

Ergodic properties of dynamical systems beyond uniform hyperbolicity (38th Iranian Mathematical Conference, Zanjan-Iran)

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The goal of this survey-type article is to present some of the recent trends in ergodic theory of dynamical systems and new achievements in this rapidly developing area. I will try my best to write it accessible for graduate students and include open questions. October, 2007 ICMC-USP

1. INTRODUCTION

As it appears in the title of this article, this is the extended version of my talk in the 38th annual Iranian mathematical conference 2007. We give some examples of ergodic dynamical systems and some methods to prove the ergodicity of them. In fact, there are not too many known ergodic volume preserving dynamics. May be it is better to write that, we do not know many methods to prove ergodicity of dyncamis. So, one of the aims of this survey is to encourage people to work to find more “technologies” to prove the ergodicity of volume preserving dynamical systems.

A modern approach to study the ergodic properties of dynamical systems can be divided into two following cases:

1. Volume preserving systems: In this case, we consider a diffeomorphism $f : M \rightarrow M$ which preserves a Lebesgue probability measure (induced from a volume form of the compact Riemannian manifold M .)

2. Dissipative systems: in this case we consider diffeomorphisms $f : M \rightarrow M$ such that there exists a *trapping region*, i.e., an open set $U \subset M$, $f(U) \subset U$; the set $\Lambda = \bigcap_{n \geq 0} f^n(U)$ is said to be an attractor.

A main feature of volume preserving (or consevative) diffeomorphisms is the ergodicity of Lebesgue measure. The Boltzmann-Maxwell ergodic hypothesis, prompted a search for ergodic mechanical systems. In Geometry, the quest for ergodicity led to the study

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of geodesic flows on negatively curved manifolds, where Eberhard Hopf [2] provided the first and still only argument to establish ergodicity in the case of nonconstantly negatively curved surface.

The celebrated Birkhoff-Khinchin ergodic theorem states that whenever $f : M \rightarrow M$ (we always consider M compact metric space) preserves a measure ν and ϕ is an integrable function then for ν -almost all $x \in M$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = \hat{\phi}(x),$$

where $\hat{\phi}$ is an integrable function and moreover $\int \phi d\nu = \int \hat{\phi} d\nu$.

We just recall that μ is ergodic if for μ - almost all $x \in M$ and any $\phi \in C^0(M, \mathbb{R})$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = \int \phi d\mu.$$

It can be proved that the following definitions are equivalent to the ergodicity

- any invariant measurable subset A (i.e., $f^{-1}(A) = A$) has full or null measure.
- Whenever $\phi \in L^2(\nu)$ and $\phi \circ f = \phi$ a.e. then ϕ is constant a.e.

The main purpose of this survey is firstly to review some application of the ergodicity of some transformations. Secondly, we recall some well known arguments for the proof of the ergodicity of Anosov diffeomorphisms and finally announce some new results of the ergodicity of volume measure for conservative diffeomorphisms beyond uniformly hyperbolic ones.

In the dissipative setting there exist a similar parallel study. An invariant probability measure μ is called physical (in some cases it coincides with Sinai-Ruelle-Bowen measures) if the basin of μ has positive **Lebesgue** measure. By basin of $\mu, B(\mu)$ we mean:

$$B(\mu) := \{x \in M : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = \int \phi d\mu\}.$$

This means that for a positive Lebesgue measure sets of initial datas, the time average of any continuous observable $\phi : M \rightarrow \mathbb{R}$ coincides with its space average with respect to μ . We emphasize that, by Birkhoff theorem if μ is ergodic $B(\mu)$ has full μ -measure and we dont have a priori any information about the Lebesgue measure of the basin of a measure. Given a dynamical systems, it is a challenging problem to find such measures or to prove their finiteness. The uniqueness of such physical measures can be compared to the ergodicity of Lebesgue measure for the conservative diffeomorphisms. In fact, by definition if the Lebesgue measure is ergodic then it is physical with basin of full measure.

2. SOME EXAMPLES

2.1. Irrational Rotations

The rotation R_θ for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is an ergodic transformation with respect to the Lebesgue measure of circle $S^1 := \{z : |z| = 1\}$.

To see this, suppose that $A \subset S^1$ is an invariant set and $m(A) > 0$. Let x be a density point of A . By definition, for small ϵ there exists an arc $I := (x - \delta, x + \delta)$ (with $\delta < \epsilon$) such that $m(A \cap I) > (1 - \epsilon)m(I)$. From invariance of A and m we conclude that

$$m(R^n(I) \cap A) > (1 - \epsilon)m(R^n(I)).$$

Now we apply a well known result of Jacobi which states that the orbit of any point is dense (minimality of dynamics) in the circle. This implies that we can take n_1, n_2, \dots, n_k such that $R^{n_i}(I)$ are disjoint and cover a subset of S^1 with measure at least $1 - 2\epsilon$. So, we obtain

$$m(A) \geq \sum m(A \cap R^{n_i}(I)) \geq (1 - \epsilon)m(\bigcup R^{n_i}(I)) \geq (1 - \epsilon)(1 - 2\epsilon)$$

This proof can be generalized to obtain the ergodicity of some “irrational rotations” on the n -dimensional torus.

Remark 2. 1. It is easy to see that in general whenever an absolutely continuous (with respect to Lebesgue measure) measure is ergodic, then the orbit of almost all point is dense. The inverse is not true in general. In fact Furstenberg [8] had given an example of minimal volume preserving diffeomorphism which is not ergodic. In the above example we used the minimality to prove the ergodicity. The density of almost all orbits some times is called weak ergodicity. It is a good project to investigate when weak ergodicity implies ergodicity. In [19] we related robust transitivity of diffeomorphisms with ergodicity in some especial cases. See also [6], where weak ergodicity and essential accessibility is treated.

