CADERNOS DE MATEMÁTICA **08**, 143–172 October (2007)
ARTIGO NÚMERO SMA#279

An extension of the concept of gradient semigroups which is stable under perturbation

Alexandre N. Carvalho*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: andcarva@icmc.usp.br

José A. Langa†

Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1080, 41080-Sevilla Spain
E-mail: langa@us.es

In this article we introduce the concept of *gradient-like semigroup* as an intermediate concept between a *gradient* semigroup (those possessing a *Liapunov function*, see [13]) and a semigroup possessing a *gradient-like attractor*. We prove that a perturbation of a gradient-like semigroup is remains a gradient-like semigroup, introduce the concept of gradient-like evolution process and prove that a non-autonomous perturbation of a gradient-like semigroup is a gradient-like evolution process. For gradient-like semigroups and evolution processes, we prove continuity, characterization and exponential attraction of their attractors under perturbation extending the results of [6] on characterization and of [2] on exponential attraction. October, 2007 ICMC-USP

1. INTRODUCTION

In the theory of infinite dimensional dynamical systems, one problem that has received a lot of attention in the last few decades is the description of the geometric structure of their associated invariant sets. Any characterization of the structures inside an invariant set of a given dynamical system helps to describe the asymptotic behavior (possible chaotic) of the modeled phenomena under study. A large literature related to this problem has been developed in the finite dimensional case whereas very little is known about the structure of attractors in infinite dimensional spaces. Only in very particular examples (see [13, 2,

* Research Partially Supported by the grants CNPq 305447/2005-0 and by FAPESP 03/10042-0, Brazil.
† Research partially supported by projects PBH2006-0003-PC, MTM2005-01412 and PCI2006-1198, MEC-Spain.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

26, 22] a detailed description of such structure is available. What it is essentially known is that, if an autonomous dynamical system (semigroup) with a finite number of equilibria is gradient; that is, it has an associated Lyapunov function, then its attractor can be characterized as the union of the unstable sets of the equilibria. Moreover, if all equilibria are hyperbolic, this structure is *"stable"* under autonomous perturbation of the system.

Recently, in [6] (see also [27]), the authors consider non-autonomous perturbations of a (autonomous) gradient semigroups for which all equilibria are hyperbolic. In both articles the authors have started from the proof that the hyperbolic equilibria under perturbation give rise to the same number of hyperbolic global solutions for the perturbed problem, proving that the perturbed attractor *"contains"* the same structures as the limiting one. Then, using the properties of the Lyapunov function (backwards and forwards convergence to equilibria and non-existence of *homoclinic structures*) for the limiting problem they prove that the perturbed attractor does not contain anything else.

In this paper, without using a Lyapunov function, we translate the concept of gradient semigroups into a concept that we call *gradient-like semigroups* which strictly include the gradient semigroups. In particular, it allows that the semigroups have periodic orbits which is not allowed for gradient semigroups and are not considered in [6, 27]. In addition, this concept can be extended to non-autonomous evolution processes.

The gradient-like semigroups are shown to be stable under perturbation (autonomous or non-autonomous). That is, without talking about hyperbolicity (but assuming the continuity of the targeted global solutions) we prove that the attractor of a gradient-like semigroup under perturbation (autonomous or not) maintains its characterization.

In the autonomous case, the attractor of a gradient-like semigroup has a Morse decomposition (see [11, 19, 25]), so that we are showing that Morse decompositions of gradient-like semigroups are, under certain assumptions, stable under perturbations. The concept of Morse decomposition can be extended to evolution processes (non-autonomous dynamical systems) and if a semigroup has an attractor which has a Morse decomposition, a perturbation (autonomous or not) of this semigroup has also a pullback attractor with a Morse decomposition.

In order to state our main results we introduce some terminology. Let $\mathcal{Z}$ be a Banach space. A nonlinear evolution process is a two parameter family $\{S(t,\tau):\ t\geq\tau\in\mathbb{R}\}$ of continuous operators from $\mathcal{Z}$ into itself such that

$$\begin{aligned}&1)\ S(\tau,\tau)=I,\\&2)\ S(t,\sigma)S(\sigma,\tau)=S(t,\tau),\ \text{ for each } t\geq\sigma\geq\tau,\ \text{ and}\\&3)\ (t,\tau)\mapsto S(t,\tau)z_0 \text{ is continuous for } t\geq\tau,\ z_0\in\mathcal{Z}.\end{aligned}$$

A continuous function $z:\mathbb{R}\to\mathcal{Z}$ is called a global solution for the evolution process $\{S(t,\tau):t\geq\tau\}$ if it satisfies

$$S(t,\tau)z(\tau)=z(t),\ \text{ for all } t\geq\tau\in\mathbb{R}.$$

A nonlinear semigroup (or autonomous evolution process) is a family $\{S(t):\ t\geq 0\}$ with the property that $\{S(t,\tau)=S(t-\tau):t\geq\tau\in\mathbb{R}\}$ is an evolution process.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

DEFINITION 1.1. A set $\mathcal{A} \subseteq \mathcal{Z}$ is said to be the *global attractor* for $\{S(t) : t \geq 0\}$ if it is

(i) compact,
(ii) invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and
(iii) it attracts bounded subsets B of $\mathcal{Z}$,

$$\operatorname{dist}(S(t)B, \mathcal{A}) \to 0 \text{ as } t \to \infty.$$

Since any fixed set A will not, in general, be invariant in the above sense for a non-autonomous process, it is natural to define *invariance* in this context as follows:

• A family $\{\mathcal{A}(t) \subset \mathcal{Z} : t \in [\sigma, \infty)\}$ is invariant under $S(\cdot, \cdot)$ if $S(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau \geq \sigma$.

In the autonomous case the attractor is exactly the union of all its global orbits (see [26]),

$$\mathcal{A} = \{z : \text{ there is a bounded global solution through } z\}. \tag{1.1}$$

In the non-autonomous case, the definition of an 'attractor' that has the same characterization as the union of all globally-defined bounded orbits,

$$\{\mathcal{A}(t) : t \in \mathbb{R}\} = \{\xi(t) : \xi(\cdot) : \mathbb{R} \to \mathcal{Z} \text{ is bounded and } S(t, \tau)\xi(\tau) = \xi(t)\} \tag{1.2}$$

is the pullback attractor (see [9], [10], [16], [24]) :

DEFINITION 1.2. A family of compact sets $\{\mathcal{A}(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$, with $\overline{\cup_{t \in \mathbb{R}} A(t)}$ compact, is a *pullback attractor* for $\{S(t, \tau) : \ t \geq \tau \in \mathbb{R}\}$ if it is invariant and attracts all bounded subsets of $\mathcal{Z}$ 'in the pullback sense', i.e.

$$\lim_{\tau \to -\infty} \operatorname{dist}(S(t, \tau)B, \mathcal{A}(t)) = 0, \ \forall t \in \mathbb{R}.$$

For autonomous problems, it is clear that the concept of a pullback attractor coincides with the standard definition of the attractor, while the characterization in (1.2) shows that this notion is in some sense a 'natural' generalization (see [6] for a discussion). However, the pullback attractor will not necessarily enjoy any kind of forward attraction. Except in specific situations, the pullback behaviour and the forwards behaviour will not be related (see Theorems 4.1 and 4.2, and [8, 23, 18] for other specific cases).

DEFINITION 1.3. Let a $\Xi := \{\Xi(t) : t \in \mathbb{R}\}$ be an invariant family for the nonlinear evolution process $\{T(t, \tau) : t \geq \tau \in \mathbb{R}\}$. The set $\Gamma = \cup\{\Xi(t) : t \in \mathbb{R}\}$ is called trace of $\{\Xi(t) : t \in \mathbb{R}\}$. We say that $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$ is a set of isolated invariant families if there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Gamma_i^*) \cap \mathcal{O}_\delta(\Gamma_j^*) = \emptyset$, $1 \leq i < j \leq n$, where Γ_i^* is the trace of $\{\Xi_i^*(t) : t \in \mathbb{R}\}$ and $\mathcal{O}_\delta(\Gamma_i^*) := \{z \in \mathcal{Z} : \operatorname{dist}(z, \Gamma_i^*) < \delta\}$.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Let $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear evolution process with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ which contains a finite number of isolated invariant families $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$. Let Γ_i^* be the trace of Ξ_i^*. We define (see [12, 21])

Definition 1.4. Let δ be as in Definition 1.3 and fix $\epsilon_0 \in (0,\delta)$. For $\Xi^* \in \mathcal{S}$ and $\epsilon \in (0,\epsilon_0)$, an $\epsilon-$chain from Ξ^* to Ξ^* is a sequence $\{\Xi_{\ell_i}^*, \cdots, \Xi_{\ell_k}^*\} \subset \mathcal{S}$, a sequence of real numbers $t_1, \sigma_1, \tau_1, \cdots, t_k, \sigma_k, \tau_k$ with $t_i > \sigma_i > \tau_i$, $1 \leq i \leq k$, $k \leq n$, and a sequence of vectors u_i, $1 \leq i \leq k$, such that $u_i \in \mathcal{O}_\epsilon(\Gamma_{\ell_i}^*)$, $T(\sigma_i,\tau_i)u_i \notin \mathcal{O}_{\epsilon_0}(\cup_{i=1}^k(\Gamma_{\ell_i}^*))$ and $T(t_i,\tau_i)u_i \in \mathcal{O}_\epsilon(\Gamma_{\ell_{i+1}}^*)$, $1 \leq i \leq k$, with $\Xi^* = \Xi_{\ell_{k+1}}^* = \Xi_{\ell_1}^*$. We say that $\Xi^* \in \mathcal{S}$ is chain recurrent if there is an $\epsilon_0 \in (0,\delta)$ and $\epsilon-$chain from Ξ^* to Ξ^* for each $\epsilon \in (0,\epsilon_0)$.

Remark 1. 1. We note that the introduction of ϵ_0 in the above definition is only needed to account for the case $k = 1$. When $k > 1$ it is automatically true that the solution must leave $\mathcal{O}_{\epsilon_0}(\cup_{i=1}^k(\Gamma_{\ell_i}^*))$ while going from one isolated global solution to another.

Definition 1.5. Let $\mathcal{Z}$ be a Banach space and $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear process in $\mathcal{Z}$. Let $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ be the pullback attractor for $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$. We say that $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is a generalized gradient-like process if the following two hypotheses are satisfied:

(H1) There is a finite set $\mathcal{S} = \{\Xi_i^* : \mathbb{R} \to \mathcal{Z} : 1 \leq i \leq n\}$ of isolated invariant families in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ with the property that any global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\{A(t) : t \in \mathbb{R}\}$ satisfies

$$\lim_{t\to-\infty} \mathrm{dist}(\xi(t), \Xi_i^*(t)) = 0 \text{ and } \lim_{t\to\infty} \mathrm{dist}(\xi(t), \Xi_j^*(t)) = 0,$$

for some $1 \leq i, j \leq n$.

(H2) $\mathcal{S} = \{\Xi_1^*, \cdots, \Xi_n^*\}$ does not contain any chain recurrent invariant family.

When $\{T(t,\tau) = S(t-\tau) : t \geq \tau \in \mathbb{R}\}$ we say that $\{S(t) : t \geq 0\}$ is a generalized gradient-like nonlinear semigroup.

Next we introduce the definitions of unstable and stable sets which we will refer to as the unstable and stable manifolds, though they do not need to be a manifold (up to this point).

Definition 1.6. Let $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process. The unstable manifold of an isolated invariant family Ξ^* with trace Γ^* is the set

$$W^{\mathrm{u}}(\Xi^*) = \{ (\tau,\zeta) \in \mathbb{R} \times \mathcal{Z} : \text{ there is a global solution } \xi : \mathbb{R} \to \mathcal{Z} \text{ such that } \xi(\tau) = \zeta \text{ and } \lim_{t\to-\infty} \mathrm{dist}(\xi(t), \Gamma^*) = 0\}.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

The stable manifold of an isolated invariant family Ξ^* to $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is the set

$$W^{\mathrm{s}}(\Xi^*_\eta) = \{ (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \lim_{t\to+\infty} \mathrm{dist}(S(t,\tau)\zeta,\Gamma^*) = 0\}.$$

Also, $W^{\mathrm{u}}(\Xi^*)(\tau) := \{ \zeta \in \mathcal{Z} : (\tau,\zeta) \in W^{\mathrm{u}}(\Xi^*)\}$ and $W^{\mathrm{s}}(\Xi^*)(\tau) := \{ \zeta \in \mathcal{Z} : (\tau,\zeta) \in W^{\mathrm{s}}(\Xi^*)\}$.

The intersection of the unstable (stable) manifold with a neighborhood of $\{(t,\Xi^*(t)) : t \in \mathbb{R}\}$ in $\mathbb{R}\times\mathcal{Z}$ is called a local unstable (stable) manifold and is denoted by $W^{\mathrm{u}}_{\eta,\mathrm{loc}}$ ($W^{\mathrm{s}}_{\eta,\mathrm{loc}}$).

We are now ready to state the main results of this paper

THEOREM 1.1. *Let $\mathcal{Z}$ be a Banach space, $\eta \in [0,1]$ be a parameter and $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear evolution process in $\mathcal{Z}$ with a pullback attractor $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$. Assume that,*

(*a*) $\overline{\cup_{\eta\in[0,1]}\cup_{t\in\mathbb{R}}\mathcal{A}_\eta(t)}$ *is compact.*

(*b*) $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ *has a finite number of isolated invariant families $\mathcal{S}_\eta = \{\Xi^*_{1,\eta},\cdots,\Xi^*_{n,\eta}\}$, $\eta \in [0,1]$, which behave upper and lower semi-continuously as $\eta \to 0$* ($\sup_{1\leq i\leq n}\sup_{t\in\mathbb{R}}[\mathrm{dist}(\Xi^*_{i,\eta}(t),\Xi^*_{i,0}(t)) + \mathrm{dist}(\Xi^*_{i,0}(t),\Xi^*_{i,\eta}(t))] \overset{\eta\to 0}{\longrightarrow} 0$).

(*c*) $\|T_\eta(t+\tau,\tau)u - T_0(t+\tau,\tau)u\|_{\mathcal{Z}} \overset{\eta\to0}{\longrightarrow} 0$ *uniformly for $\tau\in\mathbb{R}$, t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$.*

(*d*) *there are $\delta > 0$ and $\eta_0 \in (0,1]$ such that, if $\eta < \eta_0$, $\xi_\eta : \mathbb{R}\to\mathcal{Z}$ is a global solution in $\{A_\eta(t) : t\in\mathbb{R}\}$, $t_0 \in \mathbb{R}$ and $\mathrm{dist}(\xi_\eta(t),\Gamma^*_{i,\eta}) < \delta$ for all $t \leq t_0$ ($t\geq t_0$), then* $\mathrm{dist}(\xi_\eta(t),\Gamma^*_i) \overset{t\to-\infty}{\longrightarrow} 0$ ($\mathrm{dist}(\xi_\eta(t),\Gamma^*_i) \overset{t\to+\infty}{\longrightarrow} 0$)

(*e*) $\{S(t) : t \geq 0\}$ *is a generalized gradient-like nonlinear semigroup and $T_0(t,\tau) = S(t-\tau)$, $t\geq\tau$.*

Then there exists $\eta_0 > 0$ such that, for all $\eta \leq \eta_0$, $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$ is a generalized gradient-like nonlinear process. Consequently, there exists $\eta_0 > 0$ such that

$$\mathcal{A}_\eta(t) = \cup_{i=1}^n W^u(\Xi^*_{i,\eta})(t),\ t\in\mathbb{R} \text{ and } \forall\eta\in[0,\eta_0].$$

Theorem 1.1 generalizes the characterization result in [6] to perturbation of autonomous generalized gradient-like nonlinear semigroups. Hence, the limit problem need not to have a Lyapunov function, nor does Ξ^*_i need to be an equilibrium point. Of course it remains to prove the continuity of the isolated global solutions $\Xi^*_{i,\eta}$ at $\eta = 0$ which is known to hold; e.g, for normally hyperbolic global solutions (see [4]). That opens the possibility for considering non-autonomous perturbations of generalized gradient-like nonlinear semigroups with periodic solutions. Indeed, for each $1\leq m\in\mathbb{N}$, the problem

$$\dot r = \begin{cases} \pi^{-1}(1-\frac{1}{2m+1}-r)^3\sin\frac{\pi}{1-r}, & r < 1-\frac{1}{2m+1}\\ -(1-\frac{1}{2m+1}-r)^2, & r\geq 1-\frac{1}{2m+1}\end{cases} \tag{1.3}$$
$$\dot\theta = 1$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

has an attractor $A_m = \{|r| \leq 1 - \frac{1}{2m+1}\}$, which is the union of the unstable manifolds of Ξ_j^*, $1 \leq j \leq 2m+1$, where Ξ_j^* is the $2\pi-$periodic solution corresponding to $r = 1 - \frac{1}{j}$, $1 \leq j \leq 2m+2$. These periodic solutions are normally hyperbolic solutions (if k is even, the orbit is unstable and if k is odd, the orbit is stable). In this case, it is easy to see that the attractor A_m is the union of the unstable manifolds of the of periodic solutions $\{\Xi_j^* : 1 \leq j \leq 2m+1\}$. Theorem 1.1 implies that any small non-autonomous perturbation of the semigroup generated by (1.3) will lead a generalized gradient-like evolution process and the perturbed attractor is characterized.

Observe that Theorem 1.1 implies that the pullback attractors for $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ are characterized as the union of the unstable manifolds of the isolated global solutions. Hyperbolicity is not required up to this point. Nonetheless, the persistence of the isolated global solutions will require some kind of hyperbolicity (in general normal hyperbolicity, see [4], [25]).

Suppose $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, is a family of nonlinear evolution processes with the property that there exist $c = c(B_0)$ and $L > 0$ such that

$$\|T_\eta(t+\tau,\tau)u - T_\eta(t+\tau,\tau)v\| \leq ce^{Lt}\|u-v\|, \text{ for all } \tau \in \mathbb{R},\ u,v \in B_0. \tag{1.4}$$

Our next result proves that a non-autonomous perturbation of a generalized gradient-like semigroup with all equilibria hyperbolic has an exponential forwards (and pullback) attractor.

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Then there exists* $\eta_0 > 0$ *such that*

$$\mathcal{A}_\eta(t) = \cup_{i=1}^n W^u(\Xi_{i,\eta}^*)(t),\ \forall t \in \mathbb{R} \text{ and } \forall \eta \in [0,\eta_0]$$

where $\mathcal{A}$ *is the global attractor for* $\{S(t) : t \geq 0\}$ *and* $\{\mathcal{A}_0(t) = \mathcal{A} : t \in \mathbb{R}\}$ *is the pullback attractor for* $\{T_0(t,\tau) : t \geq \tau \in \mathbb{R}\}$.

If we also assume that there is $\gamma > 0$ *and, for each* $1 \leq i \leq n$, *a neighborhood* V_i *of the trace* Γ_i^* *of* $\Xi_{i,\eta}^*$ *such that, for any* $u_0 \in V$, $\tau \in \mathbb{R}$, *and as long as* $T_\eta(t+\tau,\tau)u_0 \in V_i$

$$\sup_{\tau \in \mathbb{R}} \operatorname{dist}(T_\eta(t+\tau,\tau)u_0, W_\eta^u(\Xi^*)(t)) \leq Me^{-\gamma t},$$

for bounded set $B \subset \mathcal{Z}$, *there is a constant* $c(B) > 0$ *such that*

$$\sup_{\tau \in \mathbb{R}} \operatorname{dist}(T_\eta(t+\tau,\tau)u_0, \mathcal{A}_\eta(t+\tau)) \leq c(B)e^{-\gamma t},\ \text{for all } u_0 \in B. \tag{1.5}$$

This paper is organized as follows. In Section 2 we present the results for the autonomous case emphasizing the generalization of previous known results. In Section 3 we extend the results to the non-autonomous case. In Section 4 we write the consequences of the results of Section 3 for asymptotically autonomous evolution processes. Finally, Section 5 focuses on the comparison of the concept of generalized gradient-like semigroups with the concept

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

of semigroups with attractors having a Morse decomposition and to point out several problems that remain open.

2. GRADIENT SEMIGROUPS, GRADIENT-LIKE SEMIGROUPS AND SEMIGROUPS WITH 'GRADIENT-LIKE' ATTRACTORS

We firstly need to draw a distinction between gradient semigroups and semigroups with 'gradient-like' attractors.

DEFINITION 2.1. We say that a nonlinear semigroup $\{S(t) : t \geq 0\}$ is *gradient* if $\{S(t)z : t \geq 0\}$ is relatively compact for each $z \in \mathcal{Z}$ and there exists a continuous function $V : \mathcal{Z} \to \mathbb{R}$ such that

- $t \mapsto V(S(t)z) : [0, \infty) \to \mathcal{Z}$ is non-increasing for each $z \in \mathcal{Z}$.
- If $z \in \mathcal{Z}$ is such that there is a global solution $\xi(\cdot) : \mathbb{R} \to \mathcal{Z}$ through $\xi(0) = z$ and there exists a $t^* \in \mathbb{R}$ such that $V(\xi(t)) = V(z)$ for all $t \geq t^*$ or for all $t \leq t^*$, then z is a solution for $\{S(t) : t \geq 0\}$ (and so in fact $V(\xi(t)) = V(z)$ for all $t \in \mathbb{R}$).

The function $V : \mathcal{Z} \to \mathbb{R}$ is called a Lyapunov function for $\{S(t) : t \geq 0\}$.

The stable and unstable manifolds of an equilibrium y_0^*, $W^{\rm s}(y_0^*)$ and $W^{\rm u}(y_0^*)$ respectively, are defined as follows:

$$W^{\rm s}(y_0^*) = \{\, z \in \mathcal{Z} : \lim_{t \to +\infty} \|S(t)z - y_0^*\|_{\mathcal{Z}} = 0\}.$$

$$W^{\rm u}(y_0^*) = \{\, z \in \mathcal{Z} : \text{ there is a global solution } y(t) \text{ for } \{S(t) : t \geq 0\} \text{ satisfying } y(\tau) = z \text{ and such that } \lim_{t \to -\infty} \|y(t) - y_0^*\|_{\mathcal{Z}} = 0\}.$$

The following result is classical and can be found, for instance, in [13]:

THEOREM 2.1. *If $\{S(t) : t \geq 0\}$ is a gradient semigroup that has a global attractor $\mathcal{A}$ and a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots, y_n^*\}$, then $\mathcal{A}$ is given by*

$$\mathcal{A} = \bigcup_{i=1}^{n} W_0^{\rm u}(y_i^*). \tag{2.1}$$

If $V : \mathcal{Z} \to \mathbb{R}$ is Lyapunov function associated to $\{S(t) : t \geq 0\}$, denote by $\{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ the set of all distinct values of V in $\mathcal{E}$, ordered so that $\mathfrak{n}_i < \mathfrak{n}_j$, $1 \leq i < j \leq p \leq n$, and define $\mathcal{E}_k = \{y_i^* \in \mathcal{E} : V(y_i^*) = \mathfrak{n}_k\}$. If $y(\cdot) : \mathbb{R} \to \mathcal{Z}$ is a global solution for $\{S(t) : t \geq 0\}$, there are k_1, k_2 with $1 \leq k_1 < k_2 \leq p$, $y_i^* \in \mathcal{E}_{k_1}$ and $y_j^* \in \mathcal{E}_{k_2}$, such that

$$\lim_{t \to -\infty} y(t) = y_j^* \text{ and } \lim_{t \to +\infty} y(t) = y_i^*.$$

The above result shows that if $\{S(t) : t \geq 0\}$ is a gradient nonlinear semigroup the structure of $\mathcal{A}$ and its asymptotic dynamics are well understood. This is essentially the class of nonlinear semigroups in Banach spaces for which a detailed knowledge of the structure of the attractor is available. An attractor of the form (2.1) we term *'gradient-like'*. Clearly the class of *semigroups with gradient-like attractors* is larger than the class of *gradient semigroups*.

In general one does not expect that a perturbation of a gradient semigroup gives rise to a new gradient semigroup, the main difficulty being to prove that the perturbed problem has a Lyapunov function. In [6] it is proved that a perturbation of a gradient semigroup which has a global attractor gives rise to a new semigroup which possesses a gradient-like attractor. In the proofs the Lyapunov function of the unperturbed semigroup plays an essential role.

It is not difficult to see (see the example in [13] pages 2 and 3) that a gradient-like-attractor may not come from a gradient semigroup and a perturbation of a semigroup with a gradient-like attractor may have an attractor which is not gradient-like attractor, even when all equilibria of the unperturbed problem are hyperbolic. One can prove only that, if all equilibria of the unperturbed semigroup are hyperbolic, the perturbed attractor *"contains"* (possibly strictly) the same structure as the limit attractor and therefore the attractors behave upper and lower semicontinuously.

This brings up the question of what (natural) dynamical properties of semigroups ensure that they have gradient-like attractors and such that (the properties) are stable under perturbation. Of course, such properties must be satisfied by gradient semigroups and its perturbations.

With this in mind we define the following concept of *gradient-like semigroups* generalizing the concept of gradient semigroups while maintaining its essential dynamics. We will prove that the properties defining gradient-like semigroups are stable under perturbation.

DEFINITION 2.2. Consider a nonlinear semigroup $\{S(t) : t \geq 0\}$ with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$. Let

$$2\delta_0 = \min_{\substack{1 \leq i,j \leq n \\ i \neq j}} \|y_i^* - y_j^*\|_{\mathcal{Z}} > 0$$

Let $\epsilon_0 < \delta_0$, $y^* \in \mathcal{E}$ and $\epsilon \in (0, \epsilon_0)$. An $\epsilon-$chain from y^* to y^* is a sequence $\{y_{\ell_1}^*, \cdots, y_{\ell_k}^*\}$ in $\mathcal{E}$, together with $\{y_1, \cdots, y_k\}$ of points in $\mathcal{Z}$ and a sequence of numbers $\{t_1, \sigma_1 \cdots, t_k, \sigma_k\}$, $0 < \sigma_i < t_i$, $1 \leq i \leq k$, such that $\|y_i - y_{\ell_i}^*\|_{\mathcal{Z}} < \epsilon$, $1 \leq i \leq k+1$, $y^* = y_{\ell_1}^* = y_{\ell_{k+1}}^*$, $\mathrm{dist}(S(\sigma_i)y_i, \mathcal{E}) > \epsilon_0$ and $\|S(t_i)y_i - y_{\ell_{i+1}}^*\|_{\mathcal{Z}} < \epsilon$, $1 \leq i \leq k$. We say that $y^* \in \mathcal{E}$ is chain recurrent if there is an $\epsilon_0 > 0$ fixed and an $\epsilon-$chain from y^* to y^*, for each $\epsilon \in (0, \epsilon_0)$.

DEFINITION 2.3. Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. We say that $\{S(t) : t \geq 0\}$ is a *gradient-like semigroup* if the following two conditions are satisfied:

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

(G1) Given a global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\mathcal{A}$, there exists $i, j \in \{1, \cdots, n\}$ such that

$$\lim_{t\to-\infty} ||\xi(t) - y_i^*|| = 0 \text{ and } \lim_{t\to\infty} ||\xi(t) - y_j^*|| = 0.$$

(G2) $\mathcal{E} = \{y_1^*, \cdots, y_n^*\}$ does not contain any chain recurrent point.

The assumptions (G1) and (G2) carry the important dynamic properties of a semigroup with a Lyapunov function (see also Lemmas 2.3 and 2.4 below). Clearly, from (G1), we have that $\mathcal{A} = \cup_{i=1}^n W^u(y_i^*)$. Also, the hypothesis (G2) says that no finite number of orbits may produce a closed contour. The great advantage of these assumptions over Lyapunov functions is their stability under perturbation. This fact will allow us to give characterization of attractors of semigroups which are small perturbation of gradient semigroups, perturb such semigroups again and still be able to give characterization of the attractors of the newly perturbed semigroup.

Before we proceed with the analysis of the attractors of *gradient-like semigroups* under perturbation let us establish the equivalence between condition (G2) and the absence of homoclinic structures.

Definition 2.4. Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. A *homoclinic structure* in $\mathcal{A}$ is a set $\{y_{\ell_1}^*, \cdots, y_{\ell_k}^*\} \subset \mathcal{E}$ and a set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ in $\mathcal{A}$ such that, making $y_{\ell_{k+1}}^* := y_{\ell_1}^*$,

$$\lim_{t\to-\infty} \xi_i(t) = y_{\ell_i}^*, \quad \lim_{t\to+\infty} \xi_i(t) = y_{\ell_{i+1}}^*, \quad 1 \leq i \leq k.$$

Lemma 2.1. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup which has finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ and a global attractor $\mathcal{A}$. If $\{S(t) : t \geq 0\}$ satisfies* (G1)*, then* (G2) *is satisfied if and only if $\mathcal{A}$ does not have homoclinic structures.*

Proof: Assuming that $\mathcal{A}$ has a homoclinic structure it is easy to see that the stationary solutions in it are chain recurrent.

On the other hand, if $y^* \in \mathcal{E}$ is chain recurrent, there is an $\epsilon_0 < \delta_0$, $\{y_{\ell_1}^*, \cdots, y_{\ell_{r+1}}^*\} \subset \mathcal{E}$, for each $\mathbb{N} \ni k > \frac{1}{\epsilon_0}$, points $y_1^k, \cdots, y_{r+1}^k$ and positive numbers $t_1^k > \sigma_1^k, \cdots, t_r^k > \sigma_r^k$ such that

$$\|y_i^k - y_{\ell_i}^*\|_{\mathcal{Z}} < \frac{1}{k},\ \operatorname{dist}(S(\sigma_i^k)y_i^k, \mathcal{E}) > \epsilon_0,\ \|S(t_i^k)y_i^k - y_{\ell_{i+1}}^*\|_{\mathcal{Z}} < \frac{1}{k},\ 1 \leq i \leq r.$$

Choose $0 < \delta < \epsilon_0$ and choose $\tau_i^k > 0$ such that $\|S(\tau_i^k)y_i^k - y_{\ell_i}^*\|_{\mathcal{Z}} = \delta$ and $\|S(t)y_i^k - y_{\ell_i}^*\|_{\mathcal{Z}} < \delta$, for all $0 \leq t < \tau_i^k$. Note that $\tau_i^k \to +\infty$ and $t_i^k - \tau_i^k \to +\infty$ as $k \to +\infty$. For $t \in [-\tau_i^k, t_i^k - \tau_i^k]$ let $\xi_i^k(t) = S(\tau_i^k + t)y_i^k$.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Taking subsequences we define the global solutions $\xi_i : \mathbb{R} \to \mathcal{Z}$ by $\xi_i(t) = \lim_{k\to\infty} \xi_i^k(t)$. Since each $\xi_i(t)$ must converge to an equilibrium solution as $t \to +\infty$ and as $t \to -\infty$ and since $\xi_i(t) \in B_\delta(y^*_{\ell_i})$ for all $t \leq 0$ we have that $\xi_i(t) \to y^*_{\ell_i}$ as $t \to -\infty$.

Also, there is a time $s_i^k > 0$ such that $\|\xi_i^k(s_i^k) - y^*_{\ell_{i+1}}\| = \delta$ and $\xi_i^k(t) \in B_\delta(y^*_{\ell_{i+1}})$ for all $t \in [s_i^k, t_i^k - \tau_i^k]$. It is easy to see that $\{s_i^k\}$ is bounded and that $t_i^k - \tau_i^k - s_i^k \overset{k\to\infty}{\longrightarrow} +\infty$. Consequently $\xi_i(t) \to y^*_{\ell_{i+1}}$ as $t \to +\infty$.

The set $\{y^*_{\ell_1}, \cdots, y^*_{\ell_k}\} \subset \mathcal{E}$ and the set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ are such that,

$$\lim_{t\to-\infty} \xi_i(t) = y^*_{\ell_i}, \quad \lim_{t\to+\infty} \xi_i(t) = y^*_{\ell_{i+1}}, \quad 1 \leq i \leq k,$$

with $y^*_{\ell_{k+1}} := y^*_{\ell_1}$. Hence $\mathcal{A}$ has a homoclinic structure.□

We are now ready to prove that (G1) and (G2) are stable under perturbation, i.e. the concept of a gradient-like semigroup is robust under perturbation.

Theorem 2.2. *Let $\mathcal{Z}$ be a Banach space and $\{S_\eta(t) : t \geq 0\}$, $\eta \in [0,1]$, be a family of nonlinear semigroups in $\mathcal{Z}$ which satisfy*

(*a*) *for each $\eta \in [0,1]$ $\{S_\eta(t) : t \geq 0\}$ has a global attractor $\mathcal{A}_\eta$ and $\overline{\cup_{\eta\in[0,1]}\mathcal{A}_\eta}$ is compact.*

(*b*) *$\{S_\eta(t) : t \geq 0\}$ has a finite number of stationary solutions $\mathcal{E}_\eta = \{y^*_{1,\eta}, \cdots, y^*_{n,\eta}\}$, for all $\eta \in [0,1]$, and $\sup_{1\leq i\leq n} \|y^*_{i,\eta} - y^*_{i,0}\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$.*

(*c*) *there is a $\delta > 0$ such that, if a solution ξ_η in $\mathcal{A}_\eta$ is such that $\|\xi_\eta(t) - y^*_{i,\eta}\|_{\mathcal{Z}} \leq \delta$ for all $t \geq t_0$ (or for all $t \leq t_0$), $t_0 \in \mathbb{R}$, then $\xi_\eta(t) \overset{t\to\infty}{\longrightarrow} y^*_{i,\eta}$ (or $\xi_\eta(t) \overset{t\to-\infty}{\longrightarrow} y^*_{i,\eta}$).*

(*d*) *$\|S_\eta(t)u - S_0(t)u\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$ uniformly for t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$.*

(*e*) *$\{S_0(t) : t \geq 0\}$ is a gradient-like semigroup.*

Then, there exists $\eta_0 > 0$ such that, for all $\eta \in [0,\eta_0]$, the nonlinear semigroup $\{S_\eta(t) : t \geq 0\}$ is gradient-like.

Proof: Note $y^*_{i,\eta} \to y^*_{i,0} =: y^*_i$ and that there is a $\delta > 0$ such that, for suitably small η, if a solution ξ_η satisfies $\|\xi_\eta(t) - y^*_i\|_{\mathcal{Z}} \leq \delta$ for all $t \geq t_0$ and for some $t_0 > 0$, then $\xi_\eta(t) \to y^*_{i,\eta}$ as $t \to \infty$.

We argue by contradiction to prove that for all suitably small η, $\{S_\eta(t) : t \geq 0\}$ satisfies (G1). Assume that there is a sequence $\eta_k \overset{k\to\infty}{\longrightarrow} 0$ and corresponding global solutions ξ_k in $\mathcal{A}_{\eta_k}$ such that

$$\limsup_{t\to\infty} \operatorname{dist}(\xi_k(t), \mathcal{E}) > \delta. \tag{2.2}$$

Taking subsequences $\xi_k(t) \to \xi_0(t)$ uniformly in compact subsets of $\mathbb{R}$ where ξ_0 is a global solution in $\mathcal{A}_0$. Since $\xi_0(t) \overset{t\to\infty}{\longrightarrow} y^*_i$, for some $1 \leq i \leq n$, we have that, given $r \in \mathbb{N}\backslash\{0\}$ there is a $t_r > 0$ and $k_r \in \mathbb{N}$ such that $\|\xi_k(t_r) - y^*_i\|_{\mathcal{Z}} < \frac{1}{r}$, for each $k \geq k_r$. From (2.2), there exists $t'_r > t_r$ such that $\|\xi_{k_r}(t) - y^*_i\|_{\mathcal{Z}} < \delta$ for all $t \in [t_r, t'_r)$ and $\|\xi_{k_r}(t'_r) - y^*_i\|_{\mathcal{Z}} = \delta$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Taking subsequences, if necessary, let $\xi_1(t) = \lim_{r\to\infty} \xi_{k_r}(t+t_r)$. Then, since $t'_r - t_r \to \infty$ and $t_r \to +\infty$ as $r \to \infty$, $\|\xi_1(t) - y_i^*\|_{\mathcal{Z}} \le \delta$ for all $t \le 0$ and consequently $\xi_1(t) \to y_i^*$ as $t \to -\infty$. And $\xi_1(t) \to y_j^*$ as $t \to \infty$ with $i \neq j$ by (G1) and (G2) for $\eta = 0$. From the fact that $\xi_{k_r}(t) \to \xi_1(t)$ uniformly in compact subsets of $\mathbb{R}$ we have that, for each $m \in \mathbb{N}$, there is a time $t_m > 0$ and $k_m \in \mathbb{N}$ such that $\|\xi_{k_r}(t_m) - y_j^*\|_{\mathcal{Z}} < \frac{1}{m}$ for all $r \ge k_m$. Again, from (2.2), there exists $t'_m > t_m$ such that $\|\xi_{k_m}(t) - y_j^*\|_{\mathcal{Z}} < \delta$ for all $t \in [t_m, t'_m)$ and $\|\xi_{k_m}(t'_m) - y_j^*\|_{\mathcal{Z}} = \delta$. Proceeding exactly as before we obtain a global solution $\xi_2 : \mathbb{R} \to \mathcal{Z}$ such that $\xi_2(t) \to y_j^*$ as $t \to -\infty$ and $\xi_2(t) \to y_p^*$ as $t \to \infty$ with $p \notin \{i, j\}$. In a finite number of steps we arrive at a contradiction. This proves that there is a $\eta_0 > 0$ such that, for all global solution ξ_η in $\mathcal{A}_\eta$ with $\eta \le \eta_0$, we have that

$$\lim_{t\to\infty} ||\xi_\eta(t) - y_{i,\eta}^*|| = 0.$$

To prove that there is a $\eta_1 > 0$ such that, for all global solution ξ_η in $\mathcal{A}_\eta$ with $\eta \le \eta_1$, we have that

$$\lim_{t\to -\infty} ||\xi_\eta(t) - y_{j,\eta}^*|| = 0,$$

we proceed exactly in the same manner. This completes the proof that, for all suitably small η, $\{S_\eta(t) : t \ge 0\}$ satisfies (G1).

Let us prove that, for all suitably small η, $\{S_\eta(t) : t \ge 0\}$ satisfies (G2). Again we argue by contradiction. Assume that there is a sequence $y_1^*, \cdots, y_{p+1}^*$ in $\mathcal{E}$, a sequence $\eta_k \to 0$, global solutions $\xi_{k,i}$ in $\mathcal{A}_{\eta_k}$, and times $t_1^k, \cdots, t_p^k$ such that

$$\|\xi_{k,i}(0) - y_i^*\| < \frac{1}{k}, \quad \|\xi_{k,i}(t_i^k) - y_{i+1}^*\| < \frac{1}{k}, \quad 1 \le i \le p, \quad y_1^* = y_{p+1}^*.$$

Proceeding as in the proof of (G1) we construct a homoclinic structure for $\{S_0(t) : t \ge 0\}$ and arrive at a contradiction.□

As a immediate consequence of this theorem we obtain the following generalization of the characterization result in [6] for autonomous perturbation of semigroups.

Corollary 2.1. *Under the assumptions of Theorem* 2.2*, there is an* $\eta_0 > 0$ *such that*

$$\mathcal{A}_\eta = \cup_{i=1}^n W^u(y_{i,\eta}^*), \ \forall \eta \in [0, \eta_0].$$

Remark 2. 1. We observe that, up to this point, we have not explicitly used the hyperbolicity of the equilibria y_i^*, $1 \le i \le n$. Hence the results may hold in some cases for which hyperbolicity fails. In the applications hyperbolicity of all equilibrium points for $\{S_0(t) : t \ge 0\}$ is used to prove conditions (b) and (c).

We may replace the finite set of equilibria by a finite set of isolated invariant sets, changing accordingly the definition of gradient-like semigroups and completely similar proofs.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Definition 2.5. We say that $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$ is a family of isolated invariant sets if there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i^*) \cap \mathcal{O}_\delta(\Xi_j^*) = \emptyset$, $1 \le i < j \le n$, and Ξ_i^* is the maximal invariant subset of $\mathcal{O}_\delta(\Xi_i^*) := \{z \in \mathcal{Z} : \mathrm{dist}(z, \Xi_i^*) < \delta\}$.

Let $\{S(t) : t \ge 0\}$ be a nonlinear semigroup with a global attractor $\mathcal{A}$ which contains a finite family of isolated invariant sets $\mathcal{S} = \{\Xi_1^*, \cdots \Xi_n^*\}$. We define (see [12, 21]):

Definition 2.6. Let δ be as in Definition 2.5 and fix $\epsilon_0 \in (0, \delta)$. For $\Xi^* \in \mathcal{S}$ and $\epsilon \in (0, \epsilon_0)$, an $\epsilon-$chain from Ξ^* to Ξ^* is a sequence $\{\Xi_{\ell_i}^*, \cdots, \Xi_{\ell_k}^*\} \subset \mathcal{S}$, a sequence of real numbers $t_1, \sigma_1, \cdots, t_k, \sigma_k$, with $t_i > \sigma_i$, $1 \le i \le k$, $k \le n$, and a sequence of vectors u_i, $1 \le i \le k$, such that $u_i \in \mathcal{O}_\epsilon(\Xi_{\ell_i}^*)$, $S(\sigma_i)u_i \notin \mathcal{O}_{\epsilon_0}(\cup_{i=1}^k(\Xi_{\ell_i}^*))$ and $S(t_i)u_i \in \mathcal{O}_\epsilon(\Xi_{\ell_{i+1}}^*)$, $1 \le i \le k$, with $\Xi^* = \Xi_{\ell_{k+1}}^* = \Xi_{\ell_1}^*$. We say that $\Xi^* \in \mathcal{S}$ is chain recurrent if there is an $\epsilon_0 \in (0, \delta)$ and $\epsilon-$chain from Ξ^* to Ξ^* for each $\epsilon \in (0, \epsilon_0)$.

Definition 2.7. Let $\mathcal{Z}$ be a Banach space and $\{S(t) : t \ge 0\}$ be a nonlinear semigroup in $\mathcal{Z}$. Let $\mathcal{A}$ be the global attractor for $\{S(t) : t \ge 0\}$. We say that $\{S(t) : t \ge 0\}$ is generalized gradient-like if the following two hypotheses are satisfied:

(GG1) There is a finite family $\mathcal{S} = \{\Xi_i^* : \mathbb{R} \to \mathcal{Z} : 1 \le i \le n\}$ of isolated invariant sets in $\mathcal{A}$ with the property that any global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\mathcal{A}$ satisfies

$$\lim_{t \to -\infty} \mathrm{dist}(\xi(t), \Xi_i^*) = 0 \text{ and } \lim_{t \to \infty} \mathrm{dist}(\xi(t), \Xi_j^*) = 0,$$

for some $1 \le i, j \le n$.

(GG2) $\mathcal{S} = \{\Xi_1^*, \cdots, \Xi_n^*\}$ does not contain any chain recurrent invariant set.

As before introduce the definitions of unstable and stable manifolds.

Definition 2.8. Let $\{S(t) : t \ge 0\}$ be a nonlinear semigroup. The unstable manifold of an isolated invariant set Ξ^* is given by

$$W^{\mathrm{u}}(\Xi^*) = \{\, \zeta \in \mathcal{Z} : \text{ there is a global solution } \xi : \mathbb{R} \to \mathcal{Z} \text{ such that } \xi(0) = \zeta \text{ and } \lim_{t \to -\infty} \mathrm{dist}(\xi(t), \Xi^*) = 0\}.$$

The stable manifold of an isolated invariant set Ξ^* to $\{S(t) : t \ge 0\}$ is given by

$$W^{\mathrm{s}}(\Xi_\eta^*) = \{(\zeta) \in \mathcal{Z} : \lim_{t \to +\infty} \mathrm{dist}(S(t)\zeta, \Xi^*) = 0\}.$$

The intersection of the unstable (stable) manifold with a neighborhood of Ξ^* in $\mathcal{Z}$ is called a local unstable (stable) manifold and is denoted by $W^{\mathrm{u}}_{\eta,\mathrm{loc}}$ ($W^{\mathrm{s}}_{\eta,\mathrm{loc}}$).

The proof of the following result is completely analogous to the proof of Theorem 2.2:

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

THEOREM 2.3. *Let $\mathcal{Z}$ be a Banach space η be a parameter in* $[0,1]$ *and* $\{S_\eta(t) : t \geq 0\}$ *be a nonlinear semigroup in $\mathcal{Z}$ with a global attractor $\mathcal{A}_\eta$, $\eta \in [0,1]$. Assume that,*

(*a*) $\overline{\cup_{\eta\in[0,1]}\mathcal{A}_\eta}$ *is compact.*

(*b*) $\mathcal{A}_\eta$ *has a finite many of isolated invariant sets* $\mathcal{S}_\eta = \{\Xi^*_{1,\eta}, \cdots, \Xi^*_{n,\eta}\}$, $\eta \in [0,1]$, *which behave upper and lower semi-continuously as η tends to zero* ($\sup_{1\leq i\leq n}[\mathrm{dist}(\Xi^*_{i,\eta}, \Xi^*_{i,0}) + \mathrm{dist}(\Xi^*_{i,0}, \Xi^*_{i,\eta})] \stackrel{\eta\to 0}{\longrightarrow} 0$).

(*c*) $\|S_\eta(t)u - S_0(t)u\|_{\mathcal{Z}} \stackrel{\eta\to 0}{\longrightarrow} 0$ *uniformly for t in compact subsets of* $[0,\infty)$ *and for u in compact subsets of $\mathcal{Z}$.*

(*d*) *there are $\delta > 0$ and $\eta_0 \in (0,1]$ such that, if $\eta < \eta_0$, $\xi_\eta : \mathbb{R} \to \mathcal{Z}$ is a global solution in $\mathcal{A}_\eta$, and* $\mathrm{dist}(\xi_\eta(t), \Xi^*_{i,\eta}) < \delta$ *for all $t \leq 0$ ($t \geq 0$), then* $\mathrm{dist}(\xi_\eta(t), \Xi^*_i) \stackrel{t\to-\infty}{\longrightarrow} 0$ ($\mathrm{dist}(\xi_\eta(t), \Xi^*_i) \stackrel{t\to+\infty}{\longrightarrow} 0$).

(*e*) $\{S_0(t) : t \geq 0\}$ *is a generalized gradient-like nonlinear semigroup.*

Then there exists, $\eta_0 > 0$ such that, for all $\eta \leq \eta_0$, $\{S_\eta(t) : t \geq 0\}$ is a generalized gradient-like semigroup. Consequently, there exists $\eta_0 > 0$ such that

$$\mathcal{A}_\eta = \cup_{i=1}^n W^u(\Xi^*_{i,\eta}), \ \forall \eta \in [0, \eta_0].$$

Remark 2. 2. Note that Theorem 2.1 proves that a gradient semigroup is also a gradient-like semigroup. Moreover, *a perturbation of a gradient semigroup, in general, is not a gradient semigroup.* Theorem 2.2 proves that a gradient-like semigroup $\{S(t) : t \geq 0\}$ is stable under perturbation. Since gradient semigroups are also gradient-like semigroups we have that a perturbation of a gradient semigroup is a gradient-like semigroup. Section 5 in [6] shows examples of regular perturbations of gradient semigroups for which the perturbed semigroups have gradient-like-attractors but are not gradient.

On the other hand, observe that a gradient-like attractor is a global attractor with the structure shown in (2.1). But, for instance, an attractor can exhibit this structure and for $u_0 \in \mathcal{Z}$ its omega limit is not just one of the stationary points y^*_j, so that (G1) does not hold.

If we take the example shown in [13], page 2, and change the arrow directions we have an example of a semigroup with a gradient-like attractor. The perturbation of this semigroup will have an attractor which is not gradient-like (contains a periodic orbit) but behaves upper and lower semicontinuously. Note that both (G1) and (G2) fail in this case. This shows that the class of semigroups with a gradient-like attractor is larger that the one of gradient-like semigroups.

As a consequence, our concept of gradient-like semigroup is actually an intermediate one between the ones of a gradient semigroup and a semigroup possessing a gradient-like attractor.

In Section 3 we show that the concept of gradient-like nonlinear semigroups can be extended in a natural way to nonlinear evolution processes; that is, non-autonomous equations.

Next, we introduce the linearization of a semigroup around an equilibrium point. The aim is to show lower semicontinuity of attractors for gradient-like semigroups for which all equilibria are hyperbolic under perturbation (which does not follow yet from the previous results). We also pursue a property that the attractor obtained is an exponential attractor (see [2]) with the aim to obtain also the same result in Section 3 under non-autonomous perturbation.

Assume that $f_\eta : \mathcal{Z} \to \mathcal{Z}$ is a continuously differentiable, globally Lipschitz and bounded function. Assume also that, for each $r > 0$,

$$\sup_{\|z\|\leq r} \|f_\eta(z) - f_0(z)\|_{\mathcal{Z}} + \|f'_\eta(z) - f'_0(z)\|_{\mathcal{L}(\mathcal{Z})} \overset{\eta\to 0}{\longrightarrow} 0. \tag{2.3}$$

Let $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of an exponentially decaying C^0-semigroup. Define $t \mapsto S_\eta(t-\tau)y_0$ by

$$S_\eta(t-\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(S_\eta(s-\tau)y_0)\, ds. \tag{2.4}$$

A solution of (2.4) is an equilibrium solution if it satisfies

$$\mathfrak{B}y + f_\eta(y) = 0. \tag{2.5}$$

If y^*_η is solution of (2.5), $A_\eta = \mathfrak{B} + f'_\eta(y^*_\eta)$, it generates a C^0- semigroup $\{e^{A_\eta t} : t \geqslant 0\} \subset \mathcal{L}(\mathcal{Z})$.

Definition 2.9. An equilibrium solution y^*_η to (2.4) is said to be hyperbolic if the following items are satisfied:

1. $\sigma(A_\eta) \cap i\mathbb{R} = \emptyset$ and $\sigma^+_\eta = \{\lambda \in \sigma(A_\eta) : \mathrm{Re}\lambda > 0\}$ is compact.
Choose a positively oriented smooth closed simple curve γ in $\rho(A_\eta) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ which encloses σ^+ and define the projection

$$\mathcal{Q}_\eta = \mathcal{Q}_\eta(\sigma^+_\eta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - A_\eta)^{-1}\, d\lambda. \tag{2.6}$$

If $\mathcal{Z}^+_\eta = \mathcal{Q}_\eta(\mathcal{Z})$, $\mathcal{Z}^-_\eta = (I - \mathcal{Q}_\eta)(\mathcal{Z})$, and $A^\pm_\eta = (A_\eta)_{|_{\mathcal{Z}^\pm}}$, then $\mathcal{Z} = \mathcal{Z}^+_\eta \oplus \mathcal{Z}^-_\eta$, A^-_η generates a C_0-semigroup on $\mathcal{Z}^-_\eta$ and $A^+_\eta \in L(\mathcal{Z}^+_\eta)$.

2. There are constants $\bar{M} \geq 1$ and $\beta > 0$ such that

$$\begin{aligned} \|e^{A^+_\eta t}\|_{L(\mathcal{Z}^+)} &\leqslant \bar{M} e^{\beta t}, \quad t \leqslant 0, \\ \|e^{A^-_\eta t}\|_{L(\mathcal{Z}^-)} &\leqslant \bar{M} e^{-\beta t}, \quad t \geqslant 0. \end{aligned} \tag{2.7}$$

We consider the family $\{S_\eta : t \geq 0\}$ of semigroups that converge to $\{S_0(t) : t \geq 0\}$ as $\eta \to 0$. We assume that the equilibria of $\{S_0(t) : t \geq 0\}$ give rise to the same number of hyperbolic equilibria for $\{S_\eta(t) : t \geq 0\}$ for η sufficiently small, and that the corresponding stable and unstable manifolds change continuously.

The intersection of the unstable manifold with a neighborhood of y^*_η is termed the local unstable manifold, which we write $W^{\mathrm{u}}_{\mathrm{loc}}(y^*_\eta)$. The existence of local unstable manifolds as graphs near a hyperbolic equilibrium is well-known (see [5] for a proof).

Lemma 2.2. *Let y^*_η be stationary solution of* (2.4)*; that is, a solution of* (2.5)*. Assume* (2.3) *and that y^*_η is hyperbolic in the sense of Definition* 2.9*. If Q_η is the projection in Definition* 2.9*, there is a neighborhood V of y^*_η (which may be chosen independently of all suitably small η) and a function $\Sigma_\eta : R(Q_\eta) \to Ker(Q_\eta)$ such that*

$$W^u(y^*_\eta) \cap V = \{y^*_\eta + Q_\eta(u) + \Sigma_\eta(Q_\eta u) : u \in \mathcal{Z}\} \cap V,$$

$$\sup_{u \in V} \|Q_\eta(u) - Q_0 u\| + \|\Sigma_\eta(Q_\eta u) - \Sigma_0(Q_0)u\|_{\mathcal{Z}} \stackrel{\eta \to 0}{\longrightarrow} 0$$

and there exists $\gamma > 0$ such that, for any $u_0 \in V$ and as long as $S(t)u_0 \in V$,

$$\|(I - Q_\eta)(S_\eta(t)u_0) - \Sigma^u_\eta(Q_\eta(S_\eta(t)u_0))\|_{\mathcal{Z}} \leq Me^{-\gamma t}.$$

In this case, if one only assumes that $\{S_0(t) : t \geq 0\}$ has a gradient-like attractor $\mathcal{A}_0$, all its stationary solutions are hyperbolic and $\{S_\eta(t) : t \geq 0\}$ has an attractor $\mathcal{A}_\eta$, then it is shown (see for example [5]) that the attractors $\mathcal{A}_\eta$ of $\{S_\eta(t) : t \geq 0\}$ behave continuously as $\eta \to 0$; that is,

$$\mathrm{dist}_H(\mathcal{A}_\eta, \mathcal{A}_0) \to 0 \quad \text{as} \quad \eta \to 0,$$

where $\mathrm{dist}_H(X, Y) = \max[\mathrm{dist}(X, Y), \mathrm{dist}(Y, X)]$. In fact this continuity result is proved by showing that the attractors for the perturbed problem *contain* (possibly strictly) the union of the unstable manifolds of the hyperbolic equilibria, while the remaining part of the attractor for the perturbed problem (if it exists) is small.

The main result in Carvalho et al. [6] is that under the additional assumption that the unperturbed problem is *gradient* (has a Lyapunov function), then there is no 'remainder', and the attractor has the same structure as the autonomous attractor; that is

$$\mathcal{A}_\eta = \bigcup_{j=1}^{n} W^{\mathrm{u}}(y^*_{i,\eta}),$$

where the $y^*_{i,\eta}$ are the hyperbolic equilibria corresponding to the hyperbolic equilibria y^*_j in the original problem. We also show in [6] that every solution converges to one of the $y^*_{i,\eta}$ as $t \to \pm\infty$.

The above remarks together with Theorem 2.2 generalize the results in [6] to the class of gradient-like semigroups.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Next we show, following [2] (see also [27]), that the global attractor of a gradient-like semigroup is also an exponential attractor. This proof will be extended in Section 3 to pullback attractors of gradient-like non-autonomous evolution processes.

Suppose $\{S(t) : t \geq 0\}$ is Lipschitz continuous. In particular, there exist $c = c(B_0)$ and $L > 0$ such that

$$||S(t)u - S(t)v|| \leq ce^{Lt}||u - v||, \text{ for all } u, v \in B_0. \tag{2.8}$$

Theorem 2.4. *If $\{S(t) : t \geq 0\}$ is a gradient-like semigroup with a global attractor $\mathcal{A}$, with all its stationary solutions being hyperbolic and such that* (2.8) *holds, then*

$$\mathcal{A} = \bigcup_{j=1}^{n} W^{\mathrm{u}}(y_j^*) \tag{2.9}$$

and there exists $\gamma > 0$ such that, for all $B \subset \mathcal{Z}$ bounded there is a $c(B) > 0$ such that

$$\mathrm{dist}(S(t)u_0, \mathcal{A}) \leq c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{2.10}$$

To prove this theorem we need the following important lemmas

Lemma 2.3. *Let $\{S(t) : t \geq 0\}$ be a gradient-like nonlinear semigroup with finitely many stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ and a global attractor $\mathcal{A}$. Given $\delta < \delta_0$ and $B \subset \mathcal{Z}$ bounded, there is a $T = T(\delta, B) > 0$ such that $\{S(t)u_0 : 0 \leq t \leq T\} \cap \cup_{i=1}^n B_\delta(z_i^*) \neq \emptyset$ for all $u_0 \in B$.*

Proof: We argue by contradiction. Assume that there is a sequence u_k in B and a sequence of positive numbers t_k (with $t_k \stackrel{k\to\infty}{\longrightarrow} \infty$) such that $\{S(t)u_k : 0 \leq t \leq t_k\} \cap \cup_{i=1}^n B_\delta(z_i^*) = \emptyset$. Extracting subsequences we have that there is a global solution $\xi : \mathbb{R} \to \mathcal{Z}$ such that $S(t + \frac{t_k}{2})u_k \to \xi(t)$ uniformly in compact subsets of $\mathbb{R}$. Clearly $\xi(t) \in \mathcal{A}$, for all $t \in \mathbb{R}$ and $\xi(t) \notin \cup_{i=1}^n B_\delta(z_i^*)$ for all $t \in \mathbb{R}$ which contradicts (G1).□

Lemma 2.4. *Let $\{S(t) : t \geq 0\}$ be a gradient-like nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ which has a global attractor $\mathcal{A}$. Given $0 < \delta < \delta_0$, there is a $\delta' > 0$ such that, if for some $1 \leq i \leq n$, $\|u_0 - y_i^*\|_{\mathcal{Z}} < \delta'$ and, for some $t_1 > 0$, $\|S(t_1)u_0 - y_i^*\|_{\mathcal{Z}} \geq \delta$, then $\|S(t)u_0 - y_i^*\|_{\mathcal{Z}} > \delta'$ for all $t \geq t_1$.*

Proof: Assume that, for some $1 \leq i \leq n$, there is a sequence u_k in $\mathcal{Z}$ with $\|u_k - z_i^*\|_{\mathcal{Z}} < \frac{1}{k}$ and sequences $t_k < \tau_k$ of positive numbers such that $\|S(t_k)u_k - z_i^*\|_{\mathcal{Z}} \geq \delta$ and $\|S(\tau_k)u_k - z_i^*\|_{\mathcal{Z}} < \frac{1}{k}$. That contradicts (G2).□

Proof of Theorem 2.4: Clearly (2.9) holds.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

To prove (2.10) we first choose $\delta < \delta_0$ such that $B_\delta(y_i^*) \subset V_i$ and V_i is the neighborhood given in Lemma 2.2 for y_i^*. From Lemma 2.4, for all suitably small δ, there exists $\delta' = \delta'(\delta) < \delta$ such that, if $u_0 \in B_{\delta'}(y_i^*)$ and for some $t_1 > 0$

$$S(t_1)u_0 \notin B_\delta(y_i^*),$$

then

$$S(t)u_0 \notin B_{\delta'}(y_i^*), \text{ for all } t \geq t_1.$$

Now, let B be a bounded subset of $\mathcal{Z}$ and B_0 be a closed ball centered at $z = 0$ that contains B and $\cup_{z\in\mathcal{A}} B_\delta(z)$. From Lemma 2.3, there exists $T = T(\delta', B_0)$ such that, for all $u_0 \in B_0$

$$S(t)u_0 \in \mathcal{O}_{\delta'} = \bigcup_{i=1}^{n} B_{\delta'}(y_i^*) \text{ for some } t \leq T.$$

Thus, given $u_0 \in B_0$, there are sequences $\{t_-^i\}_{i=0}^M$ and $\{t_+^i\}_{i=0}^M$, $M \leq n$ and $\{y_i^*\}_{i=0}^M$ such that

$$t_-^0 \leq T, \qquad t_-^i - t_+^{i-1} \leq T, \quad 1 \leq i \leq M \qquad t_+^M = +\infty$$

for which $S(t)u_0 \in \mathcal{O}_\delta(y_i^*)$, for all $t \in [t_-^i, t_+^i]$ and $i \in \{0, 1, \ldots, M\}$. Then, by Lemma 2.2

$$\mathrm{dist}(S(t)u_0, \mathcal{A}) \leq c_0(B_0)e^{-\gamma t}, \text{ for all } t \in [t_-^i, t_+^i].$$

On the other hand, for $t \in [t_+^{i-1}, t_-^i]$, $t = s + t_+^{i-1}$, for some $s \leq T$, and using (2.8) we have that

$$\begin{aligned}
\mathrm{dist}(S(t)u_0, \mathcal{A}) &= \mathrm{dist}(S(s + t_+^{i-1})u_0, \mathcal{A}) \\
&= \mathrm{dist}(S(s)S(t_+^{i-1})u_0, S(s)\mathcal{A}) \\
&\leq c_1(B_0)e^{kT}\mathrm{dist}(S(t_+^{i-1})u_0, \mathcal{A}) \\
&\leq c_1(B_0)e^{kT}c_0(B_0)e^{-\gamma t_+^{i-1}} = c(B_0)e^{-\gamma t}. \square
\end{aligned}$$

We close this section with a result on exponential attractors for generalized gradient-like semigroups. The main difference here is that the exponential attraction of local unstable manifolds is not known for such general situation and must be assumed. When the isolated invariant sets are normally hyperbolic invariant manifolds there are results indicating that hypothesis of the theorem below will be satisfied and will be explored in a future work where more applications to differential equations will be considered.

THEOREM 2.5. *Let the assumptions of Theorem* 2.3 *be satisfied. Then there exists* $\eta_0 > 0$ *such that*

$$\mathcal{A}_\eta = \cup_{i=1}^n W^u(\Xi_{i,\eta}^*), \ \forall \eta \in [0, \eta_0].$$

If we also assume (2.8) *and that there is* $\gamma > 0$ *and, for each* $1 \leq i \leq n$*, a neighborhood* V_i *of* $\Xi^*_{i,\eta}$ *such that, for any* $u_0 \in V_i$*, as long as* $S_\eta(t)u_0 \in V_i$

$$\mathrm{dist}(S_\eta(t)u_0, W^u_\eta(\Xi^*)) \leq Me^{-\gamma t},$$

for bounded set $B \subset \mathcal{Z}$*, there is a constant* $c(B) > 0$ *such that*

$$\mathrm{dist}(S_\eta(t)u_0, \mathcal{A}_\eta) \leq c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{2.11}$$

The proof of this result is identical to the proof of Theorem 2.4 using the following generalizations of Lemma 2.3 and Lemma 2.4 with identical proofs.

Lemma 2.5. *Let* $\{S(t) : t \geq 0\}$ *be a generalized gradient-like nonlinear semigroup with finitely many isolated invariant sets* $\mathcal{E} = \{\Xi^*_1, \cdots \Xi^*_n\}$ *and a global attractor* $\mathcal{A}$*. Given* $\delta < \delta_0$ *and* $B \subset \mathcal{Z}$ *bounded, there is a* $T = T(\delta, B) > 0$ *such that* $\{S(t)u_0 : 0 \leq t \leq T\} \cap \cup_{i=1}^n \mathcal{O}_\delta(\Xi^*_i) \neq \emptyset$ *for all* $u_0 \in B$*.*

Lemma 2.6. *Let* $\{S(t) : t \geq 0\}$ *be a generalized gradient-like nonlinear semigroup with a global attractor* $\mathcal{A}$ *and finitely many isolated invariant sets* $\mathcal{E} = \{\Xi^*_1, \cdots \Xi^*_n\}$ *in* $\mathcal{A}$*. Given* $0 < \delta < \delta_0$*, there is a* $\delta' > 0$ *such that, if for some* $1 \leq i \leq n$*,* $\mathrm{dist}(u_0, \Xi^*_i) < \delta'$ *and, for some* $t_1 > 0$*,* $\mathrm{dist}(S(t_1)u_0, \Xi^*_i) \geq \delta$*, then* $\mathrm{dist}(S(t)u_0, \Xi^*_i) > \delta'$ *for all* $t \geq t_1$*.*

3. NON-AUTONOMOUS GRADIENT-LIKE DYNAMICAL SYSTEMS

The definition of a gradient-like semigroup given in Section 2 can be extended to an evolution process. In this section we prove that all properties observed for gradient-like semigroups can be extended also to gradient-like evolution processes.

Note that, if $T(t,\tau) = S(t-\tau)$, $t \geq \tau$, with $\{S(t) : t \geq 0\}$ being a nonlinear semigroup, Definition 1.4 and Definition 1.5 generalizes Definitions 2.2 and 2.3, by also including the possibility of ξ^*_i are global isolated solutions.

Definition 3.1. Let $\{S(t) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear semigroup with a finite number of isolated global solutions $\mathcal{S} = \{\xi^*_1, \cdots \xi^*_n\}$ and assume that it has a global attractor $\mathcal{A}$. A *homoclinic structure* in $\mathcal{A}$ is a set $\{\xi^*_1, \cdots, \xi^*_k\} \subset \mathcal{S}$ and a set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ in $\mathcal{A}$ such that, making $\xi^*_{k+1} := \xi^*_1$,

$$\lim_{t\to-\infty} \xi_i(t) = \xi^*_i(t), \quad \lim_{t\to+\infty} \xi_i(t) = \xi^*_{i+1}(t), \quad 1 \leq i \leq k.$$

The proof of the following lemma is similar to the proof of Lemma 2.1.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Lemma 3.1. *Let $\{S(t) : t \geq \tau \in \mathbb{R}\}$ be a gradient-like nonlinear semigroup in the sense of Definition 1.5. Then the attractor $\mathcal{A}$ for $\{S(t) : t \geq 0\}$ does not contain any homoclinic structures.*

As in the case of semigroups we prove that a non-autonomous perturbation of a gradient-like nonlinear semigroup is a gradient-like nonlinear evolution process.

Theorem 3.1. *Let $\mathcal{Z}$ be a Banach space, $\eta \in [0,1]$ and $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear evolution process in $\mathcal{Z}$. Assume that there exists a uniform absorbing compact set B_0 (independent of η). Let $\{A_\eta(t) : t \in \mathbb{R}\}$ be the pullback attractor for $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$. Assume that $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, has a finite number of isolated global solutions $\mathcal{S} = \{\xi^*_{1,\eta}, \cdots, \xi^*_{n,\eta}\}$ which behave continuously as $\eta \to 0$ ($\sup_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \|\xi_{i,\eta}(t) - \xi_{i,0}(t)\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$). Assume also that $\|T_\eta(t+\tau,\tau)u - T_0(t+\tau,\tau)u\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$ uniformly for $\tau \in \mathbb{R}$, t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$.*

If $\{T_0(t) : t \geq 0\}$ is a gradient-like nonlinear semigroup, $T_0(t,\tau) = T_0(t-\tau)$, $t \geq \tau$; that is, (H1)-(H2) *are satisfied for $\{T_0(t,\tau) : t \geq \tau \in \mathbb{R}\}$, then there exists $\eta_0 > 0$ such that, for all $\eta \leq \eta_0$, $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is a gradient-like nonlinear process.*

As a immediate consequence of this theorem we obtain the following generalization of the characterization result which generalizes the main one in [6] for perturbations of evolution processes.

Corollary 3.1. *Under the assumptions of Theorem* 3.1*, there exists $\eta_0 > 0$ such that*

$$A_\eta(t) = \cup_{i=1}^n W^u(\xi^*_{i,\eta})(t), \ t \in \mathbb{R} \text{ and } \forall \eta \in [0,\eta_0].$$

We will need the following important lemma (Lemma 3.1 in [6]):

Lemma 3.2. *Let η_k be a sequence of positive numbers such that $\eta_k \to 0$ as $k \to \infty$. Assume that there is a sequence $\eta_k \overset{k\to\infty}{\longrightarrow} 0$, and global solutions $\xi_{\eta_k} : \mathbb{R} \to \mathcal{Z}$ in $\{A_{\eta_k}(t) : t \in \mathbb{R}\}$, $k \in \mathbb{Z}^+$. Then, for any sequence $\{t_k\}$ in $\mathbb{R}$, there is a subsequence which we again denote by ξ_{η_k} and a global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in the attractor $\mathcal{A}$ for $\{T_0(t) : t \geq 0\}$ such that*

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k) \to \xi(t) \tag{3.1}$$

uniformly for t in compact subsets of $\mathbb{R}$.

Proof of Theorem 3.1. Note $\xi^*_{i,\eta} \to \xi^*_{i,0} =: \xi^*_i$, uniformly in $\mathbb{R}$, and that there is a $\delta > 0$ such that, for suitably small η, if a solution ξ_η satisfies $\|\xi_\eta(t) - \xi^*_i(t)\|_{\mathcal{Z}} \leq \delta$ for all $t \geq t_0$ and for some $t_0 > 0$, then $\xi_\eta(t) \to \xi^*_{i,\eta}(t)$ as $t \to \infty$.

We argue by contradiction to prove that for all suitably small η, $\{T_\eta(t,\tau) : t \geq 0\}$ satisfies (H1). Assume that there exists a sequence $\eta_k \overset{k\to\infty}{\longrightarrow} 0$ and corresponding global solutions

$\xi_k : \mathbb{R} \to \mathcal{Z}$ in $\{A_{\eta_k}(t) : t \in \mathbb{R}\}$ such that

$$\liminf_{t\to\infty} \operatorname{dist}(\xi_k(t), \xi_i^*(t)) > \delta, \quad 1 \le i \le n. \tag{3.2}$$

Since $\xi_k(t) \to \xi_0(t)$ uniformly in compact subsets of $\mathbb{R}$ and since $\|\xi_0(t) - \xi_i^*(t)\|_{\mathcal{Z}} \to 0$ as $t \to \infty$, for some $1 \le i \le n$, we have that, given $\ell > 0$ there is a $t_\ell > 0$ and $k_\ell \in \mathbb{N}$ such that $\|\xi_k(t_\ell) - \xi_i^*(t_\ell)\|_{\mathcal{Z}} < \frac{1}{\ell}$, for each $k \ge k_\ell$. From (3.2), there exists $t'_\ell > t_\ell$ such that $\|\xi_{k_\ell}(t) - \xi_i^*(t)\|_{\mathcal{Z}} < \delta$ for all $t \in [t_\ell, t'_\ell)$ and $\|\xi_{k_\ell}(t'_\ell) - \xi_i^*(t'_\ell)\|_{\mathcal{Z}} = \delta$. Taking subsequences, if necessary, let $\xi_1(t) = \lim_{\ell\to\infty} \xi_{k_\ell}(t + t_\ell)$. Then, since $t'_\ell - t_\ell \to \infty$ as $\ell \to \infty$, $\|\xi_1(t) - \xi_i^*(t)\|_{\mathcal{Z}} \le \delta$ for all $t \le 0$ and consequently $\xi_1(t) \to \xi_i^*(t)$ as $t \to -\infty$. And $\xi_1(t) \to \xi_j^*(t)$ as $t \to \infty$ with $i \ne j$. From the fact that $\xi_{k(\ell)}(t + t_\ell) \to \xi_1(t)$ uniformly in compact subsets of $\mathbb{R}$ we have that, for each $m \in \mathbb{N}$, there is a time $t_m > 0$ and $k_m \in \mathbb{N}$ such that $\|\xi_k(t_m) - \xi_j^*(t_m)\|_{\mathcal{Z}} < \frac{1}{m}$ for all $k \ge k_m$. Again, from (3.2), there exists $t'_m > t_m$ such that $\|\xi_{k_m}(t) - \xi_j^*(t)\|_{\mathcal{Z}} < \delta$ for all $t \in [t_m, t'_m)$ and $\|\xi_{k_m}(t'_m) - \xi_j^*(t'_m)\|_{\mathcal{Z}} = \delta$. Proceeding exactly as before we obtain a global solution $\xi_2 : \mathbb{R} \to \mathcal{Z}$ such that $\xi_2(t) \to \xi_j^*(t)$ as $t \to -\infty$ and $\xi_2(t) \to \xi_r^*(t)$ as $t \to \infty$ with $r \notin \{i, j\}$. In a finite number of steps we arrive at a contradiction. This proves that there is a $\eta_0 > 0$ such that, for all global solution ξ_η in $\{A_\eta(t) : t \in \mathbb{R}\}$ with $\eta \le \eta_0$, we have that

$$\lim_{t\to\infty} \|\xi_\eta(t) - \xi_i^*(t)\|_{\mathcal{Z}} = 0.$$

To prove that there is a $\eta_1 > 0$ such that, for all global solution ξ_η in $\{A_\eta(t) : t \in \mathbb{R}\}$ with $\eta \le \eta_1$, we have that

$$\lim_{t\to-\infty} ||\xi_\eta(t) - \xi_j^*(t)|| = 0,$$

we proceed exactly in the same manner. This completes the proof that, for all suitably small η, $\{T_\eta(t, \tau) : t \ge \tau \in \mathbb{R}\}$ satisfies (H1).

Let us prove that, for all suitably small η, $\{T_\eta(t, \tau) : t \ge \tau \in \mathbb{R}\}$ satisfies (H2). Again we argue by contradiction. Assume that there is a sequence $\xi_1^*, \cdots, \xi_{p+1}^*$ in $\mathcal{S}$, a sequence $\eta_k \to 0$, global solutions ξ_k in $\{A_{\eta_k} : t \in \mathbb{R}\}$, and times $t_1^k, \tau_1^k, \cdots, t_p^k, \tau_p^k$ with $t_i^k > \tau_i^k$, $1 \le i \le p$, such that

$$\|\xi_k(\tau_i^k) - \xi_i^*(\tau_i^k)\| < \frac{1}{k}, \quad \|\xi_k(t_i^k) - \xi_{i+1}^*(t_i^k)\| < \frac{1}{k}, \quad 1 \le i \le p \text{ and } \xi_1^* = \xi_{p+1}^*$$

Proceeding as in the proof of (H1) we construct a homoclinic structure for $\{T_0(t, \tau) : t \ge \tau \in \mathbb{R}\}$ and arrive at a contradiction.□

Next, we introduce the linearization of a nonlinear process around a global solution. The aim is to show lower semicontinuity of attractors for gradient-like processes for which all global isolated solutions are hyperbolic and which are perturbation of a nonlinear semigroup (which does not follow yet from the previous result). We also pursue a property that the pullback attractor obtained is an exponential pullback attractor and exponential forward attractor.

Definition 3.2. We say that a linear process $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy with exponent ω and constant M if there exists a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset L(\mathcal{Z})$ such that

1. $Q(t)U(t,s) = U(t,s)Q(s)$, for all $t \geq s$.

2. The restriction $U(t,s)_{|R(Q(s))}$, $t \geq s$ is an isomorphism from $R(Q(s))$ into $R(Q(t))$; we denote its inverse by $U(s,t) : R(Q(t)) \to R(Q(s))$.

3. There are constants $\omega > 0$ and $M \geq 1$ such that

$$\begin{aligned} &\|U(t,s)(I - Q(s))\|_{L(\mathcal{Z})} \leq Me^{-\omega(t-s)} \quad t \geq s \\ &\|U(t,s)Q(s)\|_{L(\mathcal{Z})} \leq Me^{\omega(t-s)}, \quad t \leq s. \end{aligned} \tag{3.3}$$

Assume that $f_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is continuously differentiable, globally Lipschitz in the second variable and bounded. Assume also that, for each $r > 0$,

$$\sup_{\|z\| \leq r} \sup_{t \in \mathbb{R}} \{\|f_\eta(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f'_\eta(t,z) - f'_0(z)\|_{\mathcal{L}(\mathcal{Z})}\} \xrightarrow{\eta \to 0} 0, \tag{3.4}$$

where $f'(t,z) \in \mathcal{L}(\mathcal{Z})$ denotes the derivative of f_η with respect to the second variable in (t,z). Let $t \mapsto T_\eta(t,\tau)y_0$ be the solution of

$$T_\eta(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s,\tau)y_0)\, ds. \tag{3.5}$$

Suppose that ξ^*_η is a global solution of (3.5). If $B_\eta(t) = f'_\eta(t, \xi^*_\eta(t))$, consider the linear evolution process $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defined by

$$U_\eta(t,\tau)z_0 = e^{\mathcal{A}(t-\tau)}z_0 + \int_\tau^t e^{\mathcal{A}(t-s)} B_\eta(s) U_\eta(s,\tau) z_0\, ds. \tag{3.6}$$

Definition 3.3. Let $\xi^*_\eta : \mathbb{R} \to \mathcal{Z}$ be a global solution of (3.5). We say that ξ^*_η is hyperbolic if the linear evolution process $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defined by (3.6) has exponential dichotomy. A global solution that has an exponential dichotomy will be called a *global hyperbolic solution*.

Definition 3.4. The unstable manifold of a global hyperbolic solution ξ^*_η to (3.5) is the set

$$\begin{aligned} W^{\mathrm{u}}_\eta(\xi^*_\eta) = \{\, &(\tau,\zeta) \in \mathbb{R} \times \mathcal{Z} : \text{ there is a backwards solution } z(t,\tau,\zeta) \text{ of (3.5)} \\ &\text{satisfying } z(\tau,\tau,\zeta) = \zeta \text{ and such that } \lim_{t \to -\infty} \|z(t,\tau,\zeta) - \xi^*_\eta(t)\|_{\mathcal{Z}} = 0\}. \end{aligned}$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

The stable manifold of a hyperbolic solution ξ^*_η to (3.5) is the set

$$W^{\rm s}_\eta(\xi^*_\eta) = \{\, (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \text{ there is a forwards solution } z(t,\tau,\zeta) \text{ of } (3.5) \\ \text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to+\infty} \|z(t,\tau,\zeta)-\xi^*_\eta(t)\|_{\mathcal{Z}} = 0\}.$$

The intersection of the unstable (stable) manifold with a neighborhood of the curve $(\cdot,\xi(\cdot))$ in $\mathbb{R}\times\mathcal{Z}$ is called a local unstable (stable) manifold and is denoted by $W^{\rm u}_{\eta,\rm loc}$ ($W^{\rm s}_{\eta,\rm loc}$).

The proof of the following result can be adapted from [5].

Lemma 3.3. *Assume* (3.4) *and let* ξ^*_0 *be hyperbolic global solution of* (3.5) *with* $\eta = 0$. *Then, there is* η_0 *such that* (3.5) *has a unique hyperbolic global solution* ξ^*_η *in a neighborhood of* ξ^*_0 *and* $\sup_{1\le i\le n}\sup_{t\in\mathbb{R}}\|\xi_{i,\eta}(t)-\xi_{i,0}(t)\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$. *If* $\{Q_\eta(t) : t\in\mathbb{R}\}$ *is the family of projections in Definition* 3.2 *there is a neighborhood* V *of* $(\cdot,\xi^*_\eta(\cdot))$ *(which may be chosen independently of all suitably small* η*) and a family of function* $\Sigma_\eta(t,\cdot) : R(Q_\eta(t)) \to Ker(Q_\eta(t))$ *such that*

$$W^u(\xi^*_\eta)\cap V = \{(t,\xi^*_\eta(t)+Q_\eta(t)u+\Sigma_\eta(Q_\eta(t)u) : t\in\mathbb{R},\ u\in\mathcal{Z}\}\cap V,$$

$$\sup_{(t,u)\in V} \|Q_\eta(t)u - Q_0(t)u\|_{\mathcal{Z}} + \|\Sigma_\eta(t,Q_\eta(t)u)-\Sigma_0(Q_0(t))u\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$$

and there exists $\gamma>0$ *such that, for any* $u_0\in V$ *and as long as* $T_\eta(t)u_0\in V$,

$$\sup_{\tau\in\mathbb{R}} \|(I-Q_\eta(t))(T_\eta(t+\tau,\tau)u_0) - \Sigma^u_\eta(t,Q_\eta(t)(T_\eta(t+\tau,\tau)u_0))\|_{\mathcal{Z}} \le Me^{-\gamma t}$$

In this case, if one only assumes that the attractor of $\{T_0(t,\tau) : t\ge\tau\in\mathbb{R}\}$ has a gradient-like pullback attractor $\{\mathcal{A}(t) : t\in\mathbb{R}\}$, all its isolated global solutions are hyperbolic and $\{T_\eta(t,\tau) : t\ge\tau\in\mathbb{R}\}$ has an attractor $\{\mathcal{A}_\eta(t) : t\in\mathbb{R}\}$, then it is shown (see [7]) that the attractors $\{\mathcal{A}_\eta(t) : t\in\mathbb{R}\}$ of $\{T_\eta(t,\tau) : t\ge\tau\in\mathbb{R}\}$ behave continuously as $\eta\to 0$; that is,

$$\sup_{t\in\mathbb{R}} {\rm dist}_H(\mathcal{A}_\eta(t),\mathcal{A}_0(t)) \to 0 \quad \text{as} \quad \eta\to 0.$$

In fact this continuity result is proved by showing that the attractors for the perturbed problem *contain* (possibly strictly) the union of the unstable manifolds of the global hyperbolic solutions. The main result in [6] is that under the additional assumption that the unperturbed problem is *gradient*, then there is no ‘remainder’, and the attractor has the same structure as the autonomous attractor. We also show in [6] that every solution converges to one of the $y^*_{i,\eta}$ as $t\to\pm\infty$.

Next we generalize the exponential rate of attraction in Theorem 2.4 to gradient-like nonlinear evolution process proving Theorem 1.2 (cf. [27]).

Proof of Theorem 1.2: To prove the exponential decay we proceed in three steps:
Step 1. Let η_0 be such that $\{T_\eta(t,\tau) : t \geq \tau\}$ satisfies (H1) and (H2) for all $\eta \leq \eta_0$. It holds that, for all δ small enough, there exists $\delta' = \delta'(\delta) < \delta$ and $\eta_1 > 0$ such that, for each $\eta \leq \eta_0$, if $u_0 \in B_{\delta'}(\xi_i^*(\tau))$, for some $\tau \in \mathbb{R}$ and for some $t_1 > 0$

$$T_\eta(t_1 + \tau, \tau)u_0 \notin B_\delta(\xi_{i,\eta}^*(t_1 + \tau)),$$

then

$$T_\eta(t + \tau, \tau)u_0 \notin B_{\delta'}(\xi_{i,\eta}^*(t + \tau)), \text{ for all } t \geq t_1.$$

We follow the argument in Lemma 2.4. Indeed, if not, there exist $\delta_0 > 0$ and sequences $\eta_k \to 0$, $\tau_k \in \mathbb{R}$, $t_k < t_k'$, and u_k such that, for some of the hyperbolic global solutions ξ_{i,η_k}^*,

- $u_k \in B_{1/k}(\xi_{i,\eta_k}^*(\tau_k))$
- $T_{\eta_k}(t_k + \tau_k, \tau_k)u_k \notin B_{\delta_0}(\xi_{i,\eta_k}^*(t_k + \tau_k))$
- $T_{\eta_k}(t_k' + \tau_k, \tau_k)u_k \in B_{1/k}(\xi_{i,\eta_k}^*(t_k' + \tau_k))$.

Taking subsequences, it follows from Lemma 3.2 that

$$\xi_k(t) := T_{\eta_k}(t + t_k + \tau_k, t_k + \tau_k)(t_k + \tau_k, \tau_k)u_k \to \xi_0(t)$$

uniformly in compact subsets of $\mathbb{R}$ where $\xi_0 : \mathbb{R} \to \mathcal{Z}$ is a solution in $\mathcal{A}$. Consequently, $\xi_0(t) \to y_i^*$ as $t \to -\infty$ and $\xi_0(t) \to y_j^*$ as $t \to +\infty$ with $j \neq i$. Proceeding as in the proof of Theorem 3.1 we arrive at a contradiction.

That is clearly in contradiction with (H2).

Step 2. For every bounded set $B \subset \mathcal{Z}$ and every $\delta > 0$ there exists $\eta_2 > 0$ and $T = T(\delta, B)$ such that, for all $u_0 \in B$ and $\tau \in \mathbb{R}$

$$T_\eta(t + \tau, \tau)u_0 \in O_\delta := \bigcup_{i=1}^{n} B_\delta(y_\eta^*) \text{ for some } t \leq T.$$

Indeed, if not there exist $\varepsilon > 0$ $u_k \in B$, $\eta_k \to 0$, $\tau_k \in \mathbb{R}$ and $t_k \to +\infty$ such that

$$\text{dist}(T_{\eta_k}(t_k + \tau_k, \tau_k)u_k, O_\delta) \geq \varepsilon,$$

for all $t \in [0, t_k]$. Now, if we define

$$\xi_k(t) := T_{\eta_k}(t + \tau + \frac{t_k}{2}, \tau + \frac{t_k}{2})T_{\eta_k}(\tau + \frac{t_k}{2}, \tau)u_k,$$

then

$$\text{dist}(\xi_k(t), O_\delta) \geq \varepsilon,$$

for all $t \in [-t_k/2, t_k/2]$.

Again, we have that there exists $\zeta(t)$ global solution in $\mathcal{A}$ such that

$$\lim_{k\to+\infty}\xi_k(t)=\zeta(t),$$

for all $t\in\mathbb{R}$, uniformly on bounded subsets of $\mathbb{R}$. Thus,

$$\mathrm{dist}(\zeta(t),O_\delta)\geq\varepsilon,$$

for all $t\in\mathbb{R}$, which contradicts (H1).

Step 3. Now, by the two previous steps, there is $\bar\eta>0$ such that, given $u_0\in B_0$ and $\eta\leq\bar\eta$, one can find sequences $\{t_-^i\}_{i=0}^M$ and $\{t_+^i\}_{i=0}^M$, $M\leq n$ and $\{y_i^*\}_{i=0}^M$ such that

$$t_-^0=\tau,\qquad t_+^i-t_-^{i-1}\leq T,\qquad t_-^M=+\infty$$

for which

$$T_{\eta_k}(t+\tau,\tau)u_0\in\mathcal{O}_\delta(y_i^*),$$

for all $t\in[t_+^i,t_-^i]$ and $i\in\{1,\dots,M\}$. Then, by Lemma 3.3

$$\mathrm{dist}(T_{\eta_k}(t+\tau,\tau)u_0,A_{\eta_k}(t+\tau))\leq c_0(B_0)e^{-\gamma t},\ \text{for all } t\in[t_+^i,t_-^i].$$

On the other hand, for $t\in[t_-^{i-1},t_+^i]$, $t=s+t_-^{i-1}$, for some $s\leq T$, and using (1.4) we have that

$$\begin{aligned}\mathrm{dist}(T_{\eta_k}(t+\tau,\tau)u_0,A_{\eta_k}(t+\tau))&=\mathrm{dist}(T_{\eta_k}(s+t_-^{i-1}+\tau,\tau)u_0,A_{\eta_k}(t+\tau))\\&=\mathrm{dist}(T_{\eta_k}(s+t_-^{i-1}+\tau,t_-^{i-1}+\tau)T_{\eta_k}(t_-^{i-1}+\tau,\tau)u_0,T_{\eta_k}(s+t_-^{i-1}+\tau,t_-^{i-1}+\tau)A_{\eta_k}(t_-^{i-1}+\tau))\\&\leq c_1(B_0)e^{kT}\mathrm{dist}(T_{\eta_k}(t_-^{i-1}+\tau,\tau)u_0,A_{\eta_k}(t_-^{i-1}+\tau))\\&\leq c_1(B_0)e^{kT}c_0(B_0)e^{-\gamma t_-^{i-1}}=c(T,B_0)e^{-\gamma t}.\ \square\end{aligned}$$

4. ASYMPTOTICALLY AUTONOMOUS DYNAMICAL SYSTEMS

An asymptotically autonomous (backwards and/or forwards) dynamical system is one of the important examples which can be interpreted as a non-autonomous regular perturbation of an autonomous system (see [3, 21, 15, 20]). In this section we apply our result to these cases, generalizing the results in Section 4 of [6]. In particular, if we suppose that the long-time behaviour in the past and future are unrelated (i.e., there is no uniform regular perturbation of the limit system) we are still able to give a characterization of the associated pullback and forwards attractors. However, the uniform (with respect to the initial time) exponential convergence to these sets will be lost, so that only an exponential pullback attractor or an exponential forwards attractor will exist.

Consider a Banach space $\mathcal{Z}$ and the semilinear problem

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f(t,y) \\ y(\tau) &= y_0, \end{aligned} \tag{4.1}$$

where $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of a C^0-semigroup of bounded linear operators and $f(t,\cdot)$ is a differentiable function that is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ with Lipschitz constant independent of t. If we denote by $t \mapsto T(t,\tau)y_0$ the solution for (4.1), then $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defines a nonlinear process. We will assume that the problem (4.1) has a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

Consider also the semilinear autonomous problem

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_0(y) \\ y(\tau) &= y_0 \in \mathcal{Z}. \end{aligned} \tag{4.2}$$

Assume that (4.2) gives rise to a nonlinear gradient-like semigroup $\{S(t) : t \geq 0\}$ which has a global attractor $\mathcal{A}$.

4.1. Asymptotically Autonomous Problems at $-\infty$

Assume that

$$\lim_{t\to-\infty} \sup_{z\in B(0,r)} \{\|f(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0, \qquad \text{for each } r > 0, \tag{4.3}$$

and that (4.2) has an autonomous attractor $\mathcal{A}_0$.

Theorem 4.1. *Let $f : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ be a differentiable function that satisfies* (4.3). *Consider the initial value problem* (4.1). *Assume that* (4.2) *is gradient-like and $\mathcal{A}_0$ is its global attractor,*

1. Then the attractor $\{\mathcal{A}(\tau) : \tau \in \mathbb{R}\}$ for (4.1) *is given by*

$$\mathcal{A}(\tau) = \cup_{i=1}^{n} W^{\mathrm{u}}(\xi_i^*)(\tau).$$

2. For each globally defined bounded solution $\xi(\cdot)$ of (4.1) *there is i_- with $1 \leq i_- \leq n^-$ such that*

$$\lim_{t\to-\infty} \|\xi(t) - y_{i_-}^*\|_{\mathcal{Z}} = 0. \tag{4.4}$$

3. For every $B \subset \mathcal{Z}$ bounded

$$\mathrm{dist}(T(t,t-\tau)B, \mathcal{A}(t)) \leq c(B)e^{-\gamma\tau}. \tag{4.5}$$

Proof: The proof of (1) is a consequence of Corollary 3.1 if we analyze (4.1) by considering the small non-autonomous perturbations of (4.2) obtained by replacing $f(t,y)$ by

$$f_\nu(t,y) = \begin{cases} f(t,y), & \text{if } t \leq -\nu \\ f(\nu,y), & \text{if } t > -\nu. \end{cases}$$

Indeed, for suitably large ν, there exists a pullback attractor $\{\mathcal{A}_\nu(s) : s \in \mathbb{R}\}$ for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_\nu(t,y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.6}$$

given by $\mathcal{A}_\nu(s) = \cup_{i=1}^n W_\nu^{\mathrm{u}}(\xi_{i,\nu}^*)(s)$. To obtain the pullback attractor for (4.1) we first note that (4.6) and (4.1) coincide for $t \leq \tau \leq -\nu$. Hence $\mathcal{A}(t) = \mathcal{A}_\nu(t)$ for $t \leq -\nu$. To recover $\mathcal{A}(t)$ for $t \geq -\nu$ we only have to define $\mathcal{A}(t) = T(t,\tau)\mathcal{A}(\nu)$, for all $\tau \leq -\nu \leq t$.

Now, (2) is also essentially proved since, by (H1), every global solution approaches one of the equilibria $y_{i\_}^*$ as $t \to -\infty$, so that, in particular, (2) holds.

Finally, Theorem 1.2 gives (3).□

4.2. Asymptotically Autonomous Problems at $+\infty$

Assume that

$$\lim_{t\to+\infty} \sup_{z\in B(0,r)} \{\|f(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0, \qquad \text{for each } r > 0, \tag{4.7}$$

and that (4.2) has an autonomous attractor $\mathcal{A}_0$. We note that the nonlinearity f_0 in this subsection may be different from that in the previous subsection and consequently the attractor $\mathcal{A}_0$ in this subsection may be different from that in the previous one. We assume in addition that is gradient-like; it follows from Theorem 1.2 that $\mathcal{A}_0$ is given by (2.1).

Consider $f_k(t,z)$ the function which coincides with f in $[k,\infty) \times \mathcal{Z}$ and which is equal to $f(k,z)$ for all $t < k$ and $z \in \mathcal{Z}$. Then

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \sup_{z\in B(0,r_0)} \{\|f_k(t,z) - f_0(z)\|_{\mathcal{Z}} + \|(f_k)_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0. \tag{4.8}$$

It has been proved in [6] that the family of attractors for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_k(t,y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.9}$$

behaves upper and lower semicontinuously as $k \to \infty$ with the limit attractor being the attractor for (4.2), i.e.

$$\sup_{t\in\mathbb{R}} \operatorname{dist}(\mathcal{A}_k(t), \mathcal{A}_0) \to 0 \qquad \text{as } \ k \to \infty,$$

where $\mathrm{dist}(A, B)$ is the symmetric Hausdorff distance defined in Section 2.

Let k_0 be such that for $k \geq k_0$ the pullback attractor of (4.9) coincides with the union of the unstable manifolds of all those $\{\xi^*_{i,k}\}$ with $\sup_{t\in\mathbb{R}} \|\xi^*_{i,k}(t) - y^*_i\|_{\mathcal{Z}} \to 0$ as $k \to \infty$. Define $\mathcal{A}^+(t) = \mathcal{A}_{k_0}(t)$ for $t \geq k_0$. Note that $\mathcal{A}^+(t)$ is in fact the forwards image of the global attractor of the autonomous system $\dot{y} = \mathfrak{B}y + f(k_0, y)$ under the non-autonomous process $T(t, \tau)$.

Then we have the following result:

Theorem 4.2. *Assume* (4.8). *Then, there is a* $t_0 \in \mathbb{R}$ *and a time dependent forwards attractor* $\{\mathcal{A}^+(t) : t \geq t_0\}$ *for* (4.1). *Moreover, if* $T(t, \tau)$ *is gradient-like, it holds that, for* τ *big enough and* $B \subset \mathcal{Z}$ *bounded*

$$\mathrm{dist}(T(t + \tau, \tau)u_0, \mathcal{A}^+(t + \tau)) \leq c(B)e^{-\gamma t}, \ \textit{for all } u_0 \in B.$$

5. FURTHER COMMENTS AND OPEN PROBLEMS

We have proved in Section 3 that a non-autonomous perturbation of a gradient-like semigroup is a gradient-like evolution process. It remains open if the perturbation of a gradient-like evolution process is again a gradient-like evolution process. Lemma 3.2 is the crucial fact to the extension of the results in Section 2 to non-autonomous perturbation of gradient-like semigroups. The fact that an equivalent lemma does not immediately hold for perturbations of gradient-like nonlinear processes is what impair us to extend the results of Section 2 to perturbations of gradient-like nonlinear processes.

It is also true that (we will pursue that in a future work) if the attractor of a nonlinear semigroup is the union of the unstable manifold of a finite number of *normally hyperbolic* global solutions, then it is an exponential attractor. If, in addition, the nonlinear semigroup is gradient-like, a non-autonomous perturbation of it will have an exponential pullback attractor given as union of unstable manifolds of hyperbolic global solutions. If the perturbation is autonomous, that result together with the lower semicontinuity of attractors will follow from the results of [4] whereas if the perturbation is non-autonomous is yet to be proved.

Note also that our notion of gradient-like evolution process given in Definition 1.5 resembles the concept of *Morse decomposition* ([11, 19, 25]).

Let that $\{S(t) : t \geq 0\}$ be a nonlinear semigroup. In this case, a more general notion of gradient-like nonlinear semigroup can be written as

Definition 5.1. Let $\mathcal{Z}$ be a Banach space and $\{S(t) : t \geq 0\}$ be a nonlinear semigroup in $\mathcal{Z}$. Assume that there exists an absorbing compact set B_0. Let $\mathcal{A}$ be the global attractor for $\{S(t) : t \geq 0\}$. We say that $\{S(t) : t \geq 0\}$ is a gradient-like nonlinear semigroup if the following two hypotheses are satisfied:

(G1') There exists a finite number of isolated global solutions $\{\xi_i^* : \mathbb{R} \to \mathcal{Z} : 1 \le i \le n\}$ such that, any global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\{A(t) : t \in \mathbb{R}\}$ satisfies

$$\lim_{t\to-\infty} \operatorname{dist}(\xi(t), \Gamma_i) = 0 \text{ and } \lim_{t\to\infty} \operatorname{dist}(\xi(t), \Gamma_j) = 0,$$

for some $1 \le i, j \le n$, where $\Gamma_i = \{\xi_i^*(t) : t \in \mathbb{R}\}$, $1 \le i \le n$.

(G2') $\mathcal{S} = \{\xi_1^*, \cdots, \xi_n^*\}$ does not contain any chain recurrent solution (see Definition 1.4).

On the other hand, a Morse decomposition of an attractor is defined as follows

Definition 5.2. Let $\{S(t) : t \ge 0\}$ be a nonlinear semigroup with a global attractor $\mathcal{A}$. We say that $\mathcal{A}$ has a *Morse decomposition* if the following holds:

(M1) There is a finite number of non-empty, compact and disjoint invariant sets $\Gamma_1, \cdots, \Gamma_n$ in $\mathcal{A}$. The sets Γ_i, $1 \le i \le n$, are called the Morse sets.

(M2) The $\omega-$limit of each $z \in \mathcal{Z}$ is contained in one of the invariant sets Γ_i, $1 \le i \le n$.

(M3) For each $z \in \mathcal{A}$, either $z \in \cup_{i=1}^n \Gamma_i$ or, for each global solution ϕ through z there are $i = i(z)$ and $j = j(z)$ with $1 \le i < j \le n$ such that

$$\lim_{t\to-\infty} \operatorname{dist}(\phi(t), \Gamma_i) = 0 \text{ and } \lim_{t\to\infty} \operatorname{dist}(\phi(t), \Gamma_j) = 0.$$

It is not difficult to see that a gradient-like semigroup in the sense of Definition 5.1 has a global attractor which has a Morse decomposition in the sense of Definition 5.2.

We can also define

Definition 5.3. Let $\{S(t) : t \ge 0\}$ be a nonlinear semigroup with a finite number of isolated global solutions $\mathcal{S} = \{\xi_1^*, \cdots \xi_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. A *homoclinic structure* in $\mathcal{A}$ is a sequence $\{n_1, \cdots, n_k\} \subset \{1, \cdots, n\}$, $\{\xi_{n_1}^*, \cdots, \xi_{n_k}^*\} \subset \mathcal{S}$ and a set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \le i \le k\}$ in $\mathcal{A}$ such that, making $\xi_{k+1}^* := \xi_1^*$,

$$\lim_{t\to-\infty} \operatorname{dist}(\xi_i(t), \Gamma_i), \quad \lim_{t\to+\infty} \operatorname{dist}(\xi_i(t), \Gamma_{i+1}), \quad 1 \le i \le k.$$

If in Definition 5.1 we replace (G1') by

(G2") The attractor $\mathcal{A}$ does not have any homoclinic structure,

then the new definition gives the same class of gradient-like nonlinear semigroups. Note that our results could have been written in the more general context of Morse decomposition of attractors for semigroups and processes. However, we have chosen to write the results by replacing the Morse sets by global solutions, which we think makes the arguments more clear.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

On the other hand, in the non-autonomous context we have given a definition of a gradient-like nonlinear process. We have proved that non-autonomous perturbations of autonomous gradient-like nonlinear semigroups are gradient-like nonlinear process. That gives a *Morse decomposition for nonlinear processes.* This is, to our knowledge, the first example for which a Morse decomposition for pullback attractor has been given. Finally, for semigroups, we have also proved that Morse decomposition is stable under perturbations. For processes, we conjecture that it is also the case.

REFERENCES

1. J.M. Arrieta, A.N. Carvalho, J.A. Langa, A. and Rodríguez-Bernal, Continuity Attractors for Non-Gradient Dynamical Systems under Singular Perturbations, preprint.
2. A. V. Babin and M. I. Vishik, *Attractors in Evolutionary Equations* Studies in Mathematics and its Applications **25**, North-Holland Publishing Co., Amsterdam, 1992.
3. J. M. Ball, On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations, *J. Differrential Equations*, **27** (1978), 224–265.
4. P. W. Bates, K. Lu and C. Zeng, Existence and Persistende of Invariant Manifolds for Semiflows in Banach Spaces, *Memoirs of the American Mathematical Society* **135**, 1998.
5. A. N. Carvalho and J. A. Langa, The existence and continuity of stable and unstable manifolds for semilinear problems under non-autonomous perturbation in Banach spaces, *J. Differential Equations*, **233** (2007), 622-653.
6. A. N. Carvalho, J. A. Langa and J.C. Robinson, and A. Suárez, Characterization of non-autonomous attractors of a perturbed gradient system, *J. Differential Equations*, **236** (2007), 570–603.
7. A. N. Carvalho, J. A. Langa and J.C. Robinson, Lower semicontinuity of attractors for non-autonomous dynamical systems, submitted.
8. D. Cheban, P. E. Kloeden and B. Schmalfuß, The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems, *Nonlinear Dyn. Syst. Theory*, **2** (2002), 125–144.
9. V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, Colloquium Publications **49**, American Mathematical Society, 2002.
10. H. Crauel, A. Debussche, and F. Flandoli, Random attractors. *J. Dyn. Diff. Eqn.*, **9** (1997), 397–341.
11. C. Conley, Isolated invariant sets and the Morse index. CBMS Regional Conference Series in Mathematics, **38**, American Mathematical Society, Providence, R.I., 1978.
12. C. Conley, The gradient structure of a flow I, *Ergodic Theory Dynam. Systems*, **8*** (1988), 11–26.
13. J. K. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs **25**, American Mathematical Society, 1989.
14. J. K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl.*, **154** (1989), 281–326.
15. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, 1981.
16. P. E. Kloeden, Pullback attractors in nonautonomous difference equations, *J. Differ. Equations Appl.*, **6** (2000), 33–52.
17. J. A. Langa, J. C. Robinson, A. Suárez and A. Vidal-López, The stability of attractors for non-autonomous perturbations of gradient-like systems, *J. Differential Equations*, **234** (2007), 607–625.
18. J. A. Langa, J. C. Robinson, A. Rodríguez-Bernal, A. Suárez and A. Vidal-López, Existence and nonexistence of unbounded forwards attractor for a class of non-autonomous reaction diffusion equations, *Discrete Contin. Dyn. Syst.*, **18** (2007), 483–497.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

19. J. Mallet-Paret, Morse decompositions for delay-differential equations, *J. Differential Equations*, **72** (1988), 270–315.

20. L. Markus, Asymptotically autonomous differential systems, *Ann. of Math. Stud.*, **36** (1956), 17–29.

21. K. Mischaikow, H. Smith and H. R. Thieme, Asymptotically autonomous semiflows: chain recurrent and Lyapunov functions, *Trans. Amer. Math. Soc.*, **347** (1995), 1669–1685.

22. J.C. Robinson, *Infinite-Dimensional Dynamical Systems: An introduction to dissipative parabolic PDEs and the theory of global attractors* Cambridge University Press, Cambridge UK, 2001.

23. A. Rodríguez-Bernal and A. Vidal–López, Existence, uniqueness and attractivity properties of positive complete trajectories for non-autonomous reaction-diffusion problems *Discrete Contin. Dyn. Syst.*, **18** (2007), 537–567.

24. B. Schmalfuß, Attractors for the non-autonomous dynamical systems. *International Conference on Differential Equations* **1-2** (2000), 684–689.

25. G. R. Sell and Y. You, *Dynamics of evolutionary equations* Applied Mathematical Sciences **143** Springer-Verlag, New York, 2002.

26. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* (Second edition) Springer-Verlag, Berlin, 1996.

27. M.I. Vishik and S.V. Zelik, Regular attractors and their nonautonomous perturbations, preprint.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC