CADERNOS DE MATEMÁTICA **08**, 285–303 October (2007)
ARTIGO NÚMERO SMA#285

Converse Lyapunov theorems for retarded functional differential equations

M. Federson

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: federson@icmc.usp.br

We consider retarded functional differential equations and prove converse Lyapunov theorems for these equations concerning the classical concepts of Lyapunov stability and uniform asymptotic stability of the trivial solution.
October, 2007 ICMC-USP

*Mathematical Subject Classification 2000:*34K20, 34D20.

Key Words: Lyapunov stability, variational stability, integral stability, stability by perturbations, converse theorems.

1. INTRODUCTION

In [3], the authors introduced new concepts of stability for retarded functional differential equations (RFDEs), namely variational stability and asymptotic variational stability, and established converse Lyapunov-type theorems for RFDEs concerning these new concepts. Such results apply for a very large class of RFDEs which can be described, for instance, by functions with many discontinuities.

In the present paper, we reduce the class of RFDEs so that the functions involved in the retarded system satisfy smooth conditions and, under these conditions, we prove the equivalence of the classical concepts of Lyapunov stability and uniform asymptotic stability and the corresponding concepts of variational stability and asymptotic variational stability of the trivial solution. These facts are achieved by means of the nonlinear variation-of-constants formula of Alekseev for RFDEs given in [11]. Then we get converse Lyapunov theorems for RFDEs concerning the classical concepts of stability by applying the ideas in [3].

For the sake of self-containedness of this paper, we include the terminology used in [3] and some facts from the theory of generalized ordinary differential equations (generalized ODEs) which is the key to getting the main results. We describe the correspondence

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

between RFDEs and Banach-space valued generalized ODEs (see [1]) in a very general setting and only specialize the assumptions when necessary.

Let X be a Banach space and $I \subset \mathbb{R}$ be an interval of the real line.

We denote by $G(I,X)$ be the space of locally regulated functions $f : I \to X$, that is, for each compact interval $[a,b] \subset I$, the lateral limits $f(t+) = \lim_{\rho\to 0+} f(t+\rho)$, $t \in [a,b)$, and $f(t-) = \lim_{\rho\to 0-} f(t+\rho)$, $t \in (a,b]$, exist and are finite. When $I = [a,b]$ we write $G([a,b],X)$ which is a Banach space when endowed with the usual supremum norm. In $G(I,X)$ we consider the topology of locally uniform convergence. By $G^-(I,X)$, we mean the subspace of $G(I,X)$ of left continuous functions for which $f(t-) = \lim_{\rho\to 0-} f(t+\rho) = f(t)$, $t \in I$, except for the left endpoint of I.

We denote by $BV(I,X)$ the space of functions $f : I \to X$ which are locally of bounded variation, that is, for each compact interval $[a,b] \subset I$, the restriction of f to $[a,b]$, $f\big|_{[a,b]}$, is of bounded variation. In $BV([a,b],X)$, we consider the variation norm given by $\|f\| = \|f(a)\| + \mathrm{var}_a^b f$, where $\mathrm{var}_a^b f$ stands for the variation of f in the interval $[a,b]$. Then $BV([a,b],X)$ is a Banach space and $BV([a,b],X) \subset G([a,b],X)$. When $f \in BV(I,X)$ is also left continuous ($f \in BV(I,X) \cap G^-(I,X)$), we write $f \in BV^-(I,X)$.

We write $C(I,X)$ to denote the space of continuous functions $f : I \to X$. We consider the Banach space $C([a,b],X)$ equipped with the usual supremum norm and in $C(I,X)$ we consider the topology of locally uniform convergence.

It is clear that $C(I,X) \subset G^-(I,X)$ and $BV^-(I,X) \subset G^-(I,X)$.

2. RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

Let us consider the initial value problem for a retarded functional differential equation:

$$\begin{cases} \dot{y}(t) = f(y_t, t), \\ y_{t_0} = \phi, \end{cases} \tag{2.1}$$

where $\phi \in G^-([-r,0],\mathbb{R}^n)$, $r \geq 0$, and $f(\phi,t)$ maps an open subset Ω of $G^-([-r,0],\mathbb{R}^n) \times [t_0,+\infty)$ to $\mathbb{R}^n$. Given a function $y : [t_0 - r, +\infty) \to \mathbb{R}^n$, we consider $y_t : [-r,0] \to \mathbb{R}^n$ defined, as usual, by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0],\ t \in [t_0,+\infty).$$

Let us recall the concept of a solution of problem (2.1).

Definition 2.1. Let $\sigma > 0$. A function $y \in G^-([t_0 - r, t_0+\sigma],\mathbb{R}^n)$ such that $(y_t,t) \in G^-([-r,0],\mathbb{R}^n) \times [t_0,t_0+\sigma]$ for all $t \in [t_0,t_0+\sigma]$, $y_{t_0} = \phi$ and

$$\dot{y}(t) = f(y_t,t)$$

for almost all $t \in [t_0,t_0+\sigma]$ is called a (local) *solution of* (2.1) *in* $[t_0,t_0+\sigma]$ (or sometimes also in $[t_0 - r, t_0+\sigma]$) *with initial condition* (ϕ,t_0).

We assume that $f(0,t) = 0$ for every $t \in \mathbb{R}$ so that $y \equiv 0$ is a solution of (2.1).

The system (2.1) is known to be equivalent to the "integral" equation

$$\text{(2.2)} \qquad \begin{cases} y\,(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f\,(y_s, s)\,ds, \quad t \in [t_0, +\infty), \\ y_{t_0} = \phi, \end{cases}$$

when the integral exists in the Lebesgue sense (cf. [5]). It is clear that y, being an indefinite integral of a Lebesgue integrable function, is absolutely continuous on $[t_0, t_0 + \sigma]$ (we write $y \in AC([t_0, t_0 + \sigma], \mathbb{R}^n)$).

Let $G_1 \subset G^-([t_0 - r, +\infty), \mathbb{R}^n)$ with the following property: if $y = y\,(t)$, $t \in [t_0 - r, +\infty)$, is an element of G_1 and $\bar{t} \in [t_0 - r, +\infty)$, then $\bar{y}$ given by

$$\bar{y}\,(t) = \begin{cases} y\,(t)\,, \; t_0 - r \leq t \leq \bar{t} \\ y\,(\bar{t})\,, \; \bar{t} < t < +\infty \end{cases}$$

also belongs to G_1.

Let $|\cdot|$ be a norm in $\mathbb{R}^n$.

We consider $f\,(\phi, t) : G^-\,([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$, the righthand side of the differential equation in (2.1), such that the mapping $t \mapsto f\,(y_t, t)$ is locally Lebesgue integrable for $y \in G_1$ and the following conditions are fulfilled:

(A) there is M locally Lebesgue integrable such that for all $x \in G_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} f\,(x_s, s)\,ds \right| \leq \int_{u_1}^{u_2} M\,(s)\,ds;$$

(B) there is L locally Lebesgue integrable such that for all $x, y \in G_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} [f\,(x_s, s) - f\,(y_s, s)]\,ds \right| \leq \int_{u_1}^{u_2} L\,(s)\,\|x_s - y_s\|\,ds,$$

the norm on the righthand side is the norm in $G^-\,([-r, 0], \mathbb{R}^n)$ given by

$$\|\phi\| = \sup_{t \in [-r, 0]} |\phi(t)| \text{ for } \phi \in G^-\,([-r, 0], \mathbb{R}^n).$$

Of course the functions M and L above depend on the choice of t_0.

The next definitions concern stability concepts for the solution $y \equiv 0$ of (2.1). The following three definitions are the classical definitions for Lyapunov stability, uniform (Lyapunov) stability and uniform asymptotic stability of the trivial solution of (2.1). See [5], for instance.

Definition 2.2. The trivial solution of system (2.1) is called *(Lyapunov) stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that if $\phi \in G^-([-r,0],\mathbb{R}^n)$ and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0 - r, +\infty)$ and $[\gamma, v] \ni t_0$, is a solution of (2.1) such that $\overline{y}_{t_0} = \phi$ and

$$\|\phi\| < \delta,$$

then

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [t_0, v].$$

Definition 2.3. The trivial solution of system (2.1) is called *uniformly stable* if the number δ in Definition 2.2 is independent of t_0.

Definition 2.4. The solution $y \equiv 0$ of (2.1) is called *uniformly asymptotically stable* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a $T = T(\varepsilon, \delta_0) \geq 0$ such that if $\phi \in G^-([-r,0],\mathbb{R}^n)$, and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0 - r, +\infty)$ and $[\gamma, v] \ni t_0$, is solution of (2.1) such that $\overline{y}_{t_0} = \phi$ and

$$\|\phi\| < \delta_0,$$

then

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty).$$

The next definition of stability of the solution $y \equiv 0$ of (2.1) is borrowed from [4].

Definition 2.5. The solution $y \equiv 0$ of (2.1) is said to be *integrally stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\phi \in G^-([-r,0],\mathbb{R}^n)$, $\|\phi\| < \delta$ and $p : [t_0, t_1] \to \mathbb{R}^n$ is Lebesgue integrable with $\int_{t_0}^{t_1} |p(s)| ds < \delta$, then

$$|y(t; t_0, \phi)| < \varepsilon \ \text{ for every } \ t \in [t_0, t_1],$$

where $y(t; t_0, \phi)$ is a solution of the perturbed equation

$$\begin{cases} \dot{y}(t) = f(y_t, t) + p(t), \\ y_{t_0} = \phi. \end{cases} \tag{2.3}$$

The solution of equation (2.3) has to be interpreted as a solution of the "integral" equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)\, ds + \int_{t_0}^{t} p(s) ds \\ y_{t_0} = \phi, \end{cases} \tag{2.4}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

where the integral is considered in the Lebesgue sense. The solution of (2.3), when it exists, is absolutely continuous on $[t_0, t_1]$ (i.e., $y(\cdot; t_0, \phi) \in AC([t_0, t_1], \mathbb{R}^n)$).

Now we mention a concept of stability of the trivial solution of (2.1) which generalizes Definition 2.5. This concept was introduced in [3].

Definition 2.6. The solution $y \equiv 0$ of (2.1) is said to be *variationally stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\phi \in G^-([-r, 0], \mathbb{R}^n)$, $\|\phi\| < \delta$ and $P \in BV^-([t_0, t_1], \mathbb{R}^n)$ with $\text{var}_{t_0}^{t_1} P < \delta$, then

$$|y(t; t_0, \phi)| < \varepsilon \text{ for every } t \in [t_0, t_1],$$

where $y(t; t_0, \phi)$ is a solution of

$$\begin{cases} y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\, ds + P(t) - P(t_0), & t \in [t_0, t_1] \\ y_{t_0} = \phi. \end{cases} \tag{2.5}$$

It can be seen immediately that the solution y of (2.5) is of bounded variation and left continuous, that is, $y \in BV^-([t_0, t_1], \mathbb{R}^n) \subset G^-([t_0, t_1], \mathbb{R}^n)$.

Note that (2.4) is a particular case of (2.5) for $P(t) = \int_{t_0}^{t} p(s) ds$, $t \geq t_0$. If $p : [t_0, t_1] \to \mathbb{R}^n$ is Lebesgue integrable, then $P \in AC([t_0, t_1], \mathbb{R}^n) \subset BV^-([t_0, t_1], \mathbb{R}^n)$, the derivative $\dot{P}(s) = \frac{dP}{ds}$ exists almost everywhere in $[t_0, t_1]$ and

$$\text{var}_{t_0}^{t_1} P = \int_{t_0}^{t_1} |\dot{P}(s)| ds = \int_{t_0}^{t_1} |p(s)| ds.$$

Having this in mind we can easily see that the variational stability of the trivial solution of (2.1) is a concept which is more general than that of integral stability. Therefore we consider the variational stability only.

The next two definitions are also borrowed from [3].

Definition 2.7. The solution $y \equiv 0$ of (2.1) is called *variationally attracting* if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if

$$\|\phi\| < \widetilde{\delta} \qquad \text{and} \qquad \text{var}_{t_0}^{t_1} P < \rho$$

with $P \in BV^-([t_0, t_1], \mathbb{R}^n)$, then

$$|y(t; t_0, \phi)| < \varepsilon \text{ for all } t \geq t_0 + T, \ t \in [t_0, t_1]$$

where $y(t; t_0, \phi)$ is a solution of the equation (2.5) satisfying $y_{t_0} = \phi$.

