta} H_\rho^{n+1}(X) \xrightarrow{\rho^*} \longrightarrow$$

where  $\mathcal{T}$  is the transfer homomorphism.

*Remark 2.* 4. The Smith cohomology groups are natural with respect to  $\mathbb{Z}_p$ -equivariant maps, that is, if  $f : X \rightarrow Y$  is a  $\mathbb{Z}_p$ -equivariant map then  $f$  induces homomorphism  $f_\rho^* : H_\rho^*(Y) \rightarrow H_\rho^*(X)$  which commutes with the homomorphisms in the Smith sequence.

### 3. THE $\mathbb{Z}_p$ -INDEX OF $X_\varphi$

Let us consider the free  $G$ -space  $X_\varphi$  as defined in Section 2.2. If  $H \subset G$  is a cyclic subgroup of prime order  $p$ , then  $X_\varphi$  is a free  $H \cong \mathbb{Z}_p$ -space. In these conditions, as in [7, Theorem 3], we have the following

**THEOREM 3.1.** *Let  $X$  be a compact Hausdorff space and  $\varphi : X \rightarrow S^n$  an essential map. Then, the  $\mathbb{Z}_p$ -index of  $X_\varphi$  is equal to  $n$ .*

*Proof.* For  $i = 1, 2, \dots, r$ , let us consider the maps  $\varphi_i : X \rightarrow S^n$  given by  $\varphi_i(x) = (g_i)^{-1}\varphi(x)$ , where  $g_i \in G$ . Then, we can define the map

$$\psi = \varphi_1 \times \dots \times \varphi_r : X^r \rightarrow [S^n]^r,$$

where  $[S^n]^r$  denotes the  $r$ -fold cartesian product of  $r$  copies of  $S^n$ , such that  $X_\varphi = \psi^{-1}\Delta[S^n]^r$ , where  $\Delta[S^n]^r$  is the diagonal in  $[S^n]^r$ . In these conditions, we prove the following

LEMMA 3.1. *The homomorphism  $\psi^* : H^*([S^n]^r) \rightarrow H^*(X^r)$  induced by  $\psi : X^r \rightarrow [S^n]^r$  is monomorphism in each dimension.*

*Proof.* Let  $m$  be an integer and for each  $t = 1, 2, \dots, r$  consider  $H^m([S^n]^t)$ . If  $m$  is not divisible by  $n$  then  $H^m([S^n]^t) = 0$  and the result follows. Suppose that  $m$  is divisible by  $n$ ; one then has that there exists  $\alpha = 0, 1, 2, \dots$ , such that  $m = \alpha n$ . Let us consider the following commutative diagram

$$\begin{array}{ccc} H^{\alpha n}([S^n]^t) \otimes H^0(S^n) \oplus H^{(\alpha-1)n}([S^n]^t) \otimes H^n(S^n) & \xrightarrow{\times} & H^{\alpha n}([S^n]^{t+1}) \\ \downarrow (\varphi_1 \times \dots \times \varphi_t)^* \otimes \varphi_{t+1}^* & & \downarrow (\varphi_1 \times \dots \times \varphi_{t+1})^* \\ H^{\alpha n}(X^t) \otimes H^0(X) \oplus H^{(\alpha-1)n}(X^t) \otimes H^n(X) & \xrightarrow{\times} & H^{\alpha n}(X^{t+1}) \end{array} \quad (2)$$

Applying the Künneth's formula, we have that the upper row of the above diagram is an isomorphism and the lower row is monomorphism, for each  $\alpha = 0, 1, 2, \dots$  and  $t = 1, \dots, r$ .

The proof will be done by induction on  $t$ . We assume inductively that for some  $t = 1, 2, \dots, r-1$  and for each  $\alpha = 0, 1, 2, \dots$  the homomorphism  $(\varphi_1 \times \dots \times \varphi_t)^* : H^{\alpha n}([S^n]^t) \rightarrow H^{\alpha n}(X^t)$  is monomorphism and we will show that  $(\varphi_1 \times \dots \times \varphi_t)^* \otimes \varphi_{t+1}^*$  is monomorphism. The result will follow from the commutativity of the diagram 2. By induction hypothesis it suffices to show that  $\varphi_{t+1}^*$  is monomorphism. For this, observe that  $\varphi_i^* = \varphi_j^*$ , for any  $1 \leq i, j \leq r$ . If  $\varphi_{t+1}^*$  is not monomorphism, then  $\varphi_i^*$  is not monomorphism, for any  $1 \leq i \leq t$ , which completes the proof. ■

The next step is to use Lemma 3.1 to show that the homomorphism induced by  $\psi|_{X_\varphi} : X_\varphi \rightarrow \Delta[S^n]^r$  is monomorphism. Let us consider the following commutative diagram

$$\begin{array}{ccccc} H^n(X^r, X_\varphi) & \xrightarrow{j^*} & H^n(X^r) & \xrightarrow{k^*} & H^n(X_\varphi) \\ \psi^* \uparrow & & \psi^* \uparrow & & (\psi|_{X_\varphi})^* \uparrow \\ H^n([S^n]^r, \Delta[S^n]^r) & \longrightarrow & H^n([S^n]^r) & \xrightarrow{i^*} & H^n(\Delta[S^n]^r) \end{array} \quad (3)$$

whose rows are exacts. If  $\gamma$  is a generator of  $H^n(S^n) \cong \mathbb{Z}_p$  let us denote by  $\alpha_i$  the element  $q_i^*(\gamma) \in H^n([S^n]^r)$ , where  $q_i : [S^n]^r \rightarrow S^n$  is the natural projection on the  $i$ -th

coordinate for  $i = 1, 2, \dots, r$ . Let us observe that  $q_i \circ i = q_j \circ i$  for any  $1 \leq i, j \leq r$ , where  $i : \Delta[S^n]^r \hookrightarrow [S^n]^r$  is the natural inclusion. In this way, one has that

$$i^*(\alpha_i) = i^* \circ q_i^*(\gamma) = (q_i \circ i)^*(\gamma) = (q_j \circ i)^*(\gamma) = i^*(\alpha_j). \quad (4)$$

LEMMA 3.2.  $(\psi|_{X_\varphi})^* : H^n(\Delta[S^n]^r) \rightarrow H^n(X_\varphi)$  is monomorphism.

*Proof.* Since  $\Delta[S^n]^r$  is homeomorphic to  $S^n$ , it follows from (4) that  $i^*(\alpha_i)$  is a nonzero element in  $H^n(\Delta[S^n]^r)$  for any  $i = 1, \dots, r$ . Thus, it suffices to show that  $(\psi|_{X_\varphi})^*(i^*(\alpha_1)) \neq 0$ . Let us assume that this does not happen. From diagram (3) we have that

$$k^* \circ \psi^*(\alpha_1) = (\psi|_{X_\varphi})^* \circ i^*(\alpha_1) = 0.$$

Therefore,  $\psi^*(\alpha_1) \in \text{Ker}(k^*) = \text{Im}(j^*)$  and there exists an element  $U \in H^n(X^r, X_\varphi)$  such that

$$j^*(U) = \psi^*(\alpha_1) \neq 0, \quad (5)$$

since by Lemma 3.1  $\psi^*$  is monomorphism and  $\alpha_1 = q_1^*(\gamma) \in H^n([S^n]^r)$  is a nonzero element. Let us consider the following commutative diagram,

$$\begin{array}{ccccc} H^{(r-1)n}(X^r, X^r - X_\varphi) & \xrightarrow{j^*} & H^{(r-1)n}(X^r) & \xrightarrow{k^*} & H^{(r-1)n}(X^r - X_\varphi) \\ \uparrow \psi^* & & \uparrow \psi^* & & \uparrow \psi^* \\ H^{(r-1)n}([S^n]^r, [S^n]^r - \Delta[S^n]^r) & \xrightarrow{j_1^*} & H^{(r-1)n}([S^n]^r) & \xrightarrow{k_1^*} & H^{(r-1)n}([S^n]^r - \Delta[S^n]^r) \\ \uparrow D^{-1} & & \uparrow D^{-1} & & \\ H_n(\Delta[S^n]^r) & \xrightarrow{i_*} & H_n([S^n]^r) & & \end{array} \quad (6)$$

where the first and the second rows are exact,  $D$  is the Alexander-Spanier Duality which is an isomorphism and all others maps are induced by appropriate inclusions.

Let us denote by  $a_1, a_2, \dots, a_r$  the elements of  $H_n([S^n]^r)$  which are conjugated to  $\alpha_1, \alpha_2, \dots, \alpha_r$  in  $H^n([S^n]^r)$ . More precisely,

$$\langle \alpha_j, a_i \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

where the map  $\langle \cdot, \cdot \rangle : H^n([S^n]^r) \times H_n([S^n]^r) \rightarrow \mathbb{Z}_p$  denotes the Kronecker product. Let us denote by  $c \in H_n(\Delta[S^n]^r)$  the conjugated element of  $i^*(\alpha_1) = \dots = i^*(\alpha_r)$  in  $H^n(\Delta[S^n]^r)$ . Then for any  $j = 1, \dots, r$  one has  $\langle i^*(\alpha_j), c \rangle = 1$ . Furthermore, it follows from properties of the Kronecker product that for any  $j = 1, \dots, r$

$$\langle i^*(\alpha_j), c \rangle = \langle \alpha_j, i_*(c) \rangle = 1,$$

which implies that

$$i_*(c) = \sum_{i=1}^r a_i. \quad (7)$$

Let us consider for each  $i = 1, \dots, r$  the elements

$$\beta_i = \alpha_1 \smile \dots \smile \alpha_{i-1} \smile \hat{\alpha}_i \smile \alpha_{i+1} \smile \dots \smile \alpha_r \in H^{(n-1)r}([S^n]^r),$$

where the symbol  $\hat{\alpha}_i$  means that the element  $\alpha_i$  is omitted. To simplify notation, here we will also denote by  $[S^n]^r$  the generator of  $H_{nr}([S^n]^r)$ , which is called *fundamental class of  $[S^n]^r$* . We will show that

$$D^{-1}(a_i) = (-1)^{i+1}\beta_i, \text{ that is, } (-1)^{i+1}\beta_i \frown [S^n]^r = a_i.$$

In fact, it follows from properties of the cup and cap products with respect to the Kronecker product and by definition of  $\beta_i$  that

$$\begin{aligned} \langle \alpha_i, (-1)^{i+1}\beta_i \frown [S^n]^r \rangle &= \langle \alpha_i \smile (-1)^{i+1}\beta_i, [S^n]^r \rangle \\ &= \langle (-1)^{i+1}(-1)^{n(n(i-1))}\alpha_1 \smile \dots \smile \alpha_r, [S^n]^r \rangle \\ &= \langle \alpha_1 \smile \dots \smile \alpha_r, [S^n]^r \rangle = 1, \end{aligned} \quad (8)$$

observing that  $\alpha_1 \smile \dots \smile \alpha_r$  is the generator of  $H^{n \cdot r}([S^n]^r)$ . Thus,

$$D^{-1}\left(\sum_{i=1}^r a_i\right) = \sum_{i=1}^r D^{-1}(a_i) = \sum_{i=1}^r (-1)^{i+1}\beta_i.$$

It follows from (7), (8) and from commutativity of diagram(6) that

$$j_1^* \circ D^{-1}(c) = D^{-1} \circ i_*(c) = \sum_{i=1}^r (-1)^{i+1}\beta_i,$$

that is,

$$\sum_{i=1}^r (-1)^{i+1}\beta_i \in \text{Im}(j_1^*).$$

Since the second row of diagram (6) is exact, one has that

$$k_1^*\left(\sum_{i=1}^r (-1)^{i+1}\beta_i\right) = 0.$$

By using again the commutativity of diagram (6)

$$k^* \circ \psi^*\left(\sum_{i=1}^r (-1)^{i+1}\beta_i\right) = \psi^* \circ k_1^*\left(\sum_{i=1}^r (-1)^{i+1}\beta_i\right) = 0,$$



which implies that

$$\psi^* \left( \sum_{i=1}^r (-1)^{i+1} \beta_i \right) \in \ker(k^*) = \text{Im}(j^*).$$

Thus, there exists an element  $V \in H^{(r-1)n}(X^r, X^r - X_\varphi)$  such that

$$j^*(V) = \psi^* \left( \sum_{i=1}^p (-1)^{i+1} \beta_i \right) \neq 0, \quad (9)$$

since  $\psi^*$  is monomorphism. By using the naturality of the cup product in the following diagram

$$\begin{array}{ccc} H^n(X^r) \otimes H^{(r-1)n}(X^r) & \xrightarrow{\smile} & H^{rn}(X^r) \\ \uparrow \psi^* & & \uparrow \psi^* \\ H^n([S^n]^r) \otimes H^{(r-1)n}([S^n]^r) & \xrightarrow{\smile} & H^{rn}([S^n]^r) \end{array}$$

and observing that

$$\sum_{i=2}^r \alpha_1 \smile (-1)^{i+1} \beta_i = 0$$

one has that

$$\begin{aligned} \psi^*(\alpha_1) \smile \psi^* \left( \sum_{i=1}^r (-1)^{i+1} \beta_i \right) &= \psi^* \left( \alpha_1 \smile \sum_{i=1}^r (-1)^{i+1} \beta_i \right) \\ &= \psi^* \left( \alpha_1 \smile \beta_1 + \sum_{i=2}^r \alpha_1 \smile (-1)^{i+1} \beta_i \right) \\ &= \psi^*(\alpha_1 \smile \beta_1) \\ &= \psi^* \left( \alpha_1 \smile \cdots \smile \alpha_r \right) \neq 0, \end{aligned} \quad (10)$$

since  $\alpha_1 \smile \cdots \smile \alpha_r$  is the generator of  $H^{rn}([S^n]^r)$  and  $\psi^*$  is monomorphism. On the other hand, from naturality of the cup product in the diagram

$$\begin{array}{ccc} H^n(X^r, X_\varphi) \otimes H^{(r-1)n}(X^r, X^r - X_\varphi) & \xrightarrow{\smile} & H^{rn}(X^r, X^r) \\ j^* \downarrow & & \downarrow j^* \\ H^n(X^r) \otimes H^{(r-1)n}(X^r) & \xrightarrow{\smile} & H^{rn}(X^r) \end{array}$$

and from equations (5) and (9) we conclude that

$$\begin{aligned}\psi^*(\alpha_1 \smile \dots \smile \alpha_r) &= \psi^*(\alpha_1) \smile \psi^*\left(\sum_{i=1}^p (-1)^{i+1} \beta_i\right) \\ &= j^*(U) \smile j^*(V) = j^*(U \smile V) = j^*(0) = 0,\end{aligned}$$

which contradicts (10). This completes the proof.  $\blacksquare$

Now, let us consider the map

$$\theta = q_1 \circ \psi|_{X_\varphi} : X_\varphi \rightarrow S^n,$$

where  $q_1 : \Delta[S^n]^r \rightarrow S^n$  is the natural projection on the 1-th coordinate, which is an homeomorphism. Since by Lemma 3.2,  $(\psi|_{X_\varphi})^*$  is monomorphism, one then has that  $\theta^* : H^n(S^n) \rightarrow H^n(X_\varphi)$  is monomorphism. Note that, if  $(x_1, \dots, x_r) \in X_\varphi$ , we have that for each  $i$ ,  $g_i \theta(x_1, \dots, x_r) = g_i \varphi(x_1) = \varphi(x_i) = \theta g_i(x_1, \dots, x_r)$ , thus  $\theta$  is a  $G$ -equivariant map, and consequently,  $\theta$  is a  $H$ -equivariant map, where  $H \subset G$  is a cyclic subgroup of prime order. Thus, in particular for  $\rho = \sigma$ , we can consider the homomorphism induced by  $\theta$ ,  $\theta_\sigma^* : H_\sigma^n(S^n) \rightarrow H_\sigma^n(X_\varphi)$ , where  $H_\sigma^n$  denotes the  $n$ -dimensional Smith special cohomology group with coefficients in  $\mathbb{Z}_p$  in the sense of Section 2.4.

By remarks in [2, Results following 5.2] whose dual holds in cohomology,  $i^* : H^n(S^n) \rightarrow H_\sigma^n(S^n)$  is an isomorphism, and since  $\theta^* : H^n(S^n) \rightarrow H^n(X_\varphi)$  is a monomorphism it follows that  $\theta_\sigma^*$  is monomorphism. To conclude that the  $\mathbb{Z}_p$ -index of  $X_\varphi$  is equal to  $n$  it suffices to verify that the map between the orbit spaces  $\bar{\theta} : X_\varphi/H \rightarrow S^n/H$  induces a monomorphism in cohomology. From results in Section 2.4, we have that  $H_\sigma^n(S^n) \cong H^n(S^n/H)$  and  $H_\sigma^n(X_\varphi) \cong H^n(X_\varphi/H)$ , and considering the commutative diagram

$$\begin{array}{ccc} H_\sigma^n(S^n) & \xrightarrow{\theta_\sigma^*} & H_\sigma^n(X_\varphi) \\ \uparrow \cong & & \uparrow \cong \\ H^n(S^n/H) & \xrightarrow{\bar{\theta}^*} & H^n(X_\varphi/H) \end{array}$$

it follows that  $\bar{\theta}^* : H^n(S^n/H) \rightarrow H^n(X_\varphi/H)$  is monomorphism. Therefore, the  $\mathbb{Z}_p$ -index of  $X_\varphi$  is equal to  $n$ .  $\blacksquare$

#### 4. PROOF OF THEOREMS 1.2 AND 1.3

*Proof of Theorem 1.2.* By following the similar steps of [3], we first prove Theorem 1.2 in the case that  $G = H = Z_p$ , where  $p \geq 2$ . We need to show that the  $\mathbb{Z}_p$ -index of the set

$$A_f = \{x \in X; f(x) = f(gx) = \dots, f(g^{p-1}x)\}$$

is greater than or equal to  $n - pk$ . For this, let us consider  $F : X \rightarrow Y^p$  given by  $F(x) = (f(x), f(gx), \dots, f(g^{p-1}x))$ , where  $Y^p = Y \times \dots \times Y$  denotes the  $p$ -fold cartesian product of  $Y$  and  $g$  is a fixed generator of  $\mathbb{Z}_p$ . In these conditions, we prove the following

LEMMA 4.1. *The homomorphism  $F^* : H^q(Y^p) \rightarrow H^q(X)$  induced by map  $F : X \rightarrow Y^p$  is zero for any  $q \geq 1$ .*

*Proof.* We have that  $F = (f_0 \times f_1 \times \dots \times f_{p-1}) \circ d$ , where  $d : X \rightarrow X^p$  is the diagonal map and  $f_i(x) = f(g^i x)$ , for any  $x \in X$  and  $i = 0, \dots, p-1$ . In this way, it suffices to show that  $(f_0 \times f_1 \times \dots \times f_{p-1})^* : H^q(Y^p) \rightarrow H^q(X^p)$  is trivial for any  $q \geq 1$ . Let us consider the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i+j=q} H^i(Y^t) \otimes H^j(Y) & \xrightarrow{\cong} & H^q(Y^{t+1}) \\ \downarrow (f_0 \times \dots \times f_t)^* \otimes f_{t+1}^* & & \downarrow (f_0 \times \dots \times f_{t+1})^* \\ \bigoplus_{i+j=q} H^i(X^t) \otimes H^j(X) & \xrightarrow{\times} & H^q(X^{t+1}) \end{array} \quad (11)$$

Since  $Y$  is a CW-complex, applying the Künneth's formula we have that the upper row of diagram (11) is an isomorphism for any  $q = 1, 2, \dots$  and  $t = 1, \dots, p-1$ .

The proof will be done by induction on  $t$ . Suppose inductively that  $(f_0 \times \dots \times f_t)^* : H^i(Y^t) \rightarrow H^i(X^t)$  is zero for some  $t = 1, 2, \dots, p-1$  and for each  $i = 1, 2, \dots$ . By hypothesis,  $f$  induces the zero homomorphism in each dimension; in particular  $f_{t+1}^*$  is zero and thus  $(f_0 \times \dots \times f_t)^* \otimes f_{t+1}^*$  is trivial. It follows from commutativity of the diagram (11) that  $(f_0 \times \dots \times f_{t+1})^*$  is zero, which completes the proof. ■

We can define a  $\mathbb{Z}_p$ -action on  $Y^p$  as follows: for each  $(y_1, \dots, y_p) \in Y^p$  define the  $\mathbb{Z}_p$ -action on  $Y^p$  given by  $g(y_1, \dots, y_{p-1}, y_p) = (y_p, y_1, \dots, y_{p-1})$ . Since  $p$  is a prime, this action is free on  $Y^p - \Delta$ , where  $\Delta$  is the diagonal in  $Y^p$ . Let us observe that  $A_f = F^{-1}(\Delta)$ , thus  $F$  determines a  $\mathbb{Z}_p$ -equivariant map  $F_0 : X - A_f \rightarrow Y^p - \Delta$ , which induces a map between the orbit spaces  $\bar{F}_0 : [X - A_f]^* \rightarrow [Y^p - \Delta]^*$ . In these conditions, we prove that

LEMMA 4.2. *The map  $\bar{F}_0^* : H^{pk}([Y^p - \Delta]^*) \rightarrow H^{pk}([X - A_f]^*)$  is zero.*

*Proof.* Let us consider the map of pairs  $(F, F_0) : (X, X - A_f) \rightarrow (Y, Y - \Delta)$ . One then has the following commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & H^{pk}(X) & \xrightarrow{i^*} & H^{pk}(X - A_f) & \longrightarrow & H^{pk+1}(X, X - A_f) & \longrightarrow \\ & \uparrow F^* & & \uparrow F_0^* & & \uparrow (F, F_0)^* & \\ \longrightarrow & H^{pk}(Y^p) & \xrightarrow{j^*} & H^{pk}(Y^p - \Delta) & \longrightarrow & H^{pk+1}(Y^p, Y^p - \Delta) & \longrightarrow \end{array}$$

where the homomorphisms  $i^*$  and  $j^*$  are induced by appropriate inclusions. Since  $\dim(Y^p)$  is less than or equal to  $pk$  we have that  $H^{pk+1}(Y^p, Y^p - \Delta)$  is trivial and thus  $j^*$  is surjective.

On the other hand  $F_0 : (X - A_f) \rightarrow (Y^p - \Delta)$  is a  $\mathbb{Z}_p$ -equivariant map and it follows from Remark 2.4 that the diagram

$$\begin{array}{ccccc} H^{pk}(X - A_f) & \xrightarrow{\mathcal{T}} & H^{pk}([X - A_f]^*) & \longrightarrow & H_\rho^{pk+1}(X - A_f) \\ \uparrow F_0^* & & \uparrow \bar{F}_0^* & & \uparrow \\ H^{pk}(Y^p - \Delta) & \xrightarrow{\mathcal{T}} & H^{pk}([Y^p - \Delta]^*) & \longrightarrow & H_\rho^{pk+1}(Y^p - \Delta) \end{array}$$

between the Smith sequences of  $X - A_f$  and  $Y^p - \Delta$  is commutative and since  $H_\rho^{pk+1}(Y^p - \Delta)$  is zero,  $\mathcal{T}$  is surjective.

Putting together these diagrams, one obtains a new commutative diagram

$$\begin{array}{ccccc} H^{pk}(X) & \xrightarrow{i^*} & H^{pk}(X - A_f) & \xrightarrow{\mathcal{T}} & H^{pk}([X - A_f]^*) \\ \uparrow F^* & & \uparrow F_0^* & & \uparrow \bar{F}_0^* \\ H^{pk}(Y^p) & \xrightarrow{j^*} & H^{pk}(Y^p - \Delta) & \xrightarrow{\mathcal{T}} & H^{pk}([Y^p - \Delta]^*) \end{array} \quad (12)$$

where the horizontal sequences are not necessarily exacts, but the composition  $\mathcal{T} \circ j^*$  is surjective. Therefore, as  $F^*$  is zero by Lemma 4.1, it follows from commutativity of the diagram (12) that  $\bar{F}_0^*$  is zero. ■

Let  $h : X^* \rightarrow B\mathbb{Z}_p$  be a classifying map for the principal  $\mathbb{Z}_p$ -bundle  $X \rightarrow X^*$ . Then the compositions

$$h \circ i_1 : A_f^* \rightarrow B\mathbb{Z}_p \quad \text{and} \quad h \circ i_2 : [X - A_f]^* \rightarrow B\mathbb{Z}_p$$

are classifying maps for the following principal  $\mathbb{Z}_p$ -bundles  $A_f \rightarrow A_f^*$  and  $X - A_f \rightarrow [X - A_f]^*$  respectively, where the maps  $i_1 : A_f^* \rightarrow X^*$  and  $i_2 : [X - A_f]^* \rightarrow X^*$  are induced by the inclusions between the orbit spaces.

Let us consider  $G : Y^p - \Delta \rightarrow B\mathbb{Z}_p$  a classifying map for the principal  $\mathbb{Z}_p$ -bundle  $Y^p - \Delta \rightarrow [Y^p - \Delta]^*$ . Since  $F_0 : X - A_f \rightarrow Y^p - \Delta$  is a  $\mathbb{Z}_p$ -equivariant map, one has that

$$G \circ \bar{F}_0 : [X - A_f]^* \rightarrow B\mathbb{Z}_p$$

also classifies the principal  $\mathbb{Z}_p$ -bundle  $X - A_f \rightarrow [X - A_f]^*$ . In this way,

$$i_2^* \circ h^* = \bar{F}_0^* \circ G^* : H^*(B\mathbb{Z}_p) \rightarrow H^*([X - A_f]^*). \quad (13)$$

To conclude that the  $\mathbb{Z}_p$ -index of  $A_f$  is greater than or equal to  $n - pk$ , it suffices to show that  $i_1^* \circ h^*(\mu) \neq 0$ , where  $\mu$  is the generator of  $H^{n-pk}(B\mathbb{Z}_p)$ .

We first consider the case that  $k$  is odd. Let us observe that  $n$  must be necessarily odd, since  $p > 2$  is a prime. Then,  $n - pk$  is even and it follows from Remark 2.3 that  $\mu = b^{(n-pk)/2} \in H^{n-pk}(B\mathbb{Z}_p)$ . Suppose that  $i_1^* \circ h^*(\mu) = 0$ . From continuity of the cohomology, there exists a neighborhood  $V$  of  $A_f$  in  $X$  which is invariant by the action of

$\mathbb{Z}_p$  and such that  $i_1^* \circ h^*(\mu) = 0$  in  $H^{n-pk}(V^*)$ . From exact cohomology sequence of the pair  $(X^*, V^*)$  one has

$$h^*(\mu) \in \text{Im} [H^{n-pk}(X^*, V^*) \rightarrow H^{n-pk}(X^*)]. \quad (14)$$

Since  $pk$  is odd from Remark 2.3  $\eta = a \smile b^{(pk-1)/2}$  is a generator of  $H^{pk}(B\mathbb{Z}_p)$ . It follows from Lemma 4.2 and (13) that

$$i_2^* \circ h^*(\eta) = \bar{F}_0^* \circ G^*(\eta) = 0 \in H^{pk}([X - A_f]^*)$$

and from exact cohomology sequence of the pair  $(X^*, [X - A_f]^*)$  one has

$$h^*(\eta) \in \text{Im} [H^{pk}(X^*, [X - A_f]^*) \rightarrow H^{pk}(X^*)]. \quad (15)$$

Thus from (14), (15) and by the naturality of the cup product we have

$$h^*(\eta \smile \mu) = h^*(\eta) \smile h^*(\mu) \in \text{Im} [H^n(X^*, [X - A_f]^* \cup V^*) \rightarrow H^n(X^*)].$$

Let us note that the element

$$\eta \smile \mu = a \smile b^{(pk-1)/2} \smile b^{(n-pk)/2} = a \smile b^{(n-1)/2}$$

is a generator of  $H^n(B\mathbb{Z}_p)$ . Furthermore,

$$H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*) = 0$$

and then  $h^*(\eta \smile \mu) = 0 \in H^n(X^*)$ , that is,  $h^* : H^n(B\mathbb{Z}_p) \rightarrow H^n(X^*)$  is trivial which contradicts the hypothesis of the  $\mathbb{Z}_p$ -index of  $X$  is greater than or equal to  $n$ .

If  $k$  is even, then  $n - pk$  is odd and  $pk$  is even. In this case, the proof is analogous to the previous case, considering now the generators

$$\mu = a \smile b^{(n-pk-1)/2} \in H^{n-pk}(B\mathbb{Z}_p) \text{ and } \eta = b^{(pk)/2} \in H^{pk}(B\mathbb{Z}_p).$$

Let us examine the case where  $G = \mathbb{Z}_2$ . Here,  $n$  can be any positive integer and the generator of  $H^{n-2k}(B\mathbb{Z}_2)$  is  $\mu = a^{n-2k}$ . To show that the  $\mathbb{Z}_2$ -index of  $A_f$  is greater than or equal to  $n - 2k$ , it suffices to prove that  $i_1^* \circ h^*(\mu) \neq 0$ . Let us assume that  $i_1^* \circ h^*(\mu) = 0$ . Then there exists a neighborhood  $V$  of  $A_f$  in  $X$  which is invariant by the action of  $\mathbb{Z}_2$  and such that  $i_1^* \circ h^*(\mu) = 0$  in  $H^{n-2k}(V^*)$ . From exact cohomology sequence of the pair  $(X^*, V^*)$  one has that

$$h^*(\mu) \in \text{Im} [H^{n-2k}(X^*, V^*) \rightarrow H^{n-2k}(X^*)]. \quad (16)$$

On the other hand,  $\eta = a^{2k}$  is the generator of  $H^{2k}(B\mathbb{Z}_2)$  and it follows from Lemma 4.2 and (13) that

$$i_2^* \circ h^*(\eta) = \bar{F}_0^* \circ G^*(\eta) = 0 \in H^{2k}([X - A_f]^*).$$

Moreover from exact cohomology sequence of  $(X^*, [X - A_f]^*)$  one has that

$$h^*(\eta) \in \text{Im} [H^{2k}(X^*, [X - A_f]^*) \rightarrow H^{2k}(X^*)]. \quad (17)$$

Thus, from (16), (17) and by the naturality of the cup product we have

$$h^*(\eta \smile \mu) = h^*(\eta) \smile h^*(\mu) \in \text{Im} [H^n(X^*, [X - A_f]^* \cup V^*) \rightarrow H^n(X^*)].$$

Let us observe that  $\eta \smile \mu = a^{2k} \smile a^{n-2k} = a^n$  is the generator of  $H^n(B\mathbb{Z}_2)$ . Furthermore,  $H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*)$  is trivial and then  $h^*(\eta \smile \mu) = 0 \in H^n(X^*)$  which contradicts the hypothesis of the  $\mathbb{Z}_2$ -index of  $X$  is greater than or equal to  $n$ . This concludes the proof of Theorem 1.2 in the case  $G = H = \mathbb{Z}_p$ .

For the general case, suppose that  $G$  is a finite group which acts freely on  $X$  and let  $H \subset G$  be a normal cyclic subgroup of prime order  $p$ . We denote by  $s = |G|/p$ , the number of the left cosets of  $G/H$  and let  $a_1, \dots, a_s$  be a set of representatives of the cosets. We define the map  $F : X \rightarrow Y^s$  by

$$F(x) = (f(a_1x), \dots, f(a_sx)). \quad (18)$$

We need to show that

$$A(f, H, G) = A_F = \{x \in X; F(x) = F(hx), \forall h \in H\}.$$

Let  $x$  be a point in the set  $A(f, H, G)$ , then  $f$  collapses each orbit determined by the action of  $H$  on  $a_ix$  to a single point, for each  $i = 1, \dots, s$ . If  $h \in H$

$$F(hx) = (f(ha_1x), \dots, f(ha_sx)).$$

Since  $H$  is a normal subgroup of  $G$ ,  $ha_ix = a_ihx$ . Furthermore,  $a_ix$  and  $a_ihx = ha_ix$  belongs to the same  $H$ -orbit and  $f(a_ix) = f(ha_ix)$ , for each  $i = 1, \dots, s$  which implies that  $F(x) = F(hx)$ . Therefore  $x \in A_F$ . The proof of the another inclusion is entirely analogous.

To conclude, let us observe that  $H \cong \mathbb{Z}_p$  acts freely on  $X$  by restriction and by hypothesis the  $\mathbb{Z}_p$ -index of  $X$  is greater than or equal to  $n$ . By using Lemma 4.1 for the map  $F : X \rightarrow Y^s$  defined in (18) one has that  $F^* : H^q(Y^s) \rightarrow H^q(X)$  is trivial for any  $q \geq 1$ . Since dimension of  $Y^s$  is  $ks$  and Theorem 1.2 is true for  $G = \mathbb{Z}_p$  we can conclude that the  $\mathbb{Z}_p$ -index of  $A_F = A(f, H, G)$  is greater than or equal  $n - p(ks) = n - pk(|G|/p) = n - |G|k$  and this completes the proof.  $\blacksquare$

*Proof of Theorem 1.1.* Let  $\tilde{f} : X_\varphi \rightarrow Y$  given by  $\tilde{f}(x_1, \dots, x_r) = f(x_1)$ , that is  $\tilde{f} = f \circ \pi_1$  where  $\pi_1$  is the natural projection on the 1-th coordinate. By hypothesis  $f$  induces the zero homomorphism in each dimension, then we have that  $\tilde{f}^* : H^i(Y) \rightarrow H^i(X_\varphi)$  is trivial for any  $i \geq 1$ . Moreover, the  $\mathbb{Z}_p$ -index of  $X_\varphi$  is equal to  $n$  by Theorem 3.1. In this way,  $X_\varphi$  and  $\tilde{f}$  satisfy the hypotheses of Theorem 1.2 which implies that the  $\mathbb{Z}_p$ -index of the set

$A(\tilde{f}, H, G)$  is greater than or equal to  $n - |G|k$ . By Definition 2.1  $A_\varphi(f, H, G) = A(\tilde{f}, H, G)$ , and then  $\text{cohom.dim } A_\varphi(f, H, G) \geq n - |G|k$ . ■

*Remark 4. 1.* In the particular case that  $G = H = \mathbb{Z}_p$  with  $p$  prime, Volovikov in [13, Theorem 3.2] proved a version of Theorem 1.1.

### REFERENCES

1. K.Borsuk, Drei Sätze über die  $n$ -dimensionale Euklidische Sphäre, *Fund. Math.*, **20**, (1933), 177–190.
2. G.E. Bredon, *Introduction to Compact Transformation Groups*, in: Pure and Applied Mathematics, Academic Press, New York, 1972.
3. D.L. Gonçalves, J. Jaworowski and P.L.Q. Pergher, Measuring the size of the coincidence set, *Topology and its Applications*, **125**, (2002), 465–470.
4. D.L. Gonçalves, J. Jaworowski and P.L.Q. Pergher,  $G$ -coincidences for maps of homotopy spheres into CW-complexes, *Proc. Amer. Math. Soc.*, **130**, 10, (2002), 3111–3115.
5. D.L. Gonçalves, J. Jaworowski, P.L.Q. Pergher and A.Yu. Volovikov, Coincidences for maps of spaces with finite group actions, *Topology Appl.*, **145**, 1–3, (2004), 61–68.
6. D.L. Gonçalves and P.L.Q. Pergher,  $\mathbb{Z}_p$ -coincidence for maps of spheres into CW-complexes, *Kobe J. Math.*, **15**, (1998), 191–195.
7. M. Izydorek, On the Borsuk-Ulam Theorem for Free  $\mathbb{Z}_p$ -Action, *Bulletin of the Polish Academy of Sciences Math.*, **38**, (1990), 15–21.
8. M. Izydorek and J. Jaworowski, Antipodal coincidence for maps of spheres into complexes, *Proc. Amer. Math. Soc.*, **123**, 6, (1995), 1947–1950.
9. J. Jaworowski, Periodic coincidence for maps of spheres, *Kobe J. Math.*, **17**, (2000), 21–26.
10. K.D. Joshi, A non-symmetric generalization of the Borsuk-Ulam Theorem, *Fund. Math.*, **80**, (1973), 13–33.
11. S. Spież, On Generalizations of Theorems of Borsuk-Ulam, Dyson and Livesay, *Bull. L'Académie Polonaise Sciences*, **4**, (1978), 325–333.
12. P.A. Smith, Transformations of finite period, *Annals of Math.*, **39**, (1938), 127–164.
13. A.Yu. Volovikov, Coincidence points of mappings of  $\mathbb{Z}_p^n$ -spaces, *Izv. Ross. Akad. Nauk Ser. Mat.*, **69**, 5, (2005), 53–106; *translation in Izv. Math.*, **69**, 5, (2005), 913–962.