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Lyapunov stability of closed sets in impulsive semidynamical systems

Everaldo de Mello Bonotto *

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: ebonotto@icmc.usp.br

Nivaldo de Góes Grulha Jr. †

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: njunior@icmc.usp.br

In this paper, we consider impulsive semidynamical systems defined in a metric space X with impulse effects at variable times. Several types of stability concepts for sets of X are studied. The main results concern the stability of certain closed sets of X. They give necessary and sufficient conditions for the various types of stability of closed sets of X. Hence converse-type theorems are included in our results. These results are achieved by means of functionals which play the role of a Lyapunov functional indicating how the solutions behave when entering a "stable" set. October, 2007 ICMC-USP

1. INTRODUCTION

Impulsive semidynamical systems present interesting and important phenomena. In the recent years, the theory of such systems has been studied and developed intensively. See for instance [2]-[8].

In [1], the author presents several results which give necessary and sufficient conditions for the stability of closed sets in non-impulsive dynamical systems. In the present paper, we extend these results for impulsive semidynamical systems. We consider semidynamical systems defined in a metric space X and subject to impulse action which varies in time and we give necessary and sufficient conditions for various types of stability of closed sets of X. In other words, we establish necessary and sufficient conditions so that the solutions of the impulsive system become "stable" in some sense after entering a closed set of X. In this manner, converse-type theorems are included in our results.

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In the first part of the paper, we present the basis of the theory of the impulsive semidynamical systems. In Section 2.1, we give some basic definitions and notations about impulsive semidynamical systems. In Section 2.2, we discuss the continuity of a function which describes the time of reaching the impulsive set. In Section 2.3, we give some additional useful definitions.

The second part of the paper concerns the main results. In Section 3.1, we introduce two new concepts of stability of sets in impulsive semidynamical systems and we relate these concepts of stability to other known concepts. In Section 3.2, we give necessary and sufficient conditions for the various types of stability of closed sets of X. We prove that there exits a functional which plays the role of a Lyapunov functional indicating how the solutions behave when entering a "stable" closed set provided this set is "stable" and we also state the reciprocal this fact. In addition, we show that this Lyapunov functional is continuous when the impulsive set is contained in the closed set.

2. PRELIMINARIES

In this section we present the basic definitions and notation of the theory of impulsive semidynamical systems. We also include some fundamental results which are necessary for understanding the basis of the theory.

2.1. Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(\mathrm{X}, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : \mathrm{X} \times \mathbb{R}_+ \longrightarrow \mathrm{X}$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in \mathrm{X}$ and $t, s \in \mathbb{R}_+$. We denote such system by $(\mathrm{X}, \pi, \mathbb{R}_+)$ or simply (X, π). When $\mathbb{R}_+$ is replaced by $\mathbb{R}$ in the definition above, the triple $(\mathrm{X}, \pi, \mathbb{R})$ is a *dynamical system*. For every $x \in \mathrm{X}$, we consider the continuous function $\pi_x : \mathbb{R}_+ \longrightarrow \mathrm{X}$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *trajectory* of x.

Let (X, π) be a semidynamical system. Given $x \in \mathrm{X}$, the *positive orbit* of x is given by $\mathrm{C}^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$ which we also denote by $\pi^+(x)$. For $t \geq 0$ and $x \in \mathrm{X}$, we define $\mathrm{F}(x, t) = \{y : \pi(y, t) = x\}$ and, for $\triangle \subset [0, +\infty)$ and $D \subset \mathrm{X}$, we define

$$\mathrm{F}(D, \triangle) = \bigcup \{\mathrm{F}(x, t) : x \in D \text{ and } t \in \triangle\}.$$

Then a point $x \in \mathrm{X}$ is called an *initial point*, if $\mathrm{F}(x, t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X and a continuous function $\mathrm{I} : \mathrm{M} \rightarrow \mathrm{X}$ such that for every $x \in \mathrm{M}$, there exists $\varepsilon_x > 0$ such that

$$\mathrm{F}(x, (0, \varepsilon_x)) \cap \mathrm{M} = \emptyset \qquad \text{and} \qquad \pi(x, (0, \varepsilon_x)) \cap \mathrm{M} = \emptyset.$$

The points of M are isolated in every trajectory of the system (X, π). The set M is called the *impulsive set*, the function I is called *impulse function* and we write $\mathrm{N} = \mathrm{I}(\mathrm{M})$. We also define

$$\mathrm{M}^+(x) = (\pi^+(x) \cap \mathrm{M}) \setminus \{x\}.$$

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Another property of the impulsive set M is that M is a meager set in X as shown by the next lemma.

LEMMA 2.1. *Let* (X, π; M, I) *be an impulsive semidynamical system. The impulsive set* M *is a meager set in* X.

Proof. The proof is immediate because the points of M are isolated in every trajectory of the system (X, π). Therefore $int(\overline{\mathrm{M}}) = \emptyset$ in X and the result follows. □

Given an impulsive semidynamical systems (X, π; M, I) and $x \in \mathrm{X}$ with $\mathrm{M}^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. This result is stated next and a proof of it can be found in [2].

LEMMA 2.2. *Let* (X, π; M, I) *be an impulsive semidynamical system. Then for every* $x \in \mathrm{X}$*, there is a positive number* s*,* $0 < s \leq +\infty$*, such that* $\pi(x,\, t) \notin \mathrm{M}$*, whenever* $0 < t < s$*, and* $\pi(x,\, s) \in \mathrm{M}$ *if* $\mathrm{M}^+(x) \neq \emptyset$.

Let (X, π; M, I) be an impulsive semidynamical system and $x \in \mathrm{X}$. By means of Lemma 2.2, it is possible to define a function $\phi : \mathrm{X} \to (0,\, +\infty]$ in the following manner

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x,\, s) \in \mathrm{M} \text{ and } \pi(x,\, t) \notin \mathrm{M} \text{ for } 0 < t < s, \\ +\infty, & \text{if } \mathrm{M}^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Thus for each $x \in \mathrm{X}$, we call $\pi(x,\, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in (X, π; M, I) is an X−valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $\mathrm{M}^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x,t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $\mathrm{M}^+(x) \neq \emptyset$, it follows from Lemma 2.2 that there is a smallest positive number s_0 such that $\pi(x,\, s_0) = x_1 \in \mathrm{M}$ and $\pi(x,\, t) \notin \mathrm{M}$, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x,\, t), & 0 \leq t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = \mathrm{I}(x_1)$ and $\phi(x) = s_0$.

Since $s_0 < +\infty$, the process now continues from x_1^+ onward. If $\mathrm{M}^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, for $s_0 \leq t < +\infty$, and $\phi(x_1^+) = +\infty$. When $\mathrm{M}^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.2 that there is a smallest positive number s_1 such that $\pi(x_1^+,\, s_1) = x_2 \in \mathrm{M}$ and $\pi(x_1^+,\, t - s_0) \notin \mathrm{M}$, for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0,\, s_0 + s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+,\, t - s_0), & s_0 \leq t < s_0 + s_1 \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = \mathrm{I}(x_2)$ and $\phi(x_1^+) = s_1$.

Now we suppose $\widetilde{\pi}_x$ is defined on the interval $[t_{n-1},\, t_n]$ and $\widetilde{\pi}_x(t_n) = x_n^+$, where $t_n = \sum_{i=0}^{n-1} s_i$. If $\mathrm{M}^+(x_n^+) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x_n^+, t - t_n)$, $t_n \leq t < +\infty$, and $\phi(x_n^+) = +\infty$. If however $\mathrm{M}^+(x_n^+) \neq \emptyset$, then there exists $s_n \in \mathbb{R}_+$ such that $\pi(x_n^+,\, s_n) = x_{n+1} \in \mathrm{M}$ and $\pi(x_n^+,\, t - t_n) \notin \mathrm{M}$, for $t_n < t < t_{n+1}$. Besides

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_n^+,\, t - t_n), & t_n \leq t < t_{n+1} \\ x_{n+1}^+, & t = t_{n+1}, \end{cases}$$

where $x_{n+1}^+ = \mathrm{I}(x_{n+1})$ and $\phi(x_n^+) = s_n$. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n,\, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\widetilde{\pi}_x$ is defined on $[0,\, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $\mathrm{M}^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $\mathrm{M}^+(x_n^+) \neq \emptyset$, $n = 1, 2, 3, \ldots$, and if $\widetilde{\pi}_x$ is defined on the interval $[0,\, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Also given $x \in \mathrm{X}$, one of the three properties hold:

i) $\mathrm{M}^+(x) = \emptyset$ and hence the trajectory of x has no discontinuities.

ii) For some $n \geq 1$, each x_k^+, $k = 1, 2, \ldots, n$, is defined and $\mathrm{M}^+(x_n^+) = \emptyset$. In this case, the trajectory of x has a finite number of discontinuities.

iii) For all $k \geq 1$, x_k^+ is defined and $\mathrm{M}^+(x_k^+) \neq \emptyset$. In this case, the trajectory of x has infinitely many discontinuities.

Let $(\mathrm{X},\, \pi;\, \mathrm{M},\, \mathrm{I})$ be an impulsive semidynamical system. Given $x \in \mathrm{X}$, the *impulsive positive orbit* of x is defined by the set

$$\widetilde{\mathrm{C}}^+(x) = \{\widetilde{\pi}(x,\, t) : t \in \mathbb{R}_+\},$$

which we also denote by $\widetilde{\pi}^+(x)$. We denote the closure of $\widetilde{\mathrm{C}}^+(x)$ in X by $\widetilde{\mathrm{K}}^+(x)$.

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies standard properties which follow straightforwardly from the definition. See the next proposition and [3] for a proof of it.

Proposition 2.1. *Let* $(\mathrm{X},\, \pi;\, \mathrm{M},\, \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \mathrm{X}$. *The following properties hold:*

i) $\widetilde{\pi}(x,\, 0) = x$,

ii) $\widetilde{\pi}(\widetilde{\pi}(x,\, t),\, s) = \widetilde{\pi}(x,\, t + s)$, *for all* $t, s \in [0,\, T(x))$ *such that* $t + s \in [0,\, T(x))$.

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2.2. Semicontinuity and continuity of ϕ

The result of this section is borrowed from [5]. It concerns the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system. Such result is applied sometimes intrinsically in the proofs of the main theorems of the next section.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a $\lambda-$*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

a) $F(L, \lambda) = S$;
b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [5], Theorem 3.8.

THEOREM 2.1. *Consider an impulsive semidynamical system* $(X, \pi; M, I)$. *Assume that no initial point in* (X, π) *belongs to the impulsive set* M *and that each element of* M *satisfies the condition* (TC). *Then* ϕ *is continuous at* x *if and only if* $x \notin M$.

2.3. Additional definitions

Let us consider a metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in X$ and ratio δ. Let $B(A, \delta) = \{x \in X : \rho_A(x) < \delta\}$ and $B[A, \delta] = \{x \in X : \rho_A(x) \leq \delta\}$, where $\rho_A(x) = \inf\{\rho(x, y) : y \in A\}$. Throughout this paper, we use the notation ∂A, $int(A)$ and $\overline{A}$ to denote respectively the boundary, interior and closure of A in X.

In what follows, $(X, \pi; M, I)$ is an impulsive semidynamical system and $x \in X$.

We define the *prolongation set* of x in $(X, \pi; M, I)$ by

$$\widetilde{D}^+(x) = \{y \in X : \widetilde{\pi}(x_n, t_n) \to y, \text{ for some } x_n \to x \text{ and } t_n \in [0, +\infty)\}.$$

For a set $A \subset X$ we consider $\widetilde{D}^+(A) = \bigcup\{\widetilde{D}^+(x) : x \in A\}$.

If $\widetilde{\pi}^+(A) \subset A$, we say that A is $\widetilde{\pi}-$*invariant*.

A point $x \in X$ is called *stationary* or *rest point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for all $t \geq 0$, it is a *periodic point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for some $t > 0$ and x is not stationary, and it is a *regular point* if it is neither a rest point nor a periodic point.

Let $A \subset \mathrm{X}$. If for every $\varepsilon > 0$ and every $x \in A$, there is $\delta = \delta(x, \varepsilon) > 0$ such that $\widetilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \varepsilon)$, then A is called $\widetilde{\pi}-$stable. The set A is orbitally $\widetilde{\pi}-$stable if for every neighborhood U of A, there is a positively $\widetilde{\pi}-$invariant neighborhood V of A, $V \subset U$. If for all $x \in A$ and all $y \notin A$, there exist a neighborhood V of x and a neighborhood W of y such that $W \cap \widetilde{\pi}(V, [0, +\infty)) = \emptyset$, we say that A is $\widetilde{\pi}$-stable according to Bhatia-Hajek. We define the set

$$\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A}) = \{x \in \mathrm{X} : \text{for every neighborhood } U \text{ of A, there is a sequence } \{t_n\} \subset \mathbb{R}_+,\ t_n \to +\infty \text{ such that } \widetilde{\pi}(x, t_n) \in U\}.$$

The set $\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A})$ is called *region of weak attraction* of A with respect to $\widetilde{\pi}$. If $x \in \widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A})$, then we say that x is $\widetilde{\pi}-$*weakly attracted* to A. A subset $\mathrm{A} \subset \mathrm{X}$ is called a *weak* $\widetilde{\pi}-$*attractor*, if $\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A})$ is a neighborhood of A. A set $\mathrm{A} \subset \mathrm{X}$ is called *asymptotically* $\widetilde{\pi}-$*stable*, if it is both a weak $\widetilde{\pi}-$attractor and orbitally $\widetilde{\pi}-$stable.

For results concerning the stability and invariancy of sets in an impulsive system, the reader may want to consult [2], [3], [6] and [7].

3. THE MAIN RESULTS

We divide this section into two parts. The first part concerns the relations among some concepts of stability. In the second part, we discuss Lyapunov stability of closed sets in impulsive semidynamical systems. These results give necessary and sufficient conditions for various types of stability of closed sets.

Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system where X is a metric space. We assume the following additional hypotheses:

- no initial point in (X, π) belongs to the impulsive set M, that is, given $x \in \mathrm{M}$ there are $y \in \mathrm{X}$ and $t \in \mathbb{R}_+$ such that $\pi(y, t) = x$.
- ϕ is continuous in $\mathrm{X} \setminus \mathrm{M}$.
- $\mathrm{M} \cap \mathrm{I}(\mathrm{M}) = \emptyset$.
- for all $k \geq 1$, x_k^+ is defined and $\mathrm{M}^+(x_k^+) \neq \emptyset$, that is, the trajectory of $x \in \mathrm{X}$ has infinitely many discontinuities. Consequently, $\phi(x) < +\infty$ for all $x \in \mathrm{X}$.

3.1. Stability

We introduce two new concepts of stability for impulsive semidynamical systems. Then, we relate these new concepts to known ones.

Definition 3.1. Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. A set $A \subset \mathrm{X}$ is said to be

a) equi $\widetilde{\pi}$-stable, if for each $x \notin A$, there is a $\delta = \delta(x) > 0$ such that

$$x \notin \overline{\widetilde{\pi}(B(A, \delta), [0, +\infty))}.$$

b) uniformly $\widetilde{\pi}$-stable, if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \varepsilon).$$

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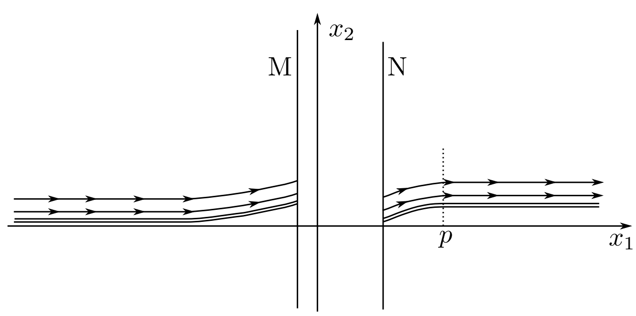
The next result deals with the equivalence between equi $\widetilde{\pi}$-stability and uniform $\widetilde{\pi}$-stability of a compact subset $A \subset X$. This result is also valid when we replace the condition of equi $\widetilde{\pi}$-stability by $\widetilde{\pi}$-stability. The proof is similar to the continuous case.

THEOREM 3.1. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system,* X *is locally compact and* $A \subset X$ *is compact. Then,* A *is equi* $\widetilde{\pi}$*-stable if and only if* A *is uniformly* $\widetilde{\pi}$*-stable. Replacing the hypotheses equi* $\widetilde{\pi}-$*stability by* $\widetilde{\pi}-$*stability, the result remains true.*

Remark 3. 1. If we consider in Theorem 3.1 flows instead of semiflows the result also is true. But if the set $A \subset X$ is closed but not compact, then the sufficiency of the theorem does not necessarily hold. Indeed, consider the discontinuous flow shown in Figure 1, where $M = \{(-1, x_2) : x_2 \in \mathbb{R}\}$, $N = \{(2, x_2) : x_2 \in \mathbb{R}\}$ and the impulsive function $I : M \to N$ is given by $I(-1, x_2) = \left(2, x_2^{'}\right)$ such that $x_2^{'} < x_2^{''} < x_2$, where $x_2^{''}$ is such that for some unique $\lambda > 0$, $\pi((2, x_2^{'}), \lambda) = (p, x_2^{''})$. Note that the trajectories for $x_1 > p$ are straight lines parallel to the axis $0x_1$. This discontinuous flow has the property that for all $x \in \mathbb{R}^2$, $\lim_{t \to -\infty} \widetilde{\pi}(x, t) = 0$. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Clearly A is $\widetilde{\pi}-$stable, but it is not uniformly $\widetilde{\pi}-$stable.



The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ is $\widetilde{\pi}$-stable, but it is not uniformly $\widetilde{\pi}$-stable.

The next result shows the equivalence between the orbital stability and the uniform stability in impulsive semidynamical systems.

THEOREM 3.2. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and* $A \subset X$ *be closed. Then* A *is orbitally* $\widetilde{\pi}-$*stable if and only if* A *is uniformly* $\widetilde{\pi}-$*stable.*

Proof. Suppose that A is orbitally $\widetilde{\pi}-$stable, then given $\varepsilon > 0$, there is a positively $\widetilde{\pi}-$invariant neighborhood V of A, $V \subset B(A, \varepsilon)$. Taking $\delta > 0$ such that $B(A, \delta) \subset V$, we have

$$\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \varepsilon)$$

and A is uniformly $\widetilde{\pi}-$stable.

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Suppose now that A is uniformly $\widetilde{\pi}-$stable. Given $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\widetilde{\pi}(B(A,\delta),[0,+\infty)) \subset B(A,\varepsilon)$. Taking $V = \widetilde{\pi}(B(A,\delta),[0,+\infty))$, note that $B(A,\delta) \subset \widetilde{\pi}(B(A,\delta),[0,+\infty))$. Thus, V is a positively $\widetilde{\pi}-$invariant neighborhood of A and $V \subset B(A,\varepsilon)$. Hence, A is orbitally $\widetilde{\pi}-$stable. □

By Theorem 4.1 from [6] and Theorems 3.1 and 3.2 above, we have the following result which relates various concepts of stability.

THEOREM 3.3. *Let* $(\mathrm{X},\pi;\mathrm{M},\mathrm{I})$ *be an impulsive semidynamical system. Assume that* X *is locally compact and* A *is a compact subset of* X*. Then the following conditions are equivalent:*

a) A is $\widetilde{\pi}-$stable.
b) A is orbitally $\widetilde{\pi}-$stable.
c) A is $\widetilde{\pi}-$stable in the sense of Bhatia and Hajek.
d) A is uniformly $\widetilde{\pi}-$stable.
e) A is equi $\widetilde{\pi}-$stable.
f) $\widetilde{\mathrm{D}}^+(A) = A$.

The $\widetilde{\pi}$-stability of a closed subset A of X implies that $\mathrm{I}(\mathrm{M}) \subset A$, for $\mathrm{M} \subset A$, as shown by the next lemma.

LEMMA 3.1. *Let* $(\mathrm{X},\pi;\mathrm{M},\mathrm{I})$ *be an impulsive semidynamical system and* $A \subset \mathrm{X}$ *be closed. If* A *is* $\widetilde{\pi}-$*stable and* $\mathrm{M} \subset A$*, then* $\mathrm{I}(\mathrm{M}) \subset A$.

Proof. Given $x \in A$ and $\varepsilon > 0$, there a $\delta = \delta(x,\varepsilon) > 0$ such that $\widetilde{\pi}(B(x,\delta),[0,+\infty)) \subset B(A,\varepsilon)$. Since ε is arbitrary, we have $\widetilde{\pi}^+(x) \subset \overline{A} = A$. Therefore, $\mathrm{I}(\mathrm{M}) \subset A$ provided $\mathrm{M} \subset A$. □

3.2. Lyapunov Stability

In this section, we shall present the results that concern the Lyapunov stability of certain closed sets of X. These results are achieved by means of functionals which play the role of a Lyapunov functional indicating how the solutions behave when entering a "stable" set. The results give necessary and sufficient conditions for the various types of stability of closed sets of X. We start by presenting a result on $\widetilde{\pi}$-stability. We need the following lemma from [3, Lemma 3.2].

LEMMA 3.2. *Given an impulsive semidynamical system* $(\mathrm{X},\pi;\mathrm{M},\mathrm{I})$*, where* X *is a metric space. Suppose* $w \in \mathrm{X}\setminus\mathrm{M}$ *and* $\{w_n\}_{n\geq 1}$ *is a sequence convergent to the point* w*. Then for any* $t \in [0, T(w))$*, there exists a sequence of real numbers* $\{\varepsilon_n\}_{n\geq 1}$*,* $\varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$*, such that* $t+\varepsilon_n < T(w_n)$ *and* $\widetilde{\pi}(w_n, t+\varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w,t)$.

THEOREM 3.4. *Let* $(\mathrm{X},\pi;\mathrm{M},\mathrm{I})$ *be an impulsive semidynamical system and* $A \subset \mathrm{X}$ *be closed.*

1. If there exists a functional $\psi : \mathrm{X} \to \mathbb{R}_+$ *with the following properties:*

a) ψ *is continuous in* $\mathrm{X} \setminus (\mathrm{M} \setminus A)$.

b) For every $\varepsilon > 0$*, there is a* $\delta > 0$ *such that* $\psi(x) \geq \delta$ *whenever* $\rho(x, A) \geq \varepsilon$ *and* $x \notin \mathrm{M}$*, and for any sequence* $\{x_n\}_{n\geq 1} \subset \mathrm{X}$ *such that* $x_n \stackrel{n\to+\infty}{\longrightarrow} x \in A$ *implies* $\psi(x_n) \stackrel{n\to+\infty}{\longrightarrow} 0$.

c) $\psi(\pi(x, t)) \leq \psi(x)$ *if* $x \in \mathrm{X} \setminus \mathrm{M}$ *and* $0 \leq t \leq \phi(x)$*, and,* $\psi(\mathrm{I}(x)) \leq \psi(x)$ *if* $x \in \mathrm{M}$.

Then A *is* $\widetilde{\pi}$*-stable.*

2. Reciprocally, if A *is* $\widetilde{\pi}$*-stable, then there is a functional* $\psi : \mathrm{X} \to \mathbb{R}_+$ *satisfying conditions a), b) and c) above.*

Proof. Let us prove the necessary condition. Given $\varepsilon > 0$ and $x \in A$, set

$$\mu = \inf \left\{ \psi(w) : w \notin \mathrm{M} \text{ and } \rho(w, A) \geq \frac{\varepsilon}{2} \right\}.$$

Note that $\mu > 0$, because by item b), there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \frac{\varepsilon}{2}$ and $x \notin \mathrm{M}$. We have two cases to consider: when $x \in int(A)$ and when $x \in \partial A$.

Firstly, suppose $x \in int(A)$. Then, by the second part of item b) and by the continuity of ψ in A, there is a $\delta_1 > 0$ such that

$$\psi(y) < \mu \quad \text{for all} \quad y \in B(x, \delta_1) \subset A. \tag{3.1}$$

We suppose by contradiction that $\widetilde{\pi}(B(x, \delta_1), [0, +\infty))$ is not contained in $B(A, \varepsilon)$. Thus, there are $z \in B(x, \delta_1)$ and $t_1 \in (0, +\infty)$ such that

$$\widetilde{\pi}(z, t_1) \notin B(A, \varepsilon). \tag{3.2}$$

Note that $\widetilde{\pi}(z, t_1) \notin \mathrm{M}$ because $\mathrm{M} \cap \mathrm{I}(\mathrm{M}) = \emptyset$. By equation (3.2), $\rho(\widetilde{\pi}(z, t_1), A) \geq \varepsilon$ and this implies

$$\psi(\widetilde{\pi}(z, t_1)) \geq \inf \left\{ \psi(w) : w \notin \mathrm{M} \text{ and } \rho(w, A) \geq \frac{\varepsilon}{2} \right\} = \mu. \tag{3.3}$$

We have two cases to consider: when $z \in \mathrm{M}$ and when $z \notin \mathrm{M}$. First suppose that $z \notin \mathrm{M}$. Note that as $z \in B(x, \delta_1)$, then $\psi(z) < \mu$ by (3.1). Hence for $0 \leq t < \phi(z)$, we have

$$\psi(\widetilde{\pi}(z, t)) = \psi(\pi(z, t)) \stackrel{c)}{\leq} \psi(z) < \mu.$$

If $t = \phi(z)$ and remembering from the definition of $\widetilde{\pi}$ that $z_1 = \pi(z, \phi(z))$, then

$$\psi(\widetilde{\pi}(z, t)) = \psi(\widetilde{\pi}(z, \phi(z))) = \psi(\mathrm{I}(z_1)) \stackrel{c)}{\leq} \psi(z_1) = \psi(\pi(z, \phi(z))) \stackrel{c)}{\leq} \psi(z) < \mu. \tag{3.4}$$

Now if $\phi(z) < t < \phi(z) + \phi(z_1^+)$, then

$$\psi(\widetilde{\pi}(z, t)) = \psi(\pi(z_1^+, t - \phi(z))) \overset{c)}{\leq} \psi(z_1^+) = \psi(\widetilde{\pi}(z, \phi(z))) \overset{(3.4)}{<} \mu.$$

Repeating this argument, we get $\psi(\widetilde{\pi}(z, t)) < \mu$ for all $t \geq 0$. In particular for $t = t_1$, $\psi(\widetilde{\pi}(z, t_1)) < \mu$ which is a contradiction by (3.3). Hence, $\widetilde{\pi}(B(x, \delta_1), [0, +\infty)) \subset B(A, \varepsilon)$. Now, suppose $z \in \mathrm{M}$. Taking $\nu > 0$, $\nu < t_1$, such that $\widetilde{\pi}(z, \nu) = \pi(z, \nu) \in B(x, \delta_1) \setminus \mathrm{M}$. By the same argument used above for $z \notin \mathrm{M}$, we get $\psi(\widetilde{\pi}(\widetilde{\pi}(z, \nu), t)) < \mu$ for all $t \geq 0$. In particular for $t = t_1 - \nu$, $\psi(\widetilde{\pi}(z, t_1)) = \psi(\widetilde{\pi}(\widetilde{\pi}(z, \nu), t_1 - \nu)) < \mu$ which is a contradiction by (3.3). Therefore, we get again $\widetilde{\pi}(B(x, \delta_1), [0, +\infty)) \subset B(A, \varepsilon)$.

Now we assume that $x \in \partial A$. As ψ is continuous in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$, M is a meager set in X and by the second part of item b), there is a $\delta_2 > 0$, $\delta_2 < \varepsilon$, such that $\psi(y) < \mu$ for all $y \in B(x, \delta_2) \setminus \mathrm{M}$. Supposing that $\widetilde{\pi}(B(x, \delta_2), [0, +\infty))$ is not contained in $B(A, \varepsilon)$, there are $z \in B(x, \delta_2)$ and $t_2 \in (0, +\infty)$ such that $\widetilde{\pi}(z, t_2) \notin B(A, \varepsilon)$. Thus $\rho(\widetilde{\pi}(z, t_2), A) \geq \varepsilon$, $\widetilde{\pi}(z, t_2) \notin \mathrm{M}$ because $\mathrm{M} \cap \mathrm{I}(\mathrm{M}) = \emptyset$ and therefore

$$\psi(\widetilde{\pi}(z, t_2)) \geq \inf \left\{ \psi(w) : w \notin \mathrm{M} \text{ and } \rho(w, A) \geq \frac{\varepsilon}{2} \right\} = \mu. \tag{3.5}$$

If $z \in B(x, \delta_2) \setminus \mathrm{M}$, then it can be shown that $\psi(\widetilde{\pi}(z, t)) < \mu$ for all $t \geq 0$ as we did before. Hence, $\psi(\widetilde{\pi}(z, t_2)) < \mu$ which is a contradiction by (3.5). Also, if $z \in B(x, \delta_2) \cap \mathrm{M}$, then z is an initial point for the impulsive system and there is a time $\tau > 0$ such that $\widetilde{\pi}(z, (0, \tau)) = \pi(z, (0, \tau)) \subset B(x, \delta_2) \setminus \mathrm{M}$. Taking t^*, $0 < t^* < \tau$. By the previous case, $\psi(\widetilde{\pi}(\pi(z, t^*), t)) < \mu$ for all $t \geq 0$. As result, $\psi(\widetilde{\pi}(z, t_2)) = \psi(\widetilde{\pi}(\pi(z, t^*), t_2 - t^*)) < \mu$ and this is a contradiction by (3.5). Therefore, $\widetilde{\pi}(B(x, \delta_2), [0, +\infty)) \subset B(A, \varepsilon)$.

Consequently, A is $\widetilde{\pi}-$stable.

Let us prove the sufficient condition. Define the function $\psi : \mathrm{X} \to \mathbb{R}_+$ by

$$\psi(x) = \begin{cases} \sup\limits_{t \geq 0} \dfrac{\rho(\widetilde{\pi}(x, t), A)}{1 + \rho(\widetilde{\pi}(x, t), A)}, & \text{if } x \in \mathrm{X} \setminus \mathrm{M}, \\ \psi(\mathrm{I}(x)), & \text{if } x \in \mathrm{M}. \end{cases}$$

We shall verify that ψ satisfies conditions a), b) and c).

a) Taking $x \in \mathrm{X} \setminus \mathrm{M}$. Since $\{x\}$ is compact and M is closed, there is a $\eta > 0$, such that $B(x, \eta) \cap \mathrm{M} = \emptyset$. Given a sequence $\{x_n\}_{n \geq 1} \subset \mathrm{X}$ such that $x_n \overset{n \to +\infty}{\longrightarrow} x$, there is a integer $n_0 > 0$ such that $x_n \in B(x, \eta)$ for $n > n_0$. By Lemma 3.2, there exists a sequence of real numbers $\{\varepsilon_n\}_{n \geq 1}$, $\varepsilon_n \overset{n \to +\infty}{\longrightarrow} 0$, such that

$$\widetilde{\pi}(x_n, t + \varepsilon_n) \overset{n \to +\infty}{\longrightarrow} \widetilde{\pi}(x, t),$$

for all $t \geq 0$. Then, for $n > n_0$

$$\psi(x_n) = \sup_{t \geq 0} \frac{\rho(\widetilde{\pi}(x_n, t), A)}{1 + \rho(\widetilde{\pi}(x_n, t), A)} = \sup_{t \geq -\varepsilon_n} \frac{\rho(\widetilde{\pi}(x_n, \varepsilon_n + t), A)}{1 + \rho(\widetilde{\pi}(x_n, \varepsilon_n + t), A)}.$$

Therefore, $\psi(x_n) \overset{n\to+\infty}{\longrightarrow} \psi(x)$ and ψ is continuous in $\mathrm{X} \setminus \mathrm{M}$. Since $\psi(x) = 0$ for all $x \in A$ (because A is closed and $\widetilde{\pi}-$stable and hence $\widetilde{\pi}^+(A) \subset \overline{A} = A$), M is a meager set in X, it follows that ψ is continuous in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$.

b) Consider $x \in \mathrm{X} \setminus \mathrm{M}$. Given $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{1+\varepsilon}$. Thus, if $\rho(x, A) \geq \varepsilon$ then $\dfrac{\rho(x, A)}{1+\rho(x, A)} \geq \delta$. Therefore, $\psi(x) \geq \delta$.

For the second part of item b), let us assume that $x \in A$. If $x \notin \mathrm{M}$, as M is closed and $\{x\}$ is compact, there is a $\delta > 0$ such that $B(x, \delta) \cap \mathrm{M} = \emptyset$. Thus, if $\{x_n\}_{n\geq 1}$ is any sequence in X such that $x_n \overset{n\to+\infty}{\longrightarrow} x$, there exists a positive integer $N > 0$ such that $x_n \in B(x, \delta)$ for $n > N$, by continuity of ψ in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$,

$$\psi(x_n) \overset{n\to+\infty}{\longrightarrow} \psi(x).$$

Now, suppose $x \in \mathrm{M}$. First of all, we should note that if $\{z_n\}_{n\geq 1} \subset \mathrm{X}\setminus\mathrm{M}$ and $z_n \overset{n\to+\infty}{\longrightarrow} x$, then the continuity of ψ in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$ implies

$$\psi(z_n) \overset{n\to+\infty}{\longrightarrow} \psi(x).$$

Also, if $\{z_n\}_{n\geq 1} \subset \mathrm{M}$ and $z_n \overset{n\to+\infty}{\longrightarrow} x$, since the impulsive operator I is continuous, we have

$$\mathrm{I}(z_n) \overset{n\to+\infty}{\longrightarrow} \mathrm{I}(x).$$

Thus, since $\mathrm{I}(z_n) \notin \mathrm{M}$ for all $n \in \mathbb{N}$, $\mathrm{I}(x) \notin \mathrm{M}$ and ψ is continuous in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$, it follows that

$$\psi(\mathrm{I}(z_n)) \overset{n\to+\infty}{\longrightarrow} \psi(\mathrm{I}(x)),$$

then by the definition of ψ,

$$\psi(z_n) \overset{n\to+\infty}{\longrightarrow} \psi(x).$$

Consequently, if $\{x_n\}_{n\geq 1} \subset \mathrm{X}$ is any sequence such that $x_n \overset{n\to+\infty}{\longrightarrow} x$, then $\psi(x_n) \overset{n\to+\infty}{\longrightarrow} \psi(x)$. Since $x \in A$, we have $\psi(x) = 0$ and therefore, $\psi(x_n) \overset{n\to+\infty}{\longrightarrow} 0$.

c) Let $x \in \mathrm{X} \setminus \mathrm{M}$ and $0 \leq s < \phi(x)$. Note that $\widetilde{\pi}(\pi(x, s), t) = \widetilde{\pi}(x, t+s)$ for all $t \geq 0$. Then,

$$\psi(\pi(x, s)) = \sup_{t\geq 0} \frac{\rho(\widetilde{\pi}(\pi(x, s), t), A)}{1+\rho(\widetilde{\pi}(\pi(x, s), t), A)} = \sup_{t\geq 0} \frac{\rho(\widetilde{\pi}(x, t+s), A)}{1+\rho(\widetilde{\pi}(x, t+s), A)} =$$

$$= \sup_{t\geq s} \frac{\rho(\widetilde{\pi}(x, t), A)}{1+\rho(\widetilde{\pi}(x, t), A)} \leq \sup_{t\geq 0} \frac{\rho(\widetilde{\pi}(x, t), A)}{1+\rho(\widetilde{\pi}(x, t), A)} = \psi(x). \tag{3.6}$$

Now, we shall prove that $\psi(\pi(x, \phi(x))) \leq \psi(x)$. Consider a positive increasing sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$, such that $t_n \stackrel{n\to+\infty}{\longrightarrow} \phi(x)$, and take $z_n = \pi(x, t_n)$. Thus, $z_n \stackrel{n\to+\infty}{\longrightarrow} \pi(x, \phi(x)) = x_1$ and $\phi(z_n) \stackrel{n\to+\infty}{\longrightarrow} 0$. Note that $\psi(z_n) \leq \psi(x)$ by (3.6). Then,

$$\begin{aligned}\psi(z_n) &= \sup_{t\geq 0} \frac{\rho(\widetilde{\pi}(z_n, t), A)}{1+\rho(\widetilde{\pi}(z_n, t), A)} \geq \sup_{t\geq\phi(z_n)} \frac{\rho(\widetilde{\pi}(z_n, t), A)}{1+\rho(\widetilde{\pi}(z_n, t), A)}\\ &= \sup_{t\geq\phi(z_n)} \frac{\rho(\widetilde{\pi}(\widetilde{\pi}(z_n, \phi(z_n)), t-\phi(z_n)), A)}{1+\rho(\widetilde{\pi}(\widetilde{\pi}(z_n, \phi(z_n)), t-\phi(z_n)), A)}\\ &= \sup_{t\geq\phi(z_n)} \frac{\rho(\widetilde{\pi}(\mathrm{I}(x_1), t-\phi(z_n)), A)}{1+\rho(\widetilde{\pi}(\mathrm{I}(x_1), t-\phi(z_n)), A)}\\ &= \sup_{t\geq 0} \frac{\rho(\widetilde{\pi}(\mathrm{I}(x_1), t), A)}{1+\rho(\widetilde{\pi}(\mathrm{I}(x_1), t), A)} = \psi(\mathrm{I}(x_1)).\end{aligned}$$

Hence, $\psi(x_1) = \psi(\mathrm{I}(x_1)) \leq \psi(z_n) \leq \psi(x)$, and consequently $\psi(\pi(x, t)) \leq \psi(x)$ for $0 \leq t \leq \phi(x)$. Now we prove the second part of c). Let $x \in \mathrm{M}$. Then by the definition of ψ, $\psi(\mathrm{I}(x)) = \psi(x)$ and the theorem is proved. □

The next result is a corollary of Theorem 3.4. It says that if $\mathrm{M} \subset A$, then we get the continuity of the function ψ.

Corollary 3.1. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. A closed subset* $A \subset \mathrm{X}$ *such that* $\mathrm{M} \subset A$ *is* $\widetilde{\pi}$*-stable if and only if there exists a functional* $\psi : \mathrm{X} \to \mathbb{R}_+$*, with the following properties:*

a) ψ *is continuous.*

b) For every $\varepsilon > 0$*, there is a* $\delta > 0$ *such that* $\psi(x) \geq \delta$ *whenever* $\rho(x, A) \geq \varepsilon$*, and for any sequence* $\{x_n\}_{n\geq 1} \subset \mathrm{X}$ *such that* $x_n \stackrel{n\to+\infty}{\longrightarrow} x \in A$ *implies* $\psi(x_n) \stackrel{n\to+\infty}{\longrightarrow} 0$.

c) $\psi(\pi(x, t)) \leq \psi(x)$ *if* $x \in \mathrm{X} \setminus \mathrm{M}$ *and* $t \geq 0$*, and,* $\psi(\mathrm{I}(x)) \leq \psi(x)$ *if* $x \in \mathrm{M}$.

Proof. The necessary condition is obvious. Consider $\psi : \mathrm{X} \to \mathbb{R}_+$ given by

$$\psi(x) = \sup_{t\geq 0} \frac{\rho(\widetilde{\pi}(x, t), A)}{1+\rho(\widetilde{\pi}(x, t), A)}.$$

Then the sufficient condition follows. □

Lemma 3.3, will be necessary to prove Theorem 3.5.

Lemma 3.3. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $A \subset \mathrm{X}$ *be closed. Let* $\psi : \mathrm{X} \to \mathbb{R}_+$ *be continuous on* $\mathrm{X} \setminus (\mathrm{M} \setminus A)$ *such that* $\psi(x) > 0$ *for* $x \notin A \cup \mathrm{M}$ *and* $\psi(x) = 0$ *for all* $x \in A$*. Suppose there is a* $\widetilde{\delta} > 0$ *such that* $\psi(\widetilde{\pi}(w, t)) \leq \psi(w)$ *for all* $t \geq 0$ *and* $w \in \overline{B(A, \widetilde{\delta})} \setminus \mathrm{M}$*. Then, there is a* $\delta > 0$*,* $0 < \delta \leq \widetilde{\delta}$*, such that* $\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \widetilde{\delta})$.

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Proof. We shall suppose that for each $\delta_n = \dfrac{\widetilde{\delta}}{n} > 0$, $n \in \mathbb{N}$, there are $x_n \in B\left(A, \dfrac{\widetilde{\delta}}{n}\right)$ and $t_0^n \in (0, +\infty)$ such that

$$\widetilde{\pi}(x_n, t_0^n) \notin B\left(A, \widetilde{\delta}\right).$$

Taking $\mu = \inf\left\{\psi(x) : x \notin \mathrm{M} \text{ and } \rho(x, A) \geq \widetilde{\delta}\right\}$. Note that $\mu > 0$ by the properties of ψ. If there are countably many elements of $(\{x_n\}_{n\geq 1} \setminus \mathrm{M})$, say $\{x_{n_k}\}_{k\geq 1} \subset \mathrm{X} \setminus \mathrm{M}$, $n_1 < n_2 < n_3 < \cdots$, since $\dfrac{\widetilde{\delta}}{n}$ converges to zero as $n \to +\infty$, $\psi(x) = 0$ for all $x \in A$ and ψ is continuous on $\mathrm{X} \setminus (\mathrm{M} \setminus A)$, then there is a positive integer n_{k_0} such that $\psi(x_{n_{k_0}}) < \mu$. Note that $x_{n_{k_0}} \in \overline{B(A, \widetilde{\delta})} \setminus \mathrm{M}$. Thus, $\psi(\widetilde{\pi}(x_{n_{k_0}}, t_0^{n_{k_0}})) \leq \psi(x_{n_{k_0}}) < \mu$, which is a contradiction because $\rho(\widetilde{\pi}(x_{n_{k_0}}, t_0^{n_{k_0}}), A) \geq \widetilde{\delta}$ and $\widetilde{\pi}(x_{n_{k_0}}, t_0^{n_{k_0}}) \notin \mathrm{M}$ inasmuch as $\mathrm{M} \cap \mathrm{I}(\mathrm{M}) = \emptyset$. But, if there are countably many elements of $(\{x_n\}_{n\geq 1} \cap \mathrm{M})$, say $\{x_{n_k}\}_{k\geq 1} \subset \mathrm{M}$, then for each $k \in \mathbb{N}$ there is a $0 < \epsilon_k < t_0^{n_k}$ such that

$$\widetilde{\pi}(x_{n_k}, \epsilon_k) = \pi(x_{n_k}, \epsilon_k) \in B\left(A, \frac{\widetilde{\delta}}{n_k}\right) \setminus \mathrm{M}.$$

By the continuity of ψ on $\mathrm{X} \setminus (\mathrm{M} \setminus A)$ and by a similar argument used in the end of the previous paragraph, there exists a n_{k_1} such that

$$\psi(\widetilde{\pi}(x_{n_{k_1}}, \epsilon_{k_1})) < \mu.$$

Thus,

$$\psi(\widetilde{\pi}(x_{n_{k_1}}, t_0^{n_{k_1}})) = \psi(\widetilde{\pi}(\widetilde{\pi}(x_{n_{k_1}}, \epsilon_{k_1}), t_0^{n_{k_1}} - \epsilon_{k_1})) \leq \psi(\widetilde{\pi}(x_{n_{k_1}}, \epsilon_{k_1})) < \mu,$$

and this is a contradiction. Therefore, there exists a $\delta > 0$ with $0 < \delta \leq \widetilde{\delta}$ such that $\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \widetilde{\delta})$. □

Theorem 3.5 deals with the equi $\widetilde{\pi}$-stability of a closed set of X.

THEOREM 3.5. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $A \subset \mathrm{X}$ *be closed.*

1. *If there exists a functional* $\psi : \mathrm{X} \to \mathbb{R}_+$ *with the following properties:*
   - *a)* ψ *is continuous in* $\mathrm{X} \setminus (\mathrm{M} \setminus A)$.
   - *b)* $\psi(x) = 0$ *for* $x \in A$, $\psi(x) > 0$ *for* $x \notin A \cup \mathrm{M}$.
   - *c)* $\psi(\pi(x, t)) \leq \psi(x)$ *for* $x \in \mathrm{X} \setminus \mathrm{M}$ *and* $0 \leq t \leq \phi(x)$, *and,* $\psi(\mathrm{I}(x)) \leq \psi(x)$ *for* $x \in \mathrm{M}$.

Then A *is equi* $\widetilde{\pi}$*-stable.*

2. Reciprocally, if A is equi $\widetilde{\pi}$-stable, then there is a functional $\psi : \mathrm{X} \to \mathbb{R}_+$ satisfying conditions a), b) and c) above.

Proof. 1. Let $x \notin A$. Set $\rho(x, A) = \varepsilon$. Let $\widetilde{\delta} > 0$ such that $\widetilde{\delta} < \varepsilon$. Given $w \in \overline{B(A, \widetilde{\delta})} \setminus \mathrm{M}$ we have $\psi(\pi(w, t)) \leq \psi(w)$ for $0 \leq t \leq \phi(w)$ and $\psi(w_1^+) = \psi(\mathrm{I}(w_1)) \leq \psi(w_1) = \psi(\pi(w, \phi(w))) \leq \psi(w)$. Now if $\phi(w) < t < \phi(w) + \phi(w_1^+)$, it follows that

$$\psi(\widetilde{\pi}(w, t)) = \psi(\pi(w_1^+, t - \phi(w))) \leq \psi(w_1^+) = \psi(\widetilde{\pi}(w, \phi(w))) \leq \psi(w).$$

For $t = \phi(w) + \phi(w_1^+)$,

$$\psi(\widetilde{\pi}(x, t)) = \psi(w_2^+) = \psi(\mathrm{I}(w_2)) \leq \psi(w_2) = \psi(\pi(w_1^+, \phi(w_1^+))) \leq \psi(w_1^+) \leq \psi(w).$$

Thus, $\psi(\widetilde{\pi}(w, t)) \leq \psi(w)$ for all $t \geq 0$ and $w \in \overline{B(A, \widetilde{\delta})} \setminus \mathrm{M}$. By Lemma 3.3, there is a δ, $0 < \delta \leq \widetilde{\delta}$, such that

$$\overline{\widetilde{\pi}(B(A, \delta), [0, +\infty))} \subset \overline{B(A, \widetilde{\delta})}.$$

Therefore, $x \notin \overline{\widetilde{\pi}(B(A, \delta), [0, +\infty))}$ and the result follows.

2. Consider the function $\psi(x)$ defined in Theorem 3.4. The result follows similarly as in Theorem 3.4 . □

We have the following corollary where we obtain the continuity of the function ψ, provided $\mathrm{M} \subset A$.

Corollary 3.2. *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. A closed subset $A \subset \mathrm{X}$ such that $\mathrm{M} \subset A$ is equi $\widetilde{\pi}$-stable if and only if there exists a functional $\psi : \mathrm{X} \to \mathbb{R}_+$, with the following properties:*

a) ψ is continuous.
b) $\psi(x) = 0$ for $x \in A$, $\psi(x) > 0$ for $x \notin A$.
c) $\psi(\pi(x, t)) \leq \psi(x)$ for $x \in \mathrm{X} \setminus \mathrm{M}$ and $t \geq 0$, and, $\psi(\mathrm{I}(x)) \leq \psi(x)$ for $x \in \mathrm{M}$.

For the case of uniformly $\widetilde{\pi}$-stability, we have the following result.

Theorem 3.6. *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $A \subset \mathrm{X}$ be closed.*

1. If there exists a functional $\psi : \mathrm{X} \to \mathbb{R}_+$ with the following properties:

a) ψ is continuous in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$.

b) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$ and $x \notin \mathrm{M}$.

c) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ wherever $\rho(x, A) \leq \delta$.

d) $\psi(\pi(x, t)) \leq \psi(x)$ if $x \in \mathrm{X} \setminus \mathrm{M}$ and $0 \leq t \leq \phi(x)$, and, $\psi(\mathrm{I}(x)) \leq \psi(x)$ if $x \in \mathrm{M}$.

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Then A is uniformly $\widetilde{\pi}$-stable.

2. Reciprocally, if A is uniformly $\widetilde{\pi}$-stable, then there is a functional $\psi : \mathrm{X} \to \mathbb{R}_+$ satisfying conditions a), b), c) and d) above.

Proof. To prove 2., note since A is uniformly $\widetilde{\pi}-$stable, then A is $\widetilde{\pi}-$stable. Thus, the proof follows as in Theorem 3.4. The proof of the item 1. follows by an argument similar to that used in the proof of item 1. of the Theorem 3.5. □

Corollary 3.3. *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. A closed subset $A \subset \mathrm{X}$ such that $\mathrm{M} \subset A$ is uniformly $\widetilde{\pi}$-stable if and only if there exists a functional $\psi : \mathrm{X} \to \mathbb{R}_+$, with the following properties:*

a) ψ is continuous.
b) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$.
c) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ wherever $\rho(x, A) \leq \delta$.
d) $\psi(\pi(x, t)) \leq \psi(x)$ if $x \in \mathrm{X} \setminus \mathrm{M}$ and $t \geq 0$, and, $\psi(\mathrm{I}(x)) \leq \psi(x)$ if $x \in \mathrm{M}$.

Now, we present the result concerning asymptotically $\widetilde{\pi}$-stability to closed sets.

Theorem 3.7. *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $A \subset \mathrm{X}$ be closed.*

1. If there exists a functional $\psi : \mathrm{X} \to \mathbb{R}_+$ with the following properties:

a) ψ is continuous in $\mathrm{X} \setminus (\mathrm{M} \setminus A)$.
b) $\psi(x) = 0$ for $x \in A$, and $\psi(x) > 0$ for $x \notin A \cup \mathrm{M}$.
c) $\psi(\pi(x, t)) \leq \psi(x)$ if $x \in \mathrm{X} \setminus \mathrm{M}$ and $0 \leq t \leq \phi(x)$, and, $\psi(\mathrm{I}(x)) \leq \psi(x)$ if $x \in \mathrm{M}$.
d) there is a $\delta > 0$ such that if $x \in B(A, \delta) \setminus A$, then $\psi(\pi(x, t)) \to 0$ as $t \to +\infty$.

Then A is asymptotically $\widetilde{\pi}$-stable.

2. Reciprocally, if A is asymptotically $\widetilde{\pi}$-stable, then there is a functional $\psi : \mathrm{X} \to \mathbb{R}_+$ satisfying conditions a), b), c) and d) above.

Proof. 1. Item b) of this theorem implies items b) and c) of Theorem 3.6. Thus, by Theorem 3.6, A is uniformly $\widetilde{\pi}-$stable, and by Theorem 3.2 A is orbitally $\widetilde{\pi}-$stable. The condition d) says that $B(A, \delta) \subset \widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{A})$, then A is a weak $\widetilde{\pi}-$attractor. Hence, A is asymptotically $\widetilde{\pi}-$stable.

2. Clearly the functional ψ given by $\psi(x) = \sup_{t \geq 0} \dfrac{\rho(\widetilde{\pi}(x, t), A)}{1 + \rho(\widetilde{\pi}(x, t), A)}$, for $x \in \mathrm{X} \setminus \mathrm{M}$ and $\psi(x) = \psi(\mathrm{I}(x))$, satisfies the conditions of the theorem. □

Corollary 3.4. *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. A closed subset $A \subset \mathrm{X}$ is asymptotically $\widetilde{\pi}$-stable if and only if there exists a functional $\psi : \mathrm{X} \to \mathbb{R}_+$, with the following properties:*

a) ψ *is continuous.*
b) $\psi(x) = 0$ *for* $x \in A$, *and* $\psi(x) > 0$ *for* $x \notin A$.
c) $\psi(\pi(x, t)) \leq \psi(x)$ *if* $x \in \mathrm{X} \setminus \mathrm{M}$ *and* $t \geq 0$, *and,* $\psi(\mathrm{I}(x)) \leq \psi(x)$ *if* $x \in \mathrm{M}$.
d) *there is a* $\delta > 0$ *such that if* $x \in B(A, \delta) \setminus A$, *then* $\psi(\pi(x, t)) \to 0$ *as* $t \to +\infty$.

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