APPLICATIONS OF THE NON-STANDARD VERSION OF THE BORSUK-ULAM THEOREM

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Abstract. Let us consider a topological space X and a continuous map $\varphi : X \to X$. The non-standard version of the Borsuk-Ulam theorem proved in [1], states that, under certain conditions on X and φ, for a given map $f : X \to \mathbb{R}^2$, although the existence of a point $x \in X$ such that x and $\varphi(x)$ are collapsed by f cannot always be assured, a very interesting phenomenon concerning the three points $f(x)$, $f(\varphi(x))$, and $f(\varphi^2(x))$ occurs when $f(x) \neq f(\varphi(x))$, for any $x \in X$. In this paper, we establish efficient methods which allows to apply the non-standard version of the Borsuk-Ulam theorem for graphs and for the suspension SX of a space X.

Key words: Borsuk-Ulam theorem, free maps, coincidence points, graphs.

1. Introduction

Given topological spaces X and Y, a continuous map $f : X \to Y$ is called a *free map* if there exists a continuous map $\varphi : X \to X$ which has the property that $f(x) \neq f(\varphi(x))$, for any $x \in X$. This definition was introduced by Hopf in 1937. In [3], among other results, he proved that for any $n \geq 3$ there exists a *free map* $f : S^n \to \mathbb{R}^n$. In the same paper, Hopf asked whether the following questions holds:

1. There exists a *free map* $f : S^2 \to \mathbb{R}^2$.
2. In the affirmative case, for which natural numbers $n > 2$ there exists a *free map* from n-sphere S^n into Euclidean 2-space $\mathbb{R}^2$.

The Hopf's questions had been answered in 1952 by Erika Pannwitz [1]. In [5], she showed the existence of a *free map* $f : S^n \to \mathbb{R}^2$, for any $n \geq 0$.

Let us consider a topological space X and a continuous map $\varphi : X \to X$. In [1] we proved that, under certain conditions on X and φ, for a given map

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[1]Erika Pannwitz (1904-1975) was a PhD student of Heinz Hopf in Berlin in the 1930s. She was the first person to show that knotted polygonal curves have quadrisecants.

$f : X \to \mathbb{R}^2$, although by Pannwitz's result, the existence of a point $x \in X$ such that x and $\varphi(x)$ are collapsed by f cannot always be assured, a very interesting phenomenon concerning the three points $u = f(\varphi(x))$, $v = f(x)$ and $w = f(\varphi^2(x))$ occurs when $f(x) \neq f(\varphi(x))$, for any $x \in X$: for at least a point x in a special subset of X, u belongs to the line segment determined by v and w and we obtained a non-standard version of the Borsuk-Ulam theorem.

Specifically, was proved the following

Theorem A. *Let X be Hausdorff space and A a nonempty compact, connected, and locally pathwise connected subset of X. Let $\varphi : X \to X$ be a continuous map such that $\varphi(A) \subset A$. Suppose that*

$$Id_* - \varphi^1_* : i_*(H_1(A, \mathbb{Q})) \to i_*(H_1(A, \mathbb{Q})) \tag{1.1}$$

is a surjective map. Then for any continuous map $f : X \to \mathbb{R}^2$, either there exists a point $x \in X$ such that $f(x) = f(\varphi(x))$, or there exists a point $x \in A$ such that $f(\varphi(x))$ belongs to the closed line segment in $\mathbb{R}^2$ determined by $f(x)$ and $f(\varphi^2(x))$, that is,

$$f(\varphi(x)) \in [f(x), f(\varphi^2(x))]. \tag{1.2}$$

Remark 1.1. Let us observe that when $A = X$ in Theorem A, one has $i_*(H_1(A, \mathbb{Q})) = H_1(X, \mathbb{Q})$.

The following consequence of Theorem A was proved in [1, Corollary 2.10].

Corollary A. *Let X be a Hausdorff space and A a nonempty compact, connected, and locally pathwise connected subset of X. Let $\varphi : X \to X$ be a continuous map such that $\varphi(A) \subset A$ and $(\varphi|_A)^3 = Id_A$. Suppose that*

$$Id_* - \varphi^1_* : i_*(H_1(A, \mathbb{Q})) \to i_*(H_1(A, \mathbb{Q}))$$

is a surjective map. Then for any continuous map $g : X \to \mathbb{R}$, there exists a point $x \in X$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

In this paper we establish efficient methods which allows to verify the condition (1.1) given in the Theorem A for two important classes of topological spaces; the class of the graphs and the class of the spaces suspension SX of a space X. Using such methods, we construct interesting examples where the non-standard version of the Borsuk-Ulam theorem can be applied.

The paper is organized as follows. In Section 2, we extended the Pannwitz's construction, replacing the sphere S^n by more general topological space. As a consequence, using Corollary A, we obtain a result about coincidence points. In Section 3, we describe an useful method to verify the hypotheses of Theorem A. Further, we construct examples of topological spaces of dimension one for which Theorem A can be applied. In particular, in Section 4, when X is a connected graph, we obtain an interesting physical interpretation, by using the condition of alignment of three points, given by (1.2).

Throughout the paper, we assume that homology groups will always have rational coefficients. Thus the pth homology group $H_p(X)$ of a topological space X is considered a vector space over the field of rational numbers; the symbol $\tilde{H}_*(X)$ denotes the reduced homology groups.

2. A generalization of the Pannwitz's construcion

The first result of this section was motivated upon ideas of Pannwitz [5]. For each $n \geq 0$, Pannwitz constructed maps $\varphi_n : S^n \to S^n$ and $f_n : S^n \to \mathbb{R}^2$ such that $f_n(x) \neq f_n(\varphi_n(x))$, for any $x \in S^n$. Such construction was made by induction on n. Just observing that the suspension of S^n is S^{n+1}, a similar construction can be made if we replace the sphere S^n by more general topological space X and S^{n+1} by its suspension SX.

For a given topological space X, we will denote by SX the suspension of X, that is, the quotient of $X \times I$ obtained by collapsing $X \times \{1\}$ to one point and $X \times \{-1\}$ to another point. We prove the following

Theorem 2.1. *Suppose that there exists a free map $g : X \to \mathbb{R}^2$, that is, there is a map $\rho : X \to X$ such that $g(p) \neq g(\rho(p))$, for any $p \in X$. For any $p \in X$, let us denotes $g(p) = (u(p), v(p))$ and $g(\rho(p)) = (u(\rho(p)), v(\rho(p)))$. If $v(p) \leq v(\rho(p))$ for any $p \in X$ then there exists a free map $f : SX \to \mathbb{R}^2$.*

Proof. Denote by O_1 and O_2 the points $(1, p)$ and $(-1, p)$ in SX, respectively. Thus, we can define maps $\varphi : SX \to SX$ and $f : SX \to \mathbb{R}^2$ by

$$\begin{aligned} \varphi(t,p) &= (-t|t|, \rho(p)) \\ f(t,p) &= tf(O_1) + (1-|t|)g(p), \ \forall (t,p) \in SX, \end{aligned} \tag{2.1}$$

where $f(O_1) = (1, 0)$, $f(O_2) = (-1, 0)$ and $-1 \leq t \leq 1$. We write

$$f(t,p) = (x(t,p), y(t,p)) \text{ and } f(\varphi(t,p)) = (x(\varphi(t,p)), y(\varphi(t,p))).$$

Without loss of generality, we can assume that the image $g(X)$ is entirely contained in the upper half-plane $v > 0$. Now we prove that f is a *free map.* For any $(t, p) \in SX$, with $0 < |t| < 1$, one has that

$$\begin{aligned} y(t,p) &= (1-|t|)v(p); \\ y(\varphi(t,p)) &= y(-t|t|, \rho(p)) = (1-t^2)v(\rho(p)). \end{aligned}$$

From our preceding assumption, $g(\rho(X))$ is contained in the half-plane $v > 0$ and by hypotheses we have that $0 < v(p) \leq v(\rho(p))$. Furthermore, it follows from $(1 - |t|) < (1 - t^2)$ that for any $(t, p) \in SX$, with $0 < |t| < 1$

$$y(t,p) = (1 - |t|)v(p) < (1 - t^2)v(\rho(p)) = y(\varphi(t,p)),$$

which implies $f(t,p) \neq f(\varphi(t,p))$, for any $(t,p) \in SX$, with $0 < |t| < 1$. Moreover, $f(O_1) \neq f(O_2)$ and since f coincides with the *free map* g in points of kind $(0,p)$, we conclude that $f(t,p) \neq f(\varphi(t,p))$, for any $(t,p) \in SX$ and therefore f is a *free map*. ☐

The following corollary is an important application of Corollary A and it is based in the definition of the map φ, given in the proof of Theorem 2.1 (step (2.1)).

A $\mathbb{Z}_p$-space is a pair (X, ρ), where X is a topological space equipped with a free action of the cyclic group $\mathbb{Z}_p$, generated by a periodic homeomorphism $\rho : X \to X$ of period p.

Corollary 2.2. *Let (X, ρ) be a $\mathbb{Z}_3$-space. Suppose that X is a compact, connected, locally pathwise connected and Hausdorff space. Then there exists $\varphi : SX \to SX$ satisfying:*
(i) $\varphi|_X = \rho$ and $\varphi(x) \neq x$, for any $x \in SX$;
(ii) for any continuous map $g : SX \to \mathbb{R}$ there exists a point $x \in SX$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

Proof. It follows from step (2.1) of the proof of Theorem 2.1 that there exists $\varphi : SX \to SX$ such that $\varphi|_X = \rho$ and $\varphi(x) \neq x$, for any $x \in SX$. On the other hand, since (X, ρ) is a $\mathbb{Z}_3$-space, one has $\varphi(X) \subset X$ and $(\varphi|_X)^3 = Id_X$. Moreover, the inclusion map $i : X \to SX$ induces trivial homomorphism in homology, which implies that $i_*(H_1(X; \mathbb{Q})) = 0$, consequently $Id_* - \varphi_*$ is a surjective map. It follows from Corollary A, that for any continuous map $g : SX \to \mathbb{R}$, there exists $x \in SX$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$. ☐

Example 2.3. Let $\rho : S^1 \to S^1$ be a rotation through an angle $2\pi/3$. We have that (S^1, ρ) is a $\mathbb{Z}_3$-space. It follows from Corollary 2.2 that there exists $\varphi : S^2 \to S^2$ such that for any continuous map $g : S^2 \to \mathbb{R}$, there exists $x \in S^2$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

Remark 2.4. It is important to emphasize that in Corollary 2.2, is not necessary that the cyclic group $\mathbb{Z}_3$ acts freely on SX, we only need to assume that it acts freely on X. Let us note that, under these conditions, the classical Borsuk-Ulam theorems (see, for example [4]) cannot be applied to prove that any continuous map $g : SX \to \mathbb{R}$ sends three points of some orbit of φ to a single point, that is, $g(x) = g(\varphi(x)) = g(\varphi^2(x))$, for some $x \in SX$. Therefore, Corollary 2.2 is a special result about coincidence points in which the group only acts freely on some subspace of a certain topological space. This situation is illustrated by Example 2.3, since the cyclic group $\mathbb{Z}_3$ acts freely on the sphere S^1, but does not acts freely on sphere S^2.

3. Applications of Theorem A to graphs

In order to apply Theorem A, we need to know the homomorphism $Id_* - \varphi_*^1$. The Propositions 3.1 and 3.3 establish efficient methods which allows to verify when this map is surjective.

Proposition 3.1. *Let $X = \cup_{i=1}^k X_i$ be a topological space, where $X_1, X_2, \ldots, X_k$ are the path components of X. Let $\rho : X \to X$ a continuous map and $\varphi : S(X) \to S(X)$ a continuous map such that $\varphi_{|X} = \rho$. Let us denote by $[\varphi_*^1]$ and $[\rho_*^0]$ the matrices of the linear transformations $\varphi_*^1 : H_1(SX;\mathbb{Q}) \to H_1(SX;\mathbb{Q})$ and $\rho_*^0 : H_0(X;\mathbb{Q}) \to H_0(X;\mathbb{Q})$, respectively and let I be the identity matrix. Then*

$$\det(I - [\varphi_*^1]) = (1/2) \cdot \det(I + [\rho_*^0]).$$

Proof. Let us observe that a basic property of the induced homomorphism $\varphi_*^p : H_p(S(X);\mathbb{Q}) \to H_p(S(X);\mathbb{Q})$ is the following

$$\varphi_*^p = \begin{cases} (-1)^p \rho_*^{p-1}, & \text{if} \quad p > 1 \\ (-1)^p \tilde{\rho}_*^{p-1}, & \text{if} \quad p = 1, \end{cases}$$

where $\tilde{\rho}_*$ denotes the induced homomorphism by ρ between the reduced homology groups. In particular, if $p = 1$, one has

$$\varphi_*^1 = -\tilde{\rho}_*^0. \tag{3.1}$$

Now, let us describe how the matrices of the induced homomorphisms

$$\rho_*^0 : H_0(X;\mathbb{Q}) \to H_0(X;\mathbb{Q})$$
$$\tilde{\rho}_*^0 : \tilde{H}_0(X;\mathbb{Q}) \to \tilde{H}_0(X;\mathbb{Q})$$

are related: if $x_1, x_2, \ldots, x_k$ denote points in each path component $X_1, X_2, , \ldots, X_k$ of X, we may choose a basis $\{u_1, u_2, \ldots, u_k\}$ for the finite-dimensional rational vector space $H_0(X,\mathbb{Q})$, with $u_i = [x_i]$ a generator of $H_0(X_i,\mathbb{Q})$, for $i = 1, 2, \ldots, k$. From $\{u_1, u_2, \ldots, u_k\}$, we can produce a basis $\beta = \{v_1, v_2, \cdots, v_k\}$ for $H_0(X,\mathbb{Q})$ by defining

$$v_i = \begin{cases} u_i, & \text{if} \quad i = 1 \\ u_i - u_1, & \text{if} \quad i = 2, \ldots, k. \end{cases}$$

Thus, the matrix of ρ_*^0 with respect to the basis β is the $k \times k$ block matrix

$$[\rho_*^0]_\beta = \begin{bmatrix} I_1 & O_{1\times k-1} \\ C_{k-1\times 1} & [\tilde{\rho}_*^0]_{\tilde{\beta}} \end{bmatrix}$$

where I_1 denotes the 1×1 identity matrix, $O_{1\times k-1}$ denotes the $1 \times k-1$ zero matrix and $[\tilde{\rho}_*^0]_{\tilde{\beta}}$ denotes the $k-1 \times k-1$ matrix of $\tilde{\rho}_*^0$ with respect to the basis $\tilde{\beta} = \{v_2, v_3, \cdots, v_k\}$ of $\tilde{H}_0(X;\mathbb{Q})$. Therefore,

$$I + [\rho_*^0]_\beta = \begin{bmatrix} 2I_1 & O_{1\times k-1} \\ C_{k-1\times 1} & I + [\tilde{\rho}_*^0]_{\tilde{\beta}} \end{bmatrix}. \tag{3.2}$$

On the other hand, from (3.1), one has $[\varphi_*^1] = -[\tilde{\rho}_*^0]_{\tilde{\beta}}$, which implies that

$$\det(I - [\varphi_*^1]) = \det(I + [\tilde{\rho}_*^0]_{\tilde{\beta}}). \tag{3.3}$$

From (3.2) and (3.3), we can compute the determinant

$$\det(I + [\rho_*^0]_\beta) = 2 \cdot \det(I + [\tilde{\rho}_*^0]_{\tilde{\beta}}) = 2 \cdot \det(I - [\varphi_*^1]),$$

and we obtain the desired formula

$$\begin{aligned} \det(I - [\varphi_*^1]) &= (1/2) \cdot \det(I + [\rho_*^0]_\beta) \\ &= (1/2) \cdot \det(I + [\rho_*^0]). \end{aligned}$$

□

In the next example, we construct a *free map* from an 1-dimensional topological space into Euclidean 2-space $\mathbb{R}^2$ and using Proposition 3.1, we show that Theorem A can be applied in this case.

Example 3.2. Let $X = \{x_1, x_2, x_3\}$ and define a map $\rho : X \to X$ by

$$\rho(x_1) = x_2,\ \rho(x_2) = x_3 \text{ and } \rho(x_3) = x_1.$$

Then $\rho(x) \neq x$, for any $x \in X$. It follows from (2.1), that there exist $\varphi : SX \to SX$ such that $\varphi(x) \neq x$, for any $x \in SX$. Let $f : SX \to \mathbb{R}^2$ be a topological embedding and observe that $f(x) \neq f(\varphi(x))$, for any $x \in SX$, that is, f is a *free map*. We affirm that, there exists a point $x \in SX$ such that

$$f(\varphi(x)) \in [f(x), f(\varphi^2(x))],$$

where $[f(x), f(\varphi^2(x))]$ denotes the closed line segment in $\mathbb{R}^2$ joining the points $f(x)$ and $f(\varphi^2(x))$. To verify this assertion, it suffices to show that

$$Id_* - \varphi_*^1 : H_1(SX;\mathbb{Q}) \to H_1(SX;\mathbb{Q})$$

is a surjective map and in this case, the result follows from Theorem A. Since $Id_* - \varphi_*^1$ is a linear transformation between vector spaces, let us check that

$\det(I-[\varphi_*^1]) \neq 0$. Let u_1, u_2, u_3 be generators of $H_0(X;\mathbb{Q})$, then $\rho_*^0(u_1) = u_2$, $\rho_*^0(u_2) = u_3$ and $\rho_*^0(u_3) = u_1$. The matrix of ρ_*^0 has the form

$$[\rho_*^0] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and thus, } I + [\rho_*^0] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore, from Proposition 3.1,

$$(1/2).\det(I + [\rho_*^0]) = 1 = \det(I - [\varphi_*^1]) \neq 0,$$

and the result follows.

Proposition 3.3. *Let X be a connected graph and let $\varphi : X \to X$ be a continuous map such that φ sends each vertex of X to a vertex of X. Then*

$$\det(Id - \varphi_\sharp^1) = \det(Id - \varphi_*^1) \cdot \det(Id - \varphi_\sharp^0)$$

Proof. Let us consider the homomorphisms

$$\begin{aligned} \varphi_\sharp^i &: C_i(X) \to C_i(X) \\ \varphi_*^i &: H_i(X) \to H_i(X), \text{for } i = 0, 1. \end{aligned}$$

Since X is a connected 1-dimensional CW complex, we have that $C_2(X) = 0$ and therefore the boundary operator $\partial_2 : C_2(X) \to C_1(X)$ is trivial, which implies that

$$H_1(X) = \frac{Z_1(X)}{\mathrm{Im}(\partial_2)} = Z_1(X)$$

One can write $C_1(X) = H_1(X) \oplus L_1(X)$, where $\partial_1|_{L_1(X)} : L_1(X) \mapsto B_0(X)$ is an isomorphism. Let us consider the following commutative diagram

$$\begin{array}{ccc} C_1(X) & \xrightarrow{\partial_1} & B_0(X) \\ \downarrow{\scriptstyle \varphi_\sharp^1} & & \downarrow{\scriptstyle \tilde{\varphi}_\sharp^0} \\ C_1(X) & \xrightarrow[\partial_1]{} & B_0(X) \end{array} \tag{3.4}$$

Choose $B = B_1 \cup B_2$ as the basis for $C_1(X)$, where $B_1 = \{u_1, \ldots u_n\}$ is a basis for $H_1(X)$ and $B_2 = \{v_1, \ldots, v_m\}$ is a basis for $L_1(X)$. Thus, the linear transformation $\varphi_\sharp^1$ is given by formulas:

$$\begin{aligned} \varphi_\sharp^1(u_j) &= \sum_{i=1}^{n} a_{ij} u_i + \sum_{i=1}^{m} b_{ij} v_i, \;\; j = 1, \ldots, n \\ \varphi_\sharp^1(v_j) &= \sum_{i=1}^{n} c_{ij} u_i + \sum_{i=1}^{m} d_{ij} v_i, \;\; j = 1, \ldots, m. \end{aligned}$$

If we write $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$, then the matrix of $\varphi^1_\sharp$ is the block matrix

$$[\varphi^1_\sharp] = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

One has that $A = (a_{ij})$ is the matrix of $\varphi^1_* : H_1(X) \to H_1(X)$ and since that $\partial_1|_{L_1(X)}$ is an isomorphism between $L_1(X)$ and $B_0(X)$, it follows from diagram 3.4 that C is the zero matriz and $D = (d_{ij}) = [\tilde{\varphi}^0_\sharp]$. Therefore

$$[\varphi^1_\sharp] = \begin{bmatrix} [\varphi^1_*] & O \\ B & [\tilde{\varphi}^0_\sharp] \end{bmatrix}$$

Thus, the determinant of the matrix $I - [\varphi^1_\sharp]$ is equal to

$$\det(I - [\varphi^1_\sharp]) = \det(I - [\varphi^1_*]) \cdot \det(I - [\tilde{\varphi}^0_\sharp]),$$

which completes the proof of the proposition. □

Remark 3.4. Let X be a connected graph and $\varphi : X \to X$ such that the determinant of $I - [\varphi^1_\sharp]$ is not zero. It follows from Proposition 3.3 that the determinant of $I - [\varphi^1_*]$ is not zero. Then, if $f : X \to \mathbb{R}^2$ is a continuous map, from Theorem A, either there exists a point $x \in X$ such that $f(x) = f(\varphi(x))$, or there exists $x \in X$ such that the three points $f(x), f(\varphi(x))$ and $f(\varphi^2(x))$ are lined up, with $f(\varphi(x))$ between $f(x)$ and $f(\varphi^2(x))$.

In the next example, we construct a connected graph X and $\varphi : X \to X$ such that the determinant of $I - [\varphi^1_\sharp]$ is not zero.

Example 3.5. Consider X an oriented connected graph as indicated in Figure 1 and let us denote by $A = \{a_1, a_2, a_3\}$ the set of vertices of X and by $L = \{l_1, l_2, l_3, l_4\}$ the set of edges of X.

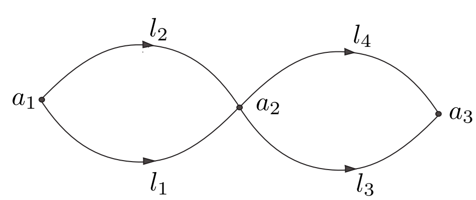

FIGURE 1.

Let $\varphi : X \to X$ be a continuous map defined on vertices of X as follows

$$\varphi(a_1) = a_2, \;\; \varphi(a_2) = a_3, \;\; \varphi(a_3) = a_1$$

and let $\varphi^1_\sharp : C_1(X) \to C_1(X)$ be the chain map induced by φ given by

$$\varphi^1_\sharp(l_1) = l_4,\ \ \varphi^1_\sharp(l_2) = l_3,\ \ \varphi^1_\sharp(l_3) = -l_1 - l_4,\ \ \varphi^1_\sharp(l_4) = -l_2 - l_3. \tag{3.5}$$

Then φ sends vertex to vertex and $\varphi(x) \neq x$, for any $x \in X$. Let us check that $\det(I - \varphi^1_\sharp) \neq 0$. From (3.5), the matrix of $\varphi^1_\sharp$ is given by

$$[\varphi^1_\sharp] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \text{ and thus } I - [\varphi^1_\sharp] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

which implies that $\det(I - [\varphi^1_\sharp]) = -3 \neq 0$ and it follows from Proposition 3.3 that $\det(I - [\varphi^1_*]) \neq 0$.

Remark 3.6. Consider $\varphi : S^1 \to S^1$ a rotation through an angle $2\pi/3$ and let us denote by $f : S^1 \to \mathbb{R}^2$ the natural inclusion. Then, there is not a point $x \in S^1$ with the property that the three points $f(x)$, $f(\varphi(x))$ and $f(\varphi^2(x))$ are lined up. In this case, it is possible to verify that the hypothesis of Theorem A is not satisfied, since $\det(I - [\varphi^1_\sharp]) = 0$.

4. A physical interpretation

It has been known the following physical interpretation of the classical Borsuk-Ulam theorem [2]: *there exists at least one pair of antipodal points on the surface of the earth having identical atmospheric pressures and temperatures.*

Next, we give an interesting physical interpretation of the non-standard version of the Borsuk-Ulam theorem.

Imagine three particles P_1, P_2 and P_3 moving in 2-space in such a way that its positions at time t relative to some coordinate system are given by

$$x(t), \varphi(x(t)) \text{ and } \varphi^2(x(t)),$$

respectively, where φ is a continuous map and when t varies through a time interval, the path traced out by the particles is a connected graph X.

We only assume that φ and X satisfy nice properties as in Remark 3.4. The results previously obtained guarantee that *there exists an instant t_0 such that the particles P_1, P_2 and P_3 are lined up. Moreover, P_2 is between P_1 and P_3.*

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