2.1.1. Distributions of first digits of 2^n

As a corollary of the ergodicity of irrational rotation we can prove some nice results in number theory. The last digits of a natural number indicates the congruence modulo 10^k . However, the first digits of a number is not so easy to detect. For example the first two digits of 2^{16} is equal to 65 because $2^{16} = 65536$. The first digits of 2^n is equal to k if and only if $k \times 10^r \leq 2^n < (k + 1)10^r$, for some $r \in \mathbb{N}$. This is equivalent to

$$r + \log(k) \leq n \log(2) < r + \log(k + 1).$$

Let $\alpha = \log(2) \in \mathbb{R} \setminus \mathbb{Q}$. So, the above inequality is equivalent to

$$r + \log(k) \leq \{n\alpha\} < r + \log(k + 1).$$

Take $A = [\log(k), \log(k + 1)]$. By (unique) ergodicity of rotation R_α we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} F(n, A) = \log\left(1 + \frac{1}{k}\right).$$

Where $F(n, A)$ is the number of times that the orbit of (up to iterate n) a typical point belongs to A . It is easy to see that this is nothing but the number of elements of the sequence $1, 2, \dots, 2^{n-1}$ which first digit is k . As a corollary we obtain even more interesting result which states that “7” occurs more frequently as a first digit of powers of 2 than “8”!

2.2. Lagrange Problem

Consider a system of $d + 1$ points labelled $0, 1, \dots, d$ that perform a planar motion according to the following rule: the point 0 is fixed while for $k = 1, \dots, d$ the point k circle around the point labelled $k - 1$ with radius r_k and angle velocity ω_k . We would like to describe the motion of the last point which by means of complex numbers may be written as

$$z(t) = a_1 e^{i\omega_1 t} + \dots + a_k e^{i\omega_k t}$$

with $a_k \in \mathbb{C}$, $|a_k| = r_k$ for $k = 1, \dots, d$. Lagrange asked whether an average angular velocity could be assigned to the system. By average angular velocity we mean $\omega_\infty := \lim_{t \rightarrow \infty} \frac{\phi(t)}{t}$ where $\phi(t)$ is the argument of $z(t)$. Lagrange found that in the special case

$$r_{k_0} > \sum_{i \neq k_0} r_i \quad \text{for some } k_0 \in \{1, \dots, d\},$$

the mentioned limit exists and in fact $\omega_\infty = \omega_{k_0}$. Here, we use the ergodicity of certain torus map to show a more general result about the average ω_∞ .

For this purpose we recall an irrational translation in \mathbb{T}^d . For every $v \in \mathbb{R}^d$, the v -translation ρ_v of \mathbb{T}^d is defined by $\rho_v(x) = x + \pi(v)$ where π is projection to torus. We may define a continuous translation generating by v (velocity vector) as $\phi_t(x, v) := \rho_{tv}(x)$. The numbers x_1, x_2, \dots, x_d are said to be independent over \mathbb{Q} if they are independent vectors in \mathbb{R} considered as a vector space over rational field. For $v = (x_1, \dots, x_d)$, the ergodicity (of Lebesgue measure) of ρ_v is equivalent to the independence of $1, x_1, x_2, \dots, x_d$. The proof is an easy corollary of the following: There exists an orthonormal base $\{\phi_k | k \in \mathbb{Z}^d\}$ of $L^2(\mathbb{T}^d)$ such that $\phi_0 = 1$ and $\phi_k \circ \rho_{\pi(x)} = e^{i\langle k, 2\pi x \rangle} \phi_k$. So, if $\phi \in L^2$ is such that $\phi \circ \rho_{\pi(x)} = \phi$ we conclude that

$$\begin{aligned} \langle \phi, \phi_k \rangle &= \langle \phi \circ \rho_{\pi(x)}, \phi_k \rangle \\ &= \langle \phi, \phi_k \circ \rho_{\pi(-x)} \rangle = e^{-i\langle k, 2\pi x \rangle} \langle \phi, \phi_k \rangle \end{aligned}$$

and so if $\langle k, x \rangle \neq 0 \pmod{\mathbb{Z}}$ for all $k \in \mathbb{Z}^d$ implies that $\langle \phi, \phi_k \rangle = 0$ and so ϕ is constant.

Now, we will back to the Lagrange problem on the mean rotation of moving bodies. Let $z(t) := r(t)e^{i\phi(t)}$. We have that

$$\phi' = \mathcal{R}\left(\frac{z'}{iz}\right) = \mathcal{R}\left(\frac{\sum a_k \omega_k e^{i\omega_k t}}{\sum a_k e^{i\omega_k t}}\right).$$

Now, let us define $f : \mathbb{T}^d \rightarrow \mathbb{R}$ as

$$f(x_1, \dots, x_d) := \mathcal{R} \left(\frac{\sum |a_k| \omega_k e^{2\pi i x_k}}{\sum |a_k| e^{2\pi i x_k}} \right).$$

Taking $a_k = |a_k| e^{2\pi i \alpha_k}$ introduce

$$\alpha := \pi(\alpha_1, \dots, \alpha_d) \quad \text{and} \quad \omega := \frac{1}{2\pi}(\omega_1, \dots, \omega_d).$$

If $z(t) \neq 0$ we have

$$\phi(T) - \phi(0) = \int_0^T f(\rho_{t\omega}(\alpha)) dt.$$

By the **unique ergodicity** of the irrational translation (so provided ω is irrational, i.e, $1, \omega_1, \dots, \omega_d$ are independent over rational numbers.) we have that

$$\omega_\infty := \lim_{T \rightarrow \infty} \frac{\phi(T)}{T} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\rho_{t\omega}(\omega)) dt = \int_{\mathbb{T}^d} f dm_{\mathbb{T}^d}.$$

Observe that in the above proof we used the convergence of time average to space average for an specified point which is permitted by unique ergodicity (in the case of uniquely ergodic transformation, the Birkhoff average converges for **all** points.)

So, it has come out that $\omega_\infty = \sum \omega_k p_k$ where

$$p_k = \mathcal{R} \left(\int_{[0,1]^d} \frac{|a_k| a^{2\pi i x_k}}{\sum_l |a_l| e^{2\pi i x_l}} dx_1 \cdots dx_d \right)$$

Finally we leave as an exercise to prove that

$$p_k = m_{\mathbb{T}^d}(\{x \in \mathbb{T}^d : |a_k| > |\sum_{l \neq k} |a_l| e^{2\pi i x_l}|\}).$$

This in particular, implies the Lagrange result.

2.3. Linear Automorphisms

Let A be a matrix in $SL(N, \mathbb{Z})$. Then A induces in a canonical way a diffeomorphism on the torus \mathbb{T}^n which will be denoted by A . It can be proved that A is ergodic if and only if no root of unity is an eigenvalue of A . We recall a proof, because of its nature. It uses Fourier analysis. However, this method is not used for the proof of the ergodicity of larger classes of transformations.

If $\sum_{k \in \mathbb{Z}^N} c_k e^{2\pi i \langle k, x \rangle}$ is the Fourier series of a function $f : \mathbb{T}^N \rightarrow \mathbb{R}$ then

$$f \circ A = \sum_{k \in \mathbb{Z}^N} c_k e^{2\pi i \langle A^t k, x \rangle}.$$

If $e^{2\pi i \frac{p}{q}}$ is an eigenvalue of A then $(A^t)^q(k_0) = k_0$ for some $k \in \mathbb{Z}^N \setminus \{0\}$. Consequently, $f(x) = \sum_{j=0}^{q-1} e^{2\pi i \langle (A^t)^j(k_0), x \rangle}$ is invariant $f \circ A = f$ and is not constant.

Now assume that no root of unity is eigenvalue of A . If f is an invariant function then $c_k = c_{(A^t)^n k}$ for all n . So, if f is not constant then for infinitely many l we have c_l is constant and contradicts $\sum |c_k|^2 < \infty$.

A more interesting question is whether an ergodic automorphism of torus is stable ergodic. A volume preserving diffeomorphism f is stably ergodic iff any volume preserving diffeomorphism g , C^1 -close to f is also ergodic. For a deep result in this direction see [13].

3. ERGODICITY OF CONSERVATIVE ANOSOV DIFFEOMORPHISMS AND HOPF ARGUMENT

Anosov [1] proved that C^2 -globally hyperbolic diffeomorphism (so called Anosov diffeomorphisms) are ergodic. A diffeomorphism $f : M \rightarrow M$ of a compact manifold is Anosov if the whole manifold M is a hyperbolic set, i.e. there exists a decomposition $TM = E^s \oplus E^u$ such that

$$\|Df|E^s\| < \lambda < 1 < \lambda^{-1} < \|Df^{-1}|E^u\|^{-1}$$

for an appropriate choice of norm. This means that the derivative is direct sum of a uniform expansion and a uniform contraction. In fact any C^1 -perturbation of Anosov diffeomorphism is also Anosov and consequently, they are *stably ergodic*. Here we give the sketch of the proof of the ergodicity of a volume preserving Anosov diffeomorphism. After it we discuss the developments after Anosov and the stable ergodicity beyond uniform hyperbolicity.

3.1. Ergodicity of Anosov diffeomorphisms

Let $f : M \rightarrow M$ be a diffeomorphism preserving a Lebesgue probability measure μ . We denote by \mathcal{B} the Borel σ -algebra. In what follows, we will outline the Hopf's argument [2] and show how it can be applied to prove the ergodicity of C^2 -volume preserving Anosov diffeomorphisms. By ergodicity of a diffeomorphism here we mean the ergodicity of the fixed Lebesgue measure.

Observe that as μ is also invariant for f^{-1} one can apply the Birkhoff-Khinchin theorem for f^{-1} . It comes out that the function $\hat{\phi}$ obtained as above is the same for both f and f^{-1} in a full measure set. See [7] for a proof of Birkhoff-Khinchin theorem in more general forms and more consequences.

Let f be a C^2 diffeomorphism of a smooth compact Riemannian manifold M that is partially hyperbolic and that preserves a smooth measure μ . To study ergodicity of f one uses a version of the Hopf argument adapted to the case of partially hyperbolic systems. Let \mathcal{B} be the Borel σ -algebra of M . Say that $x, y \in M$ are stably equivalent if

$$d(f^n(x), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and unstably equivalent if

$$d(f^n(x), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow -\infty.$$

Stable and unstable equivalence classes induce two partitions of M and we denote by S and U , the Borel σ -algebras they generate. In the case of an Anosov diffeomorphism the partition of stably equivalence and unstably equivalence classes are respectively partition by stable and unstable foliations. Recall that for an algebra $\mathcal{A} \subset \mathcal{B}$ its saturated algebra is the set

$$\text{Sat}(\mathcal{A}) = \{B \in \mathcal{B} : \text{there exists } A \in \mathcal{A} \text{ with } \mu(A \Delta B) = 0\}.$$

Now we briefly recall the Hopf argument which states that f is ergodic if

$$(3.1) \quad \text{Sat}(S) \cap \text{Sat}(U) = \mathcal{T}$$

where \mathcal{T} is the trivial algebra. To see this observe the following:

1. For any continuous function ϕ , let $\hat{\phi}^+$ (resp. $\hat{\phi}^-$) denotes the Birkhoff average of ϕ obtained in Birkhoff-Khinchin theorem for iterates of f (resp. f^{-1}). Then $\hat{\phi}^+$ (resp. $\hat{\phi}^-$) is constant along the elements of stably (resp. unstably) equivalence classes.

2. As we have mentioned before, there exists a full μ -measure subset of M on which $\hat{\phi}^+ = \hat{\phi}^-$.

The above two observations imply that for any $\alpha \in \mathbb{R}$, the level set

$$\phi_\alpha := \{x \in M : \hat{\phi}^+(x) \geq \alpha\}$$

is an invariant set which belongs to $\text{Sat}(S) \cap \text{Sat}(U)$. Recall that the ergodicity of the Lebesgue measure is equivalent to prove that for some α and almost all $x \in M$ one has $\hat{\phi}^+(x) = \alpha$. Now by 3.1 and routine measure theory argument we obtain the ergodicity of the Lebesgue measure.

It remains to show the identity 3.1 for an Anosov diffeomorphism. Observe that, up to the moment, we have not used more than C^1 regularity for the diffeomorphism. In fact we just used the existence of the invariant manifolds. But, now we need a $C^{1+\epsilon}$, $\epsilon > 0$ regularity of f . This is because we will use the absolute continuity of the stable and unstable foliations of an Anosov diffeomorphisms. Indeed, let $\mathcal{A} \in \text{Sat}(S) \cap \text{Sat}(U)$. By definition there exists \mathcal{A}_1 with $m(\mathcal{A}_1 \Delta \mathcal{A}) = 0$ and \mathcal{A}_1 is formed by stable leaves. On the other hand, there exists a subset \mathcal{B} with $m(\mathcal{B} \Delta \mathcal{A}_1) = 0$ such that \mathcal{B} is u -saturated. By absolute continuity \mathcal{B} is almost equal to an s and u saturated subset of M . The only non empty s and u saturated subset of M is exactly M .

As we pointed out, at this point the C^2 (or even $C^{1+\alpha}$) regularity of the Anosov diffeomorphism is crucial. In fact C. Robinson and L.S. Young gave an example of a C^1 Anosov diffeomorphism with non-absolutely continuous stable foliation. See subsection ‘‘Fubini Nightmare’’ to see examples of non-absolutely continuous foliations. ♠

4. STABLE ERGODICITY BEYOND UNIFORM HYPERBOLICITY

Although uniform hyperbolic behavior has proved to be a powerful tool to get different types of chaotic properties, as early as the late 60’s or early 70’s the need of relaxing the

full hyperbolicity hypothesis became apparent. Among other important attempts, partial hyperbolicity and non-uniform hyperbolicity have shown to be very useful in order to obtain such kind of properties, for instance the ergodic ones. Let M be a compact manifold and $f: M \rightarrow M$ a C^1 diffeomorphism. We say that the splitting $TM = E^{cs} \oplus E^{cu}$ is dominated if it is Df invariant and there exist $C > 0$ and $\lambda < 1$ such that:

$$\|Df|_{E_x^{cs}}\| \cdot \|Df^{-1}|_{E_{f(x)}^{cu}}\| \leq \lambda \quad \text{for all } x \in M.$$

A diffeomorphism is called **partially hyperbolic** if the tangent bundle admits a dominated splitting and at least one of the sub-bundles E^{cu} or E^{cs} is uniformly hyperbolic. Some times people call f a partially hyperbolic diffeomorphism if $TM = E^s \oplus E^c \oplus E^u$ where E^s and E^u are respectively uniformly contracting and expanding and E^c is called central bundle.

Ergodicity is one of the paradigms of chaotic dynamics and is quite well understood for the conservative uniformly hyperbolic diffeomorphisms. Recently, we have focused on some ergodic properties of diffeomorphisms in the midway between non-uniform and partial hyperbolicity.

In very recent results we give new criteria to obtain ergodicity based on properties of the Lyapunov exponents and periodic points of the diffeomorphism f . These criteria should be seen as complementary to the method of local ergodicity. Roughly speaking, local ergodicity means that each ergodic component is contained and of full measure in some open set. This fact plus transitivity implies ergodicity of f . Probably there are many examples where both criteria, ours and local ergodicity, can be applied. In a forthcoming paper ([15] and the research announcement [14]) we present some cases where it seems difficult to obtain (a priori) local ergodicity and our criteria apply. For instance we prove that a transitive non-uniformly hyperbolic conservative diffeomorphisms of a surface is ergodic.

4.1. Ergodicity and partial hyperbolicity

An important tool in proving ergodicity of these systems is the accessibility property i.e. f is accessible if any two points of M can be joined by a curve that is a finite union of arcs tangent to the strong bundles.

In the last years, since the pioneer work of Grayson, Pugh and Shub [9], many advances have been made in the ergodic theory of partially hyperbolic diffeomorphisms. In particular, we want to mention the very recent works: by Burns and Wilkinson [5] proving that (essential) accessibility plus a bunching condition (trivially satisfied if center bundle is one dimensional) implies ergodicity and by R. Hertz, R. Hertz, and Ures [16] obtaining the Pugh-Shub Conjecture (see about density of stable ergodicity for conservative partially hyperbolic diffeomorphisms with one dimensional center. That is, ergodic diffeomorphisms contain an open and dense subset of the conservative partially hyperbolic ones. See [17] for a recent survey on the subject.

Pugh and Shub have splitted their conjecture into two conjectures (see [11]): (essential) accessibility implies ergodicity and stable (essential) accessibility is a dense property among partially hyperbolic diffeomorphisms.

Let us define accessibility and essential accessibility. For a partially hyperbolic diffeomorphism there are two invariant foliations W^s, W^u which are respectively stable and unstable foliations of a partially hyperbolic diffeomorphism. The existence of such foliations are guaranteed by previous works of Hirsch-Pugh-Shub. For any point $x \in M$ we define the accessibility class of x as the set of all points y which satisfies the following: $y \in Acc(x)$ if there exists a piecewise smooth curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x, \gamma(1) = y$ and $\gamma(t)$ is tangent to $E^s \cup E^u$. (It is not $E^s \oplus E^u$.) A diffeomorphism is called accessible if the unique accessibility class is the whole M and a partially hyperbolic diffeomorphism $f : M \rightarrow M$ is essentially accessible if every measurable set that is a union of entire accessibility classes has either full or zero volume.

The question of accessibility is closely related to problems in control theory. In fact, analogous density theorems in control theory initially suggested the Conjectures of Pugh-Shub. The sole reason that the results in control theory cannot be directly transported to this setting is that we do not perturb the bundles E^u and E^s directly, but rather the diffeomorphism f .

Many of the arguments of [5] and [16] seem to have technical difficulties to be generalized to prove, in its full generality, Pugh-Shub Conjecture for center bundle of any dimension. Just to mention one of them, some center bunching condition (see [5]) seems to be needed to obtain that strong holonomies restricted to center manifolds are Lipschitz. This fact is used in an essential way in their Juliennes arguments.

From here up to the subsection of Fubini Nightmare we follow Pugh-Shub-Grayson paper:

Hirsch, Pugh and Shub showed that f (a partially hyperbolic diffeomorphism) leaves invariant foliations W^u, W^s . Some times, It also leaves invariant a center foliation W^c . For example time one map of the geodesic flow on a manifold with negative curvature and any C^1 -perturbation of it satisfies this peoperties. According to Pugh and Shub, the foliations W^u, W^s are absolutely continuous and their Radon-Nikodym derivatives are uniformly bounded on reasonable transversals. The absolute continuity property in a more general setting is verified. To analyze ergodicity of f one adapts the ideas from Hopf and consider the forward, backward, and bi-directional Birkhoff averages of functions $g : M \rightarrow \mathbb{R}$ along f -orbits, If g is any L^1 - function, these Birkhoff averages converge almost everywhere to limit functions B^+g and B^-g . These three functions are invariant under f and are equal almost everywhere. In fact the map is a continuous linear projection from L^1 onto the closed linear subspace $Inv(f)$ of f -invariant functions. See Mañé's book on ergodic theory.

Suppose that f is not ergodic. Then $Inv(f)$ includes essentially nonconstant functions- functions which are nonconstant on every set of full measure. The set C of continuous functions is dense in L^1 and a continuous transformation carries dense subsets of the domain to dense subsets of the range. Thus, $B(C)$ is dense in $Invf$, and for some continuous function $g, G = B(g)$ is essentially nonconstant. However, except for a zero set, G is constant along stable leaves and along unstable leaves. Since we know that $B^+g(y)$ exists almost everywhere, it follows from the absolute continuity of W^s that for almost all W^s -leaves, $W^s(y), B^+g$ exists everywhere on $W^s(y)$ and is constant along the leaf. The same is valid for unstable leaves respecting backward Birkhoff averages. Since $B^+g(y) = B^-g(y) =$

G almost everywhere, we deduce that our essentially nonconstant function G is essentially constant along stable and unstable leaves.

For any value $v \in \mathbb{R}$, divide M , modulo a zero set, as $M = A \cup B$ where $A = \{p \in M : G(p) < v\}$ and $B = \{p \in M : G(p) > v\}$. Since G is integrable, A and B are measurable, and since G is essentially nonconstant there exists a choice of v so that A and B both have positive measure. Since G is essentially constant along unstable and stable leaves, A and B consist (modulo zero sets) of essentially complete unstable and stable leaves. That is, A and B are essentially W^{us} -saturated.

We suppose that any two point of M may be joint with curves tangent to E^s or E^u . This is the case for time one map of the geodesic flow. Let a be a density point of A and b be a density of B . They are joint by a W^{us} -path, and we claim that this leads to a contradiction. Here are false proof of this fact.

Assume that the center foliation W^c is absolutely continuous. Since A and B have positive measure they meet many center leaves in sets of positive leaf measure. Thus, we may assume that a is a density point of $A \cap W^c(a)$ and b is a density point of $B \cap W^c(b)$. The natural thing to do is to start at a and slide along the foliations W^u, W^s , using W^u along the unstable arcs and W^s along the stable ones. Sliding along the unstable foliation sends center manifolds to center manifolds because the unstable manifolds foliate the center unstable manifolds. (this is not the case for a general partially hyperbolic diffeomorphism). The same is true for the stable manifolds. The concatenated holonomy map $h : W^c(a) \rightarrow W^c(b)$ is absolutely continuous with bounded Radon-Nikodym derivative. It carries $A \cap W^c(a)$ onto $A \cap W^c(b)$ since A is essentially W^{us} -saturated. The map is a C^1 diffeomorphism (It is proved rigorously) and therefore does not affect density.

So we see that b is a density point of both $A \cap W^c(b)$ and $B \cap W^c(b)$. Since A and B are disjoint this is impossible.

The first defect in this proof is the assumption that W^c is absolutely continuous. Shub-Wilkinson has given a dynamical example for non-absolutely continuous foliation. The second defect is that essential W^{us} -saturation of A does not imply that the holonomy carries $A \cap W^c(a)$ to $A \cap W^c(b)$.

Here is a second false proof. Instead of using the unstable and stable foliations to define holonomy maps on transverse center manifolds, one can use them to slide a three-dimensional neighborhood U of a . Points of U slide along unstable arcs neighboring the first leg of the path jointing a to b , then along stable arcs neighboring the second leg, etc. This gives a homeomorphism $O : U \rightarrow V$ with $O(a) = b$ and V a neighborhood of b . By absolute continuity of W^u, W^s , O is absolutely continuous. Since A is essentially W^{us} -saturated, $O(A \cap U) = A \cap V$ except for a zero set. Since $O(a) = b$, b is a density point of both A and B , a contradiction.

The defect of this proof is the tacit assumption that O carries density points to density points. Although a Lipeomorphism between open sets of Euclidean space has this property, the existence of a bounded Radon-Nikodym derivative does not imply Lipschitzness in dimensions > 1 . To settle these problems Pugh-Shub propose a dynamical way to define the density points, called julienne density points and turns out that sliding along W^u, W^s preserves the julienne density points.

4.2. Fubini’s Nightmare

The non-absolute continuity of a foliation is a “pathological” phenomenon which has been mentioned by some dynamicists and has been shown that it is typical in some senses (so may be it is better not to call pathological!). A first example which is not a dynamical foliation has been given by Katok-Milnor, See [12]. We just give the idea of the construction of this example.

Let

$$f_c(x) = \begin{cases} \frac{x}{c}, & 0 \leq x < c; \\ \frac{x-1}{c-1}, & c \leq x \leq 1. \end{cases}$$

We know that by ergodicity of f_c , for almost all point $x \in (0, 1)$ we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} \chi_{[0,c]}(f_c^i(x))}{n} = c.$$

Now consider the following foliation

$$[0, 1]^2 = \bigcup_{\alpha} \mathcal{F}_{\alpha} = \bigcup_{\alpha} \{(c, x) : F_{[0,c]}(x) = \alpha\}$$

where $F_{[0,c]}(x)$ stands for the the expression inside the limit above. It can be shown that each leaf is an analytic curve. But we can show that the foliation (of these curves) is not even absolutely continuous. We consider two vertical interval in the square and transversal to our foliation, $T_0 = \{c = c_0\}, T_1 = \{c = c_1\}$. By the above observations we obtain that, almost all (Lebesgue one dimensional measure) points in T_0 belong to a unique \mathcal{F}_{c_0} . On the other hand almost all points in T_1 belong to \mathcal{F}_{c_1} . This means that a full measure subset of T_0 is sent to a null measure set in T_1 by the holonomy of the foliation (recall that on each leaf $F_{[0,c]}$ is constant.) As an exercise show that we can find a full (two dimensional Lebesgue measure) measure subset of $[0, 1]^2$ which intersects each leaf of the foliation in a unique point. This means that we can not apply Fubini’s theorem for this foliation.

The example of Shub-Wilkinson is as follows. Let A be a linear Anosov diffeomorphism of \mathbb{T}^2 and $f = A \times Id : \mathbb{T}^3 \rightarrow \mathbb{T}^3$. Then, f is partially hypeerbolic with zero Lyapunov exponents in the central direction. By a C^1 -pertubation one can gain a new partially hyperbolic diffeomorphism with non-zero integrant central Lyapunov exponent, i.e, $\int_{\mathbb{T}^3} \lambda_c(x) > 0$. As f is partially hyperbolic with a central foliation, this foliation is normally hyperbolic and by Hirsh-Pugh-Shub theory we know that this foliation is structurally stable. This means that for any C^1 -close diffeomorphism g a new invariant foliation and equivalent to the foliation by circles (which is invariant by f) is available. So, for instance the leaves of the new foliation are topological circles. It can be shown that under the condition of absolute continuity of this foliation, the integral Lyapunov exponents can not be positive. As, th example of C^1 -close enough g with positive central Lyapunov expoent is given by Shub-Wilkinson, their example is also an example of pathological phenomenon of non-absolute continuity.

4.3. Ergodicity beyond partial hyperbolicity

The Pugh-Shub program for ergodicity is a deep program which is in the context of partially hyperbolic diffeomorphisms. Still, stable ergodicity of Lebesgue measure implies some dynamical properties (weak forms of Hyperbolicity). Tahzibi and Horita [10] have proved that a stably ergodic symplectic diffeomorphism is partially hyperbolic. This means that the robustness of ergodicity implies a weak hyperbolicity property. A similar (but topological) result to their result is a result of Bonatti-Diaz-Pujals which proves that robustly transitive diffeomorphisms admit dominated splitting. In fact, the proof of Tahzibi-Horita's result uses the arguments of Bonatti-Diaz-Pujals.

We may divide the problems and further studies in the theory in two category:

- Give sufficient conditions for ergodicity (or even stable ergodicity).
- Understanding necessary conditions for stable ergodicity in terms of dynamics in tangent space or topological properties of the ambient manifold.

Pugh-Shub program provides an approach to the first question. There are not many results outside partially hyperbolic world.

Tahzibi, in his thesis [18] showed the existence of an open class of ergodic diffeomorphisms which are not partially hyperbolic. However, they enjoy dominated splitting property. Later, Tahzibi and Horita [10] proved that in fact partial hyperbolicity is a sufficient condition for stable ergodicity inside symplectic diffeomorphisms. Saghin and Xia, in the asame time and independently prove that if a symplectic diffeomorphism is not partially hyperbolic, then with an arbitrarily small C^1 perturbation we can create a totally elliptic periodic point inside any given open set. As a consequence, a C^1 -generic symplectic diffeomorphism is either partially hyperbolic or it has dense elliptic periodic points and so, stably ergodic diffeomorphisms are partially hyperbolic. This is because, a diffeomorphism with totally elliptic point can be perturbed to break ergodicty. This perturbation can be made in the symplectic world and this argument (which is simple and wellknown in symplectic diffeomorphisms) is common between Tahzibi-Horita and Saghin-Xia results.

Let us introduce the class of diffeomorphisms for which we proved stable ergodicity without partial hyperbolicity [18]. The class $\mathcal{V} \subset \text{Diff}^1(\mathbb{T}^n)$ under consideration consists of diffeomorphisms which are deformations of an Anosov diffeomorphism. To define \mathcal{V} , let f_0 be a linear Anosov diffeomorphism of the n -dimensional torus \mathbb{T}^n (in fact, we need f_0 only to be an Anosov diffeomorphism on $M = \mathbb{T}^n$ whose foliations lifted to \mathbb{R}^n are global graphs of C^1 functions over the corresponding invariant subbundles). Denote by $TM = E_0^s \oplus E_0^u$ the hyperbolic splitting for f_0 with $\dim(E_0^s) = s$, $\dim(E_0^u) = u$ and let $V = \bigcup V_i$ be a finite union of small pairwise disjoint balls in \mathbb{T}^n . We suppose that f_0 has at least one fixed point outside V . By definition $f \in \mathcal{V}$ if it satisfies the following C^1 open conditions:

1. TM admits a dominated decomposition and there exist small continuous cone fields C^{cu} and C^{cs} invariant for Df and Df^{-1} containing respectively E_0^u and E_0^s .
2. f is C^1 close to f_0 in the complement of V , so that for $x \notin V$ there is $\sigma < 1$:

$$\|(Df|_{T_x D^{cu}})^{-1}\| < \sigma \quad \text{and} \quad \|Df|_{T_x D^{cs}}\| < \sigma.$$

3. There exists some small $\delta_0 > 0$ such that for $x \in V$:

$$\|(Df|_{T_x D^{cu}})^{-1}\| < 1 + \delta_0 \quad \text{and} \quad \|(Df|_{T_x D^{cs}})\| < 1 + \delta_0$$

where D^{cu} and D^{cs} are disks tangent to C^{cu} and C^{cs} .

THEOREM 4.1. *Every $f \in \mathcal{V} \cap \text{Diff}_\omega^2(\mathbb{T}^n)$ is stably ergodic.*

For non-conservative diffeomorphisms in \mathcal{V} we prove uniqueness of SRB measure. In fact we assume the diffeomorphisms are **volume hyperbolic** (in the conservative case this is automatic from the domination property):

DEFINITION 4.1. Let $f: M \rightarrow M$ be a C^1 diffeomorphism and $TM = E^1 \oplus E^2$; we say that this decomposition is volume hyperbolic if for some $C > 0$ and $\lambda < 1$:

$$|\det(Df^n(x)|_{E^1})| \leq C\lambda^n \quad \text{and} \quad |\det(Df^{-n}(x)|_{E^2})| \leq C\lambda^n.$$

THEOREM 4.2. *Any $f \in \mathcal{V} \cap \text{Diff}^2(\mathbb{T}^n)$ for which the dominated decomposition $TM = E^{cs} \oplus E^{cu}$ is volume hyperbolic has a unique SRB measure with a full Lebesgue measure basin.*

4.4. Example

Here we give a C^1 open set of diffeomorphisms that satisfy the hypothesis of Theorems 4.1 and 4.2.

In particular in the above theorem, if we work in $\mathcal{V} \subset \text{Diff}_\omega(M)$ (the volume preserving subset), we obtain stably ergodic diffeomorphisms. A diffeomorphism f is called *robustly transitive* if any C^1 nearby diffeomorphism to f is also transitive. The first non-partially hyperbolic and robustly transitive example is constructed in [4] on \mathbb{T}^4 . We will construct the example of robustly transitive diffeomorphisms without hyperbolic subbundles in higher than 4 dimensions. Let f_0 be a volume preserving linear Anosov diffeomorphism on the \mathbb{T}^n for which

$$T_x(\mathbb{T}^n) = \mathbb{R}^n = E_1^s \prec E_2^s \cdots \prec E_{n-2}^s \prec E^u$$

where $\dim(E^u) = 2$ and $\dim(E_i^s) = 1$ and E^u is uniformly expanding and all E_i^s are uniformly contracting.

We may suppose that f_0 has fixed points p_1, p_2, \dots, p_{n-2} . Let $V = \bigcup B(p_i, \delta)$ be a union of balls centered at p_i and radius sufficiently small $\delta > 0$. By iteration, we may also suppose that f_0 has a fixed point out of V . The idea is to deform the Anosov diffeomorphism inside V , passing first through a pitchfork bifurcation along $(E_i^s \oplus E_{i+1}^s)(p_i)$ inside $B_i = B(p_i, \delta)$ and then another deformation (see fig 1) to get complex eigenvalue for a fixed point near to p_i .

More precisely, first we modify along $E_i^s(p_i) \oplus E_{i+1}^s(p_i)$ for $1 \leq i \leq n - 3$ until the stable index of p_i drops one and two fixed points q_i, r_i are created. These new fixed points have

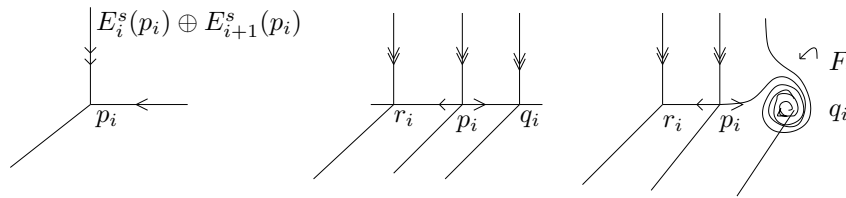


FIG. 1. Deformation of Anosov

stable index equal to $n-2$. In the next step we mix the two contracting subbundles of $T_{q_i}M$ corresponding to $E_i^s(q_i)$ and $E_{i+1}^s(q_i)$ and get a complex eigenvalue. These modifications can be done by an isotopy and in a way to obtain volume preserving diffeomorphisms (see [4]).

After these deformations we get a new diffeomorphism which we call also f and we have the following Df invariant decomposition for the tangent bundle of q_i :

$$T_{q_i}M = E_1 \prec \dots \prec E_{i-1} \prec F_i \prec \dots \prec E^u$$

where E_i is one dimensional, E^u is two dimensional and uniformly expanding and F_i is the two dimensional subbundle corresponding to the complex eigenvalue. Finally we do the same for p_{n-2} , but in the unstable direction of it. That is, after the modifications along the unstable subbundle of p_{n-2} we get a new fixed point q_{n-2} such that:

$$T_{q_{n-2}}M = E_1 \prec \dots \prec E_{n-2}^s \prec F_{n-2}$$

and F_{n-2} is the subbundle corresponding to the complex expanding eigenvalue of q_{n-2} .

In this way we get a C^1 open set $\tilde{\mathcal{V}}$ of diffeomorphisms satisfying the conditions 1-3 mentioned in the Introduction. Another important fact is about the mentioned hyperbolic fixed point outside V .

We supposed that there exists a hyperbolic fixed point q of f_0 outside V with stable index $s = n - 2$ in our example. For any $f \in \tilde{\mathcal{V}}$, as f is C^1 near to Anosov diffeomorphism outside V , it has a fixed point outside V which is the continuation of q which we will call it also by q .

It is easy to see that the stable manifold of the continuation of q intersects any disk tangent to C^{cu} with radius more than ϵ_0 , for some small $\epsilon_0 > 0$. The similar thing for the unstable manifold and disks tangent to C^{cs} happens. This is just because of the denseness of stable and unstable leaves of f_0 . Indeed, take a compact part of $W^s(q, f_0)$ to be ϵ_0 dense. Now, if V is small enough this compact part of the stable manifold continues to be a part of the stable manifold for the continuation of q .

LEMMA 4.1. $f \in \tilde{\mathcal{V}}$ is robustly transitive.

Proof: The proof goes as in \mathbb{T}^4 case in [4, Lemma 6.8] and we just remember the steps. The main idea to prove robust transitivity is to show the robust density of the stable and unstable manifold of an hyperbolic fixed point.

Let U and V be to open subsets. Using λ -Lemma and the density of invariant manifolds of q we intersect some iterate of U with V and get transitivity of f . \square

LEMMA 4.2. $f \in \tilde{\mathcal{V}}$ is not partially hyperbolic.

Proof: This is just because of the definition of partially hyperbolic systems. f is partially hyperbolic if $TM = E^s \oplus E^c \oplus E^u$ is a decomposition into continuous subbundles where at least two of them are nonzero where E^s and E^u are respectively uniformly contracting and expanding. Suppose that f is partially hyperbolic. By the continuity of the decomposition of TM and existence of a dense orbit, the dimension of E^s and E^u is constant.

We claim that $\dim(E^s) = n - 2$ and this results a contradiction, because in $T_{p_i}M$ there does not exist $n - 2$ contracting invariant directions. To prove the claim observe that if we suppose that $\dim(E^s) = j < n - 2$, then by the decomposition of $T_{q_j}M$:

$$T_{q_j}M = E_1 \prec \cdots \prec E_{j-1} \prec F_j \cdots \prec E^u.$$

By definition, $E^s(q_j)$ must contain $E_1 \oplus \cdots \oplus E_{j-1}$ and then as F_j does not have any invariant subbundle we conclude that $\dim(E^s(q_j)) \geq j + 1$ and this is a contradiction, because $\dim(E^s) = j$. This shows that $\dim(E^s) = 0$. By investigating $T_{p_{n-2}}M$, it is obvious that TM can not have invariant unstable subbundle, too. \square

4.5. Recent Progresses in Pugh-Shub Conjecture

In a recent work joint with F. Rodriguez Hertz, M.A. Rodriguez Hertz and R. Ures we proved a new criterium for ergodicity based on transversality condition between some invariant manifolds (See www.icmc.usp.br/~tahzibi/publication.html). We used this criterium to prove the Pugh-Shub conjecture in C^1 -topology for partially hyperbolic diffeomorphisms with two dimensional central bundle. More precisely we proved that there exists an open and dense subset of $\text{Diff}_\omega^2(M, 2)$ whose elements are ergodic. By $\text{Diff}_\omega^2(M, 2)$ we mean the set of volume preserving partially hyperbolic diffeomorphisms whose central direction is two dimensional. The proof of this conjecture is not in the scope of this survey.

4.6. More open Problems

- Prove that C^2 -volume preserving diffeomorphisms is dense among C^1 diffeomorphisms. This is an old question, see Zehnder [20] for the history.
- Give an example of a non-ergodic C^1 -Anosov diffeomorphism.
- Give more examples of stably ergodic diffeomorphisms which are not partially hyperbolic. We mean except for the examples in [18].

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