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Index of an implicit differential equation

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In this paper we introduce the concept of the index of an implicit differential equation $F(x, y, p) = 0$, where F is a smooth function, $p = \frac{dy}{dx}$, $F_p = 0$ and $F_{pp} = 0$ at an isolated singular point. We also apply the results to study the geometry of surfaces in $\mathbb{R}^5$. May, 2008 ICMC-USP

1. INTRODUCTION

Let $F(x, y, p) = 0$ be an implicit differential equation (IDE), where F is a smooth function and $p = \frac{dy}{dx}$. If $F_p(q_0) \neq 0$ at $q_0 \in \mathbb{R}^3$, the IDE can be written locally in the form $p = g(x, y)$ and studied using methods from the theory of ordinary differential equations. When $F_p(q_0) = 0$ the equation may define locally more than one direction in the plane. The cases that have been most intensively studied are the IDEs that define at most two directions in the plane. This is the case, for example, when:

$$F(x, y, p) = 0, F(q_0) = F_p(q_0) = 0, F_{pp}(q_0) \neq 0. \qquad (1)$$

A natural way to study these equations is to lift the multi-valued direction field determined by the EDI to a single field ξ on the surface $M = F^{-1}(0)$. (This field is determined by the restriction to the manifold of the contact planes associated with the standard contact form $dy - pdx$ in $\mathbb{R}^3$). If 0 is a regular value of F then M is smooth and the projection to

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the plane is a fold or cusp. The critical set of this projection is called the criminant and its image is the discriminant of the equation.

In [13] Davydov classified (following the work of Dara [12]) generic bi-valued fields when the discriminant is smooth and showed that the topological normal form of the IDE acquires moduli when the discriminant is a cusp.

Implicit differential equations have extensive applications to differential geometry of surfaces, partial differential equations, control theory and singularity theory. For example, lines of curvature, asymptotic and characteristic lines on a smooth surface in $\mathbb{R}^3$ are given by IDEs ([8]) and the characteristic lines of a general linear second-order differential equation are also given by an IDE ([23]).

Bruce and Tari introduced in [7] the multiplicity of an IDE, at a singular point, as the maximum number of singular points of the implicit differential equation which emerge when perturbing the equation F. In [9] and [10] the first author defined the index of an IDE (1) in terms of generic perturbations of the IDE and showed that this index is independent of the choice of a generic perturbation. A formula that expresses the index in terms of the gradient of F and the index of the 1-form $dy - pdx$, defined on a surface with isolated singularities M was also obtained. One of the main results in [9] and [10] is the invariance of the index by smooth equivalences.

In this work we define the index of an EDI at an isolated singular point at which $F_{pp}(q_0) = 0$. This definition extends the definition of the index of an EDI given in [9] and [10]. We also apply the results to study the geometry of surfaces in $\mathbb{R}^5$.

2. IMPLICIT DIFFERENTIAL EQUATION

As mentioned in the introduction, an implicit differential equation is of the form

$$F(x, y, p) = 0, \tag{2}$$

where $p = \frac{dy}{dx}$ and F is a smooth function in $\mathbb{R}^3$. An integral curve of the IDE (2) is a smooth curve $\alpha = (\alpha_1, \alpha_2) : (-1, 1) \longrightarrow \mathbb{R}^2$ such that $F(\alpha(t), \frac{\alpha_2'(t)}{\alpha_1(t)}) = 0$.

Consider the surface $M = F^{-1}(0)$, and the projection $\pi : M \longrightarrow \mathbb{R}^2$, given by $\pi(x, y, p) = (x, y)$. Generically M is a smooth surface and the projection π is generically a submersion or has a singularity of type fold, cusp or two transverse folds. The critical set $F = F_p = 0$ of this projection is called the criminant and its image is the discriminant of the EDI.

The multi-valued direction field in the plane determined by the EDI lifts to a single vector field tangent to M given by

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}.$$

Equivalently, this vector field is determined by the restriction of the canonical 1-form $dy - pdx$ to the surface M. Note that the vector field ξ may generically have an elementary zero, it is of type saddle, node or focus. One of the properties of this vector field is that the image by the projection π, of the integral curves of ξ on M, corresponds to integral curves of the IDE.

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We denote by C_F the criminant set of the EDI. We also denote by $\omega_{|_{C_F}}$ the restriction of the 1-form $\omega = dy - pdx$ on C_F.

DEFINITION 2.1. We say that $z_0 \in \mathbb{R}^2$ is a singular point of the IDE (2) if there exists $p_0 \in \mathbb{R}$ such that $q_0 = (z_0, p_0)$ is a zero of the 1-form $\omega_{|_{C_F}}$.

It is easy to verify that this definition reduces to Definition 2.3 given in [9] when the IDE define at most two directions in the plane.

PROPOSITION 2.1. ([7]) *Let $q_0 \in \mathbb{R}^3$ be a point on the criminant. Then q_0 is a zero of the 1-form $\omega_{|_{C_F}}$ if and only if q_0 is a zero of the vector field ξ or $F_{pp}(q_0) = 0$.*

From Proposition 2.1 it follows that the singular points of the EDI correspond to zeros of the vector field ξ or cusps of the natural projection π. We denote by (F, z_0) the germ of the IDE at an isolated singular point z_0.

DEFINITION 2.2. We say that (F, z_1) and (G, z_2) are equivalent if there exist a germ of diffeomorphim $h = (h_1, h_2) : (\mathbb{R}^2, z_1) \longrightarrow (\mathbb{R}^2, z_2)$ and a germ of function $\rho : (\mathbb{R}^2, z_2) \longrightarrow \mathbb{R}$, $\rho(z_2) \neq 0$ such that $G = \rho \cdot (F \circ H)$, where $H(x, y, p) = (h(x, y), \frac{h_{2x}(x,y)+h_{2y}(x,y)p}{h_{1x}(x,y)+h_{1y}(x,y)p})$.

The equivalence above, via diffeomorphism h, takes integral curves of $G = 0$ to integral curves of $F = 0$. We say that (F, z_1) and (G, z_2) are topologically equivalent if there exists a germ of homeomorphism $h : (\mathbb{R}^2, z_2) \longrightarrow (\mathbb{R}^2, z_1)$ taking integral curves of (G, z_2) to integral curves of (F, z_1).

DEFINITION 2.3. ([12]) Let $q_0 \in \mathbb{R}^3$ be a point on the criminant, such that $F_{pp}(q_0) = 0$ and $F_{ppp}(q_0)F_x(q_0) \neq 0$.

(i) We say that q_0 is an elliptic cusp if $(F_xF_{py} - F_yF_{px})(q_0) > 0$.
(ii) We say that q_0 is a hyperbolic cusp if $(F_xF_{py} - F_yF_{px})(q) < 0$.

If $q_0 = (z_0, p_0)$ is a fold singularity of the projection π and is an elementary zero of ξ, then (F, z_0) is topologically equivalent to a well-folded singularity $(p^2 - y + \lambda x^2, 0)$, $\lambda \neq 0, \frac{1}{4}$. We have a well-folded saddle if $\lambda < 0$, well-folded node if $0 < \lambda < \frac{1}{4}$ and a well-folded focus if $\lambda > \frac{1}{4}$ (see [13]). When π has a cusp singularity at q_0, the equation has functional moduli with respect to topological equivalence. There are two types of cusp singularity, the elliptic cusp and the hyperbolic cusp [13].
We denote by $C^r(\mathbb{R}^3)$ the set of all smooth functions of $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with the Whitney C^r-topology.

THEOREM 2.1. ([12]) *There exists an open and dense subset $\Lambda \subset C^3(\mathbb{R}^3)$ such that any germ of IDE of this subset is topologically equivalent to a well-folded singularity, an elliptical cusp or a hyperbolic cusp.*

Bruce and Tari introduced in [7] the multiplicity $M(F, 0)$ of an IDE as the maximum number of singular points of the implicit differential equation which emerge when perturbing the equation $F = 0$. Let $f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^m$ be an analytic map-germ. We shall denote by $m(f) = \dim \mathcal{O}_n / \langle f \rangle$, where $\mathcal{O}_n$ is the ring of germs of real analytic functions

$(\mathbb{R}^n, 0) \longmapsto \mathbb{R}$ and $\langle f \rangle$ is the ideal generated by the component functions of the map-germ f.

THEOREM 2.2. ([7]) *If $m(F, F_p, F_x + pF_y)$ and $m(F, F_p, F_{pp})$ are finite then*

$$M(F, 0) = m(F, F_p, F_x + pF_y) + m(F, F_p, F_{pp}).$$

A particular class of implicit differential equations are the binary differential equations (BDE's) of degree n, that is differential equations of the form

$$a_0(x, y)dy^n + a_1(x, y)dy^{n-1}dx + \ldots + a_n(x, y)dx^n = 0, \qquad (3)$$

where a_i are smooth functions defined on $U \subset \mathbb{R}^2$. If $dx = 0$ is not a solution of the Equation (3), we can set $p = \frac{dy}{dx}$ and reduce (3) to the IDE

$$F(x, y, p) = a_0(x, y)p^n + a_1(x, y)p^{n-1} + \ldots + a_n(x, y) = 0. \qquad (4)$$

We say that the IDE of degree n given by (4) is totally real if $a_i(0, 0) = 0$ (for any $i = 1, \ldots, n$) and for all $(x, y) \in U$, $(x, y) \neq 0$, the Equation (3) has exactly n different integral curves. An IDE of degree 1 is always totally real. In the case $n = 2$, an IDE is totally real if it is positive in the sense of [19]. In [17] Fukui and Nuño-Ballesteros introduce the concept of index for totally real IDE and produced a classification of generic singularities of this type of equations. Also, a generalization of the Poincaré-Hopf theorem and the Bendixon formula is obtained in [17].

In [10] the first author defined the index of an IDE of degree 2 not nece-ssarily totally real. One of the main results in [10] is the invariance of the index by smooth equivalences. We set $\delta = a_1^2 - 4a_0a_2$.

THEOREM 2.3. *([10]) Let $(F, 0)$ be the germ of an IDE of degree 2. Then the index of $(F, 0)$ at 0 is given by*

$$I(F, 0) = \frac{1}{2}Ind_0(\delta, (a_0\delta_x - a_1\delta_y)a_0\delta_y) - \frac{1}{2}Ind_0(a_0, a_1) - \frac{1}{2}Ind_0(\delta\delta_x, \delta_y) + \frac{1}{2}Ind_0\nabla\delta.$$

When $F_{pp}(0) \neq 0$, the formula of the index simplifies. We denote by $\mathrm{Ind}_{q_0}\xi$ the index of the vector field ξ at $q_0 \in M$, introduced by W. Ebeling and S. M. Gusein-Zade in [15].

THEOREM 2.4. ([10]) *Let $(F, 0)$ be the germ of an IDE of degree 2, and let 0 be a zero of the the vector field ξ. If $F_{pp}(0) \neq 0$ then*

$$I(F, 0) = sign\,[F_{pp}(0)] \cdot \frac{1}{2}Ind_0\nabla F + \frac{1}{2}Ind_0\xi.$$

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LEMMA 2.1. ([9]) *Let q_0 be a zero of the vector field ξ. Then,*

$$Ind_{q_0}\xi = Ind_{q_0}(F_yF, F_p, F_x + pF_y).$$

A 1-parameter perturbation $a_i^t(x,y)$ of the coefficients $a_i(x,y)$ of the IDE (4) determines a perturbation of $(F,0)$ in the space of all the IDE's of degree n given by

$$F^t(x,y,p) = a_0^t(x,y)p^n + a_1^t(x,y)p^{n-1} + \ldots + a_n^t(x,y) = 0, \tag{5}$$

where $a_i^0(x,y) = a_i(x,y)$

DEFINITION 2.4. We say that F^t is a good perturbation of $(F,0)$, if F^t is an element of Λ, for all t sufficiently close to zero, $t \neq 0$.

3. INDEX OF AN IMPLICIT DIFFERENTIAL EQUATION

We refer to [27] for the basic properties of indices of vector fields used in this section. Given $N \in GL(n,\mathbb{R})$, we denote by sgn$[N]$ the sign of the determinant of the matrix N.

LEMMA 3.1. ([16]) *Let $f,g : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ be map-germs. If there exist a germ of diffeomorphim $R : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ and a map-germ $S : (\mathbb{R}^n,0) \longrightarrow GL(n,\mathbb{R})$ such that $f = S \cdot g \circ R$ then*

$$Ind_0 f = sgn[S(0)] \cdot Ind_0 g \cdot sgn[d_0R].$$

DEFINITION 3.1. We say that z_0 is a non-degenerate singular point of the IDE (2) if (F,z_0) is topologically equivalent to a well folded singularity, an elliptic cusp or a hyperbolic cusp.

If z_0 is a non-degenerate singular point of (2), then there exists $p_0 \in \mathbb{R}$ such that (z_0,p_0) is a saddle, node or focus of the vector field $\xi_{|_M}$ or elliptic cusp or hyperbolic cusp. So we can associate a number $K_F(z_0)$ to each non-degenerate singular point z_0 of (2), as follows:

(*i*) $K_F(z_0) = -1$ if (z_0,p_0) is a saddle of the vector field $\xi_{|_M}$
(*ii*) $K_F(z_0) = 1$ if (z_0,p_0) is a node or focus of the vector field $\xi_{|_M}$
(iii) $K_F(z_0) = 1$ if (z_0,p_0) is an elliptic cusp
(iv) $K_F(z_0) = -1$ if (z_0,p_0) is a hyperbolic cusp.

Let $(F,0)$ be the germ of an IDE of degree 3 given by

$$F(x,y,p) = a(x,y)p^3 + b(x,y)p^2 + c(x,y)p + d(x,y) = 0, \tag{6}$$

such that $F(0) = F_p(0) = F_{pp}(0) = 0$.

Using Equation (6) we deduce that

$$\begin{aligned} 27a^2F &= [3ap+b]^3 + 9ap[3ac-b^2] + 27a^2d - b^3 \\ 3aF_p &= [3ap+b]^2 + 3ac - b^2 \\ F_{pp} &= 2[3ap+b]. \end{aligned}$$

From the above equation, we obtain that the discriminant of the EDI (6) is given by

$$\delta = (27a^2d - 9abc + 2b^3)^2 + 4(3ac-b^2)^3.$$

Remark 3. 1. If 0 is an elliptic or a hyperbolic cusp of the germ $(F,0)$, then

$$sgn[F_{ppp}(0)]\mathrm{Ind}_0(F, F_p, F_{pp}) = \mathrm{Ind}_0(9ad - bc, 3ac - b^2) = K_F(0).$$

Let F^t be a 1-parameter perturbation of the IDE (6) given by

$$F^t(x,y,p) = a^t(x,y)p^3 + b^t(x,y)p^2 + c^t(x,y)p + d^t(x,y) = 0. \tag{7}$$

To show that F^t is a good perturbation of the IDE (6), it is sufficient to prove that 0 is a regular value of the map $(9a_td_t - b_tc_t, 3a_tc_t - b_t^2)$, for all $t \neq 0$. We denote by $\mathcal{P}_k(\mathbb{R}^2)$ the set of all polynomials of 2 variables and degree less than or equal to k. Let $\Phi : \mathbb{R}^2 \times \mathcal{P}_k^4(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ be a smooth map defined by

$$\Phi(x,y,r) = [9(a+\tilde{a})(d+\tilde{d}) - (b+\tilde{b})(c+\tilde{c}), 3(a+\tilde{a})(c+\tilde{c}) - (b+\tilde{b})^2](x,y),$$

where $r = (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) \in \mathcal{P}_k^4(\mathbb{R}^2)$. We set $\Phi_r(x,y) = \Phi(x,y,r)$

LEMMA 3.2. *There exists an open and dense set Δ of $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in \Delta$, 0 is a regular value of Φ_r.*

Proof. By Thom transversality lemma, there exists a dense set Δ of $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in \Delta$, 0 is a regular value of Φ_r, that is Φ_r intersect 0 transversally. It is not difficult to show that $H : \mathcal{P}_k^4(\mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$H(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = [9(a+\tilde{a})(d+\tilde{d}) - (b+\tilde{b})(c+\tilde{c}), 3(a+\tilde{a})(c+\tilde{c}) - (b+\tilde{b})^2]$$

is continuous. Let $r_0 = (a_0, b_0, c_0) \in \Delta$. As the set of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which intersect 0 transversally is open, we have that there exists a neighborhood U of r_0 in $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in U$, $H(r) = \Phi_r$ intersect 0 transversally. The result now follows. ∎

THEOREM 3.1. *If a, b, c, d are smooth functions then there exists a good perturbation F^t of $(F,0)$.*

Proof. By Lemma 3.2, there exists an open and dense set Δ of $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in \Delta$, 0 is a regular value of Φ_r. It is not difficult to show that there exists a smooth curve $\alpha : (-1,1) \to \mathcal{P}_k^4(\mathbb{R}^2)$ such that $\alpha[(-1,1) - \{0\}] \subseteq \Delta$ and $\alpha(0) = 0$. Let $(a^t, b^t, c^t, d^t) = (a,b,c,d) + \alpha(t)$. Then, $F^t = a^t p^3 + b^t p^2 + c^t p + d^t = 0$ is a good perturbation of (6). ▮

Definition 3.2. Let F^t be a good perturbation of the germ $(F,0)$ given by (6). Then the cusp index of $(F,0)$ at 0 is defined by

$$J(E,0) = \frac{1}{3} \sum_i K_{F^t}(z_i)$$

where z_i are non-degenerate singular points of F^t of elliptic or hyperbolic cusp type.

The next lemma shows that the index $J(F,0)$ can be expressed in terms of the coefficients a, b, c e d.

Theorem 3.2. *Let $(F,0)$ be the germ of EDI given by (6). If $F_{pp}(0) = 0$ then*

$$J(F,0) = \frac{1}{3}[Ind_0(9ad - bc, 3ac - b^2) - Ind_0(a,b)]$$

Proof. It follows from Theorem 3.1 that there exists a good perturbation F^t of the IDE (6) such that 0 is a regular value of $(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2)$, $t \neq 0$. Then using Proposition 2.2 in [11] we obtain

$$\mathrm{Ind}_0(9ad - bc, 3ac - b^2) = \sum \mathrm{Ind}_{z_i}(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2). \tag{8}$$

We denote by $W_t = (9a_t d_t - b_t c_t, 3a_t c_t - b_t^2)$. By Remark 3.1,

$$\sum_{a_t(z_i) \neq 0} \mathrm{Ind}_{z_i} W_t \;=\; K_{F_t}(z_i). \tag{9}$$

As 0 is a regular value of W_t, we get

$$\begin{aligned} \sum_{a_t(z_i)=0} \mathrm{Ind}_{z_i} W_t &= \mathrm{Ind}_0(a_t, b_t) \\ &= \mathrm{Ind}_0(a,b). \end{aligned}$$

The result now follows. ▮

Theorem 3.3. *Let $(F,0)$ and $(G,0)$ be the germs of IDE's of degree 3. If $(F,0)$ and $(G,0)$ are equivalent then $Ind_0 F = Ind_0 G$.*

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Proof. It follows from the hypothesis that there exist a germ of diffeomorphim $h = (h_1, h_2) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and a germ of function $\rho : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$, $\rho(0) \neq 0$ such that $G = \rho \cdot (F \circ H)$, where $H(x, y, p) = (h(x, y), \frac{h_{2x}(x,y)+h_{2y}(x,y)p}{h_{1x}(x,y)+h_{1y}(x,y)p})$.

We set $T(x, y, p) = \frac{h_{2x}(x,y)+h_{2y}(x,y)p}{h_{1x}(x,y)+h_{1y}(x,y)p}$. Then,

$$\begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho T_p & 0 \\ 0 & T_{p^2} & \rho T_p^2 \end{pmatrix} \begin{pmatrix} F \circ H \\ F_p \circ H \\ F_{p^2} \circ H \end{pmatrix} = \begin{pmatrix} G \\ G_p \\ G_{p^2} \end{pmatrix}, \qquad (10)$$

$G_{p^3} = \rho[(F_{p^3} \circ H)T_p^3 + 3(F_{p^2} \circ H)T_pT_{p^2} + (F_p \circ H)T_{p^3}]$ and $T_p = \frac{det[dh]}{[h_{1x}+h_{1y}p]^2}$.

By Theorem 3.1, there exists a good perturbation F^t of $(F, 0)$. Using the equations above, it is not difficult to show that $G^t = \rho \cdot (F^t \circ H)$ is a good perturbation of $(G, 0)$ and

$$sgn[F^t_{p^3}(r_i)]\mathrm{Ind}_{r_i}(F^t, F^t_p, F^t_{p^2}) = sgn[G^t_{p^3}(q_i)]\mathrm{Ind}_{q_i}(G^t, G^t_p, G^t_{p^2}),$$

where $H(q_i) = r_i$. The result then follows by Remark 3.1 and Definition 3.2. ∎

Corollary 3.1. *Let $(F, 0)$ be the germ of EDI given by (6). If $F_{ppp}(0) \neq 0$ and $F_{pp}(0) = 0$ then*

$$J(F, 0) = sgn[F_{ppp}(0)]Ind_0(F, F_p, F_{pp}).$$

Proof. The proof follows by using Theorem 3.2 and Remark 3.1. ∎

Note that Theorem 2.4 introduce a definition of the index at a fold point of the projection π corresponding to a zero of vector field $\xi_{|M}$. Also from Corollary 3.1 we have a definition of the index at a non-fold singularity of the projection. So we introduce a definition of the index to any germ of IDE, that extends the two definitions given by Theorem 2.4 and Corollary 3.1.

Definition 3.3. Let $(F, 0)$ be the germ of an IDE of degree n, $n \geq 2$ and suppose that the number of zeros of the 1-form $\omega_{|M}$ in $(0, 0) \times \mathbb{R}$ is finite. Then the index of $(F, 0)$ at 0 is defined by

$$\mathrm{Ind}_0 F = \frac{1}{n}[\sum_{i=1}^{n_0} \mathrm{Ind}_{s_i}\xi_{|M} + \sum_{i=1}^{n_0} \mathrm{Ind}_{s_i}(F_{pp}F_y, F_p, F_x + pF_y) + \sum_{i=1}^{n_1} \mathrm{Ind}_{s_i}(F_{ppp}F, F_p, F_{pp})],$$

where $s_i = (0, 0, p_i)$.

Theorem 3.4. *Let $(F, 0)$ be the germ of an IDE of degree n. Then,*

(1) *If $n = 2$, $F_{pp}(0) \neq 0$ and 0 is a zero of $\xi_{|M}$ then $Ind_0 F = I(F, 0)$*

(2) *If* $n = 3$, $F_{pp}(0) = 0$, $F_{ppp}(0) \neq 0$ *and* 0 *is not a zero of* $\xi_{|M}$ *then* $Ind_0 F = J(F, 0)$.

Proof. The proof is a straightforward calculation. ∎

Let $G(x, y, p) = 0$ be an IDE and $N = G^{-1}(0)$.

THEOREM 3.5. *Let* $(F, 0)$ *and* $(G, 0)$ *be the germs of IDE's and suppose that the number of zeros of the 1-form* $\omega_{|M}$ *in* $(0, 0) \times \mathbb{R}$ *is finite. If* $(F, 0)$ *and* $(G, 0)$ *are equivalent then* $Ind_0 F = Ind_0 G$.

Proof. It follows from the hypothesis that there exist a germ of diffeomorphim $h = (h_1, h_2) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and a germ of function $\rho : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$, $\rho(0) \neq 0$ such that $G = \rho \cdot (F \circ H)$, where $H(x, y, p) = (h(x, y), \frac{h_{2x}(x,y)+h_{2y}(x,y)p}{h_{1x}(x,y)+h_{1y}(x,y)p})$.

We set $T(x, y, p) = \frac{h_{2x}(x,y)+h_{2y}(x,y)p}{h_{1x}(x,y)+h_{1y}(x,y)p}$, and to simplify the arguments, we take $\rho = 1$. Then,

$$\begin{aligned} G_x &= (F_x \circ H)h_{1x} + (F_y \circ H)h_{2x} + (F_p \circ H)T_x \\ G_y &= [(F_x + pF_y) \circ H]h_{1y} + (F_y \circ H)\frac{\det[dh]}{h_{1x} + ph_{1y}} + (F_p \circ H)T_y \\ G_p &= (F_p \circ H)T_p \end{aligned}$$

and $\det[dH] = \det[dh] \cdot T_p$. Using these equations, we get

$$\begin{pmatrix} S_1 & GT_y & h_{1y}G \\ 0 & T_p & 0 \\ 0 & S_2 & h_{1x} + ph_{1y} \end{pmatrix} \begin{pmatrix} (F_y F) \circ H \\ F_p \circ H \\ (F_x + pF_y) \circ H \end{pmatrix} = \begin{pmatrix} G_y G \\ G_p \\ G_x + pG_y \end{pmatrix} \tag{11}$$

where $S_1 = \frac{\det[dh]}{h_{1x}+ph_{1y}}$ and $S_2 = T_x + pT_y$. Suppose that $q_0 = (0, 0, p_0)$ is a zero of $\omega_{|M}$. Then, using Equation (11) and Lemma 3.1, we obtain

$$\mathrm{Ind}_{q_0}(F_y F, F_p, F_x + pF_y) = \mathrm{Ind}_{q_1}(G_y G, G_p, G_x + pG_y), \tag{12}$$

where $q_1 = (0, 0, T(q_0))$. Following the same arguments as above we show that

$$\begin{aligned} \mathrm{Ind}_{q_0}(F_{pp} F_y, F_p, F_x + pF_y) &= \mathrm{Ind}_{q_1}(G_{pp} G, G_y, G_x + pG_y) \\ \mathrm{Ind}_{q_0}(F_{ppp} F, F_p, F_{pp}) &= \mathrm{Ind}_{q_1}(G_{ppp} G, G_p, G_{pp}), \end{aligned}$$

and the result follows. ∎

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4. INDEX OF A DIFFERENTIAL N-FORM

Let M be a C^∞-compact, connected, oriented, 2-dimensional surface. A (symmetric) differential n-form on M is a differentiable section of the symmetric tensor fiber bundle $S^n(T^*M)$. Let ω be a differential n-form on M. An integral curve of ω is a smooth curve $\alpha : (-1,1) \longrightarrow M$ such that $\omega(\alpha(t))[\alpha'(t)] = 0$.

We denote by $\mathbb{H}$ the set of points p in M such that the subset $\omega(p)^{-1}(0)$ is the union of n transversal lines. Through every point p of $\mathbb{H}$ pass n transverse integral curves of ω. Under the orientability hypothesis imposed on M, the differential n- form ω defines n line fields $L_{\omega,1}, \ldots, L_{\omega,n}$ on $\mathbb{H}$ such that the integral curves of $L_{\omega,i}$ correspond to integral curves of ω.

Let PM be the projectivized tangent bundle of M, and $\mathcal{H} \subset PM$ the set of points $(p,[v])$ such that $\omega(p)[v] = 0$. Let $\pi : PM \longrightarrow M$ be the natural projection given by $\pi(p,[v]) = p$. Then, the restriction of the projection π to $\mathcal{H}$ covers the closure of $\mathbb{H}$. Lifting to $\mathcal{H}$ the line fields $L_{\omega,1}, \ldots, L_{\omega,n}$ define a single line field L on $\pi^{-1}(\mathbb{H})$ which, under the condition of regularity, uniquely extends to a smooth line field L defined on the whole $\mathcal{H}$.

Let $h : U \longrightarrow \mathbb{R}^2$ be a local chart of M at p_0 such that $h(p_0) = 0$. Then, the pull-back of ω defines an implicit differential equation of degree n on $\mathbb{R}^2$ given by

$$h^*(\omega)(x,y,p) = a_0(x,y)p^n + a_1(x,y)p^{n-1} + \ldots + a_n(x,y) = 0, \tag{13}$$

where $p = \frac{dy}{dx}$ and $a_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are smooth functions. We say that $p_0 \in M$ is a singular point of ω if 0 is a singular point of $h^*(\omega)$.

Definition 4.1. Let p_0 be a singular point of ω. Then, the index of ω at p_0 is defined by $\mathrm{Ind}_{p_0}\omega = \mathrm{Ind}_0 h^*(\omega)$.

The next lemma shows that the index is independent of the choice of a local chart of M.

Lemma 4.1. *Let p_0 be a singular point of ω and $h_1 : V \longrightarrow \mathbb{R}^2$ a local chart of M at p_0 such that $h_1(p_0) = 0$. Then, $Ind_0 h^*(\omega) = Ind_0 h_1^*(\omega)$.*

Proof. Note that $h^*(\omega)(z)[u] = \omega(h^{-1}(z))[dh_z^{-1}(u)]$, where $z, u \in \mathbb{R}^2$. Then, the map $h_1 \circ h^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ sends integral curves of $h^*(\omega)$ to integral curves of $h_1^*(\omega)$. Therefore, by Theorem 3.5 the result follows. ▌

We denote by (ω, p) the germ of the differential n-form ω at a singular point p. Let (ω_1, p_1), (ω_2, p_2) be two germs of differential n-forms defined on 2-dimensional surfaces M, N, respectively. We say that they are equivalent if there exists a germ of diffeomorphism $k : (M, p_1) \longrightarrow (N, p_2)$ that sends integral curves of (ω_1, p_1) to integral curves of (ω_2, p_2).

Theorem 4.1. *Suppose that $\pi_{|\mathcal{H}}$ is a stable map without cusp and transverse folds. If $\mathbb{H} \neq \emptyset$, then $\chi(\mathcal{H}) = n \cdot \chi(\pi(\mathcal{H}))$.*

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Proof. By hypothesis, there exist n line fields $L_{\omega,1},\dots,L_{\omega,n}$ on $\mathbb{H}$ which are linearly independent, and $\chi(L_{\omega,i}(\bar{\mathbb{H}})\cap L_{\omega,j}(\bar{\mathbb{H}}))=0$, $i\neq j$, where $\bar{\mathbb{H}}$ is the closure of $\mathbb{H}$. Note that the singular points of ω belong to the boundary of $\bar{\mathbb{H}}$ and $\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}})$ is a surface with boundary.

As $\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}})=\cup_{i=1}^{n}L_{\omega,i}(\bar{\mathbb{H}})$, it follows that $\chi(\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}}))=n\cdot\chi(\pi(\bar{\mathbb{H}}))$. Analogously one proves that $\chi(\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}}^m))=m\cdot\chi(\pi(\bar{\mathbb{H}}^m))$, where $\mathbb{H}^m$ is the set of points p in M such that $\omega(p)^{-1}(0)$ is the union of m transversal lines. As the restriction of the line field L to the surface $\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}}^m)$ has no zeros, we have that $\chi(\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}}^m))=0$ and $\chi(\pi(\bar{\mathbb{H}}^m))=0$. Since $\mathcal{H}=\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}})\cup(\cup_{i=1}^{n-1}\pi^{-1}_{|\mathcal{H}}(\bar{\mathbb{H}}^i))$ and $\pi(\mathcal{H})=\bar{\mathbb{H}}\cup(\cup_{i=1}^{n-1}\bar{\mathbb{H}}^i)$, the result follows. ▮

THEOREM 4.2. *Suppose that $\pi_{|\mathcal{H}}$ is a stable map without cusp and transverse folds. If $\mathbb{H}\neq\emptyset$ and n is odd, then*

$$\chi(M)=\sum_{i=1}^{s}Ind_{p_i}\omega,$$

where $p_1,...,p_s$ are the singular points of ω.

Proof. The proof follows by using Definition 3.3 and Theorem 4.1. ▮

5. APPLICATIONS

Let M be a 2-dimensional smooth surface M in $\mathbb{R}^5$, and denote by TM and NM its tangent and normal bundles. The contact of the surface with 4-dimensional planes is measured by the singularities of the height function

$$H:M\times S^4\longrightarrow\mathbb{R}\times S^4$$

given by $H(q,v)=(h_v(q),v)$, where $h_v(q)=\langle q,v\rangle$. A height function h_v has a singularity at $q\in M$ if and only if $v\in N_qM$. It follows from Looijenga's theorem ([4]) that h_v has generically a singularity of type $A_{k\leq 5}$, $D_4^{\pm}$ or D_5 (see [2] for notation).

A direction $v\in N_qM$ is said to be degenerate if q is a non-stable singularity of h_v. In this case, the kernel of the Hessian of h_v, $ker(Hess(h_v)(q))$, contains non zero vectors. Any direction $u\in ker(Hess(h_v)(q))$ is called a contact direction associated to v.

A unit vector $v\in N_qM$ is called a binormal direction if h_v has a singularity of type A_3 or worse at q.

DEFINITION 5.1. ([26]) Let $q\in M$ and $v\in N_qM$ be a binormal direction. An asymptotic direction at q is any contact direction associated to v.

THEOREM 5.1. ([29]) *There is at least one and at most five asymptotic curves passing through any point on a generic immersed surface in $\mathbb{R}^5$. These curves are solutions of the* *implicit differential equation*

$$A_0dy^5 + A_1dy^4dx + A_2dy^3dx^2 + A_3dy^2dx^3 + A_4dydx^4 + A_5dx^5 = 0$$

where the coefficients A_i ($i = 0, \ldots, 5$) depend on the coefficients of the second fundamental form and their first order partial derivatives.

For a generic surface, at least one of the coefficients in the Theorem 5.1 is not zero at any point $q \in M$. We then set $p = \frac{dy}{dx}$ (as dx=0 is not solution of the equation) so that the equation of the asymptotic curves in a neighbourhood U of q is an IDE of degree 5 in the form

$$F(x, y, p) = A_0p^5 + A_1p^4 + A_2p^3 + A_3p^2 + A_4p + A_5 = 0. \quad (14)$$

It follows that there exists a differential 5-form η on M such that $\eta = F$ in U. The discriminant of the asymptotic curves Δ_η is given by the discriminant of the IDE (14).

Theorem 5.2. *Let M be a closed orientable surface generically immersed in $\mathbb{R}^5$. Then*

$$\chi(M) = \sum_{i=1}^{s} Ind_{q_i}\eta,$$

where $q_1, ..., q_s$ are the singular points of η.

Proof. Since Δ_η is a smooth curve, the result then follows by Theorem 4.2. ∎

A folded singularity of the asymptotic curves is a folded-singularity of the IDE (14). The following result also appears in [29].

Corollary 5.1. *Let M be a closed orientable surface generically immersed in $\mathbb{R}^5$ with $\chi(M) \neq 0$. Then Δ_η is not empty and the asymptotic curves have folded singularities.*

Proof. The proof follows by using Theorem 5.2. ∎

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