CADERNOS DE MATEMÁTICA **09**, 1–18 May (2008)
ARTIGO NÚMERO SMA#272

# Dissipative wave problems with fast growing nonlinearities

Alexandre N. Carvalho*

*Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil*
E-mail: andcarva@icmc.usp.br

Jan Cholewa

*Institute of Mathematics, Silesian University, 40-007 Katowice, Poland*
E-mail: jcholewa@ux2.math.us.edu.pl

Tomasz Dłotko

*Institute of Mathematics, Silesian University, 40-007 Katowice, Poland*
E-mail: tdlotko@ux2.math.us.edu.pl

Let $a > 0$, $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $-A$ denote the Laplace operator with Dirichlet boundary conditions in $L^2(\Omega)$. We study the damped wave problem

$$\begin{cases} u_{tt} + au_t + Au = f(u), \ t > 0, \\ u(0) = u_0 \in H^1_0(\Omega), \quad u_t(0) = v_0 \in L^2(\Omega), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying the growth condition $|f(s) - f(t)| \leq C|s - t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$, $1 < \rho < \frac{N+2}{N-2}$, $(N \geq 3)$, and the dissipativeness condition $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$ with $\lambda_1$ being the first eigenvalue of $A$. We construct the global weak solutions of this problem as the limits as $\eta \to 0^+$ of the solutions of wave equations involving a strongly damping term $\eta A^{\frac{1}{2}} u$ with $\eta > 0$. We define a subclass $\mathcal{LS} \subset C([0,\infty), L^2(\Omega) \times H^{-1}(\Omega)) \cap L^\infty([0,\infty), H^1_0(\Omega) \times L^2(\Omega))$ of the 'limit' solutions such that through each initial condition from $H^1_0(\Omega) \times L^2(\Omega)$ passes at least one solution of the class $\mathcal{LS}$. We show that the class $\mathcal{LS}$ is bounded dissipative in $H^1_0(\Omega) \times L^2(\Omega)$ and we construct a closed bounded invariant subset $\mathbf{A}$ of $H^1_0(\Omega) \times L^2(\Omega)$, which is compact in $H^s_{\{I\}}(\Omega) \times H^{s-1}(\Omega)$, $s \in [0,1)$, and attracts in any of these spaces bounded subsets of $H^1_0(\Omega) \times L^2(\Omega)$. For $N = 3, 4, 5$ we also prove a local uniqueness result for the case of smooth initial data. May, 2008 ICMC-USP

* Research partially supported by CNPq grant 305447/2005-0 and by FAPESP grant 03/10042-0, Brazil.



Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

## 1. INTRODUCTION

In this paper we study damped wave equations of the form

$$\begin{cases} u_{tt} + au_t - \Delta u = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1}$$

where $a > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 3$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying for some $1 < \rho < \frac{N+2}{N-2}$ the growth condition

$$|f'(s)| \leq C(1 + |s|^{\rho-1}) \tag{2}$$

and the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} =: \mu_1 < \lambda_1, \tag{3}$$

with $\lambda_1$ being the first eigenvalue of the negative Dirichlet Laplacian in $\Omega$.

Assuming (3) and (2) with $\rho \in (1, \frac{N+2}{N-2})$, we consider a *parabolic type perturbations* of the problem (1),

$$\begin{cases} u_{tt} + 2\eta(-\Delta)^{\frac{1}{2}} u_t + au_t - \Delta u = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{4}$$

and we approximate solutions of (1) for initial data $u_0 \in H^1_0(\Omega)$, $v_0 \in L^2(\Omega)$ with the solutions of (4) where $\eta > 0$. In particular we construct a bounded in $H^1_0(\Omega) \times L^2(\Omega)$ uniformly absorbing set for the resulting solutions of (1) and show the existence of the bi-space global attractor.

Damped wave equations with fast growing nonlinearities have been considered before by many authors and much progress has been achieved; see [1, 2, 3, 6, 13, 15, 16, 17, 19, 20] and references therein.

In [15] the author studied the damped wave problem in $H^1(\Omega)\times L^2(\Omega)$, when $\Omega$ was a *N-dimensional Riemannian compact manifold without boundary* and $f$ satisfied (2) with $\rho < \frac{N+2}{N-2}$, $N \geq 3$.

In [16] the wave problems were locally well posed in the whole of $\mathbb{R}^N$, $N \geq 3$, for $u_0 \in H^1(\mathbb{R}^N)$, $v_0 \in L^2(\mathbb{R}^N)$ and $f$ satisfying (2) with $\rho \leq \frac{N+2}{N-2}$ and $f(0) = 0$. Note that the solutions were unique thanks to the *Strichartz inequality*, being the main tool to obtain suitable a priori estimates in the case of these special domains, and that to this date such results have not been extended to bounded domains with Dirichlet or Neumann boundary conditions.

In [20] the author studied asymptotic behavior of weak energy solutions to (1) in the non-autonomous case. Some sort of the monotonicity assumption was considered for $f$ but

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

no growth restriction was used to prove existence of the solutions of (1). The approach of [20] comes back to the classical monograph [17], where the solutions were obtained for the problem (1) with $|f(s)|$ growing not faster than $|s|^{\frac{N}{N-2}}$ $(N \geq 3)$.

We remark that when the nonlinear term $f$ satisfies (3) and $|f(s)| \leq c(1 + |s|^{\frac{N+2}{N-2}})$ for $s \in \mathbb{R}$, no further assumption is needed to construct a suitable notion of a global solution of (1). This can be found for example in consideration of [4, Theorem 4.3], where the approach of [17] was followed.

Some mild assumptions on the nonlinearity were considered in [19, Corollary] to construct the *finite-energy solution* of (1) although simultaneously the initial data therein was taken from $H^1_0(\Omega) \cap L^\infty(\Omega) \times L^2(\Omega)$ and not from $H^1_0(\Omega) \times L^2(\Omega)$. On the other hand, in [19, Theorem 2] strong dissipativeness condition was assumed, which in the non-conservative case we deal with would lead to the asymptotic decaying of the solutions to zero steady state.

In [4] previous results of [5] concerning generalized semiflows in metric spaces were used to show the existence of an attractor for (1). In this consideration a hypothesis (see [4, p. 46]) that *"every weakly continuous solution satisfies the energy equation"* was assumed.

Since the question whether weakly continuous solutions fulfil the energy equation has not been answered yet we investigate the properties of the solutions of (1) based on the regularity and dissipativeness properties of solutions of the perturbed (parabolic type) problems replacing the Galerkin approximation technique by a variant of *Viscosity Method*, which seems to lead to some interesting results.

In order to explain the procedure better we introduce some terminology.

Denote by $-A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ the Laplacian in $L^2(\Omega)$ with the domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$; that is $A$ is a positive self-adjoint operator with compact resolvent. Consider the closed extension of $A$ to $(H^1_0(\Omega))'$ with the domain $H^1_0(\Omega)$ and still denote it by $A$. Also, let $A^{\frac{1}{2}} : H^1_0(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ be the half fractional power of $A$.

We consider the family of problems

$$\begin{cases} u_{tt} + 2\eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u), \\ u(0) = u_0 \in H^1_0(\Omega), \ u_t(0) = v_0 \in L^2(\Omega), \end{cases} \tag{5}$$

which we rewrite as a first order system

$$\tfrac{d}{dt} \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right), \ t > 0, \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega), \tag{6}$$

where $\eta$ is a non-negative parameter, $\mathcal{A}_\eta$ denotes the strongly damped wave operator;

$$\mathcal{A}_\eta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} + aI \end{bmatrix}, \quad D(\mathcal{A}_\eta) = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega), \tag{7}$$

and

$$F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 \\ f(u) \end{smallmatrix}\right], \ \ f(u)(x) = f(u(x)) \text{ for } u \in H^1_0(\Omega),\ x \in \Omega.$$

For $f$ satisfying (2) with $1 < \rho \leq \frac{N+2}{N-2}$, $N \geq 3$, it has been shown in [7] (exploiting the fact that $\mathcal{A}_\eta$ generates analytic semigroup) that (6) is locally well posed for any $\eta > 0$ and that its solutions are regular. Also, it is shown in [10] that bootstrapping methods can be used to obtain additional regularity properties of solutions of (6).

For $\eta > 0$ and $f$ satisfying (3) and (2) with $1 < \rho < \frac{N+2}{N-2}$ it has been shown in [8] that (6) is globally well posed. In addition, it was shown in [8] that the nonlinear semigroup of global solutions associated to (6) has a compact global attractor, which results were extended in [9] to almost critical nonlinearities; that is maps $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$\limsup_{|s|\to\infty} s^{-\rho+1} f'(s) = 0 \ \text{ with } \rho = \frac{N+2}{N-2} \text{ for } N \geq 3. \tag{8}$$

In this paper we show that the solutions of (6) with $\eta > 0$ and with dissipative nonlinearity $f$ satisfying (2) with $\rho \in (1, \frac{N+2}{N-2})$ converge along subsequences to a function which is, in a sense that is to be explained below, a *global weak solution of* (6) with $\eta = 0$.

Here, if $\eta \geq 0$, we will say that

Definition 1.1. A bounded function $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) : [0, \infty) \to H^1_0(\Omega) \times L^2(\Omega)$ is a *global weak solution* of (6) provided that the maps $[0, \infty) \ni t \mapsto \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $[0, \infty) \ni t \mapsto F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)) \in L^2(\Omega) \times H^{-1}(\Omega)$ are continuous and that for each $t \in [0, \infty)$

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) = e^{-\mathcal{A}_\eta t} \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_\eta (t-s)} F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)) ds. \tag{9}$$

To be more specific about our technique and results note first that for any $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$ the mild solution of the problem (6) with $\eta > 0$ is globally defined and, moreover, its $H^1_0(\Omega) \times L^2(\Omega)-$norm is bounded uniformly for $\eta > 0$ and $t \geq 0$ due to the properties of the energy functional

$$\mathcal{L}_0(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) := \frac{1}{2}\|w_1\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|w_2\|^2_X - \int_\Omega \int_0^{w_1} f(s) ds dx, \ \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega). \tag{10}$$

Hence with the same $f$ and for the same initial conditions a global weak solution of (6) exists with $\eta = 0$ and can be viewed as a limit of the solutions of (6) corresponding to positive $\eta$'s. Furthermore, the existence of an absorbing set $B_0$ for (6) follows, which set is bounded in $H^1_0(\Omega) \times L^2(\Omega)$ and possesses the *entrance time* independent of $\eta \in (0, 1]$ as well as of initial data in bounded subsets of $H^1_0(\Omega) \times L^2(\Omega)$. This naturally leads to a construction of the bi-space global attractor in the form of the limit set

$$\{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] : \exists_{t_n \nearrow \infty} \ \exists_{\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0} \ \exists_{\eta_n \searrow 0} \ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) \overset{\text{w}-H^1_0(\Omega)\times L^2(\Omega)}{\rightharpoonup} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\},$$

enjoying suitably defined attracting, compactness and invariance properties.

To describe a large class of global weak solutions of (6) with $\eta = 0$ which are obtained as limits of the solutions of (6) with $\eta = \eta_n \to 0^+$ we will prove the following result.

THEOREM 1.1. *Assume that* $f \in C^1(\mathbb{R},\mathbb{R})$ *satisfies* (3) *and* (2) *with some* $\rho \in (1, \frac{N+2}{N-2})$. *Hence, if* $\eta_n \to 0^+$, $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ *belongs to* $H_0^1(\Omega)\times L^2(\Omega)$, $\left\{\left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right]\right\}_{n=1}^{\infty}$ *is a sequence in* $H_0^1(\Omega)\times L^2(\Omega)$, *convergent weakly in* $H_0^1(\Omega)\times L^2(\Omega)$ *to* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, *and if for* $n \in \mathbb{N}$ *the function* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(\cdot, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n\right)$ *is a mild solution of* (6) *with* $\eta = \eta_n$, *then there is a subsequence* $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(\cdot, \left[\begin{smallmatrix} u_0^{n_k} \\ v_0^{n_k} \end{smallmatrix}\right], \eta_{n_k}\right)\}$ *and a bounded function* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] : [0,\infty) \to H_0^1(\Omega)\times L^2(\Omega)$, *being a global weak solution of* (6) *with* $\eta = 0$, *to which this subsequence converges in* $L^2(\Omega)\times H^{-1}(\Omega)$ *uniformly on compact subsets of* $[0,\infty)$; *that is*

$$\sup_{t\in[0,\tau]} \left\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(t, \left[\begin{smallmatrix} u_0^{n_k} \\ v_0^{n_k} \end{smallmatrix}\right], \eta_{n_k}\right) - \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](t) \right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \longrightarrow 0 \quad \text{for each } \tau > 0. \tag{11}$$

If the initial data in (6) are smooth our result is as follows.

THEOREM 1.2. *Suppose that* $N = 3, 4, 5$, $f \in C^2(\mathbb{R},\mathbb{R})$ *satisfies with a certain* $\rho \in \left[2, \frac{N+2}{N-2}\right)$ *the condition*

$$|f''(s)| \le c(1 + |s|^{\rho-2}), \tag{12}$$

*and* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X^1 \times X^{\frac{1}{2}}$. *Then there exists* $\tau > 0$ *such that, for each* $\eta \in [0,1]$, *the problem* (6) *has a unique solution* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in C([0,\tau], X^1 \times X^{\frac{1}{2}})$. *Furthermore,*

$$\sup_{\eta\in[0,1]} \| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \|_{C([0,\tau], X^1\times X^{\frac{1}{2}})} < \infty \tag{13}$$

*and*

$$\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) - \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], 0) \|_{C([0,\tau], X^1\times X^{\frac{1}{2}})} \to 0 \quad \text{as} \quad \eta \to 0^+. \tag{14}$$

Concerning the asymptotic properties of the global weak solutions of (6) with $\eta = 0$ we will show that

THEOREM 1.3. *Under the assumptions of Theorem 1.1 there is a subclass* $\mathcal{LS}$ *of the set of global weak solutions of* (6) *with* $\eta = 0$ *possessing the following properties:*

(*i*) *(existence) for each* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H_0^1(\Omega)\times L^2(\Omega)$ *at least one* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in \mathcal{LS}$ *exists with* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$,

(*ii*) *(bounded dissipativeness) there is a bounded subset* $B_0$ *of* $H_0^1(\Omega)\times L^2(\Omega)$ *such that for any* $B$ *bounded in* $H_0^1(\Omega)\times L^2(\Omega)$, *each* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]$ *from the class* $\mathcal{LS}$ *with* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](0) \in B$ *enters* $B_0$ *in a certain time* $\tau_B \ge 0$ *and stays in* $B_0$ *for all* $t \ge \tau_B$,

(*iii*) *(attractor) there is a bounded closed subset* $\mathbf{A}$ *of* $H_0^1(\Omega)\times L^2(\Omega)$ *such that*

(*a*) *(compactness)* $\mathbf{A}$ *is compact in* $H^s_{\{I\}}(\Omega)\times H^{s-1}(\Omega)$ *for any* $s \in [0,1)$, (*b*) *(invariance)* $\{\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right](t):\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in\mathcal{LS},t\geq 0\}\subset\mathbf{A}$ *and for each* $\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]\in\mathbf{A}$ *and any* $t\geq 0$ *there is a certain* $\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right]\in\mathcal{LS}$ *with* $\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right](0)\in\mathbf{A}$ *such that* $\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right](t)=\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]$,

(*c*) *(attracting property)*

$$\sup_{\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\in\mathcal{LS},\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](0)\in B}\ \inf_{\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]\in\mathbf{A}}\|\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t)-\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]\|_{H^s_{\{I\}}(\Omega)\times H^{s-1}(\Omega)}\overset{t\to\infty}{\to}0 \tag{15}$$

*for any set* $B$ *bounded in* $H^1_0(\Omega)\times L^2(\Omega)$ *and each* $s\in[0,1)$.

Note that the unified semigroup approach involving parabolic approximation procedure essentially influences the resulting properties of the solutions, which in particular are shown to satisfy the variation of constants formula (9). We remark that, although it is true from [3] that a suitable notion of weak solution satisfies variation of constants formula, this usually requires some additional attention. For example, it remains unknown if the finite-energy solution constructed in [19] possesses such property.

On the other hand any global weak solution $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]$ resulting from Theorem 1.1 with the aid of approximating sequence of the form $\left[\begin{smallmatrix}u_{\eta_n}\\ v_{\eta_n}\end{smallmatrix}\right]=\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](\cdot,\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right],\eta_n)$ has finite energy property as it satisfies the energy inequality. Namely, we have for each $t\geq 0$

$$\mathcal{L}_0(\left[\begin{smallmatrix}u_{\eta_n}\\ v_{\eta_n}\end{smallmatrix}\right])=-a\int_0^t\|v_{\eta_n}(s)\|^2_X ds-2\eta_n\int_0^t\|v_{\eta_n}(s)\|^2_{X^{\frac14}}ds+\mathcal{L}_0(\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right])$$

and hence $\{\mathcal{L}_0(\left[\begin{smallmatrix}u_{\eta_n}\\ v_{\eta_n}\end{smallmatrix}\right])\}$ is a sequence of decreasing functions bounded for every $t\geq 0$. Consequently, there is a subsequence $\{n_k\}$ such that $\mathcal{L}_0(\left[\begin{smallmatrix}u_{\eta_{n_k}}\\ v_{\eta_{n_k}}\end{smallmatrix}\right])$ is convergent pointwise and by weak convergence property we have

$$\mathcal{L}_0(\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right])\leq\lim_{\eta_{n_k}\to 0}\mathcal{L}_0(\left[\begin{smallmatrix}u_{\eta_{n_k}}\\ v_{\eta_{n_k}}\end{smallmatrix}\right])\leq\mathcal{L}_0(\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]).$$

In fact, we can choose a subsequence $u_{\eta_{n_k}}$ convergent to $u$ in $L^s(\Omega)$ for any $s>1$ less but arbitrarily close to $\frac{2N}{N-2}$ and with the growth as in Theorem 1.1 we infer that

$$\int_\Omega\int_0^{u_{\eta_{n_k}}}f(s)dsdx\to\int_\Omega\int_0^u f(s)dsdx.$$

This leads to the conclusion that there exists a 'limit norm'

$$\lim_{\eta_{n_k}\to 0}\|\left[\begin{smallmatrix}u_{\eta_{n_k}}\\ v_{\eta_{n_k}}\end{smallmatrix}\right]\|_{H^1_0(\Omega)\times L^2(\Omega)}$$

of the approximating solutions; nonetheless, we can merely ensure that the above limit is bigger than the $H^1_0(\Omega)\times L^2(\Omega)-$norm of the corresponding limit solution.

Let us emphasize that we consider a nonconservative case $a > 0$ as we are interested in the stability properties of (1) in the large; the properties being described in Theorem 1.3 with the aid of the bi-space global attractor. As for the existential part of the consideration it just brings the construction of a suitable notion of solution and it can be applied as well in the conservative case $a = 0$.

The paper is organized as follows. In Section 2 we are establishing global well posedness of $(6)_{\eta>0}$. We show boundedness of the solutions of $(6)_{\eta>0}$ uniform for $\eta \in (0, 1]$ and prove Theorem 1.1. We are also briefly discussing local well posedness of (6) with $\eta = 0$ for $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ and complete the proof of Theorem 1.2. In Section 3 we are constructing a bounded absorbing set for $(6)_{\eta>0}$ with the *entrance time* independent of $\eta \in (0, 1]$ and of $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ in bounded subsets of $H^1_0(\Omega) \times L^2(\Omega)$ and we complete the proof of Theorem 1.3. In the closing Section 4 we are viewing some additional properties of the solutions resulting from Theorem 1.1, that come back to consideration of [4, 3, 17, 20].

**Acknowledgement:** This work has been carried out while the first author visited the Institute of Mathematics of the Silesian University in Katowice, Poland. The authors wish to thank the Institute for supporting this cooperation. Also, the first author wishes to thank the people at the Institute for their hospitality.

## 2. EXISTENCE RESULTS

Let $X^\alpha$, $\alpha \in \mathbb{R}$, denote the fractional power spaces associated to $A$. Then,

$$X^\alpha = [H^2(\Omega) \cap H^1_0(\Omega), L^2(\Omega)]_\alpha =: H^{2\alpha}_{\{I\}}(\Omega) \quad \text{and} \quad X^{-\alpha} = (H^{2\alpha}_{\{I\}}(\Omega))' \quad \text{for } \alpha \in [0, 1],$$

where $[\cdot, \cdot]_\alpha$ denotes the complex interpolation functor (see [18]). In particular $X := X^0 = L^2(\Omega)$, $X^{\frac{1}{2}} = H^1_0(\Omega)$, $X^{-\frac{1}{2}} = (H^1_0(\Omega))'$ and $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$.

Let $a > 0$ and $Y^1 = X^{\frac{1}{2}} \times X$ with the norm $\| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y^1} = \|\phi\|_{X^{\frac{1}{2}}} + \|\psi\|_X$ . Let also $Y^2 = X^1 \times X^{\frac{1}{2}}$. For $\eta \geq 0$ consider a family of the damped wave operators (7)

$$\mathcal{A}_\eta : D(A_\eta) \subset Y^1 \to Y^1$$

and note that

$$\mathcal{A}_\eta^{-1} : Y^1 \to Y^1, \quad \mathcal{A}_\eta^{-1} = \begin{bmatrix} 2\eta A^{-\frac{1}{2}} + aA^{-1} & A^{-1} \\ -I & 0 \end{bmatrix} \quad \text{for} \quad \eta \geq 0. \tag{16}$$

The completion of the normed space $(Y^1, \|\mathcal{A}_\eta^{-1} \cdot \|_{X^{\frac{1}{2}} \times X})$ is $Y^0 := Y = X \times X^{-\frac{1}{2}}$. Indeed, since there is a constant $d > 0$ (independent of $\eta \in [0, 1]$) such that

$$d^{-1} \|\mathcal{A}_\eta^{-1} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{X^{\frac{1}{2}} \times X} \leq \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{X \times X^{-\frac{1}{2}}} \leq d \|\mathcal{A}_\eta^{-1} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{X^{\frac{1}{2}} \times X}, \; \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in X^{\frac{1}{2}} \times X, \tag{17}$$

the completions of $(X^{\frac{1}{2}} \times X, \|\mathcal{A}_\eta^{-1} \cdot \|_{X^{\frac{1}{2}} \times X})$ and $(X^{\frac{1}{2}} \times X, \| \cdot \|_{X \times X^{-\frac{1}{2}}})$ coincide with equivalent norms (independent of $\eta \in [0, 1]$).

In what follows we are still denoting by $\mathcal{A}_\eta$ the closed extension to $Y$ of the operator $\begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}}+aI \end{bmatrix}$ with the domain $Y^1$.

For $\eta \geq 0$, $\mathcal{A}_\eta : Y^1 \subset Y \to Y$ is a maximal accretive operator (see [7] and references therein). Therefore it possesses bounded imaginary powers, generates a semigroup of contractions $\{e^{-\mathcal{A}_\eta t} : t \geq 0\}$ and its fractional power spaces are given by $Y^\alpha = X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}}$, $\alpha \geq 0$. If $\eta > 0$, then $\mathcal{A}_\eta$ is a sectorial operator.

With this set-up we will consider problem (5) in the form (6).

Let $F : Y^1 \to Y^\alpha$, $\alpha \geq 0$, be a locally Lipschitz continuous map. Recall that a mild solution of (6) on $[0,\tau]$ is a function $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in C([0,\tau], Y^1)$ which satisfies (9) for $t \in [0,\tau]$. We say that (6) is *locally well posed in* $Y^1$ if for any $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^1$ there is a unique mild solution $t \mapsto \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ of (6) defined on a maximal interval of existence $[0, \tau_{u_0,v_0})$ and depending continuously on the initial data $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$.

The following two propositions can be found in [7]-[9]. Their proofs are included here for completeness.

PROPOSITION 2.1. *The problem* (6) *with* $\eta > 0$ *is locally well posed in* $Y^1$ *whenever* $f$ *satisfies* (2) *for some* $1 < \rho < \frac{N+2}{N-2}$.

**Proof:** $F$ is Lipschitz continuous on bounded sets from $Y^1$ into $X^{\frac{\sigma}{2}} \times X^{\frac{\sigma-1}{2}} = Y^\sigma$ whenever $0 < \sigma \leq \tilde{\sigma}$ and $\tilde{\sigma} = \min\{1, \frac{N}{2} + 1 - \rho(\frac{N}{2} - 1)\}$. Indeed, if $B$ is a bounded subset of $Y^1$ and $\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \in B$, we have

$$\|F\left(\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}\right) - F\left(\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\right)\|_{Y^\sigma} \leq c_1 \|f(\phi_1) - f(\phi_2)\|_{X^{\frac{\sigma-1}{2}}} \leq c_2 \|f(\phi_1) - f(\phi_2)\|_{L^{\frac{2N}{N+(1-\sigma)2}}(\Omega)}$$
$$\leq c_3 \|\phi_1 - \phi_2\|_{X^{\frac{1}{2}}} \big(1 + \|\phi_1\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2-\sigma}}(\Omega)} + \|\phi_2\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2-\sigma}}(\Omega)}\big) \leq c_4 \|\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\|_{Y^1}.$$

The proof now follows from Theorem 3.3.3 in [14]. ∎

PROPOSITION 2.2. *Under the assumptions of Proposition 2.1 and* (3)

($i$) *the problem* (6) *is globally well posed in* $Y^1$,
($ii$) *positive orbits of bounded subsets of* $Y^1$ *are bounded in* $Y^1$ *uniformly for* $\eta \in (0,1]$.

**Proof:** The functional $\mathcal{L}_0 : Y^1 \to \mathbb{R}$ defined in (10) satisfies for $\mu \in (\mu_1, \lambda_1)$ the relation

$$\mathcal{L}_0(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) \geq \frac{\lambda_1 - \mu}{2\lambda_1} \|w_1\|^2_{X^{\frac{1}{2}}} + \frac{1}{2}\|w_2\|^2_X - c_\mu,$$

where (3) and the Poincaré inequality $\lambda_1 \|w_1\|^2_X \leq \|w_1\|^2_{X^{\frac{1}{2}}}$ were used to obtain that

$$\int_\Omega \int_0^{w_1} f(s)ds\, dx \leq \frac{\mu}{2}\|w_1\|^2_X + c_\mu.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

If $\eta > 0$ and $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ is a local solution of (6) through $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^1$, then

$$\frac{d}{dt}\mathcal{L}_0(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) = -\eta\|v\|^2_{X^{\frac{1}{4}}} - a\|v\|^2_X, \tag{18}$$

that is $\mathcal{L}_0$ decreases along $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$. Hence we have the estimate

$$\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \|_{Y^1} \leqslant c_{\mathcal{L}_0}\sqrt{1 + \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])}, \tag{19}$$

where $c_{\mathcal{L}_0}$ does not depend on $\eta > 0$. The proof now follows easily. ∎

The existence of mild solutions of (6) with $\eta = 0$ when $F : Y^1 \to Y^1$ is locally Lipschitz continuous is well known. However, in order for $F$ to take $Y^1$ into itself, $f$ should satisfy (2) with $1 \leq \rho \leq \frac{N}{N-2}$. Next we are proving the existence of the *global weak solutions* of (6) with $\eta = 0$ when (2) holds with $1 < \rho < \frac{N+2}{N-2}$. The following lemmas will be useful in this proof.

Lemma 2.1. *([6, Lemma 2.4]) If* $\mathrm{Re}\lambda > 0$ *then* $\lambda$ *is in the resolvent set of* $\mathcal{A}_\eta$ *for each* $\eta \geq 0$. *There is also a constant* $c_\lambda$, *independent of* $\eta \in [0,1]$ *and of* $m = 0, 1, 2$, *such that*

$$\|(\lambda + \mathcal{A}_0)^{-1} - (\lambda + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} \leqslant \eta c_\lambda\, d \quad \textit{for} \quad \eta \in (0,1], \; m = 0, 1, 2, \tag{20}$$

*where* $d$ *is given in* (17). *Furthermore, if* $J$ *is a compact subset of* $Y^m$, *then*

$$\sup_{\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in J} \sup_{t \in [0,\tau]} \|e^{-\mathcal{A}_\eta t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - e^{-\mathcal{A}_0 t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^m} \to 0 \quad as \quad \eta \to 0^+. \tag{21}$$

**Proof:** If $\mathcal{R}_\eta = \begin{bmatrix} 0 & 0 \\ 0 & \eta A^{\frac{1}{2}} \end{bmatrix}$, then $\|\mathcal{R}_\eta\|_{L(Y^{m+1}, Y^m)} \leqslant \eta$, $\eta \in [0,1]$. Consequently, we have

$$\lambda I + \mathcal{A}_\eta = (\lambda I + \mathcal{A}_0)(I + (\lambda I + \mathcal{A}_0)^{-1}\mathcal{R}_\eta) \tag{22}$$

for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ and

$$\begin{aligned} &\|(\lambda I + \mathcal{A}_0)^{-1} - (\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} = \|(\lambda I + \mathcal{A}_0)^{-1}\mathcal{R}_\eta(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} \\ &\leq \|(\lambda I + \mathcal{A}_0)^{-1}\|_{L(Y^m)}\|\mathcal{R}_\eta\|_{L(Y^{m+1}, Y^m)}\|(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m, Y^{m+1})}. \end{aligned}$$

From (17),

$$\|(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m, Y^{m+1})} \leq d\|\mathcal{A}_\eta(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} = d\|I - \lambda(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)}.$$

This proves (20), where

$$c_\lambda = \sup_{\eta \in [0,1]} \|(\lambda I + \mathcal{A}_0)^{-1}\|_{L(Y^m)}\|I - \lambda(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)}.$$

To prove (21) fix $\delta > 0$, $\tau > 0$. Choose $\epsilon = \frac{\delta}{4}$, $n \in \mathbb{N}$ and $\begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix}$, $1 \leq i \leq n$, such that $\mathcal{J} \subset \cup_{i=1}^n \mathcal{B}_\epsilon^i$, where $B_\epsilon^i$ is the ball of radius $\epsilon$ centered in $\begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix}$, $1 \leq i \leq n$ . From Trotter-Kato theorem, there exists $\eta_\delta = \eta(\delta) > 0$ such that

$$\|e^{-\mathcal{A}_\eta t} \begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix} - e^{-\mathcal{A}_0 t} \begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix} \|_{Y^m} < \frac{\delta}{2} \text{ for all } t \in [0,\tau],\ \eta \in (0,\eta_\delta),\ i = 1,\dots,n. \tag{23}$$

If $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{J}$ and $i$ is such that $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{B}_\epsilon^i$, we conclude that

$$\sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_\eta t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - e^{-\mathcal{A}_0 t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^m} \leqslant \sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_\eta t}(\begin{bmatrix} \phi \\ \psi \end{bmatrix} - \begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix})\|_{Y^m}$$
$$+ \sup_{t\in[0,\tau]} \|(e^{-\mathcal{A}_\eta t} - e^{-\mathcal{A}_0 t}) \begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix} \|_{Y^m} + \sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_0 t}(\begin{bmatrix} \phi_i \\ \psi_i \end{bmatrix} - \begin{bmatrix} \phi \\ \psi \end{bmatrix})\|_{Y^m} < 2\epsilon + \frac{\delta}{2} = \delta.$$

The proof is complete. ▮

Lemma 2.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies* (2) *with some $\rho \in (1, \frac{N+2}{N-2})$ and $c > 0$. Then for $\theta = \frac{(\rho-1)(N-2)}{4}$, certain $\bar{c} > 0$ and all $u_1, u_1 \in X^{\frac{1}{2}}$ we have*

$$\|f(u_1) - f(u_2)\|_{X^{-\frac{1}{2}}} \leq \bar{c}\, \|u_1 - u_2\|_{X^{-\frac{1}{2}}}^{1-\theta} \|u_1 - u_2\|_{X^{\frac{1}{2}}}^{\theta} \left(1 + \|u_1\|_{X^{\frac{1}{2}}}^{\rho-1} + \|u_2\|_{X^{\frac{1}{2}}}^{\rho-1}\right). \tag{24}$$

**Proof:** Using Sobolev embedding and Hölder inequality we obtain

$$\begin{aligned}
&\|f(u_1) - f(u_2)\|_{X^{-\frac{1}{2}}} \leq \|f(u_1) - f(u_2)\|_{L^{\frac{2N}{N+2}}(\Omega)} \\
&\leq \tilde{c}\, \|u_1 - u_2\|_{L^{\frac{2Nr}{N+2}}(\Omega)} \left(1 + \|u_1\|^{\rho-1}_{L^{\frac{2Nr'(\rho-1)}{N+2}}(\Omega)} + \|u_2\|^{\rho-1}_{L^{\frac{2Nr'(\rho-1)}{N+2}}(\Omega)}\right) \\
&\leq \tilde{c}\, \|u_1 - u_2\|_{X^{\frac{s}{2}}} \left(1 + \|u_1\|_{X^{\frac{1}{2}}}^{\rho-1} + \|u_2\|_{X^{\frac{1}{2}}}^{\rho-1}\right),
\end{aligned}$$

where $r = \frac{N+2}{N+2-(\rho-1)(N-2)}$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $s = \frac{(\rho-1)(N-2)-2}{2}$. Since the *moment inequality* implies that

$$\|u_1 - u_2\|_{X^{\frac{s}{2}}} \leq \hat{c} \|u_1 - u_2\|_{X^{\frac{1}{2}}}^{\frac{s+1}{2}} \|u_1 - u_2\|_{X^{-\frac{1}{2}}}^{\frac{1-s}{2}},$$

the result now follows easily. ▮

Lemma 2.3. *Suppose $\mathcal{F}$ is a Banach space and $\mathcal{E}$ is a compact metric space. A necessary and sufficient condition for $\mathcal{H} \subset C(\mathcal{E}, \mathcal{F})$ to be relatively compact in $C(\mathcal{E}, \mathcal{F})$ is that $\mathcal{H}$ is equicontinuous and, for every $e \in \mathcal{E}$, $\{h(e), h \in \mathcal{H}\}$ is relatively compact in $\mathcal{F}$.*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

**Proof:** See [11, §7.5] (also [12]). ∎

**Proof of Theorem 1.1.** Let $B$ be a bounded subset of $Y^1$. Then, due to Proposition 2.2, for $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$, $\eta \in (0,1]$ and $t \geq 0$ there is a constant $C_B > 0$ such that

$$\|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)\|_{Y^1} \leq C_B. \tag{25}$$

From this and from (5), there is $C'_B > 0$ such that, for $t \geq 0$, $\eta \in (0,1]$ and $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$

$$\|\left[\begin{smallmatrix} u_t \\ v_t \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)\|_{Y} \leq C'_B. \tag{26}$$

It follows from the Arzela-Ascoli theorem (see Lemma 2.3) that for each sequence $\{\eta_n\}_{n=1}^{\infty}$ convergent to 0 and $\left\{ \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right] \right\}_{n=1}^{\infty} \subset B$ convergent in $Y$ to $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, there is a subsequence (which we denote the same) and a function $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in C([0,\infty), Y)$ such that (11) holds. Furthermore, as a consequence of (25) and weak convergence

$$\sup_{t\in[0,\infty)} \left\| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](t) \right\|_{Y^1} \leq C_B. \tag{27}$$

It remains to prove that $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]$ satisfies (9) with $\eta = 0$. Since

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) = e^{-\mathcal{A}_{\eta_n} t} \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_{\eta_n}(t-s)} F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n)) ds, \tag{28}$$

then using Lemma 2.1 it is enough to prove that

$$\int_0^t e^{-\mathcal{A}_{\eta_n}(t-s)} F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n)) ds \xrightarrow{Y} \int_0^t e^{-\mathcal{A}_0(t-s)} F(\left[\begin{smallmatrix} \phi(s) \\ \psi(s) \end{smallmatrix}\right]) ds$$

uniformly for $t \in [0,\tau]$. For such convergence, using again Lemma 2.1, it is sufficient that

$$F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n)) \xrightarrow{Y} F(\left[\begin{smallmatrix} \phi(s) \\ \psi(s) \end{smallmatrix}\right])$$

uniformly for $s \in [0,\tau]$. This is obtained in the following manner. If $r > 0$ is such that

$$\sup_{n\in\mathbb{N}} \sup_{s\in[0,\tau]} \max \left\{ \left\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) \right\|_{Y^1}, \left\| \left[\begin{smallmatrix} \phi(s) \\ \psi(s) \end{smallmatrix}\right] \right\|_{Y^1} \right\} \leq r,$$

then with the aid of Lemma 2.2 for $\theta = \frac{(\rho-1)(N-2)}{4}$ we get

$$\begin{aligned}
&\left\| F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n)) - F(\left[\begin{smallmatrix} \phi(s) \\ \psi(s) \end{smallmatrix}\right]) \right\|_Y = \| f(u(s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) - f(\phi(s)) \|_{X^{-\frac{1}{2}}} \\
&\leq \bar{c} \left\| u(s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) - \phi(s) \right\|_{X^{-\frac{1}{2}}}^{1-\theta} \left\| u(s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) - \phi(s) \right\|_{X^{\frac{1}{2}}}^{\theta} \left( 1 + \left\| u(s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) \right\|_{X^{\frac{1}{2}}}^{\rho-1} + \|\phi(s)\|_{X^{\frac{1}{2}}}^{\rho-1} \right) \\
&\leq \bar{c} \left\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n) - \left[\begin{smallmatrix} \phi(s) \\ \psi(s) \end{smallmatrix}\right] \right\|_Y^{1-\theta} (2r)^{\theta} \left( 1 + 2r^{\rho-1} \right).
\end{aligned}$$

Note that $\theta \in (0,1)$ if and only if $\rho \in (1, \frac{N+2}{N-2})$. In the light of (11), the result is proved. ∎

**Proof of Theorem 1.2.** Using Sobolev embedding and Hölder inequality we obtain

$$\begin{aligned}
&\|f(u_1)-f(u_2)\|_{X^{\frac12}} \le \|(f'(u_1)-f'(u_2))\nabla u_1\|_X + \|f'(u_2)\nabla(u_1-u_2)\|_X \\
&\le \|f'(u_1)-f'(u_2)\|_{L^N(\Omega)}\|\nabla u_1\|_{L^{\frac{2N}{N-2}}(\Omega)} + \|f'(u_2)\|_{L^N(\Omega)}\|\nabla(u_1-u_2)\|_{L^{\frac{2N}{N-2}}(\Omega)} \\
&\le \hat{c}\|u_1-u_2\|_{X^1}(1+\|u_1\|_{X^1}^{\rho-2}+\|u_2\|_{X^1}^{\rho-2})\|u_1\|_{X^1} + \hat{c}\|u_1-u_2\|_{X^1}(1+\|u_2\|_{X^1}^{\rho-1}).
\end{aligned}$$

Consequently, for some $\bar{c}>0$ and all $u_1, u_1 \in X^1$ we have

$$\|f(u_1)-f(u_2)\|_{X^{\frac12}} \le \bar{c}\,\|u_1-u_2\|_{X^1}\left(1+\|u_1\|_{X^1}^{\rho-1}+\|u_2\|_{X^1}^{\rho-1}\right), \tag{29}$$

which ensures that $F: Y^2 \to Y^2$ is Lipschitz continuous in bounded subsets of $Y^2$.

Note that the mild solutions of (6) are fixed points of a map constructed in accordance to (9). Since the Lipschitz constant of $F$ is independent of $\eta \ge 0$ and $\|e^{-\mathcal{A}_\eta t}\|_{L(Y^2)} \le 1$ for all $\eta \ge 0$, the time of existence of these solutions can be estimated independently of $\eta$. In particular (13) is proved.

It can be seen from Lemma 2.1 that $e^{-\mathcal{A}_\eta t}$ converges to $e^{-\mathcal{A}_0 t}$ in the strong topology of $L(Y^2)$ uniformly in bounded time intervals and on compact subsets of $Y^2$. Therefore, to conclude the proof, we need to ensure that the set $K = \{F([\begin{smallmatrix}u\\v\end{smallmatrix}](t,[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}],0)) : t \in [0,\tau]\}$ is compact in $Y^2$, which comes from continuity of the map $[0,\tau] \ni t \mapsto F([\begin{smallmatrix}u\\v\end{smallmatrix}](t,[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}],0)) \in Y^2$.

The proof of (14) now follows from Gronwall's Inequality. ∎

*Remark 2. 1.* If $[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}] \in Y^1$ and $\rho \le \frac{N}{N-2}$ the weak solutions of (6) with $\eta = 0$ belong to $C([0,\tau], Y^1)$, for with this growth $F: Y^1 \to Y^1$ is Lipschitz continuous in bounded sets. If, in addition to the assumptions of Theorem 1.2, $\eta > 0$ and $f$ satisfies (3), then (following [8]) the solutions $[\begin{smallmatrix}u\\v\end{smallmatrix}](\cdot,[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}],\eta)$, $[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}] \in Y^2$, $\eta > 0$, obtained in Theorem 1.2 are globally defined. However, one can hardly assure that they are bounded in $Y^2$ uniformly for $\eta \in (0,1]$ so that the global weak solutions of (6) with $\eta = 0$ starting in $[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}] \in Y^2$ may have values in $Y^2$ only for $t \in [0,\tau]$ with a certain $\tau > 0$. Note that, although the problem (6) with $\eta = 0$ is locally well posed in $Y^2$, its global well posedness in $Y^2$ is unknown.

## 3. UNIFORMLY ABSORBING SET AND THE PROOF OF THEOREM 1.3

In this section, following the procedure adapted from [8], we prove the existence of a bounded absorbing set for (6) with the *entrance time* independent of $\eta \in (0,1]$ and initial data in bounded subsets of $Y^1$. We then complete the proof of Theorem 1.3.

We define the functional $\mathcal{L}_\delta : Y^1 \to \mathbb{R}$;

$$\mathcal{L}_\delta([\begin{smallmatrix}w_1\\w_2\end{smallmatrix}]) := \mathcal{L}_0([\begin{smallmatrix}w_1\\w_2\end{smallmatrix}]) + \delta\int_\Omega w_1 w_2 dx \quad \text{for } [\begin{smallmatrix}w_1\\w_2\end{smallmatrix}] \in Y^1,\ \delta \ge 0, \tag{30}$$

where $\mathcal{L}_0 : Y^1 \to \mathbb{R}$ was given in (10). For $\mu \in (\mu_1, \lambda_1)$, choosing $\delta \le \min\{\frac{\lambda_1-\mu}{2}, \frac{1}{2}\}$, we have

$$\mathcal{L}_\delta([\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}]) \ge \frac{\lambda_1 - \mu}{4\lambda_1}\|w_1\|^2_{X^{\frac{1}{2}}} + \frac{1}{4}\|w_2\|^2_X - c_\mu, \quad [\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}] \in Y^1. \tag{31}$$

Writing $f(s) = f_0(s) + f(0)$ (see [2, 1, 8]) with

$$|f_0(s)| \le c(|s|^\rho + |s|), \; s \in \mathbb{R},$$

for some $c > 0$ we can find a constant $\bar{c} > 1$ such that

$$-\int_\Omega \int_0^{w_1} f_0(s) ds\, dx \le \bar{c}\|w_1\|^2_{X^{\frac{1}{2}}}(1 + \|w_1\|^{\rho-1}_{X^{\frac{1}{2}}})$$

and

$$-\bar{d}\int_\Omega \int_0^{w_1} f_0(s) ds\, dx \le \|w_1\|^2_{X^{\frac{1}{2}}} \tag{32}$$

for $\|w_1\|_{X^{\frac{1}{2}}} \le r$ and $\bar{d} = \frac{1}{\bar{c}(1+r^{\rho-1})}$ (thus $\bar{d} < 1$).

These properties of $\mathcal{L}_\delta$ and $f$ lead to the following result.

THEOREM 3.1. *Suppose that $f$ satisfies (2) with some $\rho \in (1, \frac{N+2}{N-2})$ and (3) holds. Then there is a bounded set $B_0 \subset Y^1$ with the property that for any $B$ bounded in $Y^1$ there is a time $\tau_B > 0$ such that*

$$[\begin{smallmatrix} u \\ v \end{smallmatrix}](t, [\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}], \eta) \in B_0 \;\; \text{for all} \;\; t \ge t_B, \; [\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}] \in B, \; \eta \in (0, 1].$$

**Proof:** Let $[\begin{smallmatrix} u \\ v \end{smallmatrix}](t, [\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}], \eta)$ ($[\begin{smallmatrix} u \\ v \end{smallmatrix}]$ for short) be the solution of (6) with $\eta > 0$. Then

$$\frac{d}{dt}\int_\Omega u\, v dx = \|v\|^2_X - a\int_\Omega u\, v dx - \eta\int_\Omega A^{\frac{1}{2}} u\, v dx - \|u\|^2_{X^{\frac{1}{2}}} + \int_\Omega u\, f(u) dx. \tag{33}$$

Fix $\mu \in (\mu_1, \lambda_1)$. Using Young's inequality, for any $\epsilon > 0$, we have

$$\frac{d}{dt}\int_\Omega u\, v dx \le (1 + \frac{a}{2\epsilon} + \frac{\eta}{2\epsilon})\|v\|^2_X - (1 - \frac{\mu}{\lambda_1} - \frac{\epsilon a}{2\lambda_1} - \frac{\epsilon\eta}{2})\|u\|^2_{X^{\frac{1}{2}}} + \bar{c}_\mu, \tag{34}$$

where we have used the dissipativeness condition (3) to get

$$\int_\Omega u\, f(u) dx \le \mu\|u\|^2_X + \bar{c}_\mu$$

and the Poincaré inequality $\lambda_1\|u\|^2_X \le \|u\|^2_{X^{\frac{1}{2}}}$. From (18) and (34) we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_\delta([\begin{smallmatrix} u \\ v \end{smallmatrix}]) &\le -(a - \delta - \frac{\delta a}{2\epsilon} - \frac{\delta\eta}{2\epsilon})\|v\|^2_X - \delta(\frac{\lambda_1 - \mu}{\lambda_1} - \frac{a\epsilon}{2\lambda_1} - \frac{\eta\epsilon}{2})\|u\|^2_{X^{\frac{1}{2}}} + \delta\bar{c}_\mu \\ &\le -\frac{a}{2}\|v\|^2_X - \delta\frac{\lambda_1 - \mu}{2\lambda_1}\|u\|^2_{X^{\frac{1}{2}}} + \delta\bar{c}_\mu \le -\omega\|[\begin{smallmatrix} u \\ v \end{smallmatrix}]\|^2_{Y^1} + \delta\bar{c}_\mu, \end{aligned} \tag{35}$$

where $\omega = \frac{1}{2}\min\{a, \delta\frac{\lambda_1-\mu}{\lambda_1}\}$, $\epsilon$ was chosen so that $\frac{\lambda_1-\mu}{2\lambda_1} \geq \frac{a\epsilon}{2\lambda_1} + \frac{\eta\epsilon}{2}$ and then $\delta$ such that $\frac{a}{2} \geq \delta + \frac{\delta a}{2\epsilon} + \frac{\delta\eta}{2\epsilon}$. Hence, for $\| \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \|_{Y^1} \leq r$, using (19), (32) and (35), we have

$$\frac{d}{dt}\mathcal{L}_\delta(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) \leq -\frac{\bar{d}\omega}{4}\mathcal{L}_\delta(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) + \bar{C}_\mu. \tag{36}$$

From (31) and (36) the result follows easily. ▮

**Proof of Theorem 1.3.** Let $B_0$ be a uniformly absorbing set from Theorem 3.1. We define the set $\mathbf{A}$ consisting of the (weak in $H^1_0(\Omega) \times L^2(\Omega)$) limit points of the solutions of (6) with $\eta \searrow 0$

$$\mathbf{A} := \{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] : \exists_{t_n \nearrow \infty}\ \exists_{\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0}\ \exists_{\eta_n \searrow 0}\ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) \overset{\mathrm{w}-H^1_0(\Omega)\times L^2(\Omega)}{\rightharpoonup} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\}. \tag{37}$$

We also define a following *subclass* $\mathcal{LS}$ of the set of global weak solutions of (6) with $\eta = 0$.

Definition 3.1. $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{LS}$ if and only if one of the following conditions holds

- $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) \in \mathbf{A}$ and $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ is a global weak solution of (6) with $\eta = 0$ being (uniform for $t$ in compact subsets of $[0,\infty)$) limit in $Y$ of a sequence of solutions of (6) of the form $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n), \eta_n)$, where $t_n \nearrow \infty$, $\eta_n \searrow 0$ and $\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0$,
- $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) \in (H^1_0(\Omega) \times L^2(\Omega)) \setminus \mathbf{A}$ and $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ is a global weak solution of (6) with $\eta = 0$ being (uniform for $t$ in compact subsets of $[0,\infty)$) limit in $Y$ of a sequence of solutions of (6) of the form $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(\cdot, \begin{bmatrix} \phi \\ \psi \end{bmatrix}(0), \eta_n\right)$, where $\eta_n \searrow 0$.

The following lemma is a consequence of Theorems 1.1 and 3.1.

Lemma 3.1. *For each* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$ *there is (at least one)* $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{LS}$ *with* $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$. *Furthermore, for any $B$ bounded in $Y^1$, each weak solution* $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ *of (6) with $\eta = 0$ from the class $\mathcal{LS}$ with* $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) \in B$ *enters $B_0$ in a certain time $\tau_B \geq 0$ and stays in $B_0$ for all $t \geq \tau_B$.*

If $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\} \subset \mathbf{A}$, then there exist sequences $t_n \to \infty$, $\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0$, $\eta_n \searrow 0$ such that $\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) - \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \|_Y \to 0$. Since, by Theorem 3.1, almost all elements of the sequence $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n)\}$ are in $B_0$ and $Y^1$ is compactly embedded in $Y$, there exist subsequences $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \eta_{n_k})\}$ and $\{\left[\begin{smallmatrix} w_{n_k} \\ z_{n_k} \end{smallmatrix}\right]\}$ both convergent in $Y$ to a certain $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in Y$. Since $B_0$ is weakly sequentially pre-compact in $Y^1$, a subsequence of $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \eta_{n_k})\}$ converges to $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$ weakly in $Y^1$; i.e. $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathbf{A}$. This proves that $\mathbf{A}$ is compact in $Y$ and consequently in $Y^s$ for any $s \in [0,1)$, since $\mathbf{A}$ is bounded in $Y^1$. Closedness of $\mathbf{A}$ in $Y^1$ is straightforward.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Suppose now that (15) fails and choose $t_n \nearrow \infty$, $\{\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}\} \subset \mathcal{LS}$ with $\{\begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}(0)\} \subset B$ such that, for some $s \in [0,1)$,

$$\inf_{\begin{bmatrix} w \\ z \end{bmatrix} \in \mathbf{A}} \| \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}(t_n) - \begin{bmatrix} w \\ z \end{bmatrix} \|_{Y^s} > \epsilon, \tag{38}$$

where $B$ is a certain bounded subset of $Y^1$ and $\epsilon$ is some positive number. By Definition 3.1 there are sequences $\{\begin{bmatrix} u_n \\ v_n \end{bmatrix}\}$ bounded in $Y^1$ and $\eta_n \searrow 0$ such that

$$\sup_{t \in [0,t_n]} \| \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}(t) - \begin{bmatrix} u \\ v \end{bmatrix}(t, \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \eta_n) \|_Y \leq \frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}. \tag{39}$$

From Theorem 3.1 we know that, for some $t_{B \cup B_0} > 0$, $\begin{bmatrix} u \\ v \end{bmatrix}(t, \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \eta_n) \in B_0$ for $t \geq t_{B \cup B_0}$. Therefore, we can choose a subsequence $\{\begin{bmatrix} u \\ v \end{bmatrix}(t_{n_k}, \begin{bmatrix} u_{n_k} \\ v_{n_k} \end{bmatrix}, \eta_{n_k})\}$ convergent weakly in $Y^1$ to a certain $\begin{bmatrix} \tilde{w} \\ \tilde{z} \end{bmatrix} \in Y^1$. Since $\begin{bmatrix} u \\ v \end{bmatrix}(t_{n_k}, \begin{bmatrix} u_{n_k} \\ v_{n_k} \end{bmatrix}, \eta_{n_k}) = \begin{bmatrix} u \\ v \end{bmatrix}\big(t_{n_k} - t_B, \begin{bmatrix} u \\ v \end{bmatrix}(t_B, \begin{bmatrix} u_{n_k} \\ v_{n_k} \end{bmatrix}, \eta_{n_k}), \eta_{n_k}\big)$ we have $\begin{bmatrix} \tilde{w} \\ \tilde{z} \end{bmatrix} \in \mathbf{A}$, whereas from Lemma 3.1 we infer that $\{\begin{bmatrix} \phi_{n_k} \\ \psi_{n_k} \end{bmatrix}(t_{n_k})\} \subset B_0$. Therefore, via (39), after choosing subsequences $\begin{bmatrix} u \\ v \end{bmatrix}(t_{n_{k_l}}, \begin{bmatrix} u_{n_{k_l}} \\ v_{n_{k_l}} \end{bmatrix}, \eta_{n_{k_l}})$ convergent in $Y$ and $\begin{bmatrix} \phi_{n_{k_l}} \\ \psi_{n_{k_l}} \end{bmatrix}(t_{n_{k_l}})$ convergent in $Y^s$, we obtain that $\begin{bmatrix} \phi_{n_{k_l}} \\ \psi_{n_{k_l}} \end{bmatrix}(t_{n_{k_l}}) \stackrel{Y^s}{\to} \begin{bmatrix} \tilde{w} \\ \tilde{z} \end{bmatrix}$, which contradicts (38).

For the proof of the invariance property choose $\begin{bmatrix} w \\ z \end{bmatrix} \in \mathbf{A}$ and $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{LS}$ with $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) = \begin{bmatrix} w \\ z \end{bmatrix}$. From Definition 3.1 we infer that

$$\begin{cases} \text{there is a sequence of the solutions of (6)} \\ \text{of the form } \begin{bmatrix} u \\ v \end{bmatrix}(t, \begin{bmatrix} u \\ v \end{bmatrix}(t_n, \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \eta_n), \eta_n), \text{ where} \\ t_n \nearrow \infty,\ \eta_n \searrow 0 \text{ and } \{\begin{bmatrix} u_n \\ v_n \end{bmatrix}\} \subset B_0, \text{ such that} \\ \begin{bmatrix} u \\ v \end{bmatrix}(t + t_n, \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \eta_n) \stackrel{L^2(\Omega) \times H^{-1}(\Omega)}{\longrightarrow} \begin{bmatrix} \phi \\ \psi \end{bmatrix}(t) \\ \text{uniformly for } t \text{ in compact subsets of } [0,\infty). \end{cases} \tag{40}$$

Note that for each fixed $t \geq 0$ almost all elements of $\{\begin{bmatrix} u \\ v \end{bmatrix}(t + t_n, \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \eta_n)\}$ are in $B_0$, so that for each fixed $t \geq 0$ the element $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(t)$ can be viewed as a weak limit in $Y^1$ of a certain subsequence of $\{\begin{bmatrix} u \\ v \end{bmatrix}(t + t_n, \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \eta_n)\}$. Recalling (37) we thus conclude that $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(t) \in \mathbf{A}$.

As for the negative invariance of $\mathbf{A}$ we remark (coming back to (40)) that, whenever $s > 0$ is fixed, a suitable subsequence $\{n_k\}$ can be chosen for which

$$\begin{aligned} &\begin{bmatrix} u \\ v \end{bmatrix}\big(t_{n_k}, \begin{bmatrix} u_{n_k} \\ v_{n_k} \end{bmatrix}, \eta_{n_k}\big) \longrightarrow \begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) \text{ strongly in } Y \text{ and weakly in } Y^1, \\ &\begin{bmatrix} u \\ v \end{bmatrix}\big(t_{n_k} - s, \begin{bmatrix} u_{n_k} \\ v_{n_k} \end{bmatrix}, \eta_{n_k}\big) \longrightarrow \begin{bmatrix} \tilde{w} \\ \tilde{z} \end{bmatrix} \text{ strongly in } Y \text{ and weakly in } Y^1. \end{aligned} \tag{41}$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Recalling (37) we have $\left[\begin{smallmatrix}\tilde{w}\\ \tilde{z}\end{smallmatrix}\right] \in \mathbf{A}$. We also infer from Theorem 1.1 and Definition 3.1 that there is a certain $\begin{bmatrix}\tilde{\phi}\\ \tilde{\psi}\end{bmatrix} \in \mathcal{LS}$ with $\begin{bmatrix}\tilde{\phi}\\ \tilde{\psi}\end{bmatrix}(0) = \left[\begin{smallmatrix}\tilde{w}\\ \tilde{z}\end{smallmatrix}\right]$ and a suitable subsequence $\{n_{k_l}\}$ for which

$$\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(t, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_{k_l}} - s, \begin{bmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{bmatrix}, \eta_{n_l}), \eta_{n_k}\Big) \overset{Y}{\longrightarrow} \begin{bmatrix}\tilde{\phi}\\ \tilde{\psi}\end{bmatrix}(t) \ \text{ for each } t \geq 0. \tag{42}$$

Evidently $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(s, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_{k_l}} - s, \begin{bmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{bmatrix}, \eta_{n_l}), \eta_{n_k}\Big) = \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(t_{n_{k_l}}, \begin{bmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{bmatrix}, \eta_{n_l}\Big) \overset{Y}{\longrightarrow} \begin{bmatrix}\tilde{\phi}\\ \tilde{\psi}\end{bmatrix}(s)$ and hence we obtain from (41) that $\begin{bmatrix}\tilde{\phi}\\ \tilde{\psi}\end{bmatrix}(s) = \begin{bmatrix}\phi\\ \psi\end{bmatrix}(0)$. Theorem 1.3 is thus proved. ▌

## 4. OTHER PROPERTIES OF WEAK SOLUTIONS

In this section we view the global weak solutions of (6) with $\eta = 0$ from another perspective.

Recall that $A$ (resp. $A^{\frac{1}{2}}$) is an isomorphism from $X^{\frac{1}{2}}$ onto $X^{-\frac{1}{2}}$ (resp. from $X$ onto $X^{-\frac{1}{2}}$) and denote by $\langle \cdot, \cdot \rangle_H$ the scalar product in a Hilbert space $H$. Suppose that the assumptions of Theorem 1.1 are satisfied and let $\left[\begin{smallmatrix}u^{\eta_n}\\ v^{\eta_n}\end{smallmatrix}\right] = \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(\cdot, \begin{bmatrix}u_0^n\\ v_0^n\end{bmatrix}, \eta_n\Big)$ be the solution of (6) with $\eta_n \in (0,1]$, $\begin{bmatrix}u_0^n\\ v_0^n\end{bmatrix} \in B$; $B$ being a bounded subset of $Y^1$. By (25)-(26), we have that

$$\begin{gathered}\{u^{\eta_n}\} \ \text{ is bounded in } L^\infty((0,\infty), X^{\frac{1}{2}}),\\ \{u_t^{\eta_n}\} \ \text{ is bounded in } L^\infty((0,\infty), X^0),\ \{u_{tt}^{\eta_n}\} \ \text{ is bounded in } L^\infty((0,\infty), X^{-\frac{1}{2}}).\end{gathered} \tag{43}$$

Fix arbitrary $\tau > 0$. If $\eta_n \to 0^+$ and $\left[\begin{smallmatrix}u_{0n}\\ v_{0n}\end{smallmatrix}\right] \overset{Y}{\to} \left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]$, then there is a subsequence, which we denote the same, and elements $u, v, w$ of an appropriate Banach spaces (as below) such that for $p \in (1,\infty)$ and suitably small $\epsilon > 0$ (see [17, Chapter 1, Theorem 5.1])

$$u^{\eta_n} \overset{L^p((0,\tau),X^{\frac{1}{2}-\epsilon})}{\longrightarrow} u, \qquad u_t^{\eta_n} \overset{L^p((0,\tau),X^{-\epsilon})}{\longrightarrow} v.$$

Also, by the property of weak convergence,

$$u_t^{\eta_n} \overset{\mathrm{w}-L^2((0,\tau),X^0)}{\rightharpoonup} v, \quad u^{\eta_n} \overset{\mathrm{w}-L^2((0,\tau),X^{\frac{1}{2}})}{\rightharpoonup} u, \quad u_{tt}^{\eta_n} \overset{\mathrm{w}-L^2((0,\tau),X^{-\frac{1}{2}})}{\rightharpoonup} w.$$

We remark that there exist $X^{-\frac{1}{2}}$-valued distribution derivatives $u_t = v$, $u_{tt} = w$ and

$$\begin{gathered}u \in C([0,\tau], X^{\frac{1}{2}-\epsilon}) \cap B((0,\tau), X^{\frac{1}{2}}),\\ u_t \in C([0,\tau], X^{-\frac{1}{2}}) \cap B((0,\tau), X^0), \quad u_{tt} \in B((0,\tau), X^{-\frac{1}{2}}),\end{gathered} \tag{44}$$

where $B((0,\tau), Z)$ denotes a space of bounded functions from $(0,\tau)$ into $Z$. These properties and properties of $A$ and $f$ allow us to show that for each $t > 0$, $\phi \in X$

$$\langle A^{-\frac{1}{2}}[f(u^{\eta_n}) - f(u)], \phi\rangle_X \to 0, \quad \langle A^{-\frac{1}{2}}[\eta_n A^{\frac{1}{2}} u_t^{\eta_n}], \phi\rangle_X = \eta_n \langle u_t^{\eta_n}, \phi\rangle_X \to 0,$$

$$\langle A^{-\frac{1}{2}}[au_t^{\eta_n} - au_t], \phi\rangle_X \to 0, \quad \langle A^{-\frac{1}{2}}[Au^{\eta_n} - Au], \phi\rangle_X \to 0, \quad \langle A^{-\frac{1}{2}}[u_{tt}^{\eta_n} - u_{tt}], \phi\rangle_X \to 0,$$

as $\eta_n \to 0^+$. We can thus pass to the limit in the equation

$$\langle A^{-\frac{1}{2}}[u_{tt}^{\eta_n} + \eta_n A^{\frac{1}{2}} u_t^{\eta_n} + au_t^{\eta_n} + Au^{\eta_n}], \phi\rangle_X = \langle A^{-\frac{1}{2}}[f(u^{\eta_n})], \phi\rangle_X, \quad \phi \in X,$$

and obtain

$$\langle A^{-\frac{1}{2}}[u_{tt} + au_t + Au], \phi\rangle_X = \langle A^{-\frac{1}{2}} f(u), \phi\rangle_X, \quad \phi \in X.$$

In particular, $u$ fulfils

$$\langle u_{tt} + au_t + Au, \psi\rangle_{X^{-\frac{1}{2}}} = \langle f(u), \psi\rangle_{X^{-\frac{1}{2}}} \quad \text{for every } \psi \in X^{-\frac{1}{2}},$$

or equivalently, $u$ fulfils $u_{tt} + au_t + Au = f(u)$ in $X^{-\frac{1}{2}}$ for a. e. $t > 0$, or

$$\langle u_{tt} + au_t + Au, \chi\rangle_{X^{-\frac{1}{2}}, X^{\frac{1}{2}}} = \langle f(u), \chi\rangle_{X^{-\frac{1}{2}}, X^{\frac{1}{2}}} \quad \text{for each } \chi \in X^{\frac{1}{2}}.$$

With our growth we have justified that $u_t \in X$ and $u_{tt} + Au - f(u) = -au_t$ in $X^{-\frac{1}{2}}$ for all $t > 0$. Therefore we conclude that

$$u_{tt} + Au - f(u) = -au_t \quad \text{in } X,\ t > 0,$$

and thus

$$\langle u_{tt} + Au - f(u), u_t\rangle_X = -a\|u_t\|_X^2,\ t > 0.$$

Nevertheless, it remains unknown whether the energy equation (as in [4, (E)])

$$\mathcal{L}_0([\begin{smallmatrix} u \\ u_t \end{smallmatrix}]) + a \int_0^t \|u_t(s)\|_X^2 ds = \mathcal{L}_0([\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}]), \quad t \geq 0,$$

holds for the global weak solutions discussed above.

## REFERENCES


1. J. M. Arietta, A. N. Carvalho and J. K. Hale, A damped hyperbolic equation with critical exponent, *Commun. in Partial Differential Equations*, **17**, (1992), 841–866.
2. A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland Publishing Co., Amsterdam, 1992.
3. J. M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, *Proc. Amer. Math. Soc.*, **63**, (1977), 370–373.
4. J. M. Ball, Global attractors for damped semilinear wave equations, *Discrete Contin. Dyn. Syst.*, **10**, (2004), 31–52.
5. J. M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, *J. Nonlinear Sci.*, **7**, (1997), 475–502. Erratum: Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, *J. Nonlinear Sci.*, **8**, (1998), 233.


Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

6. S. M. Bruschi, A. N. Carvalho and J. W. Cholewa, Tomasz Dlotko, Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations, *J. Dynam. Differential Equations*, **18**, (2006), 767–814.
7. A. N. Carvalho and J. W. Cholewa, Local well posedness for strongly damped wave equations with critical nonlinearities, *Bull. Austral. Math. Soc.*, **66**, (2002), 443–463.
8. A. N. Carvalho and J. W. Cholewa, Attractors for strongly damped wave equations with critical nonlinearities, *Pacific J. Math.*, **207**, (2002), 287–310.
9. A. N. Carvalho and J. W. Cholewa, Continuation and asymptotics of solutions to semilinear parabolic equations with critical nonlinearities, *J. Math. Anal. Appl.*, **310**, (2005), 557–578.
10. A. N. Carvalho, J. W. Cholewa, Strongly damped wave equations in $W_0^{1,p}(\Omega)\times L^p(\Omega)$, *Discrete Contin. Dyn. Syst.*, to appear.
11. J. Dieudonné, *Éléments d'analyse. Tome I: Fondements de l'analyse moderne*, Gauthier-Villars, Paris 1968.
12. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1978.
13. J. K. Hale and G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, *J. Differential Equations*, **73**, (1988), 197–214.
14. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.
15. L. Kapitanski, Minimal compact global attractor for a damped semilinear wave equation, *Comm. Partial Differential Equations*, **20**, (1995), 1303–1323.
16. L. Kapitanski, Weak and yet weaker solutions of semilinear wave equations, *Comm. Partial Differential Equations*, **19**, (1994), 1629–1676.
17. J.-L. Lions, *Quelques Méthodes de Résolution des Problemes aux Limites non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
18. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Veb Deutscher, Berlin 1978.
19. W. A. Strausss, On weak solutions of semi-linear hyperbolic equations, *An. Acad, Brasil, Cienc.*, **42**, (1970), 645–651.
20. S. Zelik, Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities, *Discrete Contin. Dyn. Syst.*, **11**, (2004), 351–392.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC