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Dynamics of the viscous Cahn-Hilliard equation

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In this we study generalized viscous Cahn-Hilliard equations with critically growing nonlinearities in $W_0^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ for $n \geq 3$. We prove that the problem with such critically growing nonlinearity is locally well posed and obtain a bootstrapping procedure assuring that the solutions are classical. For $p = 2$ and *almost critical dissipative nonlinearities* we obtain a global well posedness result, existence of global attractors in $H_0^1(\Omega)$ and uniform $L^\infty(\Omega)$ bounds for the attractors. Finally, we obtain a result on continuity of exponential attractors which shows that, in dimensions $n = 3$ and $n = 4$, the attractor of the original Cahn-Hilliard equation coincides (in a sense to be specified) with the attractor for the corresponding semilinear heat equation.
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1. INTRODUCTION

Our aim is to study viscous Cahn-Hilliard equations of the form,

$$
(1.1)\qquad
\begin{aligned}
&(1-\nu)u_t = -\Delta(\Delta u + f(u) - \nu u_t), \quad t \geq 0, \ x \in \Omega,\\
&u(t,x) = \Delta u(t,x) = 0 \text{ on } \partial\Omega,\\
&u(0,x) = u_0(x),
\end{aligned}
$$

where $\nu > 0$, $f \in C^2(\mathbb{R},\mathbb{R})$ with suitable dissipation properties (in particular; $f(z) = z - z^3$) and Ω is a smooth bounded domain in $\mathbb{R}^n, n \geq 3$. This model equation was introduced by A. Novick-Cohen in [18] to describe the dynamics of viscous first order phase transitions. Equation (1.1) is interesting not only because of its physical background but also because

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it contains important particular cases; when $\nu = 0$ the original Cahn-Hilliard equation and when $\nu = 1$ the semilinear heat equation. The transition of the asymptotic behavior as parameter ν varies from 0 to 1 was studied in [16] where (1.1) is written in the form of a system

$$(1.2) \qquad \begin{aligned} &(1-\nu)u_t = \Delta w, \ x \in \Omega, \ t > 0, \\ &\nu u_t = \Delta u + f(u) + w, \end{aligned}$$

with suitable initial-boundary conditions. Our paper extends and improves the results of [16] where asymptotic properties of (1.1) were first examined in details.

We are interested in studying the semigroup generated by (1.1) on the phase space $H^1_0(\Omega)$ (and on $W^{1,p}_0(\Omega)$ in case of local solutions), with the equation satisfied in $H^{-1}(\Omega)$ when $\nu > 0$ or in $H^{-3}(\Omega)$ when $\nu = 0$. Assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ is a real function satisfying the following set of assumptions.

- For the local existence of solutions of (1.1), we assume there are constants $\tilde{C} > 0$ and $1 < q \leq \frac{n+p}{n-p}$, such that

$$(1.3) \qquad |f'(s)| \leq \tilde{C}(1 + |s|^{q-1}), \ \forall\, s \in \mathbb{R},$$

- For the global solvability we need to add the dissipativeness condition

$$(1.4) \qquad \limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$$

where $-\lambda_1$ is the first eigenvalue of the Dirichlet Laplacian in Ω.

- Studying existence of the global attractor for (1.1) we need to assume that the nonlinearity is *almost critical*, which can be obtained if f satisfies

$$(1.5) \qquad \lim_{|s|\to\infty} \frac{|f'(s)|}{|s|^{q-1}} = 0.$$

Clearly (1.5) implies (1.3) required for local existence of ϵ-regular solutions to (1.1).

Remark 1. 1. If $q > 1$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ fulfils (1.5) then for arbitrary $\eta > 0$ there is a positive constant C_η such that

$$(1.6) \qquad |f(s_1) - f(s_2)| \leq |s_1 - s_2| \left(C_\eta + \eta|s_1|^{q-1} + \eta|s_2|^{q-1}\right)$$

and

$$(1.7) \qquad |f(s)| \leq C_\eta + \eta|s|^q.$$

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In the present paper, under critical growth condition (1.3) on f, we show local well posedness of the viscous Cahn-Hilliard equation in $W_0^{1,p}(\Omega)$. By local well posedness we understand the existence, uniqueness and continuity wit respect to initial data of ϵ-regular solutions, as introduced in [1]. We prove that

THEOREM 1.1. *Let $f \in C^1(\mathbb{R},\mathbb{R})$ fulfils the condition* (1.3). *Then, to arbitrary $u_0 \in W_0^{1,p}(\Omega)$ corresponds a unique ϵ-regular solution u of* (1.1). *Moreover, if the condition* (1.5) *is satisfied, given a bounded subset B of $W_0^{1,p}(\Omega)$, there is a time $\tau = \tau(B) > 0$ such that any solution starting at a point of B must exist in the interval $[0,\tau]$ and the image of B through the local semigroup generated by* (1.1) *in $W_0^{1,p}(\Omega)$ will be compact for all $0 < t \leq \tau$.*

When $p = 2$, assuming additionally the dissipativeness condition (1.4), local solutions constructed in Theorem 1.1 are in fact global solutions. We have

THEOREM 1.2. *If* (1.3) *and* (1.4) *are satisfied, then the unique local ϵ-regular solution to* (1.1) *in $H_0^1(\Omega)$ exists for all $t \geq 0$.*

The same condition (1.4) together with the almost critical growth assumption (1.5) are sufficient to assure existence of the global attractors $\mathcal{A}_\nu, \nu \in [0,1]$, for the semigroups generated by (1.1) on $H_0^1(\Omega)$ (see Theorem 4.1 in the main body of the paper).

The main result of our paper states that the dynamics generated by (1.1) on the space $H_0^1(\Omega)$ is the same (in a sense to be specified) for all values of ν in $[0,1]$. Thus, in particular, the dynamics generated (under our assumptions) by the semilinear heat equation (corresponding to $\nu = 1$) is the same as the dynamics generated by the original Cahn-Hilliard equation ($\nu = 0$). This result, stated precisely in Theorem 1.3 below, is obtained through the sequence of auxiliary theorems describing mostly spectral properties of the operators $B_\nu := A\big((1-\nu)I + \nu A\big)^{-1}$ (see (2.3)) and $A_\nu = (-\Delta)B_\nu$ (here $A = (-\Delta)$ with Dirichlet boundary condition). We consider first convergence of linear semigroups corresponding to various $\nu \in [0,1]$, then convergence of nonlinear semigroups and finally, assuming that all the equilibria of (1.1) are hyperbolic (the set of equilibria $\mathcal{E}$ is independent of ν) also continuity of local unstable manifolds of the set of equilibria when $\nu \in [0,1]$. Since each attractor $\mathcal{A}_\nu$ is characterized as unstable set of the set of equilibria,

$$\mathcal{A}_\nu = W_\nu^u(\mathcal{E}),$$

we have thus *continuity* of the family $\{\mathcal{A}_\nu\}_{\nu\in[0,1]}$ of global attractors to (1.1). The main result of the paper states that

THEOREM 1.3. *Let $\{T_\nu(t) : t \geq 0\}$ be the gradient nonlinear semigroup associated to* (1.1), *$\mathcal{A}_\nu$ be its global attractor and $\mathcal{V} : H_0^1(\Omega) \to \mathbb{R}$ is its Lyapunov function (independent of ν, see* (3.2)). *If $\{T_1(t) : t \geq 0\}$ has a finite number of hyperbolic equilibria u_i^*, $1 \leq i \leq n$, then $\mathcal{A}_\nu$ is given by*

$$\mathcal{A}_\nu = \bigcup_{i=1}^{n} W_\nu^u(u_i^*), \tag{1.8}$$

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where $W^u_\nu(u_i^*)$ denotes the unstable set of u_i^* related to the semigroup $\{T_\nu(t) : t \geq 0\}$. Furthermore if we denote by $\{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ the set of all distinct values of $\mathcal{V}(u_j^*)$, ordered so that $\mathfrak{n}_i < \mathfrak{n}_j$, $1 \leq i < j \leq p \leq n$, and define $\mathcal{E}_k = \{u_i^* \in \mathcal{E} : \mathcal{V}(u_i^*) = \mathfrak{n}_k\}$, then if $y_\nu(\cdot) : \mathbb{R} \to X^1$ is a global solution for $\{T_\nu(t) : t \geq 0\}$, there are k_1, k_2 with $1 \leq k_1 < k_2 \leq p$, $u_i^* \in \mathcal{E}_{k_1}$ and $u_j^* \in \mathcal{E}_{k_2}$, such that

$$\lim_{t\to-\infty} y_\nu(t) = u_j^* \text{ and } \lim_{t\to+\infty} y_\nu(t) = u_i^*.$$

For all $\nu, \mu \in [0,1]$,

$$\dim_H(\mathcal{A}_\nu) = \max_{j=1,\dots,n} \operatorname{rank}(\mathcal{Q}_{j,\nu}) = \max_{j=1,\dots,n} \operatorname{rank}(\mathcal{Q}_{j,0}) = \dim_H(\mathcal{A}_1), \tag{1.9}$$

$$\operatorname{dist}_H(\mathcal{A}_\nu, A_\mu) \stackrel{\nu\to\mu}{\longrightarrow} 0 \tag{1.10}$$

where $\dim_H$ denotes the Hausdorff dimension, dist_H is the symmetric Hausdorff distance and $\mathcal{Q}_{j,\nu} := \mathcal{Q}_\nu(\sigma_+^\nu(u_j^*))$ is the projection in $L(H_0^1(\Omega))$ defined by the part of the spectrum of the operator $L_\nu = (\nu I + (1-\nu)(-\Delta)^{-1})^{-1}(-\Delta - f'(u^*))$ to the right of the imaginary axis. In addition, there exists $\gamma > 0$ (independent of ν) such that, for all $B \subset H_0^1(\Omega)$ bounded

$$\operatorname{dist}(T_\nu(t)u_0, \mathcal{A}_\nu) \leq c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{1.11}$$

Certain problems related to (1.1) were recently considered in literature. In [13] the two parameter *hyperbolic* extension of (1.1):

$$\omega u_{tt} + u_t - \Delta(-\Delta u + f(u) + \delta u_t) = 0,$$

introduced earlier in [21] in one dimensional case ($n = 1$), was studied. The presentation includes regularity, existence of global attractors and their *upper semicontinuity* as $(\omega, \delta) \to (0,0)$ ($\omega, \delta \in [0,1], \delta \geq \mu\omega$ for some $\mu \in (0,1]$). The function $f(s)$ growth like $|s|^3$ in dimension 3 and the phase space considered in [13] is $[H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$ when $\omega > 0$ and $H^2(\Omega) \cap H_0^1(\Omega)$ when $\omega = 0$.

The problem

$$u_t - \Delta\left(\alpha(u_t - \Delta u + u^3 - u)\right) = 0, \tag{1.12}$$

where $\alpha : D(\alpha) \subset \mathbb{R} \to \mathbb{R}$ is a strictly increasing differentiable function, was considered in [19] (see also references there). Using J.M. Ball's recent approach (see [5]), she studies weak solvability and existence of the global attractor for (1.12) on large phase space (like $L^4(\Omega)$).

This paper is organized as follows. In Section 2 we obtain the local well posedness for the problem (1.1) in $W_0^{1,p}(\Omega)$ and prove a bootstrapping result ensuring that the solutions

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are classical. In Section 3 we prove global well posedness, In section 4 we obtain the existence of global attractors for (1.1). In Section 5 we study the continuity of the family of semigroups generated by $-AB_\nu$. In Section 6 we use the results of Section 5 and the variation of constants formula to obtain the convergence of nonlinear semigroups. In Section 7 we study the linear semigroups associated to the linearizations around a hyperbolic equilibrium. Using the continuity of the linear unstable manifolds obtained in Section 7 we prove in Section 8 that the local unstable manifolds of hyperbolic equilibrium points behave continuously for all values of the parameter and in particular we obtain that the Hausdorff dimension of the attractors $\mathcal{A}_\nu$ is independent of the parameter ν. Finally, in Section 9, we use the results of the previous sections to obtain upper and lower semicontinuity of attractors.

2. SETTING OF THE PROBLEM, LOCAL SOLVABILITY OF (1.1) AND BOOTSTRAPPING

We first rewrite (1.1) in a more suitable form to our purposes. If Δ denotes the Dirichlet Laplacian, inverting the operator $((1-\nu)I - \nu\Delta)$ we have that (1.1) can be rewritten as

$$\begin{aligned} u_t &= ((1-\nu)I - \nu\Delta)^{-1}(-\Delta^2 u - \Delta f(u)) \\ &= -\Delta^2((1-\nu)I - \nu\Delta)^{-1}u - \Delta((1-\nu)I - \nu\Delta)^{-1}(f(u)). \end{aligned} \tag{2.1}$$

Therefore (1.1) will be written abstractly as:

$$u_t = -A^2((1-\nu)I + \nu A)^{-1}u + A((1-\nu)I + \nu A)^{-1}(f(u)), \tag{2.2}$$

where $-A$ denotes the realization in $W^{-1,p}(\Omega)$ of the Laplace operator with Dirichlet boundary condition with domain $D(A) = W_0^{1,p}(\Omega)$.

To simplify further notation let us introduce an auxiliary operator

$$B_\nu := (-\Delta)\big((1-\nu)I - \nu\Delta\big)^{-1} = A\big((1-\nu)I + \nu A\big)^{-1}. \tag{2.3}$$

The operator B_ν has several nice properties; first $B_\nu : X^s \to X^s, s \in \mathbb{R}$ (X^s denotes here the domain of the fractional power $A^{\frac{s}{2}}$), is bounded and invertible, it is also self-adjoint and positive. With the use of this operator equation (2.2) will be written as:

$$u_t = -AB_\nu u + B_\nu(f(u)), \tag{2.4}$$

or, inverting, as

$$B_\nu^{-1}u_t = -Au + f(u). \tag{2.5}$$

Remark 2. 1. The formula (2.4) express important property that the viscous Cahn-Hilliard equation ($\nu > 0$) behaves as a parabolic equation with second order elliptic main part. Thus important properties of its solutions will be similar as for solutions of such parabolic problems. In order, the limiting original Cahn-Hilliard equation ($\nu = 0$) is again a parabolic equation, but with forth order elliptic operator in the main part.

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Remark 2. 2. In a number of references (e.g. [20]) the authors consider the "bi-Neumann" boundary value problem for the original Cahn-Hilliard equation, that is the case of boundary conditions:

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \text{ on } \partial\Omega.$$

That could be considered here as well with few changes.

When the nonlinear term is subcritical in $W_0^{1,p}(\Omega)$ (which means that $q < \frac{n+p}{n-p}$), then the problem (2.2)

$$u_t = -A^2((1-\nu)I + \nu A)^{-1}u + A((1-\nu)I + \nu A)^{-1}f(u) = -AB_\nu u + B_\nu(f(u)),$$

will be studied using the standard approach (see [15]) since $A^2((1-\nu)I + \nu A)$ is a sectorial operator in $W^{-1,p}(\Omega)$.

In this paper, for critically growing nonlinearities, we need to consider the concept of *ϵ-regular solutions* for (1.1). The notion was introduced in [1] and studied further, in particular, in [8]. For completeness of the presentation we will recall now basic definitions and properties of ϵ-regular solutions referring to the above mentioned works for more complete information. We start with an abstract formulation of the problem.

In Banach space X consider sectorial operator $A : D(A) \subset X \to X$ having the additional property that $Re\sigma(A) > 0$ (*"positive operator"*). Denoting by $X^\alpha := D(A^\alpha), \alpha \geq 0$, fractional power spaces connected with A we consider semilinear differential equation

$$\begin{aligned} &u_t + Au + F(u),\ t > 0, \\ &u(0) = u_0 \in X^1, \end{aligned} \tag{2.6}$$

where $F : D(F) \subset X^1 \to X^\alpha$ with certain $\alpha \geq 0$. With that notation we recall ([1]):

DEFINITION 2.1. The map F is called a *critical* ε-regular map relative to the pair (X^1, X), if there are positive constants c, η, C_η, $q > 1$ and $\epsilon \in (0, \frac{1}{q})$ such that for each $v, w \in X^{1+\epsilon}$

$$\begin{aligned} &\|F(v) - F(w)\|_{X^{q\epsilon}} \leq c\|v - w\|_{X^{1+\epsilon}} \left(C_\eta + \eta\|v\|^{q-1}_{X^{1+\epsilon}} + \eta\|w\|^{q-1}_{X^{1+\epsilon}}\right), \\ &\|F(v)\|_{X^{q\epsilon}} \leq c\left(C_\eta + \eta\|v\|^q_{X^{1+\epsilon}}\right) \quad \text{for } v \in X^{1+\epsilon}. \end{aligned} \tag{2.7}$$

In addition, if for each $\eta > 0$ there is $C_\eta > 0$ such that (2.7) holds with c, q and ϵ independent of η, then F is called an *almost critical* ε-regular map relative to the pair (X^1, X).

We have also a variant of Theorem 2.1 in [8]:

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PROPOSITION 2.1. *Let F be a critical ϵ-regular map. Fixing $v_0 \in X^1$ then, there is $r > 0$ and $\tau_0 > 0$ such that for each $u_0 \in B_{X^1}(v_0, r)$ there exists a unique ϵ-regular solution u of (2.6), defined in $[0, \tau_0]$. In addition,*
(i) $t^\xi \|u(t, u_0)\|_{X^{1+\xi}} \to 0$ as $t \to 0^+$, $0 < \xi < q\epsilon$,
(ii) $t^\xi \|u(t, u_1) - u(t, u_2)\|_{X^{1+\xi}} \leq C' \|u_1 - u_2\|_{X^1}$ for $t \in [0, \tau_0]$, $0 \leq \xi \leq \xi_0 < q\epsilon$, $u_1, u_2 \in B_{X^1}(v_0, r)$,
(iii) $u(t, u_0) \in C((0, \tau_0], X^{1+q\epsilon}) \cap C^1((0, \tau_0], X^{1+\xi})$ for $0 \leq \xi < q\epsilon$; in particular the solution $u(t, u_0)$ satisfies (2.6) for each $t \in (0, \tau_0]$.

If F is an almost *critical ϵ-regular map, then all the above holds for arbitrarily large $r > 0$. In addition, if the solution $u(t, u_0)$ is bounded in X^1 in its maximal interval of existence, then it must exist for all $t \geq 0$.*

Now we apply these abstract results to obtain local well posedness of (1.1) in $W_0^{1,p}(\Omega)$. First we check that the condition (2.7) of Definition 2.1 is satisfied.

LEMMA 2.1. *Let $n \geq 3$, $p < n$, $q := \frac{n+p}{n-p}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.5). When $\nu > 0$, then for each $\epsilon \in [0, \frac{1}{2q})$ there is a constant $c > 0$ and, for each $\eta > 0$, a constant $\tilde{C}_\eta > 0$ such that*

$$\|f(w_1) - f(w_2)\|_{X^{q\epsilon}} \leq c\|f(w_1) - f(w_2)\|_{W^{-1+2q\epsilon,p}(\Omega)}$$
$$\leq c\|w_1 - w_2\|_{W_0^{1+2\epsilon,p}(\Omega)} \left(\tilde{C}_\eta + \eta\|w_1\|^{q-1}_{W_0^{1+2\epsilon,p}(\Omega)} + \eta\|w_2\|^{q-1}_{W_0^{1+2\epsilon,p}(\Omega)}\right),$$

for $w_1, w_2 \in W_0^{1+2\epsilon,p}(\Omega) = X^{1+\epsilon}$. Also the second estimate of (2.7) is satisfied between the same spaces.

If instead of (1.5) we assume that (1.3) is satisfied, then the above estimate is satisfied with certain *positive constants η and $\tilde{C}_\eta$.*

Proof: It follows from Sobolev imbedding theorem that for $r := \frac{np}{n-p(2\epsilon+1)}$,

$$W_0^{1+2\epsilon,p}(\Omega) \subset L^r(\Omega), \quad L^{\frac{r}{q}}(\Omega) \subset W^{-1+2q\epsilon,p}(\Omega). \tag{2.8}$$

Next, from (1.6) and from Hölder inequality,

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(2.9)
$$
\begin{aligned}
&\|f(w_1) - f(w_2)\|_{X^{q\epsilon}} \le c\|f(w_1) - f(w_2)\|_{L^{\frac{r}{q}}(\Omega)}\\
&\le c\left[\int_\Omega [|w_1 - w_2|(C_\eta + \eta|w_1|^{q-1} + \eta|w_2|^{q-1})]^{\frac{1}{q}\frac{np}{n-p(2\epsilon+1)}}\right]^{q\frac{n-p(2\epsilon+1)}{np}}\\
&\le c\left[\int_\Omega |w_1 - w_2|^{\frac{np}{n-p(2\epsilon+1)}}\right]^{\frac{n-p(2\epsilon+1)}{np}}\left[\int_\Omega (C_\eta + \eta|w_1|^{q-1} + \eta|w_2|^{q-1})^{\frac{np}{n-p(2\epsilon+1)}}\right]^{\frac{n-p(2\epsilon+1)}{np}}\\
&\le c\|w_1 - w_2\|_{L^r(\Omega)}\left(\tilde{C}_\eta + \eta\|w_1\|^{q-1}_{L^r(\Omega)} + \eta\|w_2\|^{q-1}_{L^r(\Omega)}\right)\\
&\le c\|w_1 - w_2\|_{W_0^{1+2\epsilon,p}(\Omega)}\left(\tilde{C}_\eta + \eta\|w_1\|^{q-1}_{W_0^{1+2\epsilon,p}(\Omega)} + \eta\|w_2\|^{q-1}_{W_0^{1+2\epsilon,p}(\Omega)}\right).
\end{aligned}
$$

The proof of the second estimate in (2.7) is similar, but simpler. The lemma is thus proved.

The case of parameter $\nu = 0$ in (1.1) requires separate treatment. In that case we will use for the scale $\{X^\alpha\}$ notation with subscript 0 to point out the difference with the previous case; $X_0^1 = W_0^{1,p}(\Omega)$, $X_0^{q\epsilon} = W^{-3+4q\epsilon,p}(\Omega)$. We have:

LEMMA 2.2. *Let $n \ge 3$, $p < n$, $q := \frac{n+p}{n-p}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfying* (1.5). *When $\nu = 0$, then for each $\epsilon \in [0, \frac{1}{4q})$ and $\eta > 0$ there are $c > 0$ and $\bar{C}_\eta > 0$ such that*

$$
\begin{aligned}
\|A(f(w_1)) - A(f(w_2))\|_{X_0^{q\epsilon}} &\le c\|f(w_1) - f(w_2)\|_{W^{-1+4q\epsilon,p}(\Omega)}\\
&\le c\|w_1 - w_2\|_{W_0^{1+4\epsilon,p}(\Omega)}\left(\bar{C}_\eta + \eta\|w_1\|^{q-1}_{W_0^{1+4\epsilon,p}(\Omega)} + \eta\|w_2\|^{q-1}_{W_0^{1+4\epsilon,p}(\Omega)}\right),
\end{aligned}
$$

for $w_1, w_2 \in W_0^{1+4\epsilon,p}(\Omega) = X_0^{1+\epsilon}$.

Under weaker assumption (1.3) *the conclusion is as in the previous lemma.*

The proof is precisely as in Lemma 2.1.

Remark 2. 3. The above Lemmas allow us to conclude that if $f \in C^1(\mathbb{R}, \mathbb{R})$ fulfils conditions (1.3), then the corresponding to it Nemitskiĭ operator F is a critical ϵ-regular map relative to the pair $(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$. As a consequence of Proposition 2.1 we thus conclude existence of the unique local ϵ-regular solution to (1.1) in $W_0^{1,p}(\Omega)$. If in addition condition (1.5) is satisfied, then the corresponding to f Nemitskiĭ operator F will be an almost critical ϵ-regular map relative to the pair $(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$ and the results of Proposition 2.1 will hold with arbitrary positive r.

The last remark concludes the proof of Theorem 1.1.

2.1. Bootstrapping property

It is known (see [1], [9]) that the local solution obtained above regularizes in time. We will give now a slightly more complete description of this property. Let us focus on the case $\nu > 0$ in (1.1). From Theorem 1.1, to any $u_0 \in W_0^{1,p}(\Omega)$ corresponds a unique local

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ϵ-regular solution to (1.1),

$$u(t) \in C((0,\tau_0], X^{1+q\epsilon}) \cap C^1((0,\tau_0], X^{1+\epsilon}). \tag{2.10}$$

It was assumed in Lemma 2.1 that $2q\epsilon < 1$, hence in case $\nu > 0$ the solution

$$u(t) \in W^{2^-,p}(\Omega) \text{ for small } t > 0,$$

and for such t will vary continuously in $W^{2^-,p}(\Omega)$ (by "r^-" we denote an arbitrary number smaller than r).

Similarly, for $\nu = 0$, the local ϵ-regular solution described in Lemma 2.2 starting from $u_0 \in W_0^{1,p}(\Omega)$ will enter and for small $t > 0$ vary continuously in the space $W^{3^-,p}(\Omega)$.

Fixing now small $t_0 > 0$ and using the Sobolev inclusions

$$\begin{aligned} &W^{2^-,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset W_0^{1,p_1}(\Omega),\ p_1 \le \frac{np}{n-p^-}, \\ &W^{3^-,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset W_0^{1,p_1^0}(\Omega),\ p_1^0 \le \frac{np}{n-2p^-}, \end{aligned} \tag{2.11}$$

we will consider local ϵ-regular solutions $u(t)$ in $W_0^{1,p_1}(\Omega)$ or $W_0^{1,p_1^0}(\Omega)$ for $\nu > 0$ or $\nu = 0$, respectively, existing locally for $t \le t_0$ as shown in Theorem 1.1. This procedure can be iterated. Our task is to show that after a few iterations the iterated exponent p will exceed n. An induction argument will be used to prove the claimed. We have:

LEMMA 2.3. *Consider a sequence $\{p_m\}$ of real numbers given by the recurrent formula:*

$$p_{m+1} = p_m \frac{n}{n - p_m^-},\ \ p_0 := p < n,\ m \in \mathbb{N}.$$

As long as its elements are less than n, this sequence is strictly increasing and there is a minimal index $m = m_0$ such that $p_{m_0} > n$.

It is evident from the definition that $p_{m+1} > p_m$ (if p_m^- is taken sufficiently close to $p_m < n$). Therefore the sequence is strictly increasing. Consider next the quotient:

$$\frac{p_{m+1}}{p_m} = \frac{p_m}{p_{m-1}} \frac{n - p_{m-1}^-}{n - p_m^-}.$$

Since the sequence $\{p_m\}$ is strictly increasing, the factor $\frac{n-p_{m-1}^-}{n-p_m^-}$ will be chosen greater than 1. Therefore, as long as $p_m < n$, we have the chain of inequalities:

$$\frac{p_{m+1}}{p_m} > \frac{p_m}{p_{m-1}} > \dots > \frac{p_1}{p} > 1, \tag{2.12}$$

showing there must be an element $p_{m_0} > n$. The proof is completed.

We are thus able to formulate:

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Lemma 2.4. *Let $u_0 \in W_0^{1,p}(\Omega)$ and $p < n$. Then the corresponding to it ϵ-regular solution u enters for $t > 0$ the space $W^{2^-,p_{m_0}}(\Omega)$ and, consequently, also $C^{1+\mu}(\bar{\Omega})$ with certain $\mu > 0$.*

Similar result is avaliable for the case $\nu = 0$ when the smoothing action of the equation is even stronger. Further increase of the regularity can be obtained through considerations in Hölder spaces $C^{\eta}(\bar{\Omega})$.

Remark 2. 4. The result of the above lemma allows us to strengthen slightly the compactness property of solutions reported in Theorem 1.1. Namely, if f is as in Theorem 1.1 and generates an almost critical ϵ-regular map and B is any bounded subset of $W_0^{1,p}(\Omega)$, then the local semigroup $\{S(t)\}$ generated by (1.1) in $W_0^{1,p}(\Omega)$

$$S : [0,\tau] \times B \to W_0^{1,p}(\Omega),$$

satisfies the condition $\|S(t)B\|_{C^{1+\mu}(\bar{\Omega})} \leq \infty$, $t > 0$, that means images of bounded subsets of $X^1 = W_0^{1,p}(\Omega)$ are compact.

3. GLOBAL SOLVABILITY OF (1.1)

We will limit our studies on global solvability of (1.1) and its global attractor to the case of Hilbert space $H_0^1(\Omega)$. Of course all what was said previously about the local solvability in $W_0^{1,p}(\Omega)$ applies to the present case $p = 2$. In order to conclude global in time solvability in $H_0^1(\Omega)$ of the problem (1.1), according to Proposition 2.1, we need to justify global in time $H_0^1(\Omega)$ boundedness of the ϵ-regular solution in its maximal interval of existence. To justify this Lyapunov type functional will be used which is common for (1.1) with all values of $\nu \in [0,1]$. We will use an equivalent form (2.5) of (1.1) to construct such functional. Multiplying equation (2.5) by u_t and integrating,

$$\int_\Omega B_\nu^{-1} u_t \, u_t dx = -\int_\Omega Au \, u_t dx + \int_\Omega f(u) \, u_t dx.$$

Now, thanks to the self-adjointness of B_ν^{-1} and integration by parts in the first right hand side component we get:

$$0 \leq \frac{d}{dt}\left(-\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega F(u)dx\right), \tag{3.1}$$

where $F(s) := \int_0^s f(z)dz$ is a primitive of f. We will thus introduce the *Lyapunov function* for (1.1) through the formula

$$\mathcal{L}(\phi) = \frac{1}{2}\|\nabla \phi\|^2_{L^2(\Omega)} - \int_\Omega F(\phi)dx. \tag{3.2}$$

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Remark 3. 1. It is interesting to note that the same Lyapunov function $\mathcal{L}$ works for the semilinear heat equation, for the original Cahn-Hilliard equation and for the whole range of problems (parameter $\nu \in (0,1)$) contained in (1.1). This observation suggests that the sets of equilibria and the asymptotic behavior of solutions to all this class of problems will be the same. The last remark will be extended in section discussing continuity of global attractors.

To conclude following from Proposition 2.1 global solvability of (1.1) in $H_0^1(\Omega)$, we need to have a global in time a priori estimate of the local solution in $H_0^1(\Omega)$ norm. The last is a consequence of the form of the Lyapunov function (3.2) and our assumptions. Let us analyze briefly properties of $\mathcal{L}(u(t,\cdot))$. Note first that with our growth restriction (1.3) the primitive $F(\phi)$ is well defined when $\phi \in H_0^1(\Omega)$. Indeed,

$$\int_\Omega |F(\phi)|dx \leq C|\Omega| + C\|\phi\|^{q+1}_{L^{q+1}(\Omega)},$$

which is controlled by the $H_0^1(\Omega)$ norm of ϕ. It follows next from assumption (1.4) that, for some $\epsilon > 0$, $F(s)$ satisfies

$$F(s) \leq \frac{\lambda_1 - \epsilon}{2} s^2 + \hat{d}, \tag{3.3}$$

for some $\hat{d} > 0$. Since the Lyapunov function decreases along solutions; $\mathcal{L}(u(t,\cdot)) \leq \mathcal{L}(u_0)$, we are able to estimate the $H_0^1(\Omega)$ norm of the solution using the Poincaré inequality:

$$\begin{aligned} \frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} &\leq \mathcal{L}(u_0) + \int_\Omega F(u(t,x))dx \\ &\leq \mathcal{L}(u_0) + \left(\frac{1}{2} - \frac{\epsilon}{2\lambda_1}\right) \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + \hat{d}|\Omega|. \end{aligned} \tag{3.4}$$

As a consequence of Proposition 2.1 and of the above considerations we have completed the proof of Theorem 1.2.

4. EXISTENCE OF THE GLOBAL ATTRACTOR FOR (1.1)

Proving existence of the global attractor we will strengthen the assumption concerning nonlinear term to condition (1.5). Since the problem (1.1) has a very good structure, existence of its global attractor will be shown using the classical and nice result in that direction, stating that: *a gradient, compact nonlinear semigroup with bounded set of equilibria possess a global attractor coinciding with the unstable manifold of the set of equilibria* (see [14]).

Observe first that the problem (1.1) equipped with the Lyapunov function $\mathcal{L}$ generates in the space $H_0^1(\Omega)$ a *gradient system* in the sense of [14]. Boundedness in $H_0^1(\Omega)$ of the set of equilibria satisfying

$$\begin{aligned} &\Delta v + f(v) = 0 \text{ in } \Omega, \\ &v = 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.1}$$

follows through the standard multiplication by v with the use of the assumption (1.4) and Poincaré inequality;

$$\|\nabla v\|^2_{L^2(\Omega)} \leq \int_\Omega f(v)v dx \leq \frac{\lambda_1 - \epsilon}{\lambda_1}\|\nabla v\|^2_{L^2(\Omega)} + d|\Omega|.$$

Finally, compactness of that semigroup for $t > 0$, which is a consequence of the smoothing property of ϵ-regular solutions stated in condition (ii) of Proposition 2.1, was mentioned in Theorem 1.1. We have thus all the ingredients required to formulate:

THEOREM 4.1. *Let the conditions* (1.4) *and* (1.5) *be satisfied. Then, for arbitrary* $\nu \in [0, 1]$ *there exists a global attractor* $\mathcal{A}_\nu$ *for the semigroup of global* ϵ*-regular solutions of* (1.1) *in* $H^1_0(\Omega)$.

4.1. Additional properties of the global attractors

In further studies we need uniform with respect to $\nu \in [0, 1]$ estimate of the global attractors in $L^\infty(\Omega)$.

- We need, inside this subsection, to strengthen assumption (1.5) to the form

$$\forall_{\eta>0}\exists_{C_\eta>0}\forall_{s\in\mathbb{R}}\ |f''(s)| \leq \eta|s|^{q-2} + C_\eta,\ q = \frac{n+2}{n-2}. \tag{4.2}$$

For the problem (1.1), due to its symmetry, estimates of better norms of the solution will be obtained multiplying the equation by $A^m u, m \in \mathbb{N}$:

$$\int_\Omega ((1-\nu)I + \nu A)A^{-1}u_t\ A^m u dx = -\int_\Omega Au\ A^m u dx + \int_\Omega f(u)\ A^m u dx. \tag{4.3}$$

For $m = 2$ this leads to the equation:

$$\int_\Omega ((1-\nu)I + \nu A)u_t\ Au dx = -\int_\Omega (A^{\frac{3}{2}}u)^2 dx + \int_\Omega A^{\frac{1}{2}}f(u)\ A^{\frac{3}{2}}u dx. \tag{4.4}$$

Now, with the use of assumption (1.5), the last component above will be estimated as follows:

$$\|A^{\frac{1}{2}}f(u)\|_{L^2(\Omega)} \leq c\|f'(u)\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)} \leq c(\eta\|u\|^{\frac{4}{n-2}}_{L^\infty(\Omega)} + C_\eta)\|\nabla u\|_{L^2(\Omega)}. \tag{4.5}$$

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Further, using Agmon's and interpolation inequalities, we will estimate the $L^\infty(\Omega)$ norm of u:

$$
\begin{aligned}
&\|u\|^4_{L^\infty(\Omega)} \le c(\|u\|^{\frac{1}{2}}_{H^1(\Omega)}\|u\|^{\frac{1}{2}}_{H^2(\Omega)})^4 \le c'\|u\|^3_{H^1(\Omega)}\|u\|_{H^3(\Omega)} \text{ for } n=3,\\
(4.6)\qquad &\|u\|^2_{L^\infty(\Omega)} \le c\|u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)} \text{ for } n=4,\\
&\|u\|^{\frac{4}{3}}_{L^\infty(\Omega)} \le c\|u\|^{\frac{2}{3}}_{H^2(\Omega)}\|u\|^{\frac{2}{3}}_{H^3(\Omega)} \text{ for } n=5.
\end{aligned}
$$

In all the cases $n = 3, 4, 5$, this leads to an estimate:

$$
\begin{aligned}
(4.7)\qquad &\frac{1}{2}\frac{d}{dt}\left((1-\nu)\int_\Omega (A^{\frac{1}{2}}u)^2dx + \nu\int_\Omega (Au)^2dx\right)dx\\
&\qquad\qquad \le -\|A^{\frac{3}{2}}u\|^2_{L^2(\Omega)} + c'\left(\eta\|u\|^2_{H^3(\Omega)} + const(M(u_0))\right).
\end{aligned}
$$

Denoting $\mathcal{L}_1(\phi) := (1-\nu)\int_\Omega (A^{\frac{1}{2}}\phi)^2dx + \nu\int_\Omega (A\phi)^2dx$, for sufficiently small $\eta > 0$ the above estimate together with our earlier ν-independent a priori estimate (3.4) will be extended to

$$
(4.8)\qquad \frac{d}{dt}\mathcal{L}_1(u(t,\cdot)) \le -c\mathcal{L}_1(u(t,\cdot)) + const'(M(u_0)),
$$

with $const'(M(u_0))$ independent on ν.

We proceed with the estimates of better norms of solutions to (1.1) letting $m = 3$ in (4.3);

$$
(4.9)\qquad \int_\Omega ((1-\nu)I + \nu A)u_t\ A^2u dx = -\int_\Omega (A^2u)^2dx + \int_\Omega Af(u)\ A^2u dx.
$$

The nonlinear term will be now estimated with the use of assumption (4.2):

$$
\begin{aligned}
&(4.10)\\
&\|Af(u)\|_{L^2(\Omega)} = \|f'(u)\Delta u + f''(u)|\nabla u|^2\|_{L^2(\Omega)}\\
&\qquad \le \|f'(u)\|_{L^\infty(\Omega)}\|\Delta u\|_{L^2(\Omega)} + \|f''(u)\|_{L^\infty(\Omega)}\||\nabla u|^2\|_{L^2(\Omega)}\\
&\qquad \le c\left(\eta\|u\|^{\frac{4}{n-2}}_{L^\infty(\Omega)} + C_\eta\right)\|\Delta u\|_{L^2(\Omega)} + c\left(\eta\|u\|^{\frac{6-n}{n-2}}_{L^\infty(\Omega)} + C'_\eta\right)\|\nabla u\|^2_{L^4(\Omega)}.
\end{aligned}
$$

The interpolation inequalities give us

$$
\begin{aligned}
&\|\Delta u\|_{L^2(\Omega)} \le c\|u\|^{\frac{2}{3}}_{H^1_0(\Omega)}\|u\|^{\frac{1}{3}}_{H^4(\Omega)},\\
(4.11)\qquad &\\
&\|\nabla u\|^2_{L^4(\Omega)} \le c\|u\|^{\frac{12-n}{6}}_{H^1_0(\Omega)}\|u\|^{\frac{n}{6}}_{H^4(\Omega)},
\end{aligned}
$$

and consequently, for $n = 3, 4, 5$, we obtain

$$
\|Af(u)\|_{L^2(\Omega)} \le C(\|u\|_{H^1_0(\Omega)})\left(\eta\|u\|_{H^4(\Omega)} + 1\right).
$$

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This leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left((1-\nu)\int_\Omega (Au)^2dx+\nu\int_\Omega (A^{\frac{3}{2}}u)^2dx\right)dx\\ \le -\|A^2u\|^2_{L^2(\Omega)}dx+c''\left(\eta\|u\|^2_{H^4(\Omega)}+const(M(u_0))\right),\end{aligned}\tag{4.12}$$

and with a sufficiently small $\eta>0$, for $\mathcal{L}_2(\phi):=(1-\nu)\int_\Omega(A\phi)^2dx+\nu\int_\Omega(A^{\frac{3}{2}}\phi)^2dx$, extends to the estimate

$$\frac{d}{dt}\mathcal{L}_2(u(t,\cdot))\le -c\mathcal{L}_2(u(t,\cdot))+const''(M(u_0)),\tag{4.13}$$

with $const''(M(u_0))$ independent on ν. Note that for $n=3$ equation (4.13) provides in particular uniform with respect to $\nu\in[0,1]$ estimate of the global attractors $\mathcal{A}_\nu$ in $L^\infty(\Omega)\subset H^2(\Omega)$.

Assuming existence of the third derivative of f and an appropriate almost critical growth restriction (similar to (4.2)) we will proceed obtaining a priori estimates. In particular, analyzing formula (4.3) with $m=4$, an a priori estimate for

$$\mathcal{L}_3(u(t,\cdot)):=(1-\nu)\int_\Omega(A^{\frac{3}{2}}u(t,\cdot))^2dx+\nu\int_\Omega(A^2u(t,\cdot))^2dx$$

is available. This estimate will imply uniform with respect to $\nu\in[0,1]$ estimate in $L^\infty(\Omega)$ of the global attractors $\mathcal{A}_\nu$ in dimensions $n=4,5$.

5. CONVERGENCE OF LINEAR SEMIGROUPS

Let $A=-\Delta$ with Dirichlet boundary condition with domain $D(A)=H^1_0(\Omega)=X^1$ and taking values in $X=H^{-1}(\Omega)=(H^1_0(\Omega))'=D(A^{-1})$.

Define $A_\nu=A^2((1-\nu)I+\nu A)^{-1}$ and let $B_\nu=A((1-\nu)I+\nu A)^{-1}$. Clearly, for all $\nu\in[0,1]$, the operators A_ν are self-adjoint and positive operators defined in $D(A)$ with values in $D(A^{-1})$ for $\nu\in(0,1]$ and with values in $D(A_\nu^{-1})$ if $\nu=0$. Similarly, the operator B_ν is a bounded positive and self-adjoint if $\nu\in(0,1]$ and $B_0=A:D(A)\subset X\to X$.

LEMMA 5.1. *If $\lambda\in\rho(-A_\nu)\cap\rho(-A_\mu)$ then, the following identity holds*

$$\begin{aligned}&(\lambda+A_\nu)^{-1}-(\lambda+A_\mu)^{-1}\\ &=A_\nu(\lambda+A_\nu)^{-1}(A_\nu^{-1}-A_\mu^{-1})A_\mu(\lambda+A_\mu)^{-1}.\end{aligned}\tag{5.1}$$

In addition

$$\|(\lambda+A_\nu)^{-1}-(\lambda+A_\mu)^{-1}\|\le C\|A_\nu^{-1}-A_\mu^{-1}\|\tag{5.2}$$

and for any $\alpha\in[0,1]$, $\nu\in[0,1]$, $i=0,1$,

$$\|A_\nu^ie^{-A_\nu t}-A_\mu^ie^{-A_\mu t}\|\le Mt^{-i-\alpha}\|A_\nu^{-1}-A_\mu^{-1}\|^\alpha.\tag{5.3}$$

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Proof: To prove (5.1) we simply add and subtract $-\lambda(\lambda+A_\nu)^{-1}(\lambda+A_\mu)^{-1}$ to the left hand side of it and note that $-\lambda(\lambda+A_\mu)^{-1}+I = A_\mu(\lambda+A_\mu)^{-1}$ and $-\lambda(\lambda+A_\nu)^{-1}+I = A_\nu(\lambda+A_\nu)^{-1}$; that is,

$$\begin{aligned}(\lambda+A_\nu)^{-1} - (\lambda+A_\mu)^{-1} \\ &= (\lambda+A_\nu)^{-1}(-\lambda(\lambda+A_\mu)^{-1}+I) - (-\lambda(\lambda+A_\nu)^{-1}+I)(\lambda+A_\mu)^{-1} \\ &= (\lambda+A_\nu)^{-1}A_\mu(\lambda+A_\mu)^{-1} - A_\nu(\lambda+A_\nu)^{-1}(\lambda+A_\mu)^{-1} \\ &= A_\nu(\lambda+A_\nu)^{-1}(A_\nu^{-1}-A_\mu^{-1})A_\mu(\lambda+A_\mu)^{-1}.\end{aligned}$$

It follows from (5.1), (5.2) that

$$e^{-A_\nu t} - e^{-A_\mu t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}[(\lambda+A_\nu)^{-1} - (\lambda+A_\mu)^{-1}]d\lambda \tag{5.4}$$

and therefore

$$\|e^{-A_\nu t} - e^{-A_\mu t}\| \le Mt^{-1}\|A_\nu^{-1} - A_\mu^{-1}\|. \tag{5.5}$$

Now since

$$A_\nu e^{-A_\nu t} - A_\mu e^{-A_\mu t} = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}[(\lambda+A_\nu)^{-1} - (\lambda+A_\mu)^{-1}]d\lambda,$$

we have that

$$\|A_\nu e^{-A_\nu t} - A_\mu e^{-A_\mu t}\| \le Mt^{-2}\|A_\nu^{-1} - A_\mu^{-1}\|. \tag{5.6}$$

Also, from the fact that A_μ and A_ν are positive and self-adjoint, note that

$$\|e^{-A_\nu t} - e^{-A_\mu t}\| \le M \tag{5.7}$$

and

$$\|A_\nu e^{-A_\nu t} - A_\mu e^{-A_\mu t}\| \le Mt^{-1}. \tag{5.8}$$

Combining (5.7) with (5.5) and (5.8) with (5.6) we have that, for any $\alpha \in [0,1]$ the estimate (5.3) holds.

LEMMA 5.2. *There is a constant $M > 0$ such that*

$$\|A_\nu^{-1} - A^{-2}\| \le M\nu,$$

and

$$\|A_\nu^{-1} - A_\mu^{-1}\| \le M|\nu - \mu|.$$

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Proof: In fact, since $A_\nu = A^2((1-\nu)I+\nu A)^{-1}$, we have that

$$A_\nu^{-1} - A^{-2} = ((1-\nu)I+\nu A)A^{-2} - A^{-2} = \nu[A^{-1}(I-A^{-1})]$$

and the first estimate follows. For the second estimate we proceed in a similar manner; that is,

$$A_\nu^{-1} - A_\mu^{-1} = [((1-\nu)I+\nu A) - ((1-\mu)I+\mu A)]A^{-2} = [\nu-\mu][A^{-1}(I-A^{-1})].$$

LEMMA 5.3. *There is a constant $M>0$, independent of ν, such that*

$$\|B_\nu A_\nu^{-\frac{1}{2}}\| \le M. \tag{5.9}$$

Consequently

$$\|B_\nu e^{-A_\nu t} - Ae^{-A^2 t}\| \le Mt^{-\frac{1}{2}} \tag{5.10}$$

and

$$\|B_\nu e^{-A_\nu t} - B_\mu e^{-A_\mu t}\| \le Mt^{-\frac{1}{2}}. \tag{5.11}$$

Proof: First note that

$$B_\nu A_\nu^{-\frac{1}{2}} = (\nu I + (1-\nu)A^{-1})^{-1}A_\nu^{-\frac{1}{2}} = (\nu I + (1-\nu)A^{-1})^{-\frac{1}{2}}A^{-\frac{1}{2}} = ((1-\nu)I+\nu A)^{-\frac{1}{2}}$$

and that

$$\langle[(1-\nu)I+\nu A]u,u\rangle \ge ((1-\nu)+\nu\lambda_1)\|u\|^2.$$

Consequently

$$\langle[(1-\nu)I+\nu A]^{\frac{1}{2}}u,[(1-\nu)I+\nu A]^{\frac{1}{2}}u\rangle \ge ((1-\nu)+\nu\lambda_1)\|u\|^2$$

and

$$\|[(1-\nu)I+\nu A]^{-\frac{1}{2}}\| \le \frac{1}{((1-\nu)+\nu\lambda_1)^{\frac{1}{2}}} \le \frac{1}{\min\{1,\lambda_1^{\frac{1}{2}}\}},$$

proving the first estimate. The estimates (5.10) and (5.11) follow from (5.9).

THEOREM 5.1. *For any $0<\epsilon<1$ and $\nu\in[0,1]$,*

$$\|A^{1-\epsilon}B_\nu e^{-A_\nu t}\| \le Mt^{-1+\frac{\epsilon}{2}}.$$

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Proof: Note that

$$\|AB_\nu e^{-A_\nu t}\| \leq Mt^{-1}$$

and that

$$\|B_\nu e^{-A_\nu t}\| \leq \|B_\nu A_\nu^{-\frac{1}{2}} A_\nu^{\frac{1}{2}} e^{-A_\nu t}\| \leq Mt^{-\frac{1}{2}}.$$

Consequently

$$\|A^{1-\epsilon}B_\nu e^{-A_\nu t}\| \leq M\|AB_\nu e^{-A_\nu t}\|^{1-\epsilon}\|B_\nu e^{-A_\nu t}\|^{\epsilon} \leq Mt^{-1+\frac{\epsilon}{2}},$$

and the result is proved.

THEOREM 5.2. *Given $0 < \epsilon < 1$ then for $\alpha < \frac{\epsilon}{2(1-\epsilon)}$ we have $\beta = 1 + \alpha(1-\epsilon) - \frac{\epsilon}{2} < 1$, and*

$$\|A^{1-\epsilon}B_\nu e^{-A_\nu t} - A^{2-\epsilon}e^{-A^2 t}\| \leq Mt^{-\beta}\|A_\nu^{-1} - A^{-2}\|^{\alpha(1-\epsilon)}$$

and also

$$\|A^{1-\epsilon}B_\nu e^{-A_\nu t} - A^{1-\epsilon}B_\mu e^{-A_\mu t}\| \leq Mt^{-\beta}\|A_\nu^{-1} - A^{-2}\|^{\alpha(1-\epsilon)}$$

for all $\mu, \nu \in [0,1]$.

Proof: The result follows from (5.3), (5.10) and from the moment inequality

$$\|A^{1-\epsilon}x\| \leq M\|Ax\|^{1-\epsilon}\|x\|^{\epsilon}, \quad \text{for all } x \in D(A).$$

6. CONVERGENCE OF NONLINEAR SEMIGROUPS

In this section we consider the nonlinear semigroups

$$T_\nu(t)u_0 = e^{-A_\nu t}u_0 + \int_0^t e^{-A_\mu(t-s)}B_\nu f(T_\nu(s)u_0)ds$$

and

$$T_0(t)u_0 = e^{-A^2 t}u_0 + \int_0^t e^{-A^2(t-s)}Af(T_0(s)u_0)ds.$$

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We will assume that $f: X^1 \mapsto X^\epsilon$ is globally Lipschitz continuous with Lipschitz constant L and globally bounded by N. Then

$$\begin{aligned}\|T_\nu(t)u_0 - T_0(t)u_0\|_{X^1} &\le \|e^{-A_\nu t} - e^{-A^2 t}\|_{L(X^1)}\|u_0\|_{X^1} \\ &+ \int_0^t \|e^{-A_\mu(t-s)}B_\nu f(T_\nu(s)u_0) - e^{-A^2(t-s)}Af(T_0(s)u_0)\|_{X^1}ds \\ &\le \|e^{-A_\nu t} - e^{-A^2 t}\|_{L(X^1)}\|u_0\|_{X^1} \\ &+ \int_0^t \|B_\nu e^{-A_\mu(t-s)}[f(T_\nu(s)u_0) - f(T_0(s)u_0)]\|_{X^1}ds \\ &+ \int_0^t \|[B_\nu e^{-A_\mu(t-s)} - Ae^{-A^2(t-s)}]f(T_0(s)u_0)\|_{X^1}ds.\end{aligned}$$

Hence

$$\begin{aligned}\|T_\nu(t)u_0 - T_0(t)u_0\|_{X^1} &\le \|e^{-A_\nu t} - e^{-A^2 t}\|_{L(X^1)}\|u_0\|_{X^1} \\ &+ \int_0^t \|A^{1-\epsilon}B_\nu e^{-A_\mu(t-s)}[f(T_\nu(s)u_0) - f(T_0(s)u_0)]\|_{X^\epsilon}ds \\ &+ \int_0^t \|A^{1-\epsilon}B_\nu e^{-A_\mu(t-s)} - A^{2-\epsilon}e^{-A^2(t-s)}\|\|f(T_0(s)u_0)\|_{X^\epsilon}ds \\ &\le \|e^{-A_\nu t} - e^{-A^2 t}\|_{L(X^1)}\|u_0\|_{X^1} \\ &+ ML\int_0^t (t-s)^{-1+\frac{\epsilon}{2}}\|T_\nu(s)u_0 - T_0(s)u_0\|_{X^1}ds \\ &+ MN\|A_\nu^{-1} - A^{-1}\|^\alpha \int_0^t (t-s)^{-\beta}ds.\end{aligned}$$

It follows from the Singular Gronwall's Lemma ([15]), that

$$\sup_{t\in[0,T]}\sup_{\|u_0\|_{X^1}\le rk}\|T_\nu(t)u_0 - T_0(t)u_0\|_{X^1} \le C(r,T)\|A_\nu^{-1} - A^{-1}\|^\alpha.$$

7. LINEARIZATION AROUND A HYPERBOLIC EQUILIBRIUM

Note first that the set of equilibria for (1.1) given by

$$\mathcal{E} = \{u^* \in H^2(\Omega)\cap H_0^1(\Omega) : \Delta u^* + f(u^*) = 0\}.$$

is independent of $\nu \in [0,1]$. Clearly, $\mathcal{E}$ is a bounded subset of $C(\bar{\Omega})$.

Before we proceed, let us study the hyperbolicity of the equilibria for (1.1). The linearized equation around $u^* \in \mathcal{E}$ is given by

$$v_t = -L_\nu v,$$

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where $L_\nu : X^1 \subset X \to X$ fulfils

$$L_\nu u = (\nu I + (1-\nu)(-\Delta)^{-1})^{-1}(-\Delta - f'(u^*))u, \; u \in X^1.$$

It is interesting to note that L_ν is not necessarily self-adjoint but it is the composition of two self-adjoint operators.

Choose $\delta > 0$ such that $\hat{L}_\nu = L_\nu + \delta I$ and $\sigma(\hat{L}_\nu) > 0$ for all $\nu \in [0,1]$. Then

$$L_\nu - A_\nu = B_\nu f'(u^*),$$

and

LEMMA 7.1. *There is a constant M independent of $\nu \in [0,1]$ such that*

$$\|A_\nu^{-\frac{1}{2}}(L_\nu - A_\nu)\| \le M\|f'(u^*)\|$$

Proof: Since for each $x \in D(A)$,

$$\|A_\nu^{-\frac{1}{2}}(L_\nu - A_\nu)x\| = \|A_\nu^{-\frac{1}{2}} B_\nu f'(u^*)x\| \le M\|f'(u^*)\|\|x\|,$$

the result follows.

LEMMA 7.2. *For each $\beta \in [0,1]$ there is a constant $M_\beta > 0$, independent of ν, such that*

$$\|L_\nu^{-\beta} A_\nu^\beta\| \le M_\beta.$$

Proof: First note that $\|A_\nu^\beta(\lambda + A_\nu)^{-1}\| \le C|\lambda|^{\beta-1}$, $\|\hat{L}_\nu^\beta(\lambda + \hat{L}_\nu)^{-1}\| \le C|\lambda|^{\beta-1}$ and that

$$\begin{aligned}(\lambda + A_\nu)^{-1} - (\lambda + \hat{L}_\nu)^{-1} &= (\lambda + A_\nu)^{-1}[I - (\lambda + \hat{L}_\nu + A_\nu - \hat{L}_\nu)(\lambda + \hat{L}_\nu)^{-1}] \\ &= (\lambda + A_\nu)^{-1}(\hat{L}_\nu - A_\nu)(\lambda + \hat{L}_\nu)^{-1}.\end{aligned} \tag{7.1}$$

Consequently

$$A_\nu^\beta(\lambda + A_\nu)^{-1} = [I + (\lambda + A_\nu)^{-1} A_\nu^{\frac{1}{2}} A_\nu^{-\frac{1}{2}}(\hat{L}_\nu - A_\nu)](\lambda + \hat{L}_\nu)^{-1} A_\nu^\beta,$$

and there is a constant $\hat{C}$ such that

$$\|(\lambda + \hat{L}_\nu)^{-1} A_\nu^\beta\| \le \hat{C}\lambda^{\beta-1}.$$

Now, since

$$A_\nu^{-\beta} - \hat{L}_\nu^{-\beta} = \frac{1}{\pi}\sin\pi\beta \int_0^\infty \lambda^{-\beta}(\lambda + A_\nu)^{-1}(\hat{L}_\nu - A_\nu)(\lambda + \hat{L}_\nu)^{-1} d\lambda,$$

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then we have

$$\hat{L}_\nu^{-\beta} A_\nu^\beta = I - \frac{1}{\pi}\sin\pi\beta \int_0^\infty \lambda^{-\beta}(\lambda + A_\nu)^{-1} A_\nu^{\frac{1}{2}} A_\nu^{-\frac{1}{2}}(\hat{L}_\nu - A_\nu)(\lambda + \hat{L}_\nu)^{-1} A_\nu^\beta d\lambda.$$

Hence there is a constant $\bar{C}$, independent of $\nu \in [0,1]$, such that the integrand is bounded by $\bar{C}\lambda^{-\frac{3}{2}}$, and the result is proved.

It is easy to see that the hyperbolicity of an equilibrium is independent of the parameter ν. That is, the following result holds

PROPOSITION 7.1. *For all $\nu \in [0,1]$, L_ν is sectorial and has compact resolvent. All eigenvalues of L_ν are real and, if $\lambda = 0$ is not an eigenvalue of L_0, then it is not an eigenvalue of L_ν, for any $\nu \in (0,1]$.*

Proof: If we change the inner product of the space to $\langle B_\nu \cdot, \cdot\rangle$, the operator L_ν becomes self-adjoint and therefore all its eigenvalues are real and L_ν is bounded below. Clearly L_ν has compact resolvent. To prove the remaining statement, simply note that the injectivity of L_0 implies the injectivity of $L_\nu = L_0 B_\nu$ for all $\nu \in (0,1]$.

In similarity with A_ν, the operator L_ν behaves like a second order elliptic operator (as far as regularization goes) for $\nu \in (0,1]$ and as a fourth order operator for $\nu = 0$. Also, for all $\nu \in [0,1]$,

$$L_\nu^{-1} - L_\mu^{-1} = (\nu - \mu)(\Delta + f'(u^*))^{-1}[I - (-\Delta)^{-1}],$$

showing that the resolvent of L_ν is a Lipschitz continuous function of ν. Note also that L_ν is a self-adjoint operator for each ν and since u^* is a hyperbolic equilibrium point for (1.1) we can define

$$\beta_\nu^+ = \min\{\lambda \in \sigma(L_\nu) \cap \{\lambda : \mathrm{Re}\lambda > 0\}$$

and

$$\beta_\nu^- = \max\{\lambda \in \sigma(L_\nu) \cap \{\lambda : \mathrm{Re}\lambda < 0\}.$$

Further, for $\mu \in [0,1]$ fixed, if $\lambda \in \sigma(L_\mu)$ is isolated its generalized eigenspace equals $W(\lambda, L_\mu) = Q(\lambda, L_\mu)X^1$, where

$$Q(\lambda, L_\mu) = \frac{1}{2\pi i}\int_{|\xi-\lambda|=\delta} (\xi I - L_\mu)^{-1} d\xi,$$

and δ is so that $\sigma(L_\mu) \cap \{\xi \in \mathbb{C} : |\xi - \lambda| \leqslant \delta\} = \{\lambda\}$. From Lemma 4.9 in [4], there is ϵ_{S_δ} such that $\rho(L_\nu) \supset S_\delta = \{\xi : |\xi - \lambda| = \delta\}$, $\forall |\nu - \mu| \leqslant \epsilon_{S_\delta}$. Let $W(\lambda, L_\nu) = Q(\lambda, L_\nu)X^1$ where

$$Q(\lambda, L_\nu) = \frac{1}{2\pi i}\int_{|\xi-\lambda|=\delta} (\xi I - L_\nu)^{-1} d\xi.$$

The next result says that the spectrum of L_ν, for ϵ small, *approaches* the spectrum of L_μ (see [4]). We already know that $\sigma(L_\nu)$, $\nu \in [0,1]$, consists of isolated eigenvalues only and $0 \notin \sigma(L_\nu)$.

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THEOREM 7.1. *Let L_ν, L_μ be as above. Then the following conditions hold.*

(i) If $\lambda_\mu \in \sigma(L_\mu)$, there exists a sequence $\nu_n \to \mu$ and $\lambda_{\nu_n} \in \sigma(L_{\nu_n})$, $n \in \mathbb{N}$, such that $\lambda_{\nu_n} \to \lambda_\mu$ as $n \to \infty$.

(ii) If for some sequences $\nu_n \to \mu$, $\lambda_{\nu_n} \in \sigma(L_{\nu_n})$, $n \in \mathbb{N}$, one has $\lambda_{\nu_n} \to \lambda_\mu$ as $n \to \infty$, then $\lambda_\mu \in \sigma(L_\mu)$.

(iii) There exists $\epsilon_\mu > 0$ such that $\dim W(\lambda, L_\nu) = \dim W(\lambda_\mu, L_\mu)$ for all $|\nu - \mu| < \epsilon_\mu$.

(iv) If $u \in W(\lambda_\mu, L_\mu)$, there exists a sequence $\{u_\nu\}$, $u_\nu \in W(\lambda_\mu, L_\nu)$, such that $u_\nu \longrightarrow u$.

(v) If $\nu_n \to \mu$, and $u_n \in W(\lambda, L_{\nu_n})$, satisfies $\|u_n\| = 1$ then, $\{u_n\}$ has a convergent subsequence and any limit point of this sequence is in $W(\lambda_\mu, L_\mu)$.

In particular, β_ν^+ and β_ν^- are continuous functions of $\nu \in [0,1]$. Since for each $\nu \in [0,1]$ we have that $\beta_\nu^+ > 0$ and $\beta_\nu^- < 0$, it follows that $\min_{\nu\in[0,1]} \beta_\nu^+ = \beta^+ > 0$ and $\max_{\nu\in[0,1]} \beta_\nu^- = \beta^- < 0$.

Denote by $\sigma_+^\nu(u^*) = \{\lambda \in \sigma(L_\nu) : \mathrm{Re}\lambda > 0\}$ and let Γ be a contour, independent of ν, entirely contained in $\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$, enclosing $\sigma_+^\nu(u^*)$ and set

$$\mathcal{Q}_\nu(\sigma_+^\nu(u^*)) = \frac{1}{2\pi i}\int_\Gamma (\xi I - L_\nu)^{-1} d\xi. \tag{7.2}$$

It is clear that $R(\mathcal{Q}_\nu(\sigma_+^\nu(u^*)))$ is independent of ν.

Consider the following decomposition of X^1; let $X^1 = Y_\nu \oplus Z_\nu$ where $Z_\nu = \mathcal{Q}_\nu(\sigma_+^\nu(u^*))X^1$ and $Y_\nu = (I - \mathcal{Q}_\nu(\sigma_+^\nu(u^*)))X^1$. We note that $\mathcal{Q}_\nu((\sigma_+^\nu(u^*)))$ is a compact operator and therefore Z_ν is finite dimensional. Also, since $\mathcal{Q}_\nu((\sigma_+^\nu(u^*)))$ varies continuously with the parameter ν in the uniform operator topology, the dimension of Z_ν is independent of $\nu \in [0,1]$.

We have shown that there are constants $M \geq 1$ and $\beta > 0$ such that, for all $\nu \in (0,1]$ and $\alpha \in [0,1]$,

$$\begin{aligned} &\|e^{L_\nu t} z\|_{X^1} \leq M e^{\beta t}\|z\|_{X^1},\ t \leq 0, \\ &\|e^{L_\nu t} y\|_{X^1} \leq M t^{\alpha-1} e^{-\beta t}\|y\|_{X^\alpha},\ t > 0. \end{aligned} \tag{7.3}$$

Proceeding exactly as in Lemma 5.1 we have the following result

LEMMA 7.3. *If $\lambda \in \rho(-L_\nu) \cap \rho(-L_\mu)$, then the following identity holds*

$$\begin{aligned} &(\lambda + L_\nu)^{-1} - (\lambda + L_\mu)^{-1} \\ &\qquad = L_\nu(\lambda + L_\nu)^{-1}(L_\nu^{-1} - L_\mu^{-1})L_\mu(\lambda + L_\mu)^{-1}. \end{aligned} \tag{7.4}$$

In addition

$$\|(\lambda + L_\nu)^{-1} - (\lambda + L_\mu)^{-1}\| \leq C\|L_\nu^{-1} - L_\mu^{-1}\| \tag{7.5}$$

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and for any $\alpha \in [0,1]$, $\nu \in [0,1]$, $i = 0,1$,

$$\|L_\nu^i e^{-L_\nu t} - L_\mu^i e^{-L_\mu t}\| \leq M t^{-i-\alpha} \|L_\nu^{-1} - L_\mu^{-1}\|^\alpha. \tag{7.6}$$

Also, proceeding exactly as in Lemma 5.2 we obtain the following result

LEMMA 7.4. *There is a constant* $M > 0$ *such that*

$$\|L_\nu^{-1} - L_0^{-1}\| \leq M\nu,$$

and

$$\|L_\nu^{-1} - L_\mu^{-1}\| \leq M|\nu - \mu|.$$

LEMMA 7.5. *There is a constant* $M > 0$, *independent of* ν, *such that*

$$\|e^{-\hat{L}_\nu t} B_\nu - e^{-\hat{L}_0 t} A\| \leq M t^{-\frac{1}{2}}, \tag{7.7}$$

and

$$\|e^{-\hat{L}_\nu t} B_\nu - e^{-\hat{L}_\mu t} B_\mu\| \leq M t^{-\frac{1}{2}}. \tag{7.8}$$

Proof: Note that $\hat{L}_\nu^{-\frac{1}{2}} B_\nu = \hat{L}_\nu^{-\frac{1}{2}} A_\nu^{\frac{1}{2}} A_\nu^{-\frac{1}{2}} B_\nu$. From Lemma 5.3 and Lemma 7.2 we have that (7.7) and (7.8) hold.

Before we proceed we need to introduce the adjoint of L_ν; that is, $L_\nu^* = L_1 B_\nu$ and note that, $L_\nu^* - A_\nu = f'(u^*) B_\nu$. Consequently,

THEOREM 7.2. *For any* $0 < \epsilon < 1$ *and* $\nu \in [0,1]$,

$$\|\hat{L}_1^{1-\epsilon} e^{-\hat{L}_\nu t} B_\nu\| \leq M t^{-1+\frac{\epsilon}{2}}.$$

Furthermore, for any $\alpha \in [0,1]$, $\nu \in [0,1]$,

$$\|\hat{L}_1 e^{-\hat{L}_\nu t} B_\nu - \hat{L}_1 e^{-\hat{L}_\mu t} B_\mu\| \leq M t^{-1-\alpha} \|L_\nu^{-1} - L_\mu^{-1}\|^\alpha. \tag{7.9}$$

Proof: Note that

$$\begin{aligned} L_1 e^{-L_\nu t} B_\nu &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} L_1 (\lambda + L_\nu)^{-1} B_\nu d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} L_1 B_\nu (\lambda + L_1 B_\nu)^{-1} d\lambda = L_\nu^* e^{-L_\nu^* t} = (L_\nu e^{-L_\nu t})^*. \end{aligned} \tag{7.10}$$

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Consequently, there is a constant M, independent of ν, such that $\|L_1 e^{-L_\nu t} B_\nu\| \leq M t^{-1}$ and

$$\|\hat{L}_1^{1-\epsilon} e^{-\hat{L}_\nu t} B_\nu\| \leq M \|\hat{L}_1 e^{-\hat{L}_\nu t} B_\nu\|^{1-\epsilon} \|e^{-\hat{L}_\nu t} B_\nu\|^{\epsilon} \leq M t^{-1+\frac{\epsilon}{2}},$$

and hence the first estimate is proved. Furthermore, from Lemma 7.3, for any $\alpha \in [0,1]$, $\nu \in [0,1]$,

$$\|\hat{L}_1 e^{-\hat{L}_\nu t} B_\nu - \hat{L}_1 e^{-\hat{L}_\mu t} B_\mu\| = \|\hat{L}_\nu e^{-\hat{L}_\nu t} - \hat{L}_\mu e^{-\hat{L}_\mu t}\| \leq M t^{-1-\alpha} \|L_\nu^{-1} - L_\mu^{-1}\|^{\alpha},$$

which concludes the proof.

THEOREM 7.3. *Given $0 < \epsilon < 1$, for $\alpha < \frac{\epsilon}{2(1-\epsilon)}$ we have $\beta = 1 + \alpha(1-\epsilon) - \frac{\epsilon}{2} < 1$ and*

$$\|\hat{L}_1^{1-\epsilon} e^{-\hat{L}_\nu t} B_\nu - \hat{L}_1^{1-\epsilon} e^{-\hat{L}_0 t} A\| \leq M t^{-\beta} \|\hat{L}_\nu^{-1} - L_0^{-1}\|^{\alpha(1-\epsilon)}$$

and also

$$\|\hat{L}_1^{1-\epsilon} e^{-\hat{L}_\nu t} B_\nu - \hat{L}_1^{1-\epsilon} e^{-\hat{L}_\mu t} B_\mu\| \leq M t^{-\beta} \|\hat{L}_\nu^{-1} - A^{-2}\|^{\alpha(1-\epsilon)}$$

for all $\mu, \nu \in [0,1]$.

Proof: The result follows from (7.8), (7.9) and from the moment inequality

$$\|\hat{L}_1^{1-\epsilon} x\| \leq M \|\hat{L}_1 x\|^{1-\epsilon} \|x\|^{\epsilon}, \quad \text{for all } x \in D(\hat{L}_1).$$

THEOREM 7.4. *If $Q_\nu := Q_\nu(\sigma_+^\nu(u^*)$, there is a constant M, independent of $\epsilon \in [0,1]$, such that*

$$\|Q_\nu e^{-L_\nu(t-s)} B_\nu - Q_0 e^{-L_0(t-s)} A\| \leq M t^{-\frac{1}{2}} e^{\beta^+ t}, \ t \leq 0,$$
$$\|L_1 Q_\nu e^{-L_\nu(t-s)} B_\nu - L_1 Q_0 e^{-L_0(t-s)} A\| \leq M t^{-1} e^{\beta^+ t} \|L_\nu^{-1} - L_\mu^{-1}\|^{\alpha}, \ t \leq 0,$$

and

$$\|(I - Q_\nu) e^{-L_\nu(t-s)} B_\nu - (I - Q_0) e^{-L_0(t-s)} A\| \leq M t^{-\frac{1}{2}} e^{-\beta^- t}, \ t \geq 0,$$
$$\|L_1 (I - Q_\nu) e^{-L_\nu(t-s)} B_\nu - L_1 (I - Q_0) e^{-L_0(t-s)} A\| \leq M t^{-1} e^{-\beta^- t} \|L_\nu^{-1} - L_\mu^{-1}\|^{\alpha}, \ t \geq 0.$$

Proof: Note that

$$L_1 Q_\nu e^{-L_\nu(t-s)} B_\nu = L_\nu^* Q_\nu^* e^{-L_\nu^* t}$$

and consequently, for all $t \leq 0$,

$$\|Q_\nu e^{-L_\nu(t-s)} B_\nu - Q_0 e^{-L_0(t-s)} A\| \leq 2 \sup_{\nu \in [0,1]} \left\{ \|L_\nu^{\frac{1}{2}} Q_\nu e^{-L_\nu(t-s)}\| \|L_\nu^{-\frac{1}{2}} B_\nu\| \right\} \leq M t^{-\frac{1}{2}} e^{\beta^+ t},$$

$$\|L_1 \mathcal{Q}_\nu e^{-L_\nu (t-s)} B_\nu - L_1 \mathcal{Q}_0 e^{-L_0(t-s)} A\| = \|L_\nu^* \mathcal{Q}_\nu^* e^{-L_\nu^* t} - L_0^* \mathcal{Q}_0^* e^{-L_0^* t}\|$$
$$= \|L_\nu \mathcal{Q}_\nu e^{-L_\nu t} - L_0 \mathcal{Q}_0 e^{-L_0 t}\| \leq M t^{-1} e^{-\beta^+ t} \|L_\nu^{-1} - L_\mu^{-1}\|^\alpha.$$

The other inequality is proved in a similar way.

8. CONTINUITY OF LOCAL UNSTABLE MANIFOLDS

Let $u^* \in \mathcal{E}$ be a hyperbolic equilibrium point. Rewrite (2.4) as

$$w_t = -L_\nu w + B_\nu h(w) \tag{8.1}$$

where $w = u - u^*$, $L_\nu = B_\nu(A + f'(u^*)I)$ and $h(w) = f(u^* + w) - f(u^*) - f'(u^*)w$. We decompose this equation using the projection given in (7.2); that is, if $\mathcal{Q}_\nu := \mathcal{Q}_\nu(\sigma_+^\nu(u^*)$, $z(t) = \mathcal{Q}_\nu w(t)$ and $y(t) = (I - \mathcal{Q}_\nu)w(t)$ we have

$$\begin{aligned} z(t) &= e^{-L_\nu(t-\tau)} z(\tau) + \int_\tau^t \mathcal{Q}_\nu e^{-L_\nu(t-s)} B_\nu h(z(s) + y(s)) ds, \\ y(t) &= e^{-L_\nu(t-\tau)} y(\tau) + \int_\tau^t (I - \mathcal{Q}_\nu) e^{-L_\nu(t-s)} B_\nu h(z(s) + y(s)) ds. \end{aligned} \tag{8.2}$$

First we modify the nonlinearity h in such a way that, it remains the same inside the ball $\|w\|_{X^1} \leq \delta$, and

$$\begin{aligned} &\|h(w)\|_{X^\epsilon} \leqslant \rho, \\ &\|h(w) - h(\tilde{w})\|_{X^\epsilon} \leqslant \rho \|w - \tilde{w}\|_{X^1}. \end{aligned} \tag{8.3}$$

We obtain that the unstable manifold of u^* is given as a graph near u^* and that this graph varies continuously with respect to ν.

Remark 8. 1. Observe that we are looking for a function $\Sigma_\nu{}^*$ such that, if $\tau \in \mathbb{R}$ and $(\zeta, \Sigma_\nu{}^*(\zeta)) \in \mathcal{Z}$, then the solution of (8.2) starting at $z(\tau) = \zeta$, $y(\tau) = \Sigma_\nu{}^*(\zeta)$ stays in the graph of $\Sigma_\nu{}^*$ for all positive and all negative time. This means that $y(t) = \Sigma_\nu{}^*(z(t))$ for all t and thus (8.2) becomes

$$\begin{aligned} z(t) &= e^{-L_\nu(t-\tau)} z(\tau) + \int_\tau^t \mathcal{Q}_\nu e^{-L_\nu(t-s)} B_\nu h(z(s) + \Sigma_\nu{}^*(z(s))) ds, \\ y(t) &= e^{-L_\nu(t-\tau)} y(\tau) + \int_\tau^t (I - \mathcal{Q}_\nu) e^{-L_\nu(t-s)} B_\nu h(z(s) + \Sigma_\nu{}^*(z(s))) ds. \end{aligned} \tag{8.4}$$

Also, the solution $(z(t), y(t))$ should tend to zero as $t \to -\infty$ (in particular, it should stay bounded as $t \to -\infty$). Letting $t_0 \to -\infty$ we have

$$y(t) = \Sigma_\nu{}^*(z(t)) = \int_{-\infty}^t (I - \mathcal{Q}_\nu) e^{-L_\nu(t-s)} B_\nu h(z(s) + \Sigma_\nu{}^*(z(s))) ds$$

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and, in particular,

$$\Sigma_\nu{}^*(\zeta) = \Sigma_\nu{}^*(z(\tau)) = y(\tau) = \int_{-\infty}^{\tau} (I - \mathcal{Q}_\nu)e^{-L_\nu(\tau-s)}B_\nu h(z(s) + \Sigma_\nu{}^*(z(s)))ds.$$

The unstable manifold $W^{\mathrm{u}}(u^*)$ of an equilibrium u^* is defined as follows:

$$W^{\mathrm{u}}(u^*) = \{\, z \in \mathcal{Z} : \text{ there is a global solution } y(t) \text{ for } \{T_\nu(t) : t \geq 0\} \text{ satisfying } y(\tau) = z \text{ and such that } \lim_{t\to-\infty} \|y(t) - u^*\|_{\mathcal{Z}} = 0\}.$$

PROPOSITION 8.1. *For suitably small δ and for all $\nu \in [0,1]$ there exists a function $\Sigma_\nu^* : Z_\nu \to Y_\nu$, such that the unstable manifold $W_\nu^u(u^*)$ of the equilibrium solution u^* to* (8.2) *is given by*

$$W_\nu^u(u^*) = \{u^* + (\mathcal{Q}_\nu w, \Sigma_\nu^*(\mathcal{Q}_\nu w)) : w \in Z_\nu \oplus Y_\nu\};$$

also, for any $\zeta_\nu \in Z_\nu$ and $\nu \in [0,1]$,

$$\Sigma_\nu^*(\zeta_\nu) = \int_{-\infty}^{\tau} e^{A_\nu^-(\tau-s)}G_\nu(z_\nu^+(s), \Sigma_\nu^*(z_\nu^+(s)))ds.$$

For any $r > 0$ and $\mu \in [0,1]$,

$$\sup_{\substack{z\in X^1 \\ \|z\|_{X^1}\leqslant r}} \{\|\mathcal{Q}_\nu(z) - \mathcal{Q}_\mu(z)\|_{X^1} + \|\Sigma_\nu^*(\mathcal{Q}_\nu(z)) - \Sigma_\mu^*(\mathcal{Q}_\mu(z))\|_{X^1}\} \xrightarrow{\nu\to\mu} 0,$$

and there exists $\gamma > 0$ such that, for any $u_0 \in V$ and as long as $T_\nu(t)u_0 \in V$,

$$\|(I - \mathcal{Q}_\nu)(T_\nu(t)u_0) - \Sigma_\nu^u(\mathcal{Q}_\nu(T_\nu(t)u_0))\|_{X^1} \leq Me^{-\gamma t}.$$

The proof of this result follows using the procedure as in [7] from the results obtained in Section 5 and Section 7. Crucial is the uniformity with respect to $\nu \in [0,1]$ of the estimates obtained in Sections 5 and 7. We remark that this procedure is standard but the estimates in the above sections must be obtained and cannot be claimed just from the continuity of the resolvent operators or eigenvalues.

9. CONTINUITY OF ATTRACTORS AND THEIR DIMENSION

In this section we study continuity with respect to the parameter $\nu \in [0,1]$ of the attractors $\mathcal{A}_\nu$ of (1.1). As seen in [16] the attractors behave continuously with respect to the parameter ν (under some more restrictive conditions on the nonlinearity). Besides that we also show that the dimension of the attractor is the same for all values of $\nu \in [0,1]$.

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The attractor for (1.1) in $X^1 = H_0^1(\Omega)$ is given by

$$\mathcal{A}_\nu = W_\nu^u(\mathcal{E}),$$

where $W_\nu^u(\mathcal{E})$, *the unstable set of the set of equilibria* $\mathcal{E}$, is defined as

$$W_\nu^u(\mathcal{E}) = \{v \in H_0^1(\Omega) : \text{there is a backward solution } u_\nu(t,v) \text{ of } (1.1),\ u_\nu(0,v) = v, \text{ such that } u_\nu(t,v) \to \mathcal{E} \text{ as } t \to -\infty\}.$$

We stress the dependence of $W_\nu^u(\mathcal{E})$ on the parameter ν. Even though the set $\mathcal{E}$ is independent of ν, the flow defined by (1.1) may be completely different for different values of ν. Our aim is to show that in the attractor the flow behaves *the same* for all values of $\alpha \in [0,1]$.

Before we proceed let us recall the definition of the *gradient semigroup.*

DEFINITION 9.1. We say that a nonlinear semigroup $\{T_\nu(t) : t \geq 0\}$ is *gradient* if $\{T_\nu(t)u : t \geq 0\}$ is relatively compact for each $u \in X^1$ and there exists a continuous function $\mathcal{L} : X^1 \to \mathbb{R}$ such that

- $t \mapsto \mathcal{L}(T_\nu(t)u) : [0,\infty) \to X^1$ is non-increasing for each $u \in X^1$.
- If $u \in X^1$ is such that there is a global solution $\xi(\cdot) : \mathbb{R} \to X^1$ through $\xi(0) = u$ and there exists a $t^* \in \mathbb{R}$ such that $\mathcal{L}(\xi(t)) = \mathcal{L}(u)$ for all $t \geq t^*$ or for all $t \leq t^*$, then u is a solution for $\{T_\nu(t) : t \geq 0\}$ (and so in fact $\mathcal{L}(\xi(t)) = \mathcal{L}(u)$ for all $t \in \mathbb{R}$).

The function $\mathcal{L} : X^1 \to \mathbb{R}$ is called a Lyapunov function for $\{T_\nu(t) : t \geq 0\}$.

We note that the function $\mathcal{L}$ defined in (3.2) is independent of ν and is a Liapunov function for $\{T_\nu(t) : t \geq 0\}, \nu \in [0,1]$.

Theorem 1.3 formulated in the introduction is a consequence of the results in the previous sections and of the results in Section 3.8 of [14].

Proof of Theorem 1.3. From the results in [12] we have the first equality in (1.9). For the remaining equalities in (1.9) we only have to mention that the projections vary continuously and therefore must have constant rank. The continuity of attractors follow from the continuity of unstable manifolds (Proposition 8.1) (see [2]) and the exponential decay towards the attractors is proved in [6].

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