Definition 2.8. The solution $y \equiv 0$ of (2.1) is called *variationally asymptotically stable* if it is variationally stable and variationally attracting.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

3. THE GENERALIZED ODE CORRESPONDING TO (2.5)

Let X be a Banach space and consider $\Omega \subset X \times \mathbb{R}$. Assume that $G : \Omega \to X$ is a given X-valued function with $G(x,t)$ defined for each $(x,t) \in \Omega$.

Having the concept of Kurzweil integrability in mind (see for instance [1], [2], [9] or [10]), we now present the concept of generalized ordinary differential equation (generalized ODE).

Definition 3.1. A function $x : [\alpha,\beta] \to X$ is called *a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x,t) \tag{3.1}$$

on the interval $[\alpha,\beta] \subset \mathbb{R}$ if $(x(t),t) \in \Omega$ for all $t \in [\alpha,\beta]$ and if the equality

$$x(v) - x(\gamma) = \int_{\gamma}^{v} DG(x(\tau),t) \tag{3.2}$$

holds for every γ, $v \in [\alpha,\beta]$, where the integral is considered in the sense of Kurzweil.

Let us mention that the theory of generalized ODEs presented e.g. in [9] is for the case when $X = \mathbb{R}^n$, but it is easy to check that all the basic results hold also for the case of a Banach space. See, for instance, [1] and [2].

Given an initial condition $(z_0,t_0) \in \Omega$, we define next the solution of the initial value problem for the equation (3.1).

Definition 3.2. A function $x : [\alpha,\beta] \to X$ is a *solution of the generalized ordinary differential equation* (3.1) *with the initial condition* $x(t_0) = z_0$ *on the interval* $[\alpha,\beta] \subset \mathbb{R}$ if $t_0 \in [\alpha,\beta]$, $(x(t),t) \in \Omega$ for all $t \in [\alpha,\beta]$ and if the equality

$$x(v) - z_0 = \int_{t_0}^{v} DG(x(\tau),t) \tag{3.3}$$

holds for every $v \in [\alpha,\beta]$.

Now we consider $\Omega = G_1 \times [t_0,+\infty)$ and we define a special class of functions $F : \Omega \to X$.

Definition 3.3. We say that a function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega,h)$, if there exists a nondecreasing, left continuous function $h : [t_0,+\infty) \to \mathbb{R}$ such that

$$\|G(x,s_2) - G(x,s_1)\| \leq |h(s_2) - h(s_1)| \tag{3.4}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{3.5}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

Suppose $f(\phi, t) : G_1 \times [t_0, +\infty) \to \mathbb{R}^n$ is such that for each $y \in G_1$ the mapping $t \mapsto f(y_t, t)$ is locally Lebesgue integrable and f satisfies conditions (A) and (B). Assume further that $P \in BV^-([t_0, +\infty), \mathbb{R}^n)$.

For $y \in G_1$ and $t \in [t_0 - r, +\infty)$, define

$$F(y, t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0 \text{ or } t_0 - r \leq t \leq t_0 \\ \int_{t_0}^{\vartheta} f(y_s, s)\, ds, & t_0 \leq \vartheta \leq t < +\infty; \\ \int_{t_0}^{t} f(y_s, s)\, ds, & t_0 \leq t \leq \vartheta < +\infty. \end{cases} \tag{3.6}$$

and, for $t \in [t_0 - r, +\infty)$, let

$$\overline{P}(t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0 \text{ or } t_0 - r \leq t \leq t_0 \\ P(\vartheta) - P(t_0), & t_0 \leq \vartheta \leq t < +\infty; \\ P(t) - P(t_0), & t_0 \leq t \leq \vartheta < +\infty. \end{cases} \tag{3.7}$$

Then

$$G(y, t) = F(y, t) + \overline{P}(t) \tag{3.8}$$

defines an element $G(y, t)$ of $G^-([t_0 - r, +\infty), \mathbb{R}^n)$ and $G(y, t)(\vartheta) \in \mathbb{R}^n$ is the value of $G(y, t)$ at a point $\vartheta \in [t_0 - r, +\infty)$, that is,

$$G : G_1 \times [t_0 - r, +\infty) \to G^-([t_0 - r, +\infty), \mathbb{R}^n).$$

The idea to construct the righthand side of a GODE which corresponds to a RFDE of the form (2.1) is due to C. Imaz, F. Oliva and Z. Vorel (see [6] and [7]).

Let $h : [t_0, +\infty) \to \mathbb{R}$ be defined by

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)] ds + \text{var}_{t_0}^{t} P, \quad t \in [t_0, +\infty).$$

Since $M, L : [t_0, +\infty) \to \mathbb{R}$ are nonnegative a.e. and $P \in BV^-([t_0, +\infty), \mathbb{R}^n)$, then the function h is left continuous and nondecreasing.

Under the given assumptions, the function G given by (3.8) belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = G_1 \times [t_0, +\infty)$ (see e.g. [1]).

Consider G given by (3.8). If $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \to G^-([t_0 - r, +\infty), \mathbb{R}^n)$ is a solution of (3.1) in $[\alpha, \beta]$, then x is of bounded variation in $[\alpha, \beta]$ and $\text{var}_{\alpha}^{\beta} x \leq h(\beta) - h(\alpha) <$

$+\infty$. Moreover, every point in $[\alpha,\beta]$ at which the function h is continuous is a point of continuity of the solution $x:[\alpha,\beta]\to G^{-}([t_0-r,+\infty),\mathbb{R}^n)$.

Assume $G\in\mathcal{F}(\Omega,h)$ and suppose $x:[\alpha,\beta]\to X$, $[\alpha,\beta]\subset[t_0,+\infty)$, is of bounded variation in $[\alpha,\beta]$ and $(x(s),s)\in\Omega$ for every $s\in[\alpha,\beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau),t)$ exists and the function $s\mapsto\int_\alpha^s DG(x(\tau),t)\in X$ is of bounded variation for all $s\in[\alpha,\beta]$.

For a proof of facts above, see [9].

The next result concerns the existence of a solution of (3.1) (see [1], Theorem 2.15 for a proof).

THEOREM 3.1. *Let $G:\Omega\to X$ be an element of the class $\mathcal{F}(\Omega,h)$, where the function h is left continuous (i.e. $h(t-)=h(t)$, $t\in(a,+\infty)$). Then for every $(\widetilde{x},t_0)\in\Omega$ such that for $\widetilde{x}_+=\widetilde{x}+G(\widetilde{x},t_0+)-G(\widetilde{x},t_0)$ we have $(\widetilde{x}_+,t_0)\in\Omega$ and there exists a $\Delta>0$ such that on the interval $[t_0,t_0+\Delta]$ there exists a unique solution $x:[t_0,t_0+\Delta]\to X$ of the generalized ordinary differential equation* (3.1) *for which $x(t_0)=\widetilde{x}$.*

Consider the generalized equation (3.1). We will work now with a specific initial value problem for equation (3.3) with G given by (3.8).

Let $\phi\in G^{-}([-r,0],\mathbb{R}^n)$ and $\sigma>0$ be given. A function $x(t)$ defined on the interval $[t_0-r,t_0+\sigma]$ and taking values in $G^{-}([t_0-r,t_0+\sigma],\mathbb{R}^n)$ is a (local) *solution of the generalized ordinary differential equation* (3.1) in the interval $[t_0,t_0+\sigma]$ (or in $[t_0-r,t_0+\sigma]$), with *initial condition* $x(t_0)\in G_1$ given for $\phi\in G^{-}([-r,0],\mathbb{R}^n)$ by

$$x(t_0)(\vartheta)=\begin{cases}\phi(\vartheta-t_0) & \text{for } \vartheta\in[t_0-r,t_0],\\ x(t_0)(t_0) & \text{for } \vartheta\in[t_0,t_0+\sigma]\end{cases}$$

if

$$x(v) \;=\; x(t_0)+\int_{t_0}^{v} DG(x(\tau),t)$$

for every $v\in[t_0,t_0+\sigma]$.

The next result is the key to our approach to RFDEs by the theory of generalized ODEs. It gives a one-to-one correspondence between the solutions y of (2.5) and the solutions x of (3.1). For a proof of it see [1].

PROPOSITION 3.1.

(i) *Consider equation* (2.5)*, where $f:G_1\times[t_0,t_0+\sigma]\to\mathbb{R}^n$, $t\mapsto f(y_t,t)$ is Lebesgue integrable on $[t_0,t_0+\sigma]$, $P\in BV^{-}([t_0,t_0+\sigma],\mathbb{R}^n)$ and (A), (B) are fulfilled. Let $y(t)$ be a solution of problem* (2.5) *in the interval $[t_0,t_0+\sigma]$. Given $t\in[t_0-r,t_0+\sigma]$, let*

$$x(t)(\vartheta)=\begin{cases}y(\vartheta)\,, & \vartheta\in[t_0-r,t]\\ y(t)\,, & \vartheta\in[t,t_0+\sigma]\,.\end{cases}$$

Then $x(t)\in G^{-}([t_0-r,t_0+\sigma],\mathbb{R}^n)$ is a solution of (3.1) *in $[t_0-r,t_0+\sigma]$ where the right hand side of* (3.1) *is given by* (3.8) *.*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

(ii)*Reciprocally, let* G *be given by* (3.8) *and* $x(t)$ *be a solution of* (3.1) *in the interval* $[t_0 - r, t_0 + \sigma]$ *satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), \ t_0 \le \vartheta \le t_0 + \sigma \end{cases} \tag{3.9}$$

For every $\vartheta \in [t_0 - r, t_0 + \sigma]$, *define*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \le \vartheta \le t_0 \\ x(\vartheta)(\vartheta), & t_0 \le \vartheta \le t_0 + \sigma. \end{cases}$$

Then $y(\vartheta)$ *is a solution of* (2.5) *in* $[t_0 - r, t_0 + \sigma]$ *and* $y(\vartheta) = x(t_0 + \sigma)(\vartheta)$ *for all* $\vartheta \in [t_0 - r, t_0 + \sigma]$.

4. CONCEPTS OF STABILITY FOR GENERALIZED ODE'S

In this section, $\Omega = B_c \times [t_0 - r, \infty)$, where $r > 0$, $B_c = \{y \in X; \|y\| < c\}$, $c > 0$, and X is any Banach space.

In the sequel, we assume that $F : \Omega \to X$ belongs to $\mathcal{F}(\Omega, h)$ and $F(0,t) - F(0,s) = 0$, for $t, s \in [t_0 - r, +\infty)$. Thus for every $[\gamma, v] \subset [t_0 - r, +\infty)$, we have

$$\int_\gamma^v DF(0,t) = F(0,v) - F(0,\gamma) = 0$$

which implies $x \equiv 0$ is a solution of the generalized equation

$$\frac{dx}{d\tau} = DF(x,t) \tag{4.1}$$

on $[t_0 - r, +\infty)$.

If $F \in \mathcal{F}(\Omega, h)$ and $x : [\gamma, v] \to X$ is a solution of (4.1), where $[\gamma, v] \subset [t_0 - r, +\infty)$, then x is of bounded variation in $[\gamma, v]$. Thus it is natural to measure the distance between two solutions by the variation norm.

The next stability concepts are based on the variation of the solutions around $x \equiv 0$. Such concepts were introduced by Š. Schwabik in [10]. See also [9].

Definition 4.1. The solution $x \equiv 0$ of (4.1) is called *variationally stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 - r \le \gamma < v < +\infty$ is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

and

$$\mathrm{var}_\gamma^v\left(\overline{x}(s)-\int_\gamma^s DF(\overline{x}(\tau),t)\right)<\delta,$$

then

$$\|\overline{x}(t)\|<\varepsilon,\quad t\in[\gamma,v].$$

Definition 4.2. The solution $x\equiv 0$ of (4.1) is called *variationally attracting* if there exists a $\delta_0>0$ and for every $\varepsilon>0$, there exist a $T=T(\varepsilon)\geq 0$ and a $\rho=\rho(\varepsilon)>0$ such that if $\overline{x}:[\gamma,v]\to B_c$, $t_0-r\leq\gamma<v<+\infty$, is a function of bounded variation in $[\gamma,v]$ such that

$$\|\overline{x}(\gamma)\|<\delta_0$$

and

$$\mathrm{var}_\gamma^v\left(\overline{x}(s)-\int_\gamma^s DF(\overline{x}(\tau),t)\right)<\rho,$$

then

$$\|\overline{x}(t)\|<\varepsilon,\quad \text{for } t\in[\gamma,v]\cap[\gamma+T,+\infty) \text{ and } \gamma\geq t_0-r.$$

Definition 4.3. The solution $x\equiv 0$ of (4.1) is called *variationally asymptotically stable* if it is variationally stable and variationally attracting.

The conditions in Definition 4.1 mean that the function $\overline{x}$ of bounded variation is close (in the variation norm: $\|\overline{x}(\gamma)\|+\mathrm{var}(\overline{x}(s)-\int_\gamma^s DF(\overline{x}(\tau),t)))$ to the solution $x\equiv 0$ of (4.1).

Beside the generalized ODE (4.1), let us consider the perturbed generalized equation

$$\frac{dx}{d\tau}=D[F(x,t)+\overline{P}(t)] \tag{4.2}$$

where $\overline{P}\in BV^-([t_0-r,\infty),X)$. It is easy to verify that for the function $G(x,t)=F(x,t)+\overline{P}(t)$ we have $G\in\mathcal{F}(\Omega,h_{\overline{P}})$, where $h_{\overline{P}}(t)=h(t)+\mathrm{var}_{-r}^t\overline{P}$. Therefore the solutions of (4.2) have good properties (existence, uniqueness, etc.). See Theorem 3.1, for instance.

Let us present some other definitions also borrowed from [10].

Definition 4.4. The solution $x\equiv 0$ of (4.1) is called *stable with respect to perturbations* if for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that if $\|x_0\|<\delta$ and $\overline{P}\in BV^-([\gamma,v],X)$ is continuous from the left with $\mathrm{var}_\gamma^v\overline{P}<\delta$ then

$$\|x(t,\gamma,x_0)\|<\varepsilon \ \text{ for every } \ t\in[\gamma,v]$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

where $x(t, \gamma, x_0)$ is a solution of the perturbed generalized equation (4.2) with $x(\gamma, \gamma, x_0) = x_0$ and $[\gamma, v] \subset [t_0 - r, +\infty)$.

Definition 4.5. The solution $x \equiv 0$ of (4.1) is called *attracting with respect to perturbations* if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if

$$\|x_0\| < \widetilde{\delta} \qquad \text{and} \qquad \text{var}_\gamma^v \overline{P} < \rho$$

with $\overline{P} \in BV^-([\gamma, v], X)$, then

$$\|x(t, \gamma, x_0)\| < \varepsilon \ \text{ for all } \ t \geq \gamma + T, \ t \in [\gamma, v]$$

where $x(t, \gamma, x_0)$ is a solution of the perturbed generalized equation (4.2) with $x(\gamma, \gamma, x_0) = x_0$ and $[\gamma, v] \subset [t_0 - r, +\infty)$.

Definition 4.6. The solution $x \equiv 0$ of (4.1) is called *asymptotically stable with respect to perturbations* if it is both stable and attracting with respect to perturbations.

The next result, which can be found in [9] or [10], says that variational stability and stability by perturbations are in fact equivalent and the same applies to variational attractivity and attractivity by perturbations.

Proposition 4.1. *The following statements hold.*

(i) The solution $x \equiv 0$ of (4.1) is variationally stable if and only if it is stable with respect to perturbations.

(ii) The solution $x \equiv 0$ of (4.1) is variationally attracting if and only if it is attracting with respect to perturbations.

(iii) The solution $x \equiv 0$ of (4.1) is variationally asymptotically stable if and only if it is asymptotically stable with respect to perturbations.

5. RELATING THE CONCEPTS OF STABILITY

In [3], Theorem 4.1, the authors proved that the respective concepts of variational stability and variational attractivity of the trivial solution of the retarded equation (2.1) and the trivial solution of its corresponding generalized equation (5.1) are equivalent. We present this result in the next lines.

Consider the retarded system (2.1). Let $G_1 \subset G([t_0 - r, +\infty), \mathbb{R}^n)$ be defined as in the beginning of the paper.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

We assume that $f(\phi, t) : G_1 \times [t_0, +\infty) \to \mathbb{R}^n$ is such that for each $y \in G_1$ the mapping $t \mapsto f(y_t, t)$ is locally Lebesgue integrable and conditions (A) and (B) are fulfilled. We also assume that $f(0,t) = 0$ for every $t \in [t_0, +\infty)$ so that $y \equiv 0$ is a solution of (2.1) in $[t_0 - r, +\infty)$.

For $y \in G_1$ and $t \in [t_0 - r, +\infty)$, define $F(y,t)$ as in (3.6). Then

$$F : G_1 \times [t_0 - r, +\infty) \to C([t_0 - r, +\infty), \mathbb{R}^n)$$

and by definition we have $F(0,t) = 0$, for all $t \in [t_0 - r, +\infty)$. Then $x \equiv 0$ is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) \tag{5.1}$$

in $[t_0 - r, +\infty)$.

By the results from Proposition 3.1, there is a well described one-to-one correspondence between solutions of equations (2.1) and (5.1) with F given by (3.6).

Let us consider the perturbed retarded equation (2.5) and, again by Proposition 3.1, its corresponding perturbed generalized equation

$$\frac{dx}{d\tau} = DG(x,t) = D[F(x,t) + \overline{P}(t)], \tag{5.2}$$

where F is given by (3.6) and $\overline{P}$ given by (3.7).

We have

$$\overline{P} : [t_0 - r, +\infty) \to G^-([t_0 - r, +\infty), \mathbb{R}^n).$$

and then

$$G : G_1 \times [t_0 - r, +\infty) \to G^-([t_0 - r, +\infty), \mathbb{R}^n).$$

For a proof of the next theorem, see [3], Theorem 4.1.

Theorem 5.1. *The following statements hold.*

(i) *The solution $y \equiv 0$ of (2.1) is variationally stable if and only if the solution $x \equiv 0$ of (5.1) is variationally stable.*

(ii) *The solution $y \equiv 0$ of (2.1) is variationally attracting if and only if the solution $x \equiv 0$ of (5.1) is variationally attracting.*

(iii) *The solution $y \equiv 0$ of (2.1) is variationally asymptotically stable if and only if the solution $x \equiv 0$ of (5.1) is variationally asymptotically stable.*

Now we specialize our assumptions. We consider smooth conditions for the function f in (2.1). We want to prove that under the additional assumptions that the function $f(\psi, t)$ in the righthand side of (2.1) is continuous and admits continuous Fréchet derivative with respect to ψ, the concepts of variational stability and Lyapunov stability of the trivial solution are equivalent and the same holds for asymptotic variational stability and uniform asymptotic stability of the trivial solution of the retarded equation (2.1).

Instead of $G^-([t_0 - r, +\infty), \mathbb{R}^n)$, we consider the space of continuous functions $C([t_0 - r, +\infty), \mathbb{R}^n)$ and $G_1 \subset C([t_0 - r, +\infty), \mathbb{R}^n)$ as defined in the beginning of the paper.

Let us rename equation (2.1) as

$$\begin{cases} \dot{y}\,(t) = f(y_t, t), \\ y_{t_0} = \phi, \end{cases} \tag{5.3}$$

where now we consider $\phi \in C([-r, 0], \mathbb{R}^n)$, $r \geq 0$, and $f(t, \psi)$ is a continuous function from an open subset Ω of $C([-r, 0], \mathbb{R}^n) \times [t_0, +\infty)$ to $\mathbb{R}^n$ with continuous Fréchet derivative, f', with respect to ψ. By $y(t, t_0, \phi)$ we mean any solution of (5.3) through (t_0, ϕ).

Let $J = J(t_0, \phi)$ be the maximal interval of existence of $y(t, t_0, \phi)$. Corresponding to each solution $y(t, t_0, \phi)$ of (5.3), we define a linear functional differential equation by

$$\dot{z} = f'(y_t(t_0, \phi), t)z_t, \qquad t \in J. \tag{5.4}$$

and a linear operator $T(t, t_0, \phi)$, $t \geq t_0$, associated with (5.4).

We also consider the perturbed equation

$$\begin{cases} \dot{y}\,(t) = f(y_t, t) + p(t), \\ y_{t_0} = \phi, \end{cases} \tag{5.5}$$

with $p : [t_0 - r, +\infty) \to \mathbb{R}^n$ continuous. By $\overline{y}(t, t_0, \phi)$ we mean any solution of (5.5) through (t_0, ϕ).

Suppose $I = [t_0, t_0 + \sigma]$, $\sigma > 0$, is a subinterval of J. Then for $s \in I$, $t \in [s, t_0 + \sigma]$ and $\psi \in C([-r, 0], \mathbb{R}^n)$, we have

$$T(t, s, \psi)\xi = v_t(s, \xi),$$

where

$$\dot{v}(t) = f'(y_t(s, \psi), t)v_t.$$

In particular,

$$T(t, s, \overline{y}_t(t_0, \phi))\xi = u_t(s, \xi),$$

where

$$\dot{u}(t) = f'(y_t(s, \overline{y}_s(t_0, \phi)), t)u_t.$$

Denote the $n \times n$ matrix function Y_0 by

$$Y_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0, \\ I, & \theta = 0. \end{cases}$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Then by [11], Theorem 3, we have the following analogue of Alekseev's formula for RFDEs:

$$\overline{y}_t(t_0,\phi) = y_t(t_0,\phi) + \int_{t_0}^{t} T(t,s,y_s(t_0,\phi))\, Y_0\, p(s)ds, \tag{5.6}$$

where t is in the interval of existence of $\overline{y}_t(t_0,\phi)$.

At $\theta = 0$, (5.6) becomes

$$\overline{y}(t) = y(t) + \int_{t_0}^{t} u_t(s,Y_0)(0)\, p(s)ds, \tag{5.7}$$

Let us assume further that u is bounded with $\|u(t)\| \leq M$, for all t. Then (5.7) becomes

$$\overline{y}(t) = y(t) + M\int_{t_0}^{t} p(s)ds. \tag{5.8}$$

Theorem 5.2. *Consider the retarded system* (5.3). *The following assertions hold:*

(i) *The solution* $y \equiv 0$ *of* (5.3) *is variationally stable if and only if it is Lyapunov stable.*

(ii) *The solution* $y \equiv 0$ *of* (5.3) *is variationally asymptotically stable if and only if it is uniformly asymptotically stable.*

Proof. We start by proving item (i). Suppose the trivial solution $y \equiv 0$ of (5.3) is variationally stable. Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ be from Definition 2.6. Suppose $\phi \in C([-r,0],\mathbb{R}^n)$ and $y(t,t_0,\phi)$ is a solution of (5.3) such that $\|\phi\| < \delta$. We will show that $\|y_t\| < \varepsilon$.

Suppose $\mathrm{var}_{t_0}^{t_1} P < \delta$, where $P(t) = \int_{t_0}^{t} p(s)ds$, $t \geq t_0$. Since the solution $y \equiv 0$ of (5.3) is variationally stable, we can take $\|\overline{y}(t,t_0,\phi)\| < \varepsilon/2$, where $\overline{y}(t,t_0,\phi)$ is a solution of

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s,s)\, ds + P(t) - P(t_0), & t \in [t_0,t_1] \\ y_{t_0} = \phi. \end{cases} \tag{5.9}$$

By (5.8),

$$\overline{y}(t) = y(t) + M\left[P(t) - P(t_0)\right]. \tag{5.10}$$

Hence,

$$|y(t)| \leq |\overline{y}(t)| + M|P(t) - P(0)| < \frac{\varepsilon}{2} + M\mathrm{var}_{t_0}^{t_1} P < \frac{\varepsilon}{2} + M\delta$$

and if we take $\delta \leq \varepsilon/2M$, we get the conclusion.

Conversely, let us suppose the solution $y \equiv 0$ of (5.3) is uniformly stable. Then given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that if $\phi \in C([-r, 0], \mathbb{R}^n)$ and $y(t) = y(t, t_0, \phi)$ is a solution of (5.3) with $\|\phi\| < \delta$, then $\|y_t(t_0, \phi)\| < \varepsilon/2$.

Let $\overline{y}(t, t_0, \phi)$ be a solution of the perturbed equation (5.9) and suppose $\mathrm{var}_{t_0}^{t_1} P < \delta$. Then by (5.10),

$$\begin{aligned} |\overline{y}(t)| &\leq |y(t)| + M|P(t) - P(0)| \\ &\leq |y_t(0)| + M \mathrm{var}_{t_0}^{t_1} P \\ &< \|y_t(t_0, \phi)\| + M\delta < \frac{\varepsilon}{2} + M\delta \end{aligned}$$

and we get the conclusion by taking $\delta \leq \varepsilon/2M$.

The proof of the second assertion (item (ii)) follows in an analogous way. ▮

The next corollary is a direct consequence of Theorems 5.1 and 5.2.

Corollary 5.1. *Under the conditions of this section, the following statements hold:*

(i) The solution $y \equiv 0$ of (5.3) *is Lyapunov stable if and only if the solution $x \equiv 0$ of* (5.1) *is variationally stable.*

(ii) The solution $y \equiv 0$ of (5.3) *is uniformly asymptotically stable if and only if the solution $x \equiv 0$ of* (5.1) *is variationally asymptotically stable.*

6. CONVERSE LYAPUNOV THEOREMS

In the book [9] and in [10], converse Lyapunov-type theorems for the existence of a Lyapunov functional decreasing along the solutions of a generalized ODE are given. In [3], such results are used to obtain converse theorems for equation (2.1). We will use them again here to get our results.

Let us consider the general case where $\Omega = B_c \times [t_0 - r, \infty)$, with

$$B_c = \{y \in X; \|y\| < c\},$$

where $c > 0$ and X is a Banach space. Suppose $F : \Omega \to X$ is such that $F \in \mathcal{F}(\Omega, h)$ and $F(0, t) - F(0, s) = 0$, for $t, s \in [t_0 - r, +\infty)$ and consider the generalized differential equation

$$\frac{dx}{d\tau} = DF(x, t). \tag{6.1}$$

The following two results are respectively Theorems 10.23 and 10.24 from [9]. They can also be found in [10].

Theorem 6.1. *If the trivial solution $x \equiv 0$ of the generalized differential equation* (6.1) *is variationally stable, then for every $0 < a < c$, there exists a function $V : [t_0 - r, +\infty) \times B_a \to$*

$\mathbb{R}$, where $B_a = \{y \in X; \|y\| < a\}$, such that for every $x \in B_a$, the function $V(\cdot, x)$ belongs to $BV^-([t_0 - r, +\infty), \mathbb{R})$ and the following conditions hold:

(i) $V(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;

(ii) $|V(t, z) - V(t, y)| \leq \|z - y\|$, $t \in [t_0 - r, +\infty)$, $z, y \in B_a$.

(iii) V is positive definite along every solution $x(t)$ of the generalized equation (6.1), that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that

$$V(t, x(t)) \geq b(\|x(t)\|), \quad (t, x(t)) \in [t_0 - r, +\infty) \times B_a;$$

(iv) for all solutions $x(t)$ of (6.1),

$$\dot{V}(t, x(t)) = \limsup_{\eta \to 0_+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq 0,$$

that is, the right derivative of V along every solution $x(t)$ of (6.1) is non-positive.

Theorem 6.2. *If the trivial solution $x \equiv 0$ of the generalized differential equation (6.1) is variationally asymptotically stable, then for every $0 < a < c$, there exists a function $V : [t_0 - r, +\infty) \times B_a \to \mathbb{R}$ such that for every $x \in B_a$, the function $V(\cdot, x)$ belongs to $BV^-([t_0 - r, +\infty), \mathbb{R})$ and the following conditions hold:*

(i) $V(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;

(ii) $|V(t, z) - V(t, y)| \leq \|z - y\|$, $t \in [t_0 - r, +\infty)$, $z, y \in B_a$.

(iii) V is positive definite along every solution $x(t)$ of the generalized equation (6.1), that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that

$$V(t, x(t)) \geq b(\|x(t)\|), \quad (t, x(t)) \in [t_0 - r, +\infty) \times B_a;$$

(iv) for all solutions $x(s)$ of (6.1) defined for $s \geq t$, where $x(t) = z \in B_a$, the relation

$$\dot{V}(t, x(t)) = \limsup_{\eta \to 0_+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq V(t, z)$$

holds.

Consider equation (5.1), with F given by (3.6), and the corresponding retarded system (5.3). We consider $X = C([t_0 - r, +\infty), \mathbb{R}^n)$. As in [2] and in [3], we need to relate a Lyapunov functional for (5.1) to a Lyapunov functional for (5.3) and this can be done in the following manner: if $y : [\gamma, v] \to \mathbb{R}^n$ is a solution of equation (5.3) on $[\gamma, v] \subset [t_0 - r, +\infty)$,

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

$[\gamma, v] \ni t_0$, such that $y_t = \psi$ for a given $t \geq t_0$, then we write $y_{t+\eta} = y_{t+\eta}(t, \psi)$ for every $\eta \geq 0$, and given $U : [t_0 - r, +\infty) \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}$, we define

$$D^+U(t, \psi) = \limsup_{\eta \to 0_+} \frac{U(t+\eta, y_{t+\eta}(t, \psi)) - U(t, y_t(t, \psi))}{\eta},$$

for $t \geq t_0$.

On the other hand, if x be a solution of the generalized equation (5.1) on the interval $[\gamma, v] \subset [t_0 - r, +\infty)$, $[\gamma, v] \ni t_0$, with initial condition $x(t_0) = x_0$, where

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0) & \text{for } \vartheta \in [t_0 - r, t_0], \\ x(t_0)(t_0) & \text{for } \vartheta \in [t_0, +\infty), \end{cases} \tag{6.2}$$

then $x(t)(t+\theta) = y(t+\theta)$, for all $t \in [t_0 - r, +\infty)$ and all $\theta \in [-r, 0]$ and hence $(x(t))_t = y_t$ for all t.

Given $x(t) \in C([t_0 - r, +\infty), \mathbb{R}^n) \cap BV([t_0 - r, +\infty), \mathbb{R}^n)$, we can consider $x(t)$ as a solution on $[\gamma, v] \subset [t_0 - r, +\infty)$, $[\gamma, v] \ni t_0$, of the generalized equation (5.1), with initial condition $x(t_0) = x_0$ given by (6.2). Then Proposition 3.1 implies we can find a solution $y(t; t_0, \phi)$ of (2.1) by means of the solution $x(t; t_0, x_0)$ of (5.1). Suppose $(x(t))_t = \psi$. In this case, we write $x_\psi(t)$ instead of $x(t)$ and we have $y_t = \psi$.

Thus $(t, x_\psi(t)) \mapsto (t, y_t(t, \psi))$ is a one-to-one relation and we can define $V : [t_0 - r, +\infty) \times C([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ by

$$V(t, x_\psi(t)) = U(t, y_t(t, \psi)), \quad t \geq t_0. \tag{6.3}$$

Then we have

$$D^+U(t, \psi) = \limsup_{\eta \to 0_+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta}$$

for all $t \geq t_0$, and we write $\dot{U}(t, y_t) = D^+U(t, y_t)$.

With the notation above, we can prove the next two converse Lyapunov results for equation (5.3) by applying the ideas of [3] contained, respectively, in Theorems 5.3 and 5.4 of that paper.

THEOREM 6.3. *If the trivial solution $y \equiv 0$ of the retarded differential equation* (5.3) *is Lyapunov stable, then for every $0 < a < c$, there exists a function $U : [t_0 - r, +\infty) \times E_a \to \mathbb{R}$, where $E_a = \{\psi \in C([-r, 0], \mathbb{R}^n); \|\psi\| < a\}$, such that for every $x \in E_a$, the function $U(\cdot, \psi)$ belongs to $BV([t_0 - r, +\infty), \mathbb{R}) \cap C([t_0 - r, +\infty), \mathbb{R})$ and the following conditions hold:*

(i) $U(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;

(ii) $|U(t, \psi) - U(t, \overline{\psi})| \leq \|\psi - \overline{\psi}\|$, $t \in [t_0 - r, +\infty)$, $\psi, \overline{\psi} \in E_a$.

(iii) U is positive definite along every solution $y(t)$ of the retarded equation (5.3)*, that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that*

$$U(t, y_t) \geq b(\|y_t\|), \quad (t, y_t) \in [t_0 - r, +\infty) \times E_a;$$

(iv) for all solutions $y(t)$ of (2.1),

$$\dot{U}(t, y_t) = \limsup_{\eta \to 0_+} \frac{U(t+\eta, y_{t+\eta}) - V(t, y_t))}{\eta} \leq 0,$$

that is, the right derivative of U along every solution $y(t)$ of (5.3) *is non-positive.*

Proof. If the trivial solution of (5.3) is Lyapunov stable, then by Corollary 5.1 the trivial solution of the generalized equation (5.1) with F given by (3.6) and $\overline{P}$ is given by (3.7) is variationally stable. Then by Theorem 6.1, there exists a function $V : [t_0 - r, +\infty) \times C([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ satisfying all conditions in that theorem. Thus if we define $U : [t_0 - r, +\infty) \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ by the relation in (6.3), it can be shown, as in the proof of [2], Theorem 4.3, that U has the properties (i) to (iv) above and the proof is complete. ▮

The proof of the next theorem follows as in the proof of Theorem 6.3, but applying [2], Theorem 4.5 instead of [2], Theorem 4.3.

Theorem 6.4. *If the trivial solution $y \equiv 0$ of the retarded differential equation* (5.3) *is uniformly asymptotically stable, then for every $0 < a < c$, there exists a function $U : [t_0 - r, +\infty) \times E_a \to \mathbb{R}$ such that for every $x \in E_a$, the function $U(\cdot, x)$ belongs to $BV([t_0 - r, +\infty), \mathbb{R}) \cap C([t_0 - r, +\infty), \mathbb{R})$ and the following conditions hold:*

(i) $U(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;
(ii) $|U(t, \psi) - U(t, \overline{\psi})| \leq \|\psi - \overline{\psi}\|$, $t \in [t_0 - r, +\infty)$, $\psi, \overline{\psi} \in E_a$.
(iii) U is positive definite along every solution $y(t)$ of the retarded equation (5.3)*, that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that*

$$U(t, y_t) \geq b(\|y_t\|), \quad (t, y_t) \in [t_0 - r, +\infty) \times E_a;$$

(iv) for all solutions $y(s)$ of (5.3) *defined for $s \geq t$, where $y(t) = \psi \in E_a$, the relation*

$$\dot{U}(t, y_t) = \limsup_{\eta \to 0_+} \frac{U(t+\eta, y_{t+\eta}) - U(t, y_t)}{\eta} \leq U(t, \psi)$$

holds.

REFERENCES

1. M. Federson and Š. schwabik, Generalized ODEs approach to impulsive retarded differential equations, *Differential and Integral Equations*, **19** (2006), 1201–1234.
2. M. Federson and Š. schwabik, A new approach to impulsive retarded differential equations: stability results, preprint.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

3. M. Federson and Š. schwabik, Stability for retarded functional differential equations, *Nonlinear Oscillations*, to appear.

4. A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.

5. J.K. Hale and S.V. Verduyn lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, **99**, Springer-Verlag, New York, 1993.

6. C. Imaz and Z. Vorel, Generalized ordinary differential equations in Banach spaces and applications to functional equations, *Bol. Soc. Mat. Mexicana*, **11** (1966), 47–59.

7. F. Oliva, f. and Z. Vorel, Functional equations and generalized ordinary differential equations, *Bol. Soc. Mat. Mexicana*, **11** (1966), 40–46.

8. A.M. Samoilenko and N.A. Perestyuk, *Impulsive differential equations*, World Scientific, Singapore, 1995.

9. Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific, Series in Real Anal., **5**, 1992.

10. Š. Schwabik, Variational stability for generalized ordinary differential equations, *Časopis Pěst. Mat.*, **109** (1984), 389–420.

11. G.A. Shanholt, A nonlinear variation-of-constants formula for functional differential equations, *Math. Systems Theory*, **6** (1972/73), 343–352.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC