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Phase portraits of quadratic polynomial vector fields having a rational first integral of degree 3*

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In this paper we classify all the global phase portraits of the quadratic polynomial vector fields having a rational first integral of degree 3. October, 2007 ICMC-USP

1. INTRODUCTION

Let $\mathbb{R}[x,y]$ be the ring of the polynomials in the variables x and y with coefficients in $\mathbb{R}$. We consider a system of polynomial differential equations in $\mathbb{R}^2$ defined by

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.1}$$

where $P, Q \in \mathbb{R}[x,y]$. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of system (1.1). We can associate to system (1.1) the polynomial vector field $X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$. A *quadratic polynomial vector field* is a polynomial vector field for which $m = 2$.

Let U be an open and dense subset of $\mathbb{R}^2$, we say that a nonconstant function $H : U \to \mathbb{R}$ is a *first integral* of system (1.1) on U if $H(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of system (1.1). Obviously H is a first integral of system

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(1.1) if and only if

$$P(x,y)\frac{\partial H}{\partial x}(x,y) + Q(x,y)\frac{\partial H}{\partial y}(x,y) = 0 \tag{1.2}$$

for all $(x,y) \in U$. On the other hand given $f \in \mathbb{R}[x,y]$ we say that the curve $f(x,y) = 0$ is an *algebraic invariant curve* of system (1.1) if there exists $K \in \mathbb{R}[x,y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{1.3}$$

The search of first integrals is a classic tool in order to describe the phase portraits of a bi-dimensional differential system. As usual the *phase portrait* of a system is the decomposition of the domain of definition of this system as union of all its orbits.

Quadratic polynomial vector fields and its applications have been studied intensively in the last 25 years, and more than one thousand papers have been published about the subject (see for instance [9, 10]). But the problem of classifying all the integrable quadratic vector fields remains open.

In this paper we classify all the global phase portraits in the Poincaré disc of the quadratic vector fields having a rational first integral H of degree 3, this means that H is of the form

$$H = \frac{H_1}{H_2} = \frac{\sum_{i+j=0}^{3} e_{ij}x^iy^j}{\sum_{i+j=0}^{3} a_{ij}x^iy^j}, \tag{1.4}$$

with $e_{30}^2 + e_{21}^2 + e_{12}^2 + e_{03}^2 + a_{30}^2 + a_{21}^2 + a_{12}^2 + a_{03}^2 \neq 0$ and with H_1 and H_2 different from a constant. This last condition do not allow that H and $1/H$ be a polynomial.

The quadratic vector fields having a polynomial first integral and their phase portraits have been characterized in [5, 6]. The phase portrait of all quadratic vector fields having a rational first integral of degree 2 have been characterized in [3].

Our main result in this paper is the following.

THEOREM 1.1. *The phase portrait on the Poincaré disc of any planar quadratic polynomial vector field (P,Q) with a rational first integral of degree 3 and with $(P,Q) = 1$, or the phase portrait with the sense of all orbits reversed, is topologically equivalent to one of the 13 phase portraits described in Figure 1.*

The paper is organized as follows. The basic definitions and results about the different kinds of finite and infinite singular points and the results about the Poincaré compactification are given in Section 2. In Section 3 we described the two ways used to obtain all normal forms for the systems of Theorem 1. In Section 4 we show that there are not quadratic polynomial vector fields having a rational first integral of degree 3 when we follow the first way. In Section 5 we describe the normal forms of all quadratic polynomial vector fields having a rational first integral of degree 3 when we follow the second way. Finally in Section 6 we study the global phase portraits of the vector fields of Section 5 in order to complete the proof of the main result.

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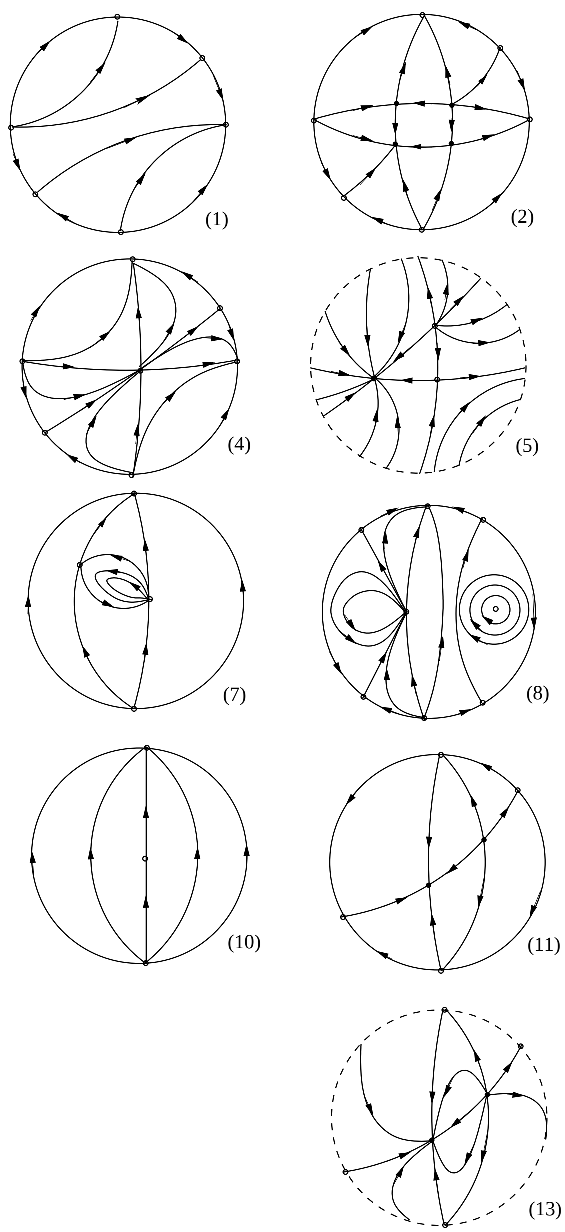

FIG. 1. The 13 non topologically equivalent phase portraits of the planar quadratic polynomial vector field with a rational first integral of degree 3. The dashed lines denotes lines filled of singular points.

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2. BASIC RESULTS

In this section we introduce the basic definitions, notations and results that we will need for the analysis of the local phase portraits of the finite and infinite singular points of the quadratic polynomial vector fields.

We denote by $P_n(\mathbb{R}^2)$ the set of all polynomial vector fields on $\mathbb{R}^2$ of the form $X(x,y) = (P(x,y), Q(x,y))$ where P and Q are real polynomials in the variables x and y of degree at most n (with $n \in \mathbb{N}$). As usual $\mathbb{N}$ denotes the set of positive integers.

2.1. Singular points

A point $p \in \mathbb{R}^2$ is said to be a *singular point* of the vector field $X = (P,Q)$ if $P(p) = Q(p) = 0$. We recall first some results which hold when P and Q are analytic functions in a neighborhood of p. As usual P_x denotes the partial derivative of P with respect to the variable x.

If $\Delta = P_x(p)Q_y(p) - P_y(p)Q_x(p)$ and $T = P_x(p) + Q_y(p)$, then the singular point p is said to be *non–degenerate* if $\Delta \neq 0$. Then p is an isolated singular point. Moreover, p is a *saddle* if $\Delta < 0$, a *node* if $T^2 \geq 4\Delta > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a *focus* if $4\Delta > T^2 > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < \Delta$; for more details see [1], p. 183.

The singular point p is called *hyperbolic* if the two eigenvalues of the Jacobian matrix $DX(p)$ have nonzero real part. So the hyperbolic singular points are the non–degenerate ones except the weak focus and the centers.

A degenerate singular point p (i.e. $\Delta = 0$) with $T \neq 0$ is called *semi–hyperbolic*, and p is isolated in the set of all singular points. Now we summarize the results on semi–hyperbolic singular points that we shall need in this paper, for a proof see Theorem 65 of [1].

PROPOSITION 2.1. *Let $(0,0)$ be an isolated point of the vector field $(F(x,y), y+G(x,y))$, where F and G are analytic functions in a neighborhood of the origin starting at least with quadratic terms in the variables x and y. Let $y = g(x)$ be the solution of the equation $y + G(x,y) = 0$ in a neighborhood of $(0,0)$. Assume that the development of the function $f(x) = F(x, g(x))$ is of the form $f(x) = \mu x^m + HOT$ (Higher Order Terms), where $m \geq 2$ and $\mu \neq 0$. When m is odd, then $(0,0)$ is either an unstable node, or a saddle depending if $\mu > 0$, or $\mu < 0$, respectively. In the case of the saddle the stable separatrices are tangent to the x–axis. If m is even, then $(0,0)$ is a saddle–node, i.e. the singular point is formed by the union of two hyperbolic sectors with one parabolic sector. The stable separatrix is tangent to the positive (respectively negative) x–axis at $(0,0)$ according to $\mu < 0$ (respectively $\mu > 0$). The two unstable separatrices are tangent to the y–axis at $(0,0)$.*

The singular points which are non–degenerate or semi–hyperbolic are called *elementary*.

When $\Delta = T = 0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is *nilpotent*. Now we summarize the results on nilpotent singular points that we shall need. For a proof see Theorems 66 and 67 and the simplified scheme of Section 22.3 of [1].

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Proposition 2.2. *Let $(0,0)$ be an isolated singular point of the vector field $(y+F(x,y), G(x,y))$, where F and G are analytic functions in a neighborhood of the origin starting at least with quadratic terms in the variables x and y. Let $y = f(x)$ be the solution of the equation $y + F(x,y) = 0$ in a neighborhood of $(0,0)$. Assume that the development of the function $G(x, f(x))$ is of the form $Kx^\kappa + HOT$ and $\Phi(x) \equiv (\partial F/\partial x + \partial G/\partial y)(x, f(x)) = Lx^\lambda + HOT$ with $K \neq 0, \kappa \geq 2$ and $\lambda \geq 1$. Then the following statements hold.*

(1) *If κ is even and*

(1.a) *$\kappa > 2\lambda + 1$, then the origin is a saddle–node. Moreover the saddle–node has one separatrix tangent to the semi–axis $x < 0$, and the other two separatrices tangent to the semi–axis $x > 0$.*

(1.b) *$\kappa < 2\lambda + 1$ or $\Phi \equiv 0$, then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors. Moreover, the cusp has two separatrices tangent to the positive x–axis.*

(2) *If κ is odd and $K > 0$, then the origin is a saddle. Moreover, the saddle has two separatrices tangent to the semi–axis $x < 0$, and the other two tangent to the semi–axis $x > 0$.*

(3) *If κ is odd, $K < 0$ and*

(3.a) *λ even, $\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ even and $\kappa > 2\lambda + 1$, then the origin is a stable (unstable) node if $L < 0$ $(L > 0)$, having all the orbits tangent to the x–axis at $(0,0)$.*

(3.b) *λ odd, $\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ odd and $\kappa > 2\lambda + 1$ then the origin is an elliptic–saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector. Moreover, one separatrix of the elliptic–saddle is tangent to the semi–axis $x < 0$, and the other to the semi–axis $x > 0$.*

(3.c) *$\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) < 0$, or $\kappa < 2\lambda + 1$, then the origin is a focus or a center, and if $\Phi(x) \equiv 0$ then the origin is a center.*

Finally if the Jacobian matrix at the singular point p is identically zero, and p is isolated inside the set of all singular points, then we say that p is *linearly zero*. The study of its local phase portrait needs a special treatment (directional blow–ups), see for more details [2]. But if a quadratic vector field has a finite linearly zero singular point, then it is equivalent to a homogeneous quadratic vector field doing if necessary a translation of the linearly zero singular point to the origin, and the global phase portraits of the quadratic homogeneous vector fields are well known, see Figure 2 and for more details [10].

2.2. Poincaré compactification

Let $X \in P_n(\mathbb{R}^2)$ be a planar polynomial vector field of degree n. The *Poincaré compactified vector field $p(X)$ corresponding to X* is an analytic vector field induced on $\mathbb{S}^2$ as follows (see for more details [7]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at point y. Identify $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of X on $\mathbb{S}^2$, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector field $Df \circ X$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly

$\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In this paper when we talk about the circle of the infinity of X we simply talk about the infinity.

In order to extend X' to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that X satisfies suitable conditions. In the case that $X \in P_n(\mathbb{R}^2)$, $p(X)$ is the only analytic extension of $y_3^{n-1}X'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \backslash \mathbb{S}^1$ there are two symmetric copies of X, and knowing the behavior of $p(X)$ around $\mathbb{S}^1$, we know the behavior of X in a neighborhood of the infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(X)$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc,* and it is denoted by $\mathbb{D}^2$.

In the rest of this paper we say that two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$.

As $\mathbb{S}^2$ is a differentiable manifold, for computing the expression for $p(X)$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \rightarrow \mathbb{R}^2$ and $G_i : V_i \rightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different things according to the local charts under consideration), then some easy computations give for $p(X)$ the following expressions:

$$z_2^n \Delta(z) \left(Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) \right) \text{ in } U_1, \tag{2.1}$$

$$z_2^n \Delta(z) \left(P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right) \text{ in } U_2, \tag{2.2}$$

$$\Delta(z) \left(P(z_1, z_2), Q(z_1, z_2) \right) \text{ in } U_3, \tag{2.3}$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. *In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$.* Thus the expression of $p(X)$ becomes a polynomial vector field in each local chart.

3. CHARACTERIZATION

Let $X = (P, Q)$ be a polynomial vector field having a rational first integral H of degree 3 written as in (1.4). Then except a common factor the polynomials P and Q are

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$$
\begin{aligned}
P =&(e_{01}a_{00} - e_{00}a_{01}) + (e_{11}a_{00} - e_{10}a_{01} + e_{01}a_{10} - e_{00}a_{11})x \\
&+ 2(-e_{00}a_{12} + e_{12}a_{00} - e_{10}a_{02}y + e_{02}a_{10})y+ \\
&(e_{21}a_{00} - e_{20}a_{01} + e_{11}a_{10} - e_{10}a_{11} + e_{01}a_{20} - e_{00}a_{21})x^2+ \\
&2(e_{12}a_{00} - e_{10}a_{02} + e_{02}a_{10} - e_{00}a_{12})xy+ \\
&(3e_{03}a_{00} + e_{02}a_{01} - e_{01}a_{02} - 3e_{00}a_{03})y^2+ \\
&(-e_{30}a_{01} + e_{21}a_{10} - e_{20}a_{11} + e_{11}a_{20} - e_{10}a_{21} + e_{01}a_{30})x^3 \\
&+ 2(-e_{20}a_{02} + e_{12}a_{10} - e_{10}a_{12} + e_{02}a_{20})x^2y+ \\
&(e_{12}a_{01} - e_{11}a_{02} - 3e_{10}a_{03} + 3e_{03}a_{10} + e_{02}a_{11} - e_{01}a_{12})xy^2 \\
&+ 2(e_{03}a_{01} - e_{01}a_{03})y^3 + (-e_{30}a_{11} + e_{21}a_{20} - e_{20}a_{21} + e_{11}a_{30})x^4 \\
&+ 2(-e_{30}a_{02} - e_{20}a_{12} + e_{12}a_{20} + e_{02}a_{30})x^3y+ \\
&(-e_{21}a_{02} - 3e_{20}a_{03} + e_{12}a_{11} - e_{11}a_{12} + 3e_{03}a_{20} + e_{02}a_{21})x^2y^2 \\
&+ 2(-e_{11}a_{03} + a_{03}a_{11})xy^3 + (e_{03}a_{02} - e_{02}a_{03})y^4+ \\
&(-e_{30}a_{21} + e_{21}a_{30})x^5 + 2(-e_{30}a_{12} + e_{12}a_{30})x^4y+ \\
&(-3e_{30}a_{03} - e_{21}a_{12} + e_{12}a_{21} + 3e_{03}a_{30})x^3y^2+ \\
&2(-e_{21}a_{03} + e_{03}a_{21})x^2y^3 + (-e_{12}a_{03} + e_{03}a_{12})xy^4,
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
Q =& -((e_{10}a_{00} - e_{00}a_{10}) + 2(e_{20}a_{00} - e_{00}a_{20})x+ \\
&(e_{11}a_{00} + e_{10}a_{01} - e_{01}a_{10} - e_{00}a_{11})y+ \\
&(3e_{30}a_{00} + e_{20}a_{10} - e_{10}a_{20} - 3e_{00}a_{30})x^2+ \\
&(2e_{21}a_{00} + 2e_{20}a_{01} - 2e_{01}a_{20} - 2e_{00}a_{21})xy \\
&+ (e_{12}a_{00} + e_{11}a_{01} + e_{10}a_{02} - e_{02}a_{10} - e_{01}a_{11} - e_{00}a_{12})y^2+ \\
&(3e_{30}a_{01} + e_{21}a_{10} + e_{20}a_{11} - e_{11}a_{20} - e_{10}a_{21} - 3e_{01}a_{30})x^2y \\
&+ (2e_{21}a_{01} + 2e_{20}a_{02} - 2e_{02}a_{20} - 2e_{01}a_{21})xy^2+ \\
&(e_{12}a_{01} + e_{11}a_{02} + e_{10}a_{03} - e_{03}a_{10} - e_{02}a_{11} - e_{01}a_{12})y^3 \\
&+ (e_{30}a_{20} - e_{20}a_{30})x^4 + 2(e_{30}a_{11} - e_{11}a_{30})x^3y+ \\
&2(e_{21}a_{02} + e_{20}a_{03} - e_{03}a_{20} - e_{02}a_{21})xy^3+ \\
&(3e_{30}a_{02} + e_{21}a_{11} + e_{20}a_{12} - e_{12}a_{20} - e_{11}a_{21} - 3e_{02}a_{30})x^2y^2+ \\
&2(e_{30}a_{10} - e_{10}a_{30})x^3 + (e_{12}a_{02} + e_{11}a_{03} - e_{03}a_{11} - e_{02}a_{12})y^4+ \\
&(e_{30}a_{21} - e_{21}a_{30})x^4y + 2(e_{30}a_{12} - e_{12}a_{30})x^3y^2+ \\
&(3e_{30}a_{03} + e_{21}a_{12} - e_{12}a_{21} - 3e_{03}a_{30})x^2y^3 \\
&+ 2(e_{21}a_{03} - e_{03}a_{21})xy^4 + (e_{12}a_{03} - e_{03}a_{12})y^5),
\end{aligned} \tag{3.2}
$$

where $P = (-\partial H/\partial y)H_2^2$ and $Q = (\partial H/\partial x)H_2^2$.

There are two ways in order that the vector field X becomes quadratic. The first is equating the coefficients of the terms of degree 3, 4 and 5 of P and Q to zero. Proposition 4.1 will show that there are not polynomial quadratic vector fields with rational first integral of degree 3 of this kind.

The second way is when P and Q have a common factor R of degree 1, 2 or 3 and the maximum of the degrees of P/R and Q/R is 2.

In order to find all systems satisfying these conditions we start trying to reduce the number of variables in the system. Without loss of generality we can assume that the numerator H_1 of H is a polynomial of degree 3 in the variables x and y. Then H_1 is either an irreducible cubic polynomial or is given by the product of a conic and a straight line. Using the classification of cubics and conics we have that H can be written as one of the following 13 cases of the next proposition.

PROPOSITION 3.1. *Let H be a rational first integral of degree 3 of the quadratic vector field X. Then H can be written after an affine change of variables in one of the following forms:*

$$(I^1) \qquad (ax^3 + xy^2 + bx^2 + cx + dy + e)/H_2(x, y)$$

$$(I^2) \qquad (ax^3 + y^2 + bx^2 + cx + d)/H_2(x, y), \quad a \neq 0,$$

$$(I^3) \qquad (ax^3 + xy + bx^2 + cx + d)/H_2(x, y), \quad a \neq 0,$$

$$(I^4) \qquad (ax^3 + y + bx^2 + cx + d)/H_2(x, y), \quad a \neq 0,$$

$$(E) \qquad (x^2 + y^2 - 1)(y - d)/H_2(x, y),$$

$$(CE) \qquad (x^2 + y^2 + 1)(y - d)/H_2(x, y),$$

$$(P) \qquad (x^2 - y)(ax + by + c)/H_2(x, y),$$

$$(H) \qquad (x^2 - y^2 - 1)(ax + by + c)/H_2(x, y),$$

$$(p) \qquad (x^2 + y^2)(y - d)/H_2(x, y),$$

$$(LV) \qquad (xy)(ax + by + c)/H_2(x, y),$$

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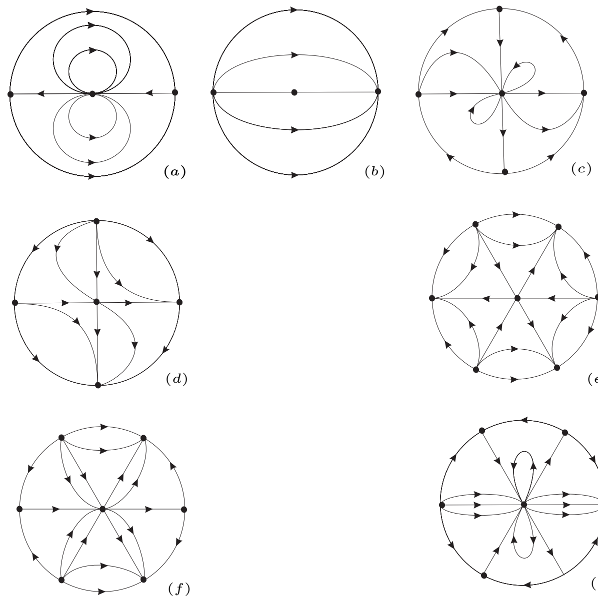

FIG. 2. The phase portraits of the homogeneous quadratic vector fields.

(PL) $$(x^2-1)(ax+by+c)/H_2(x,y),$$

(CL) $$(x^2+1)(y-d)/H_2(x,y),$$

(DL) $$x^2(ax+by+c)/H_2(x,y),$$

where H_2, given as in (1.4)*, denotes an arbitrary polynomial of degree* 3*.*

Proof. For the classification of the four irreducible cubics see the Newton's investigation of cubic algebraic curves in Section 2.5 of the book [8]. The remainder nine cases are obtained from the classification of the conics, real ellipse $x^2+y^2-1=0$, complex ellipse $x^2+y^2+1=0$, parabola $x^2-y=0$, hyperbola $x^2-y^2-1=0$, two complex straight lines intersecting in a real point $x^2+y^2=0$, two non–parallel straight lines $xy=0$, two parallel real straight lines $x^2-1=0$, two parallel complex straight lines $x^2+1=0$, and a double real straight line $x^2=0$. ∎

Now we proceed as follows, for every one of the thirteen cases of H (Proposition 3.1) find $P=-\frac{\partial H}{\partial y}H_2^2$ and $Q=\frac{\partial H}{\partial x}H_2^2$. Then suppose

(3.3) $$P-R(x,y)\overline{P}(x,y)=0,$$

$$Q - R(x,y)\overline{Q}(x,y) = 0, \tag{3.4}$$

where

$$\begin{aligned}\overline{P}(x,y) =& c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2,\\ \overline{Q}(x,y) =& d_{00} + d_{10}x + d_{01}y + d_{20}x^2 + d_{11}xy + d_{02}y^2,\\ R(x,y) =& b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 +\\ & b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.\end{aligned}$$

This means that in order to find all quadratic polynomial vector fields with rational first integral of degree 3 following the second way, we have to solve 13 systems given by 42 equations in the variables x and y with at most 37 unknowns, namely $a_{ij}, b_{ij}, c_{ij}, d_{ij}, a, b, c, d$ and e.

We solve these systems with the help of an algebraic manipulator as Mathematica. See Section 5 for details.

4. THE FIRST WAY

Let H be a rational first integral of degree 3 of the planar quadratic polynomial vector field $X = (P, Q)$. If we suppose H as in (1.4) then P and Q are given as in (3.1) and (3.2).

PROPOSITION 4.1. *All vector fields X obtained when we equate to zero the coefficients of degree 3, 4 and 5 of P and Q have not rational first integral of degree 3.*

Proof. If H is given as (1.4) then H has 20 parameters and P and also Q have 15 coefficients with degree greater than 2. So the system obtained when we equate to zero the coefficients of degree 3, 4 and 5 of P and Q has 30 equations and 20 variables.

With the help of an algebraic manipulator as Mathematica we found 9 solutions for this system. But all solutions have the first integral H given by either a polynomial function, or a rational function of degree 2. So there are no quadratic vector fields with a rational first integral of degree 3 under the assumptions of the proposition. ▮

5. THE SECOND WAY

We distinguished two cases depending on H. By Proposition 3.1 the numerator of H can be written either as an irreducible cubic, or as the product of a conic by a straight line.

5.1. Irreducible cubic

Suppose that $X = (P, Q)$ is a planar quadratic polynomial vector field with a rational first integral $H = H_1/H_2$ where H_1 is given by an irreducible cubic (cases $(I^1) - (I^4)$ of Proposition 3.1). The next proposition says about the normal forms of X and H.

PROPOSITION 5.1. *Let X be a planar quadratic polynomial vector field with $H = H_1/H_2$ as a rational first integral of degree 3 for which H_1 is an irreducible cubic. Then X can be* *written as*

$$\begin{aligned} \dot{x} &= 4(da - 2bx + 2cx^2), \\ \dot{y} &= cd + 4by + 4ay^2, \end{aligned} \tag{5.1}$$

where $a \neq 0, b, c$ *and* d *are parameters and*

$$H = \frac{2db - cdx + 4day + 4axy^2}{4b^2 - cda - 2cbx + 8bay - 4caxy + 4a^2y^2}. \tag{5.2}$$

The proof of this proposition follows from the next two lemmas.

LEMMA 5.1. *If* H *is* $(I^2), (I^3)$ *or* (I^4) *(see Proposition* 3.1*) then the quadratic vector fields having* $H = H_1/H_2$ *as a first integral also have a rational first integral* $\overline{H_1}/H_2$ *with* $\overline{H_1}$ *reducible.*

Proof. For each case we proceed as it is described in Section 3. Here we do with detail the case where H is (I^4). The other two cases follow in a similar way.

Consider the system obtained by doing zero all coefficient of the polynomials (3.3) and (3.4). In this system there are nine equations where the parameters of H_2, $a_{00}, a_{01}, a_{10}, a_{20}$, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03} can be written as linear combinations of the others variables of the system.

After doing the substitution of the $a_{i,j}$'s in the others equations, two of the resultant equations are $b_{03}c_{02} = 0, b_{03}d_{02} = 0$. Then we divide our study in two cases: $b_{03} = 0$ and $b_{03} \neq 0$.

If $b_{03} \neq 0$ we have $c_{02} = d_{02} = 0$. Under these conditions and using the other equations of the system we conclude that the coefficients $c_{01}, c_{11}, c_{10}, c_{20}, c_{00}, d_{11}, d_{01}, d_{10}, d_{00}, d_{20}$ are zero. After substituting by zero all these coefficients we obtain that H is constant. So there are no such quadratic systems when $b_{03} \neq 0$.

Suppose that $b_{03} = 0$. Then, among the other equations of the system, we can found $b_{12}c_{02} = b_{02}c_{02} = b_{12}d_{02} = 0$. So we consider 3 subcases: $b_{12} \neq 0$, $b_{12} = 0$, and $b_{02} \neq 0$ and $b_{12} = b_{02} = 0$.

If $b_{12} \neq 0$ then $c_{02} = d_{02} = 0$. Then from the others equations of the system we get $c_{11} = c_{01} = c_{20} = c_{10} = d_{11} = d_{01} = d_{20} = c_{00} = d_{10} = d_{00} = 0$. Then H is constant.

If $b_{12} = 0$ and $b_{02} \neq 0$, then $c_{02} = 0$ and using that a is also non–zero we obtain that $c_{11} = c_{01} = d_{11} = c_{10} = c_{20} = d_{01} = c_{00} = d_{00} = d_{10} = d_{20} = 0$. Again H is constant.

Now we suppose that $b_{12} = b_{02} = 0$. Here we do not have conditions about non–zero coefficients, except for a. We divide our study in several subcases.

As $b_{11}c_{02} = b_{21}c_{02} = 0$ are two equations of the system after doing $b_{12} = b_{02} = 0$, we consider two situations: $c_{02} = 0$ and $c_{02} \neq 0$. For $c_{02} \neq 0$, $b_{11} = b_{21} = 0$ and $b_{01} = b_{20} = b_{10} = b_{30} = b_{00} = 0$. Under these conditions H is constant. If $c_{02} = 0$, we proceed as above. As $b_{21}d_{02} = b_{21}c_{11} = 0$ are two equations of the system, we divide our study in two subcases: $b_{21} \neq 0$ and $b_{21} = 0$. If $b_{21} \neq 0$ we have $c_{01} = c_{20} = c_{10} =$

$c_{00} = d_{11} = d_{01} = 0$ and there are no quadratic vector field under these conditions. If $b_{21} = 0$, we have $b_{11}c_{11} = 0$ and $b_{11}c_{02} = 0$ as equations of the system so we divide the study in two subcases: $b_{11} = 0$ and $b_{11} \neq 0$. If $b_{11} \neq 0$ then $c_{11} = d_{02} = 0$ and also $c_{01} = c_{20} = c_{10} = d_{11} = c_{00} = d_{01} = d_{10} = d_{20} = d_{00} = 0$ and H is constant. If $b_{11} = 0$ we have $b_{30}(c_{20} - d_{11}) = 0$ and $b_{30}d_{20} = 0$ as equations of the system. We consider two subcases $b_{30} = 0$ and $b_{30} \neq 0$. If $b_{30} \neq 0$ then $c_{20} = d_{11}$ and $d_{20} = 0$. With the help of Mathematica we conclude that there are 16 solutions to (3.3) and (3.4). But for all these solutions the vector fields are non–quadratic, after divided by the common factor.

If $b_{30} = 0$, with the help of Mathematica we find several solutions of system (3.3) and (3.4). Eliminating the vector fields that are non–quadratic and the ones with a non–rational first integral of degree 3 we have only two distinct solutions. They are

$$
\begin{aligned}
X =&(-(b_{00} + b_{10}x)(2ab_{00} - bb_{10} - ab_{10}x), 2ab_{00}^2c - bb_{00}b_{10}c + 2bb_{10}^2d-\\
&3ab_{00}b_{10}d + 4abb_{00}^2x - 2b^2b_{00}b_{10}x - 2ab_{00}b_{10}cx + bb_{10}^2cx+\\
&3ab_{10}^2dx + 6a^2b_{00}^2x^2 - 4abb_{00}b_{10}x^2 + 2ab_{10}^2cx^2 - 3ab_{00}b_{10}y + 2bb_{10}^2y\\
&+ 3ab_{10}^2xy),\\
H =&(d + cx + bx^2 + ax^3 + y)/(2ab_{00}^3 - bb_{00}^2b_{10} + b_{10}^3d + 3ab_{00}^2b_{10}x\\
&- 2bb_{00}b_{10}^2x + b_{10}^3cx + b_{10}^3y),
\end{aligned} \tag{5.3}
$$

and

$$
\begin{aligned}
X =&(b_{10} + 2b_{20}x, -b_{10}c + 6b_{20}d - 2bb_{10}x + 4b_{20}cx - 3ab_{10}x^2+\\
&2bb_{20}x^2 + 6b_{20}),\\
H =&(d + cx + bx^2 + ax^3 + y)/(-ab_{10}^3 + 8b_{20}^3d - 6ab_{10}^2b_{20}x + 8b_{20}^3cx\\
&- 12ab_{10}b_{20}^2x^2 + 8bb_{20}^3x^2 + 8b_{20}^3y).
\end{aligned} \tag{5.4}
$$

Both systems have a first integral which can be written as the product of a conic by a straight line. So H will not be of the form (I^4). Indeed since H is a first integral of X then $H - k$ is also a first integral, for all constant k. In system (5.3) $b_{10} \neq 0$ (if $b_{10} = 0$ H is not a rational function according with our definition). So $\overline{H} = H - \dfrac{1}{b_{10}^3}$ is first integral of the same system and

$$
\overline{H} = \frac{(b_{00} + b_{10}x)^2(-2ab_{00} + bb_{10} + ab_{10}x)}{2ab_{00}^3 - bb_{00}^2b_{10} + b_{10}^3d + 3ab_{00}^2b_{10}x - 2bb_{00}b_{10}^2x + b_{10}^3cx + b_{10}^3y}.
$$

The same can be done with the system (5.4), $b_{20} \neq 0$ (if not H is not a rational function). Then $\overline{H} = H - \dfrac{1}{8b_{20}^3}$ is a first integral and

$$
\overline{H} = \frac{(b_{10} + 2b_{20}x)^3}{-ab_{10}^3 + 8b_{20}^3d - 6ab_{10}^2b_{20}x + 8b_{20}^3cx - 12ab_{10}b_{20}^2x^2 + 8bb_{20}^3x^2 + 8b_{20}^3y}.
$$

So we conclude that the quadratic vector field having a rational first integral of degree 3 with H of the form (I^4) also have a rational first integral $\overline{H_1}/H_2$ of degree 3 with $\overline{H_1}$ reducible. ∎

LEMMA 5.2. *If H is (I^1) then there is a unique quadratic differential system with $H = H_1/H_2$ as a rational first integral. The differential system and the first integral are given in* (5.1) *and* (5.2)*, respectively.*

Proof. The way to find the quadratic system in this case is the same that the one described when H is (I^4).

Here, as in that case, we find more than one differential system but all, except one has a first integral H with H_1 reducible. This system is (5.2), where we renamed the parameters as $d_{02} = a$, $d_{01} = b$ and $c_{20} = c$. ∎

From Lemmas 5.1 and 5.2 it follows the proof of Proposition 5.1.

5.2. Reducible cubic case

Now we study the quadratic systems having rational first integrals H_1/H_2 of degree 3 with H_1 reducible.

PROPOSITION 5.2. *Let X be a quadratic polynomial vector field with a rational first H_1/H_2 of degree 3 given as in Proposition 3.1 with $H \neq I^j$ for $j = 1, 2, 3, 4$. Then X and H can be written as one of the following systems:*

(S1) $$X = (ax + bx^2, -2ay - bxy + cy^2), \quad H = x^2y/(2a + 2bx - cy);$$

(S2) $$X = (abx + acx^2, -2aby + bdy^2), \quad H = x^2y/(2b^2a + 4bcax + 2c^2ax^2 - b^2dy - 2bcdxy)$$

(S3) $$X = (ax + bx^2 - cxy, -2y(a + cy)), \quad H = x^2y/(a + bx + cy)^2;$$

(S4) $$X = (2ax + bx^2 - 2cxy, -4ay + bxy - 2cy^2), \quad H = x^2y/(8a^3 + 12a^2bx + 6ab^2x^2 + b^3x^3 + 12a^2cy + 12abcxy + 6ac^2y^2 + 3bcxy^2 + c^3y^3);$$

(S5) $$X = (2ax^2 + 2bxy, 2cx + 2dx^2 + 4axy + 3by^2), \quad H = x^3/(cx + 2dx^2 + 2axy + by^2);$$

(S6) $$X = (abx + acx^2, cd + cex + 3bay + caxy), \quad H = x^3/(2bcd + 3c^2dx + 3bcex + 6c^2ex^2 + 6b^2ay + 12bcaxy + 6c^2ax^2y);$$

(S7) $$X = (2abx + 2acx^2, 3bd + ebx + 2dcx + 3aby + 2acxy) \quad H = x^3/(d^2b + 2dbx + d^2cx + e^2bx^2 + 2decx^2 + 2daby + a^2by^2 + 2aebxy + 2dacxy + 2aecx^2y + a^2cxy^2);$$

(S8) $X = (abx - bx^2, 2ac + a^2d - 3cx - 3dx - 3dx^2 + 2aby - 3bxy),$
$H = x^2(x-a)/(2c + ad + 3dx + 2by);$

(S9) $X = (-2adx + 2dx^2, -2ab + 3bx + cx^2 - 2ady + 3dxy),$
$H = x^2(x-a)/(b + cx + dy)^2;$

(S10) $X = (6acx - 6cx^2, 2ab - 3bx - adx + 4acy - 6cxy)$
$H = x^2(x-a)/(b^3 + 3b^2dx + 3bd^2x^2 + ad^3x^2 + 6b^2cy + 12bcdxy + 6cd^2x^2y + 12bc^2y^2 + 12c^2dxy^2 + 8c^3y^3);$

where a, b, c and d are parameters.

Proof. The proof follows a similar way to the proof of the irreducible case (I^4), and as in that proof H always can be written in the form (DL), see Proposition 3.1. The reduction is obtained using that if H is a first integral of X then $1/H$ and $H - k$, with k constant, are also first integrals. It is enough to choose a convenient constant k.

Now in the study of the quadratic systems having first integrals of the form (DL) three situations need to be consider: $b \neq 0$, $b = 0$ and $c \neq 0$, $b = c = 0$.

If $b \neq 0$ then doing the change of variables $(x, y) \to (x, ax + by + c)$ we can assume that $H = x^2y/H_2$ and $\overline{H} = (1/H) - a_{21}$ is also a first integral of the system so we can suppose $a_{21} = 0$. Applying the arguments of Section 3 the polynomials P and Q have x as a common factor. Dividing P and Q by x the common factor R will be at most degree 2. Consequently $b_{30} = b_{21} = b_{12} = b_{03} = 0$. Now it is possible to write a_{ij}, $i, j = 0, 1, 2$ as linear combinations of the others parameters of the system (3.3) and (3.4). Dividing the study in cases, four quadratic systems having a rational integral of degree 3 are found. They are the systems (S1)-(S3) of this proposition.

If $b = 0$ then $a \neq 0$ and we get x as a factor of P and Q again. As before dividing P and Q by x we get $b_{30} = b_{21} = b_{12} = b_{03} = 0$. From the system (3.3) and (3.4), it is possible to write the coefficients of H_2 as a linear combination of the others coefficients. We find six quadratic polynomial differential systems with a rational first integral of degree 3. They are the systems (S4)-(S10) of this proposition. ∎

6. PHASE PORTRAITS

The goal of this section is to study the eleven normal forms of the quadratic polynomial vector fields having a rational first integral of degree 3 obtained in Section 5. We shall study each normal form separately.

6.1. Irreducible first integral

In Section 5.1 it was shown that if a quadratic vector field X has a first integral $H = H_1/H_2$ of the form (I^1), then the normal forms of X and H are given by (5.1) and (5.2), respectively.

For studying the infinity of a polynomial vector field via the Poincaré compactification we need to use the local chart U_1 and the origin of the local chart U_2. In the chart U_1 the

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vector field (5.1) writes $(-4cz_1+4az_1^2+12bz_1z_2+cdz_2^2-4adz_1z_2^2,-4z_2(c-2bz_2+adz_2^2))$, having the infinite singular points $(0,0)$ and $(c/a,0)$. As $\Delta(0,0)=16c^2=-\Delta(c/a,0)$ and $T(0,0)=-8c$, $T(c/a,0)=0$ it follows that $T^2-4\Delta$ is 0 at $(0,0)$ and $64c^2$ at $(c/a,0)$. So if $c\neq 0$ then $(0,0)$ is an attractor and $(c/a,0)$ is a saddle point. If $c=0$ the origin is a linearly zero singular point.

The vector field (5.1) in the chart U_2 is $(-4az_1+4cz_1^2-12bz_1z_2+4adz_2^2-cdz_1z_2^2,-z_2(4a+4bz_2+cdz_2^2))$, having the origin as a stable node since $\Delta(0,0)=16a^2$ and $T(0,0)=-8a$ and a is non–zero, otherwise the rational first integral has degree 2.

About the finite singular points, there are four finite singular points, namely $(x_i,y_j)=((-b+(-1)^i\sqrt{b^2-acd})/c,(-b+(-1)^j\sqrt{b^2-acd})/2a)$, $i,j=1,2$ if $b^2-acd>0$ and $c\neq 0$. There is no finite singular point if $b^2-acd<0$ and $c\neq 0$ and H has degree 2 if $b^2-acd=0$ and $c\neq 0$. There are two singular points, namely $(ad/2b,0)$ and $(ad/2b,b/a)$ if $c=0$ and $b\neq 0$ and H has degree 2 if $c=b=0$.

Moreover if $c\neq 0$ and $b^2-acd>0$ then $\Delta(x_i,y_j)=(-1)^{i+j}32(b^2-acd)$ and $T(x_i,y_j)=(-1)^{i+1}4\sqrt{b^2-acd}$, if $i\neq j$ and zero if $i=j$. So for $i,j=1,2$ we have that (x_i,y_j) are nodes if $i=j$ and saddle otherwise. Besides (x_1,y_1) is a stable node and (x_2,y_2) is unstable.

If $c=0$ and $b\neq 0$ then $\Delta(ad/2b,0)=-32b^2=-\Delta(ad/2b,-b/a)$ and $T(ad/2b,0)=-4b=3T(ad/2b,-b/a)$. So the $(ad/2b,0)$ is a saddle and $(ad/2b,-b/a)$ is a stable node.

We complete the study with some invariant curves passing by the singular points. If $b^2-acd>0$ and $c\neq 0$ the straight lines $x_i=(b+(-1)^i\sqrt{b^2-acd})/c$, $y_j=(b+(-1)^j\sqrt{b^2-acd})/2a$ are invariant. If $c=0$ and $b\neq 0$ the straight lines $x=-ad/2b$, $y=-b/a$ and $y=0$ are invariant.

Using the local phase portraits at the finite and infinite singular points, some invariant straight lines and the first integral it follows the next result.

PROPOSITION 6.1. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H=H_1/H_2$ of degree 3 where H_1 is an irreducible cubic, in particular $a\neq 0$ (see Proposition 5.1), is topologically equivalent to Figure 1(1) if $c\neq 0$ and $acd-b^2>0$, Figure 1(2) if $c\neq 0$ and $acd-b^2<0$, and Figure 1(3) if $c=0$ and $b\neq 0$.*

6.2. Reducible first integral

From Proposition 5.2 there are 10 reducible systems.

System (S1) First we shall study the singular points at infinity. In the chart U_1 the vector field writes $(z_1(-2b+cz_1-3az_2),-z_2(b+az_2))$ so if $c\neq 0$, it has two infinity singular point $(0,0)$ and $(2b/c,0)$, otherwise only the origin.

If $bc\neq 0$ the origin is a stable node point since $\Delta(0,0)=2b^2,(T^2-4\Delta)(0,0)=b^2$ and $T(0,0)=-3b$. If $c\neq 0$ and $b=0$ this is a linearly zero point. The case $c=b=0$ is not possible because H has degree 2.

The second point has $\Delta(2b/c,0)=-2b^2$ and $T(2b/c,0)=b$, so it is a saddle point if $b\neq 0$ and it coincides with the origin if $b=0$.

The same vector field in the chart U_2 is $(z_1(-c+2bz_1+3az_2-1), z_2(-c+bz_1+2az_2-1))$. So the origin is a stable node point, with $\Delta(0,0)=c^2$ and $T(0,0)=-2c$ and $(T^2-4\Delta)(0,0)=0$ if $c\neq 0$ and a linearly zero point otherwise.

This vector field has at most four finite singular points depending if a, b and c are zero or not. We observe that at most one coefficient among a, b, c can be zero, otherwise the first integral has no degree 3.

If $abc \neq 0$ then there are four singular points: $(0,0)$, $(-a/b, a/c)$, $(-a/b, 0)$ and $(0, 2a/c)$. The first and the second are saddles and the others are nodes: $(-a/b, 0)$ is stable and $(0, 2a/c)$ is unstable. From the expression of the vector field we conclude that there are the invariant straight lines $x=0, y=0, x=-a/b$ and $y=(2a+2bx)/c$ passing through the singular points. If $a=0$ then $bc\neq 0$ and the unique singular point is the origin. It is a linearly zero point. Here $x=0, y=0$ and $y=2x$ are invariant straight lines. If $b=0$ or $c=0$ only two of the above four singular points exist. They have the same type described above. In the same way there are three invariant straight lines passing through the singular points. From the first integral expression we have that $x=0, y=0, x=-a/b$ and $y=(2a+2bx)/c$ are invariant straight lines depending if b, c are zero or not. The case $b=c=0$ is not possible, if not H is not rational.

The next proposition is a summary of the above study.

PROPOSITION 6.2. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H=H_1/H_2$ of degree 3 where H is as in (S1) of Proposition 5.2 is topologically equivalent to Figure 1(2) if $abc\neq 0$, Figure 1(3) if $bc=0$, $b^2+c^2\neq 0$ and $a\neq 0$, and Figure 1(4), if $a=0$ and $bc\neq 0$.*

System (S2) At infinity this system has the same behavior as system (S1) of Proposition 5.2. The same happens with the singular points. They are at most 4, depending on the coefficients a, b, c and d. If $abcd\neq 0$ they are four: $(0,0)$, $(0, 2a/d)$,$(-b/c, 0)$ and $(-b/c, 2a/d)$. If c or d are zero then there are three. The case where $c=d=0$ is not possible because H would not be rational of degree 3.

When there are four finite singular points two of them are saddles and two nodes, one stable and the other unstable. The nodes are the intersection points of three invariant curves (two straight lines and a parabola) and the saddle are the intersection points of two invariant straight lines.

When the finite singular points are two there are a saddle and a stable node. Both are the intersection points between two invariant straight lines. So the conclusion is the next result.

PROPOSITION 6.3. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H=H_1/H_2$ of degree 3 where H is as in (S2) of Proposition 5.2, in particular $ab\neq 0$, is topologically equivalent to Figure 1(2) if $cd\neq 0$, and Figure 1(3) $cd=0$ and $c^2+d^2\neq 0$.*

System (S3) The finite singular points are $(0,0)$, $(0,-a/c)$, $(-a/b, 0)$ and $(-a/b, -a/c)$, if $abc\neq 0$. So the origin and the point $(-a/b, -a/c)$ are saddles and the other two singular points are nodes: $(-a/b, 0)$ is unstable and $(0, -a/c)$ is stable. If $a = 0$ then $bc \neq 0$ and the origin is the unique singular point. It is a linearly zero point. Through it there are 3 invariant straight lines. If $a \neq 0$ and $b = 0$ or $c = 0$ only two of the four singular points exist. The case $b = c = 0$ is not possible if not H would not be rational.

From the first integral of system (S3) the straight lines $x = 0, y = 0, y = -a/c$ and $y = (-a - bx)/c$ are invariant if c is no zero. If $c = 0$ then $b \neq 0$ and $x = 0, x = -a/b$ and $y = 0$ are invariant.

At infinity the behavior is the same of system (S1) so we have the next result.

Proposition 6.4. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H = H_1/H_2$ of degree 3 where H is as in (S3) of Proposition 5.2 is topologically equivalent to Figure 1(2) if $abc \neq 0$, Figure 1(3) if $bc = 0$, $b^2 + c^2 \neq 0$ and $a \neq 0$, and Figure 1(4) if $a = 0$ and $bc \neq 0$.*

System (S4) In the chart U_1 the vector field X writes $(-6az_1z_2, -z_2(b - 2cz_1 + 2az_2))$ and in the chart U_2 is $(6az_1z_2, -z_2(2c - bz_1 + 4az_2))$. Then the infinity is filled of singular points.

The finite singular points are $(0, 0), (0, -2a/c)$ and $(-2a/b, 0)$ if $bc \neq 0$ and, if $c = 0$ or $b = 0$ then they are two. Here a is always non–zero and the case $b = c = 0$ is not possible (if not the vector field has a common factor or it is a linear system). The origin is a saddle. The point $(0, -2a/c)$ is an unstable node and $(-2a/b, 0)$ is a stable node. Besides the singular points are the intersection points between two invariant straight lines of the system.

If $c \neq 0$ then the straight lines $x = 0, y = 0$ and $y = -(2a + bx)/c$ are invariant passing through the singular points. If $c = 0$ and $b \neq 0$ then the straight lines $x = 0, y = 0$ and $x = -2a/b$ are invariants.

Summarizing we have the next result.

Proposition 6.5. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H = H_1/H_2$ of degree 3 where H is as in (S4) of Proposition 5.2, in particular $a \neq 0$, is topologically equivalent to Figure 1(5) if $bc \neq 0$ and Figure 1(6) if $bc = 0$ and $b^2 + c^2 \neq 0$.*

System (S5) In the chart U_1 the vector field (S5) writes $(2d + 2az_1 + bz_1^2 + 2cz_2, -2(a + bz_1)z_2)$ having the infinite singular points $(-a \pm \sqrt{a^2 - 2bd}/b, 0)$ if $a^2 - 2bd > 0$, $(-a/b, 0)$ if $a^2 - 2bd = 0$, it has no infinite singular points in this chart if $a^2 - 2bd < 0$.

For each singular point $\Delta = -4(a^2 - 2bd)$ and $T = 0$, indicating that if $a^2 - 2bd > 0$ the singular points are saddles. If $a^2 - 2bd = 0$ the vector field is linearly zero.

The vector field (S5) in the chart U_2 is $(-z_1(b + 2az_1 + 2dz_1^2 + 2cz_1z_2), -z_2(3b + 4az_1 + 2dz_1^2 + 2cz_1z_2))$ having the singular point $(0, 0)$ as a stable node since $\Delta(0, 0) = 3b^3$ and $T(0, 0) = -4b$ and b is always non–zero (if not the first integral has degree 2).

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Now we study the finite singular points of the vector field. As before we shall divide the study in cases. If $a^2 - 2bd \neq 0$ the finite singular points are $(0,0)$ and $(-\frac{2ac}{a^2-2bd}, \frac{2bc}{a^2-2bd})$. If $a^2 - 2bd = 0$ the unique singular point is $(0,0)$.

If $a^2 - 2bd < 0$ and $c \neq 0$ the point $(-\frac{2ac}{a^2-2bd}, \frac{2bc}{a^2-2bd})$ is a saddle, and if $a^2 - 2bd > 0$ and $c \neq 0$ it is a center. The origin has $\Delta(0,0) = T(0,0) = 0$ and the Jacobian matrix is equal to zero if $c = 0$ and non identically zero if $c \neq 0$.

Suppose now that $c \neq 0$. Then $x = 0$ and the curve $cx + 2dx^2 + 2axy + by^2 = 0$ are invariant the system passing thought the singular points. This last curve is a hyperbola, ellipse or parabola depending on $a^2 - 2bd$.

Applying Proposition 2.2 we conclude that the origin is a elliptic-saddle because $\kappa = 3$, $\lambda = 1$, so $\kappa = 2\lambda + 1$ and $L = 16b^2c^2$, $K = 24b^2c^2$, so $L^2 + 4K(\lambda + 1)$ is always positive.

If $c = 0$ then the vector field is homogeneous. Here the invariant curves will use to decide about the phase portrait. If $a^2 - 2bd > 0$ there are three invariant straight lines passing through the singular points. If $a^2 - 2bd < 0$ the line $x = 0$ is invariant. If $a^2 - 2bd = 0$ the system has a common factor and it is equivalent to a linear system.

Summarizing we have the next result.

Proposition 6.6. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H = H_1/H_2$ of degree* 3 *where H is as in* (S5) *of Proposition* 5.2, *in particular $b \neq 0$, is topologically equivalent to Figure* 1(7) *if $a^2 - 2bd < 0$ and $c \neq 0$, Figure* 1(8) *if $a^2 - 2bd > 0$ and $c \neq 0$, Figure* 1(9) *if $a^2 - 2bd = 0$ and $c \neq 0$, Figure* 1(10) *if $a^2 - 2bd < 0$ and $c = 0$, and Figure* 1(11) *if $a^2 - 2bd > 0$ and $c = 0$.*

System (S6) *and* (S7) As the behavior of systems (S6) and (S7) are very similar we shall do the study of both simultaneously.

Note that abc is always non-zero in both systems (if not the system is linear). Besides they have three invariant straight lines. The lines $x = 0, x = -b/c$ are common invariant. The line $2bcd - c^2dx + 3bcex + 6ab^2y = 0$ is invariant by system (S6) and $d + ex + ay = 0$ by system (S7). The finite singular points are the intersection points between the invariant straight lines. The singular point on $x = 0$ is a node and the singular point on $x = -b/c$ is a saddle.

At infinity the behavior also is the same for both system. In both local charts, U_1 and U_2, z_2 is a common factor, so the infinity is filled of singular points. After rescalling the system by the time we have a system with a stable node at the origin of the chart U_2. In what the we conclude the following.

Proposition 6.7. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral $H = H_1/H_2$ of degree 3 where H is as in* (S6) *or* (S7) *of Proposition* 5.2, *in particular $abc \neq 0$, is topologically equivalent to Figure* 1(6).

System (S8) *and* (S9) As in the last case systems (S8) and (S9) are also very similar.

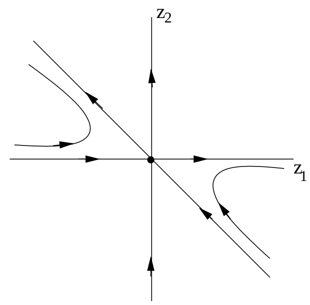

FIG. 3. The local phase portrait of the vector field (S7) and (S8) at the origin of chart U_2.

Note that because the systems can not have common factors we have $ab \neq 0$ in system (S8) and $ad \neq 0$ in (S9).

It follows from the expression of H that both systems have the straight lines $x = 0$, $x = a$ and a transversal line to them invariant.

The finite singular points are the intersection points between each two of the above invariant lines. Both points are nodes. The singular point on $x = 0$ is an unstable node and the other one is a stable node.

At infinity both systems have a unique saddle in the chart U_1 and the origin of U_2 is a degenerated point. Applying blow ups we conclude that this singular point is given by one hyperbolic and two parabolic sectors (see Figure 3). In short we have proved the following.

Proposition 6.8. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral* $H = H_1/H_2$ *of degree* 3 *where* H *is as in* (S8) *or* (S9) *of Proposition* 5.2*, in particular* $ab \neq 0$ *in* (S8) *and* $ad \neq 0$ *in* (S9)*, is topologically equivalent to Figure* 1(12).

System (S10) Outside the infinity the behavior is exactly the same than of systems (S6) and (S7).

Here c is non–zero if not the system would be linear. The lines $x = 0$, $x = a$ and $b + dx + 2cy = 0$ are invariant. The intersection between each two above lines determines one singular point. So we have two singular points. Both points are nodes, one is stable and the other unstable.

In the local chart U_1 the vector field is $(z_2(-3b - ad - 2acz_1 + 2abz_2), -6cz_2(-1 + az_2))$. So all points at infinity are singular points. In the chart U_2 the vector field is $(z_1z_2(2ac + 3bz_1 + adz_1 - 2abz_2), -z_2(-6cz_1 + 4acz_2 - 3bz_1z_2 - adz_1z_2 + 2abz_2^2))$. So the infinity is filled by singular point. Note that rescalling the system by the time we have a unique singular point, the origin of the chart U_2. This is a saddle. Hence we obtain the next result.

Proposition 6.9. *The phase portrait of a planar quadratic polynomial vector field with a rational first integral* $H = H_1/H_2$ *of degree* 3 *where* H *is as in* (S10) *of Proposition* 5.2*, in particular* $ac \neq 0$*, is topologically equivalent to Figure* 1(13).

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From Proposition 6.1 and 6.9 it follows the proof of Theorem 1.1.

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