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The classification of reversible-equivariant steady-state bifurcations on self-dual spaces

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In this paper we adapt Singularity theory methods for the classification of reversible-equivariant steady-state bifurcations depending on one real parameter. We assume that the group of symmetries and reversing symmetries is a compact Lie group Γ, and the equivalence is defined in order to preserve these symmetries and reversing symmetries in the normal forms and their unfoldings. When the representation of Γ is self-dual, we show that the classification can be reduced to the standard equivariant context. In this case, we establish a one-to-one association between the classification of bifurcations in the reversible-equivariant context and the classification of purely equivariant bifurcations related to them. As an application of the results, we obtain the classification of self-dual representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbf{D}_4$ on the plane. May, 2007 ICMC-USP

Key Words: steady-state, reversible-equivariant germ, singularities, classification, self-dual representations.

1. INTRODUCTION

Bifurcation of steady states and periodic solutions of differential systems under the variation of a distinguished paramenter has been studied in the last years by many authors in many different contexts. In this paper we adapt methods of Singularity theory to classify one-parameter steady-state bifurcations in systems of ordinary differential equations

$$\dot{x} + g(x, \lambda) = 0, \tag{1}$$

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defined on a finite-dimensional vector space V of state variables, where $g : V \times \mathbb{R} \to V$ commutes and anti-commutes with a linear action of a compact Lie group Γ. Dynamical systems with such property, that is, in the presence of symmetries (equivariances) and reversing symmetries (reversibilities) are called *reversible-equivariant* systems. We note that the simmetries form a normal subgroup H of Γ, and in the presence of reversibilities ($H \neq \Gamma$), then H is a subgroup of Γ of index 2.

There are a lot of physical models where symmetries and reversing symmetries appear and whose behaviour can be studied in a systematic way by means of representation theory of Lie groups. In this work, we assume that the group is compact. In terms of the dynamics, recall that both symmetries and reversing symmetries take trajectories into trajectories, the first ones preserving direction, whereas the others revert direction. The general forms of vector fields on the plane are obtained in [2] for different actions of various groups. A number of examples are presented therein with the help of the software P4 [1], comparing the phase portraits of each $\Gamma-$reversible-equivariant vector field with its associated standard $\Gamma-$equivariant problem, namely for which the reversing symmetries of the original problem are changed to act as symmetries; we call it standard because this is when we consider the same action of Γ on the source and target.

In recent years there has been a significant development of reversible-equivariant dynamics, but the study of local bifurcations has been less intense, and most papers on the subject restrict attention to purely reversible systems (with no equivariance) via equivariant transversality theory or Birkhoff normal form; see [3, 4, 12, 14, 15, 17, 19, 21, 24]. This paper is devoted to the systematic study of steady-state bifurcations in the presence of both reversibility and equivariance by Singularity theory methods. The starting point of this study is the observation that a Γ-reversible-equivariant problem of type (1) can be viewed, with appropriate actions of Γ on the source and target, as a Γ-equivariant problem. More specifically, the action of Γ on the target is the dual action of Γ on the source (see Subsection 2.2). This naturally leads the analysis to be separated into two cases, the so-called self-dual and non self-dual cases (Definition 2.1). In the self-dual case, there exists a $\Gamma-$reversible-equivariant linear isomorphism $L : V \to V$ which establishes a correspondence between the classification of $\Gamma-$reversible-equivariant problems and the classification of their associated standard $\Gamma-$equivariant problems. The existence of such isomorphism implies that the bifurcation equations are preserved. On the other hand, stability of solutions is not preserved in general. In fact, it is well known that in standard $\Gamma-$equivariant steady-state bifurcations, the action of Γ is generically absolutely irreducible, so asymptotically stable solutions are expected, whereas in the presence of reversibilities a stable solution may occur only if the reversing symmetries are out of its isotropy (as a consequence of [19, Lemma 1.1]). In this direction, the isomorphism L plays an important role in the deduction of stability (or instability) for one case from the other. This is illustrated in the two examples we present in Subsections 4.2 and 4.3. The non self-dual case is more subtle and the general Singularity method developed in Subsection 3.2 must be applied. In [4] the authors consider a particular class of non self-dual problems, called "separable", for which the bifurcations can still be analysed via a reduction to another associated purely equivariant problem.

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As mentioned above, a $\Gamma-$reversible-equivariant bifurcation problem is recognized as a $\Gamma-$equivariant one, by considering the same vector space on the source and target and the action of Γ on the target as the dual of the action on the source. In Subsection 3.1 we adapt the Singularity theory for the classification of one-parameter Γ-equivariant germs $g : (V \times \mathbb{R}, 0) \to W$, for any finite-dimensional vector spaces V and W and for arbitrary actions of Γ on those spaces. The reversible-equivariant case (self-dual or not) falls then into a particular class of such problem.

The remainder of this paper is organized as follows. In Section 2, we introduce the notation and summarize basic concepts. Section 3 is devoted to the theory of Singularities for the classification of reversible-equivariant bifurcation problems. In Section 4 we use the results given in the previous section to establish the classification of reversible-equivariant bifurcations on self-dual spaces and present the classifications for self-dual $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbf{D}_4-$reversible-equivariant bifurcation problems on the plane (Subsections 4.2 and 4.3, respectively). For both examples, by the existence of a reversible-equivariant isomorphism, the normal forms and their unfoldings are obtained by their associated standard equivariant classifications that appear in [20] and [11] respectively. Every solution of the self-dual $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbf{D}_4-$reversible-equivariant mapping is also a solution for the associated standard problem (and vice-versa), but with distinct stability. For the two examples we discuss in detail the effect of that isomorphism in this change of stabilities.

2. PRELIMINARIES

In this section we present the definitions, some basic results concerning the reversible-equivariant theory and the representation theory of compact Lie groups.

2.1. Reversible-equivariant theory

Let Γ be a compact Lie group acting linearly on a finite-dimensional vector space V. Let

$$\dot{x} + g(x, \lambda) = 0 \tag{2}$$

be a system of ordinary differential equations, where $g : (V \times \mathbb{R}, 0) \to V$ is a one-parameter germ of smooth mapping at the origin. Throughout we work with smooth germs at the origin and we assume that the parameter space $\mathbb{R}$ is not affected by the action of Γ.

We recall that a function germ $f : (V \times \mathbb{R}, 0) \mapsto \mathbb{R}$ is $\Gamma-$invariant if

$$f(\gamma x, \lambda) = f(x, \lambda), \quad \forall \gamma \in \Gamma, \ \forall x \in V$$

and we denote by $\mathcal{E}_{x,\lambda}(\Gamma)$ the ring of such germs. By the Hilbert-Weyl theorem [10], there exists a finite set of polynomials generating $\mathcal{E}_{x,\lambda}(\Gamma)$.

Consider a homomorphism

$$\sigma : \Gamma \to \{-1, 1\}. \tag{3}$$

The element $\gamma \in \Gamma$ is called a *symmetry* if $\sigma(\gamma) = 1$ and is called a *reversing symmetry* if $\sigma(\gamma) = -1$. We denote by Γ_+ the subset of symmetries of Γ and by Γ_- the subset of reversing symmetries. Γ_+ is a subgroup of Γ, whereas Γ_- is not.

We say that the one-parameter smooth germ g in (2) is $\Gamma-$reversible-equivariant if

$$g(\gamma x, \lambda) = \sigma(\gamma)\gamma g(x, \lambda), \quad \text{for all } \gamma \in \Gamma \text{ and } x \in V. \tag{4}$$

When Γ_- is empty, g in (2) is (purely) $\Gamma-$equivariant. When Γ_- is not empty, the group Γ is called the *reversing symmetry group* of (2).

In the reversible-equivariant context, that is, when g in (2) satisfies (4), we have the differential equation invariant by the group action under the transformation $(x, t) \mapsto (\gamma x, \sigma(\gamma)t)$. Hence, for such systems, both symmetries and reversing symmetries take trajectories onto trajectories, the first ones preserving direction in time and the others reverting direction in time.

2.2. Representation theory

Recall that to an action of Γ on a finite-dimensional vector space V corresponds a representation ρ of the group Γ on V, that is, a linear group homomorphism $\rho : \Gamma \to GL(V)$, where $GL(V)$ is the vector space of invertible linear mappings $V \mapsto V$. We define the representation $\rho_\sigma : \Gamma \to GL(V)$ by $\rho_\sigma(\gamma) = \sigma(\gamma)\rho(\gamma)$, where $\sigma : \Gamma \to \{-1, 1\}$ is the one-dimensional representation given by (3). The representation ρ_σ is called the *dual* of ρ. The action of Γ on V can be written as

$$(\gamma, x) \mapsto \rho_\sigma(\gamma)x \tag{5}$$

and the reversibility-equivariance condition (4) as

$$g\big(\rho(\gamma)x\big) = \rho_\sigma(\gamma)g(x), \quad \forall \gamma \in \Gamma,\ \forall x \in V. \tag{6}$$

Remark 2. 1. Denote by (ρ, V) the linear space V under the representation ρ and (ρ_σ, V) the linear space V under the representation ρ_σ. So (6) means that a reversible-equivariant mapping is an equivariant mapping from (ρ, V) into (ρ_σ, V). This fact is the key for the results of Subsection 3.2.

Denote by $\vec{\mathcal{F}}_{x,\lambda}(\Gamma)$ the module of $\Gamma-$reversible-equivariant germs $g : (V \times \mathbb{R}, 0) \to V$ over the ring $\mathcal{E}_{x,\lambda}(\Gamma)$. It follows from the remark above and from Poènaru Theorem [10] that there exists a finite number of generators for $\vec{\mathcal{F}}_{x,\lambda}(\Gamma)$.

Definition 2.1. A representation ρ of Γ is said to be *self-dual* if it is $\Gamma-$isomorphic to ρ_σ, namely if there exists a linear isomorphism $\Gamma-$reversible-equivariant $L : V \to V$. In this case, we also say that V is a *self-dual* space.

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Section 4 is addressed to the problem of classifying steady-states of $\Gamma-$reversible-equivariant bifurcation problems on self-dual spaces. The existence of the isomorphism L establishes an association of any $\Gamma-$reversible-equivariant mapping $g : V \mapsto V$ with a $\Gamma-$equivariant mapping $h : V \mapsto V$, namely h is $\Gamma-$equivariant if, and only if, $g = Lh$ is $\Gamma-$reversible-equivariant. In particular, their zero sets are the same. Also, we shall use the pullback of L to establish a correspondence between the Singularity theories in both contexts.

3. REVERSIBLE-EQUIVARIANT SINGULARITY THEORY

Our aim in this section is to present the general Singularity theory for the classification of steady-state bifurcations of reversible-equivariant mappings. As mentioned in the previous section, a $\Gamma-$reversible-equivariant mapping on V can be viewed as a $\Gamma-$equivariant mapping on V with appropriate actions on source and target. We now observe that the well known $\Gamma-$equivariant Singularity theory of mappings on V can be adapted in a natural way to obtain the $\Gamma-$equivariant Singularity theory of mappings between distinct vector spaces V and W under arbitrary actions of Γ. This is registered in the next subsection. Subsection 3.2 is then devoted to the statement of the results of reversible-equivariant singularities as a particular case of the results of Subsection 3.1.

3.1. The general equivariant Singularity theory

For Γ a compact Lie group acting linearly on two finite-dimensional vector spaces V and W, we denote by (ρ, V) the vector space V under a representation ρ and by (η, W) the vector space W under a representation η. In this subsection we adapt techniques of the $\Gamma-$equivariant Singularity theory presented in [10] to mappings between (ρ, V) and (η, W). We assume familiarity with basic definitions and concepts presented therein, adopting here the same sequence of ideas.

In what follows, V and W shall always mean (ρ, V) and (η, W) respectively. We say that a one-parameter germ $g : (V \times \mathbb{R}, 0) \mapsto W$ is $\Gamma-$equivariant if

$$g(\rho(\gamma)x, \lambda) = \eta(\gamma)g(x, \lambda), \quad \forall \gamma \in \Gamma, \forall x \in V. \tag{7}$$

We denote by $\vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ the module of $\Gamma-$equivariant map germs $g : (V \times \mathbb{R}, 0) \to W$ over the ring $\mathcal{E}_{x,\lambda}(\Gamma)$ and by $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ the module of $\Gamma-$equivariant map germs $h : (V \times \mathbb{R}, 0) \mapsto V$ over the ring $\mathcal{E}_{x,\lambda}(\Gamma)$. We also denote by $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ the submodule of germs in $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ that vanish at the origin and by $\mathcal{E}_\lambda$ the ring of germs at the origin of functions of λ.

A $\Gamma-$equivariant steady-state bifurcation problem is the study of solutions of $g(x, \lambda) = 0$, where $g : (V \times \mathbb{R}, 0) \mapsto W$ is a $\Gamma-$equivariant germ such that $g(0, 0) = 0$ and whose derivative $(dg)_{(0,0)}$ with respect to $x \in V$ at $(0, 0)$ is singular. By the equivariant Liapunov-

Schmidt procedure [25] we assume $(dg)_{(0,0)} \equiv 0$. Throughout this paper we shall simply say "the bifurcation problem g".

For the classification theory, the first step is to define the appropriate equivalence relation in order to preserve the symmetries of the bifurcation problems (Definition 3.1). Let $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ denote the module of families of diffeomorphisms

$$S : (V \times \mathbb{R}, 0) \to Hom(W, W) \tag{8}$$

of $\Gamma-$equivariant matrix-valued germs such that

$$S(\rho(\gamma)x, \lambda)\eta(\gamma) = \eta(\gamma)S(x, \lambda), \quad \forall \gamma \in \Gamma, \forall x \in V. \tag{9}$$

The equivariance of elements in $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ follows from the action of Γ on $Hom(W, W)$ being given by similarity. When $(\eta, W) = (\rho, V)$, we shall denote this module by $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$.

It follows again by the Poènaru Theorem [10] that $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$, $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$, $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ and $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ are all finitely generated modules over $\mathcal{E}_{x,\lambda}(\Gamma)$.

Definition 3.1. Let $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ be a $\Gamma-$equivariant bifurcation problem. We say that $h \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ is $\Gamma-$equivalent to g, or simply equivalent to g, if there exist $S \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ and an invertible change of coordinates $(x, \lambda) \longmapsto (X(x, \lambda), \Lambda(\lambda))$, with $X \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and $\Lambda \in \mathcal{E}_\lambda$ such that

$$h(x, \lambda) = S(x, \lambda)g(X(x, \lambda), \Lambda(\lambda)), \tag{10}$$

where

(a) $X(0, 0) = 0$;
(b) $\Lambda(0) = 0$ and $\Lambda'(0) > 0$;
(c) $S(0, 0)$ and $(dX)_{0,0}$ are invertible matrices in the same connected component as the identity in $GL(W)$ and $GL(V)$, respectively.

When $\Lambda(\lambda) = \lambda$ we say that h is $\Gamma-$strongly equivalent to g.

Condition (c) and $\Lambda'(0) > 0$ are necessary stability-preserving conditions. In the examples of Subsections 4.2.1 and 4.3.1, we shall also give sufficient conditions for stability of hyperbolic solutions be invariant up to equivalence.

Let $\mathcal{G}$ denote the group of equivalences defined above:

$$\mathcal{G} = \big\{(S, X, \Lambda) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W) \times \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma) \times \mathcal{E}_\lambda : S, X, \Lambda \text{ satisfy the conditions } (a) - (c)\big\},$$

the group operation being given by

$$(S_2, X_2, \Lambda_2).(S_1, X_1, \Lambda_1) = \Big(S_2.\big(S_1(X_2, \Lambda_2)\big), (X_1, \Lambda_1).(X_2, \Lambda_2)\Big),$$

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where

$$S_2.\big(S_1(X_2,\Lambda_2)\big)(x,\lambda) = S_2(x,\lambda)S_1\big(X_2(x,\lambda),\Lambda_2(\lambda)\big),$$

$$(X_1,\Lambda_1).(X_2,\Lambda_2)(x,\lambda) = \Big(X_1\big(X_2(x,\lambda),\Lambda_2(\lambda)\big),\Lambda_1(\Lambda_2(\lambda)\Big).$$

The action of $\mathcal{G}$ on $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma;V,W)$ is defined by

$$\Big((S,X,\Lambda)\cdot g\Big)(x,\lambda) \mapsto S(x,\lambda)g\big(X(x,\lambda),\Lambda(\lambda)\big).$$

Remark 3. 1. When $(\eta,W)=(\rho,V)$, $\mathcal{G}$ is the equivalence group of the purely equivariant case. When $(\eta,W)=(\rho_\sigma,V)$, it follows directly by the definition of the dual action and by the linearity of $S(x,\lambda)$, for each (x,λ), that the group $\mathcal{G}$ also coincides with the equivalence group of the purely equivariant case.

For the formulation of the general Singularity theory regarding versality and finite determinacy, the group of equivalences is required to be a *geometric subgroup* of the general contact group. The notion of geometric subgroup, introduced by Damon in [6], considers four properties: naturality, tangent space structure, exponential map and filtration condition (see [6] for details). We observe that $\mathcal{G}$ defined above is in fact a geometric subgroup. As a final remark about the equivalence relation, we observe that there is no alteration in the equivalence classes if we assume the coordinate changes in the target to be matrix-valued germs instead of general diffeomorphisms. This is obtained as an immediate adaptation of [10, XIV, Proposition 1.5] to the present context.

In what follows we define the tangent space and the restricted tangent space of a $\Gamma-$equivariant germ $g:(V\times\mathbb{R},0)\mapsto W$, both given from the $\Gamma-$equivalence (Definition 3.1). We begin with the basic concepts of the unfolding theory.

We consider the vectors $\alpha=(\alpha_1,\cdots,\alpha_l)\in\mathbb{R}^l$ and $\beta=(\beta_1,\cdots,\beta_k)\in\mathbb{R}^k$, for some l and k. Let us denote by $\mathcal{E}_{x,\lambda,\beta}(\Gamma)$ the ring of $k-$parameter families of germs in $\mathcal{E}_{x,\lambda}(\Gamma)$, by $\overrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma;V,W)$ the $\mathcal{E}_{x,\lambda,\beta}(\Gamma)-$module of $k-$parameter families of germs in $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma;V,W)$ and by $\overrightarrow{\mathcal{F}}_{x,\lambda,\beta}\,(\Gamma)$ the $\mathcal{E}_{x,\lambda,\beta}(\Gamma)-$module of $k-$parameter families of germs in $\overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma)$. Analogously, let us denote by $\overleftrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma;V,W)$ the $\mathcal{E}_{x,\lambda,\beta}(\Gamma)-$module of $k-$parameter families of germs in $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma;V,W)$, by $\mathcal{E}_{\lambda,\beta}$ the ring of $k-$parameter families of germs in $\mathcal{E}_\lambda$ and by $\mathcal{E}_{\alpha,\beta}$ the ring of germs at the origin $\Phi:(\mathbb{R}^l,0)\to\mathbb{R}^k$.

Definition 3.2. A $k-$parameter $\Gamma-$unfolding of a germ $g\in\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma;V,W)$ is a $\Gamma-$equivariant germ $G\in\overrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma;V,W)$ such that

$$G(x,\lambda,0)=g(x,\lambda).$$

Definition 3.3. If $H \in \overrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\Gamma; V, W)$ is an $l-$parameter unfolding of g and $G \in \overrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma; V, W)$ is a $k-$parameter unfolding of g, we say that H factors through G if

$$H(x, \lambda, \alpha) = S(x, \lambda, \alpha)G(X(x, \lambda, \alpha), \Lambda(\lambda, \alpha), A(\alpha)) \tag{11}$$

where

(a) $S \in \overleftrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\Gamma; V, W)$ e $S(x, \lambda, 0) = I$;
(b) $X \in \overrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\Gamma)$ e $X(x, \lambda, 0) = x$;
(c) $\Lambda \in \mathcal{E}_{\lambda,\alpha}$ e $\Lambda(\lambda, 0) = \lambda$;
(d) $A \in \mathcal{E}_{\alpha,\beta}$ e $A(0) = 0$.

Definition 3.4. We say that $G \in \overrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma; V, W)$ and $H \in \overrightarrow{\mathcal{E}}_{x,\lambda,\alpha}(\Gamma; V, W)$ are $\Gamma-$isomorphic if (11) holds for A a diffeomorphism.

When G and H are $\Gamma-$isomorphic, then H factors through G, G factors through H and $k = l$.

Definition 3.5. An unfolding G of g is versal if every unfolding H of g factors through G. It is universal if the number of parameters is minimal. This number is called the codimension of g.

The group $\mathcal{G}_{un}(k)$ of equivalences is defined as an extension of the group $\mathcal{G}$ by

$$\mathcal{G}_{un}(k) = \big\{(S, X, \Lambda, \Phi) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma; V, W) \times \overrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma) \times \mathcal{E}_{\lambda,\beta} \times \mathcal{E}_{\beta,\beta} : (S, X, \Lambda) \text{ is a } k\text{-parameter unfolding of an element in } \mathcal{G} \text{ and } \Phi \text{ is a germ of diffeomorphism such that } \Phi(0) = 0\big\},$$

the action of $\mathcal{G}_{un}(k)$ on $G \in \overrightarrow{\mathcal{E}}_{x,\lambda,\beta}(\Gamma; V, W)$ being given by

$$\Big((S, X, \Lambda, \Phi) \cdot G\Big)(x, \lambda, \beta) \mapsto S(x, \lambda, \beta)G(X(x, \lambda, \beta), \Lambda(\lambda, \beta), \Phi(\beta)).$$

We now define the *restricted tangent space* $\mathrm{RT}(g)$ of a germ $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ to be the set of perturbations $p \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ such that $g + tp$ is strongly equivalent to g, for small t. Hence

$$\mathrm{RT}(g) = \{Sg + (dg)X : (S, X) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W) \times \overrightarrow{\mathcal{M}}_{x,\lambda}(\Gamma)\}. \tag{12}$$

We define the *extended tangent space* $\mathrm{T}(g)$ of g by

$$\mathrm{T}(g) = \{Sg + (dg)X : (S, X) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W) \times \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)\} + \mathcal{E}_\lambda g_\lambda. \tag{13}$$

By definition,

$$\mathrm{T}(g) = \mathrm{RT}(g) + \mathbb{R}\{(dg)Y_1, \ldots, (dg)Y_m\} + \mathcal{E}_\lambda g_\lambda,$$

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where Y_i's, $i = 1, ...m$, span the $\Gamma-$equivariant germs that do not vanish at the origin, modulo those that do vanish at the origin.

In the following proposition we recall the algebraic structure of the submodule $\mathrm{RT}(g)$ of $\vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$. This result is the analogous of [10, XIV, Proposition 1.4] for germs from V into W.

Proposition 3.1. *For $h \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$, the generators of $\mathrm{RT}(h)$ are given by*

$$S_1 h, \ldots, S_t h; (dh)(X_1), \ldots, (dh)(X_s),$$

where $S_1, \ldots, S_t$ generate $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ and $X_1, \ldots, X_s$ generate $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$.

The following theorem generalizes [10, XV, Theorem 2.1, Corollary 2.2 and Theorem 7.4] given for purely $\Gamma-$equivariant germs. This is the main result in the unfolding theory. Its proof follows the same steps as the proof of the purely equivariant version.

Theorem 3.1. *Let Γ be a compact Lie group acting linearly on V and on W and let $g \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$.*

(a) *Let $G \in \vec{\mathcal{E}}_{x,\lambda,\beta}(\Gamma; V, W)$ be a $k-$parameter unfolding of g. Then G is versal if, and only if, $\vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W) = \mathrm{T}(g) + \mathbb{R}\{G_{\beta_1}(x, \lambda, 0), \ldots, G_{\beta_k}(x, \lambda, 0)\}$, where $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$.*

(b) *If $W \subseteq \vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$ is a vector subspace such that $\vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W) = \mathrm{T}(g) \oplus W$ and if $\{p_1(x, \lambda), \ldots, p_k(x, \lambda)\}$ is a basis for W, then $G(x, \lambda, \beta) = g(x, \lambda) + \sum_{j=1}^{k} \beta_j p_j(x, \lambda)$ is a universal $\Gamma-$unfolding of g.*

(c) *Two versal unfoldings $G(x, \lambda, \beta)$ and $H(x, \lambda, \alpha)$ are $\Gamma-$isomorphic if, and only if, they have the same number of unfolding parameters.*

It is a direct consequence of item (a) of the theorem above that the codimension of g is equal to the codimension of $\mathrm{T}(g)$ in $\vec{\mathcal{E}}_{x,\lambda}(\Gamma; V, W)$.

3.2. The reversible-equivariant Singularity theory

In this subsection we obtain the results of $\Gamma-$reversible-equivariant Singularity theory for the classification of germs $g : (\rho, V) \to (\rho_\sigma, V)$ satisfying (6). They follow directly from the results of the previous subsection, since, as pointed out in Remark 2.1, we recognize a reversible-equivariant problem as an equivariant problem under appropriate actions on source and target, i.e., just take (η, W) in Subsection 3.1 as (ρ_σ, V).

As pointed out in Remark 3.1, the group $\mathcal{G}$ of equivalences that preserves symmetries and reversing symmetries turn out to be the same group considered for the purely equivariant classification. From now on we shall adopt the following convention: We denote by g a general element in $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ and by $\tilde{g}$ a general element in $\vec{\mathcal{F}}_{x,\lambda}(\Gamma)$. Also, we shall denote by $\mathrm{RT}(g, \Gamma)$ and $\mathrm{T}(g, \Gamma)$ the restricted and extended tangent spaces of $g \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ to make

TABLE 1.
Data for restricted tangent spaces for $\Gamma = \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbf{O}(2)$, $\mathbf{D}_n$ for any $n \geq 3$ and $\mathbf{D}_n$ for n even. In the first four cases, the flips κ, κ_1 and κ_2 are reversing symmetries; in the last case, the flip κ is a symmetry and the rotating generator $\xi = 2\pi/n$ is a reversing symmetry.

Γ	$\mathbb{Z}_2$ κ $\sigma(\kappa) = -1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ κ_1 κ_2 $\sigma(\kappa_1) = -1$ $\sigma(\kappa_2) = -1$	$\mathbf{O}(2)$ $0 \leq \theta \leq 2\pi$ κ $\sigma(\theta) = 1$ $\sigma(\kappa) = -1$	$\mathbf{D}_n(n \geq 3)$ $\xi = 2\pi/n$ κ $\sigma(\xi) = 1$ $\sigma(\kappa) = -1$	$\mathbf{D}_n(n \text{ even})$ $\xi = 2\pi/n$ κ $\sigma(\xi) = -1$ $\sigma(\kappa) = 1$
V	$\mathbb{R}^2$	$\mathbb{R}^2$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
Action	$\kappa(x,y) = (x,-y)$	$\kappa_1(x,y) = (x,-y)$ $\kappa_2(x,y) = (-x,y)$	$\theta z = e^{i\theta} z$ $\kappa z = \overline{z}$	$\xi z = e^{i\xi} z$ $\kappa z = \overline{z}$	$\xi z = e^{i\xi} z$ $\kappa z = \overline{z}$
generators of $\mathcal{E}(\Gamma)$	$u = x$ $v = y^2$	$u = x^2$ $v = y^2$	$u = z\overline{z}$	$u = z\overline{z}$ $v = z^n + \overline{z}^n$	$u = z\overline{z}$ $v = z^n + \overline{z}^n$
generators of $\overrightarrow{\mathcal{E}}(\Gamma)$	$g_1 = (1,0)$ $g_2 = (0,y)$	$g_1 = (x,0)$ $g_2 = (0,y)$	$g_1 = z$	$g_1 = z$ $g_2 = \overline{z}^{n-1}$	$g_1 = z$ $g_2 = \overline{z}^{n-1}$
generators of $\overrightarrow{\mathcal{F}}(\Gamma)$	$\tilde{g}_1 = (0,1)$ $\tilde{g}_2 = (y,0)$	$\tilde{g}_1 = (0,x)$ $\tilde{g}_2 = (y,0)$	$\tilde{g}_1 = iz$	$\tilde{g}_1 = iz$ $\tilde{g}_2 = i\overline{z}^{n-1}$	$\tilde{g}_1 = \overline{z}^{\frac{n}{2}-1}$ $\tilde{g}_2 = z^{\frac{n}{2}+1}$

a distinction with the tangent spaces $\mathrm{RT}(\tilde{g})$ and $\mathrm{T}(\tilde{g})$ of $\tilde{g} \in \overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma)$. The generators of $\mathrm{RT}(\tilde{g})$ are then

$$S_1\tilde{g}, \ldots, S_t\tilde{g}; (d\tilde{g})(X_1), \ldots, (d\tilde{g})(X_s),$$

where $S_1, \ldots, S_t$ generate $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and $X_1, \ldots, X_s$ generate $\overrightarrow{\mathcal{M}}_{x,\lambda}(\Gamma)$. It turns out that generators of $\mathrm{RT}(g, \Gamma)$ and $\mathrm{RT}(\tilde{g})$ differ only due to the different symmetry conditions of g and $\tilde{g}$, the elements S_i 's and X_j 's being the same in both cases. The same holds for $\mathrm{T}(g, \Gamma)$ and $\mathrm{T}(\tilde{g})$.

In Table 1 we list specific actions of the groups $\Gamma = \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbf{O}(2)$ and $\mathbf{D}_n$ with $n \geq 3$, for the flip κ acting as a reversibility, and $\mathbf{D}_n$ with n even, when the flip κ is a symmetry and the rotating generator $\xi = 2\pi/n$ is a reversing symmetry. We note that for the dihedral groups of order n, a rotation can be a reversing symmetry only if n is even. We also present the generators of the ring $\mathcal{E}(\Gamma)$ and of the modules $\overrightarrow{\mathcal{E}}(\Gamma)$ and $\overrightarrow{\mathcal{F}}(\Gamma)$, which are used to obtain Table 2 (here we are omitting the state variable and the parameter λ in the notation of these sets). In this table we reproduce the data for the equivariants for the sake of comparison with the reversible-equivariants. In each of these examples the modules $\overrightarrow{\mathcal{E}}(\Gamma)$ and $\overrightarrow{\mathcal{F}}(\Gamma)$ are free, that is, there are finite sets of generators $g_1, \ldots, g_k$ for $\overrightarrow{\mathcal{E}}(\Gamma)$ and $\tilde{g}_1, \ldots, \tilde{g}_l$ for $\overrightarrow{\mathcal{F}}(\Gamma)$ such that any g is uniquely of the form $g = a_1 g_1 + \cdots + a_r g_k$ and any $\tilde{g}$

TABLE 2.
Generators for $\Gamma-$equivariant and $\Gamma-$reversible-equivariant restricted tangent spaces for the groups Γ of Table 1.

Γ	$\mathrm{RT}(g,\Gamma)$	$\mathrm{RT}(\tilde{g})$
$\mathbb{Z}_2$	$[p,0],[vq,0],[0,p],[0,q]$, $[up_u,uq_u],[vp_u,vq_u]$, $[\lambda p_u,\lambda q_u],[vp_v,vq_v]$	$[0,s],[0,r],[r,0],[vs,0]$, $[ur_u,us_u],[vr_u,vs_u]$, $[\lambda r_u,\lambda s_u],[vr_v,vs_v]$
$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$[p,0],[0,q],[0,up],[vq,0]$, $[up_u,uq_u],[vp_v,vq_v]$	$[0,s],[r,0],[0,ur],[vs,0]$, $[ur_u,us_u],[vr_v,vs_v]$
$\mathbf{O}(2)$	$[p],[up_u]$	$[r],[ur_u]$
$\mathbf{D}_n$ $(n\geq 3)$	$[p,q],[2up+vq,0],[u^{n-2}q,p]$, $[vp+2u^{n-1}q,0]$, $[2up_u+nvp_v$, $(n-2)q+(n+1)uq_u+nvq_v]$, $[vp_u+2nu^{n-1}p_v+(n-2)u^{n-2}q+$ $(n-1)u^{n-1}q_u,vq_u+2nu^{n-1}q_v]$	$[r,s],[2ur+vs,0],[u^{n-2}s,r]$, $[vr+2u^{n-1}s,0]$, $[2ur_u+nvr_v$, $(n-2)s+2us_u+nvs_v]$, $[vr_u+2nu^{n-1}r_v+(n-2)u^{n-2}s$, $vs_u+2nu^{n-1}s_v]$
$\mathbf{D}_n$ $(n$ even$)$	$[p,q],[2up+vq,0],[u^{n-2}q,p]$, $[vp+2u^{n-1}q,0]$, $[2up_u+nvp_v$, $(n-2)q+(n+1)uq_u+nvq_v]$, $[vp_u+2nu^{n-1}p_v+(n-2)u^{n-2}q+$ $(n-1)u^{n-1}q_u,vq_u+2nu^{n-1}q_v]$	$[r,s],[0,2u^{\frac{n}{2}-1}r+vs],[u^2s,r]$, $[vs+2u^{\frac{n}{2}-1}r,0]$, $[2ur_u-2r+nvr_v,2us_u+nvs_v]$, $[vr_u+2nu^{n-1}r_v+2u^{\frac{n}{2}}s$, $vs_u+2nu^{n-1}s_v]$

is uniquely of the form $\tilde{g}=b_1\tilde{g}_1+\cdots+b_l\tilde{g}_l$, where $a_i,b_j\in\mathcal{E}(\Gamma)$, $i=1,\ldots,k$, $j=1,\ldots,l$. We shall use below the same notation to represent elements in terms of the basis of the distinct modules $\vec{\mathcal{E}}(\Gamma)$ and $\vec{\mathcal{F}}(\Gamma)$, namely $g=a_1g_1+\cdots+a_kg_k\equiv[a_1,\ldots,a_k]\in\vec{\mathcal{E}}(\Gamma)$ and $\tilde{g}=b_1\tilde{g}_1+\cdots+b_l\tilde{g}_l\equiv[b_1,\ldots,b_l]\in\vec{\mathcal{F}}(\Gamma)$. The a_i 's and b_j 's, $i=1,\ldots,k$, $j=1,\ldots,l$, are called *invariant coordinates* of g and $\tilde{g}$, respectively.

In Table 2 we present the generators of $\mathrm{RT}(g,\Gamma)$ and $\mathrm{RT}(\tilde{g})$ for each of the groups and respective actions listed in the Table 1. We omit the calculations to obtain both tables due to their similarities with the purely equivariant case presented in detail in [10]. We also omit the dependence of the germs with respect to the bifurcation parameter λ.

Since we have set up the notation for $\Gamma-$reversible-equivariant theory, below we restate Theorem 3.1(a) in this setting:

THEOREM 3.2. *Let $\tilde{g}\in\vec{\mathcal{F}}_{x,\lambda}(\Gamma)$ and let $\tilde{G}\in\vec{\mathcal{F}}_{x,\lambda,\beta}(\Gamma)$ be a $k-$parameter unfolding of $\tilde{g}$. Then $\tilde{G}$ is versal if, and only if, $\vec{\mathcal{F}}_{x,\lambda}(\Gamma)=\mathrm{T}(\tilde{g})+\mathbb{R}\{\tilde{G}_{\beta_1}(x,\lambda,0),\ldots,\tilde{G}_{\beta_k}(x,\lambda,0)\}$, where $\beta=(\beta_1,\ldots,\beta_k)\in\mathbb{R}^k$.*

4. THE CLASSIFICATION OF REVERSIBLE-EQUIVARIANT MAPPINGS ON SELF-DUAL SPACES

The aim of this section is to establish a correspondence between the Singularity theory tools in the purely equivariant case and the self-dual reversible-equivariant case. We shall see that the classification of bifurcations of one context can be obtained in a simple way by the classification of the other. In the next subsection we present the theory. In the following two subsections we use the main result, Theorem 4.1, to derive steady-state bifurcations for self-dual actions of the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (Subsection 4.2) and $\mathbf{D}_4$ (Subsection 4.3) on the plane.

4.1. The connection between the two classifications

Let Γ be a compact Lie group acting on a vector space V as a reversing-symmetry group such that V is self-dual. Hence, there exists a reversible-equivariant linear isomorphism $L : V \to V$, that is, the representation of Γ on V is isomorphic to its dual representation. L induces the isomorphism of modules

$$\begin{array}{rccc} L^* : & \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma) & \to & \overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma) \\ & g & \mapsto & Lg \end{array} \tag{14}$$

under the ring $\mathcal{E}_{x,\lambda}(\Gamma)$. L^* is called the *pullback* of L. Hence, we obtain the following key result:

LEMMA 4.1. *In the self-dual case, the modules $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and $\overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma)$ are isomorphic.*

We also have:

LEMMA 4.2. *Let $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and let L^* be the pullback (14). Then*
(a) *the extended tangent spaces $\mathrm{T}(g, \Gamma)$ and $\mathrm{T}(L^* g)$ are isomorphic vector spaces.*
(b) *If codimension of g is finite, then the codimension of $L^* g$ in $\overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma)$ is equal to the codimension of g in $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$.*

Proof: Consider the extended tangent space of $g \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$

$$\mathrm{T}(g, \Gamma) = \{Sg + (dg)X : (S, X) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma) \times \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)\} + \mathcal{E}_\lambda g_\lambda$$

and the extended tangent space of $L^* g \in \overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma)$

$$\mathrm{T}(L^* g) = \{\bar{S}Lg + \big(d(Lg)\big)\bar{X} : (\bar{S}, \bar{X}) \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma) \times \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)\} + \mathcal{E}_\lambda L g_\lambda.$$

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For any $h \in \mathrm{T}(g, \Gamma)$,

$$\begin{aligned} L^* h &= L\big(Sg + (dg)X + \mathbb{R}\{g_\lambda, \lambda g_\lambda\}\big) \\ &= LSg + L(dg)X + \mathbb{R}\{Lg_\lambda, L\lambda g_\lambda\} \\ &= \tilde{S}Lg + \big(d(Lg)\big)X + \mathbb{R}\{Lg_\lambda, \lambda Lg_\lambda\}, \end{aligned}$$

where $\tilde{S} = LSL^{-1} \in \overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and $X \in \overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$, which implies that $L^*\big(\mathrm{T}(g,\Gamma)\big) \subseteq \mathrm{T}(L^* g)$. The other inclusion follows analogously, so $L^*\big(\mathrm{T}(g,\Gamma)\big) = \mathrm{T}(L^* g)$. Part (b) follows directly from the proof of (a) and the previous Lemma. □

The next theorem summarizes the ideas above and is the main result concerning the classification of reversible-equivariant bifurcations on self-dual spaces:

Theorem 4.1. *Consider a representation (ρ, V) of Γ on V under which V is self-dual, and let $L : V \to V$ be a reversible-equivariant linear isomorphism. Then, the pullback L^* defined in (14) determines a one-to-one correspondence between the classification of bifurcations of germs on $\overrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and on $\overrightarrow{\mathcal{F}}_{x,\lambda}(\Gamma)$.*

Therefore, in the self-dual case, if the classification of equivariant bifurcation problems is well known, then so is the classification of the corresponding reversible-equivariant bifurcation problem: normal forms, nondegeneracy conditions and universal unfoldings of the reversible-equivariant bifurcations on self-dual spaces is derived from the normal forms, nondegeneracy conditions and universal unfoldings of the associated equivariant problem via L. Regarding the bifurcation diagrams, the branching equations are preserved, but clearly this does not hold for the stability of solutions, which must be investigated in a case-by-case basis. However, there are many examples where it is possible to establish an association to derive, in a simple way, stability of one case from the other. This is discussed in detail in the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbf{D}_4$ cases presented in Subsections 4.2 and 4.3, respectively.

For the classification of germs of reversible-equivariant mappings on non-self-dual spaces there is no such association with the equivariant classification, so the techniques of Subsection 3.2 must be applied.

The representations corresponding to the actions of $\mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbf{O}(2)$, $\mathbf{D}_n$ (5th column) and $\mathbf{D}_4$ (6th column) of the Table 1 are all self-dual representations on the plane. In addition, we have that any 2×2 orthogonal matrix of order two distinct from I and $-$I is similar to

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

in other words, there exists a $\mathbb{Z}_2-$isomorphism between any representation of $\mathbb{Z}_2$ on $\mathbb{R}^2$ and the one given in Table 1. Hence, up to $\mathbb{Z}_2-$isomorphism, $\mathbb{Z}_2$ admits no other self-dual representation but the one given in Table 1. The same result holds for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, because any pair of commuting 2×2 orthogonal matrices of order two, none of them being I or $-$I, are simultaneously similar to the pair

$$\kappa_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \kappa_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows analogously that, up to $\Gamma-$isomorphism, there is only one non-self dual representation for each of the groups $\Gamma = \mathbb{Z}_2,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2$, both occurring if, and only if, $-\mathrm{I}$ is a reversing symmetry. Finally, as mentioned previously, for the dihedral groups of order n, the generating rotation $\xi = 2\pi/n$ can be taken to be a reversing symmetry only if n is even, and we notice that these cases correspond all to non-self dual representations of $\mathbf{D}_n$ for any $n \neq 4$.

For the rest of the paper, we shall denote the ring $\mathcal{E}_{x,\lambda}(\Gamma)$ and the modules $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$, $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ and $\vec{\mathcal{F}}_{x,\lambda}(\Gamma)$ simply by $\mathcal{E}(\Gamma)$, $\vec{\mathcal{E}}(\Gamma)$, $\overleftrightarrow{\mathcal{E}}(\Gamma)$ and $\vec{\mathcal{F}}(\Gamma)$, respectively.

4.2. Bifurcation problems with reversing symmetry group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

In [20], Manoel and Stewart classify degenerate bifurcation problems on the plane that commute with the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ up to topological codimension 2. In the present subsection we use those results and Theorem 4.1 to derive the behaviour of families of $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant bifurcation problems under the self-dual representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{R}^2$. We focus on the similarity between the two classifications and on the differences between the bifurcation diagrams.

For the standard action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{R}^2$ generated by the flips

$$\kappa_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \kappa_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

the ring $\mathcal{E}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ of smooth $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$invariant function germs is generated by $u = x^2$ and $v = y^2$ and the module $\vec{\mathcal{E}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is generated by $g_1(x, y, \lambda) = (x, 0)$ and $g_2(x, y, \lambda) = (0, y)$. A general $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant bifurcation problem on $\mathbb{R}^2$ is given by $g(x, y, \lambda) = 0$, where

$$g(x, y, \lambda) = p(u, v, \lambda) g_1(x, y, \lambda) + q(u, v, \lambda) g_2(u, v, \lambda), \tag{15}$$

with $p, q \in \mathcal{E}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ such that $p(0, 0, 0) = q(0, 0, 0) = 0$.

The self-dual representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on the plane is given by the flips κ_1, κ_2 acting as reversing symmetries, and

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the matrix of a $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant linear isomorphism.

Via the pullback L^*, generators of $\vec{\mathcal{F}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ are $g_{i*} = L^*(g_i)$, $i = 1, 2$, so a general $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant bifurcation problem on the plane is given by $g_*(x, y, \lambda) = 0$, where

$$g_*(x,y,\lambda) = p(u,v,\lambda)g_{1*} + q(u,v,\lambda)g_{2*}, \tag{16}$$

with $p, q \in \mathcal{E}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ such that $p(0,0,0) = q(0,0,0) = 0$. To ease notation we shall write g in (15) and g_* in (16) in the invariant coordinates as $g = pg_1 + qg_2 \equiv [p,q]$ and $g_* = pg_{1*} + qg_{2*} \equiv [p,q]$, respectively. Note that we are using the same notation for the germs g and g_* in different modules over the same ring; it was the shape of L that has induced us to adopt such convention, in order to make the expressions in the recognition problem for the reversible-equivariants be the same as for the equivariants.

As mentioned in Subsection 4.1, in self-dual cases the zero set of a reversible-equivariant bifurcation problem is identical to that of the corresponding equivariant bifurcation problem; i.e., the branching equations in both contexts are the same (see Table 3). On the other hand, typically we does have a change in the nature of the eigenvalues. This is a consequence of the general fact that presence of reversing symmetries in the group imposes distinct restrictions on the eigenvalues from those obtained when all group elements act as symmetries. In fact, in contrast to the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant case, where we can have stable trivial and pure-mode solutions, in the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant context these solutions are never stable, once their eigenvalues are either a pair of real eigenvalues of opposite sign or lie on the imaginary axis (see Lemma 4.3). However, it is important to note that the nature (real or complex) of the eigenvalues for $(dg_*)_{(x,y,\lambda)}$ on these branches can be derived from the signs of the eigenvalues of $(dg)_{(x,y,\lambda)}$ on the corresponding solution branches (see Table 3). Regarding the mixed-mode solution branches, on which the signs of the eigenvalues are determined, as usual, by the signs of the trace and determinant of the matrix $(dg_*)_{(x,y,\lambda)}$, it is only on such branches that we can find a stable regime: since $\det(dg_*)_{(x,y,\lambda)} = -\det(dg)_{(x,y,\lambda)}$, they shall appear in the corresponding bifurcation diagram for which $\det(dg)_{(x,y,\lambda)}$ is negative. We address to Proposition 4.1 the derivation of the nature of steady states for the reversible-equivariants from the associated pictures for the equivariants given by [20].

Remark 4. 1. The transition varieties for the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-reversible-equivariant bifurcation problem g_* coincide with the transition varieties for the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problem g. This is a consequence of the facts that the solution branches are the same and that there exists a relation between the eigenvalues of both contexts, so that the nature of the eigenvalues on the solution branches of g_* - although different from that for g - changes precisely where and when the stability of the corresponding branches of g is changed.

The classification of degenerate $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant bifurcations up to topological codimension 2 presented in [20, Theorem 4.1] yields seven normal forms, with their recognition conditions and unfoldings. Via L^*, the corresponding result for $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant bifurcations is then easily obtained and is stated in Theorem 4.2. The normal forms h_{i*} are given by $L^*(h_i)$ and, as already mentioned, by the adopted notation the nondegeneracy conditions and occurrence of modal parameters that characterize each

TABLE 3.
Steady-state branches and stability for $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant and $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant bifurcations

Solution Type	Equation	Equivariant stability (signs of eigenvalues)	Reversible-equivariant stability (signs of eigenvalues)
Trivial	$x = y = 0, \forall\lambda$	$p(0,0,\lambda)$, $q(0,0,\lambda)$	$\pm\sqrt{pq(0,0,\lambda)}$
x-mode	$y = p(u,0,\lambda) = 0$	$p_u(u,0,\lambda)$, $q(u,0,\lambda)$	$\pm\sqrt{p_u q(u,0,\lambda)}$
y-mode	$x = q(0,v,\lambda) = 0$	$p(0,v,\lambda)$, $q_v(0,v,\lambda)$	$\pm\sqrt{pq_v(0,v,\lambda)}$
Mixed-mode	$p = q = 0$	sign $\det(dg) =$ sign $(p_u q_v - p_v q_u)$ sign $\mathrm{tr}(dg) =$ sign $(x^2 p_u + y^2 q_v)$	sign $\det(dg_*) =$ sign $(p_v q_u - p_u q_v)$ sign $\mathrm{tr}(dg_*) =$ sign $\big(xy(q_u + p_v)\big)$

reversible-equivariant normal form h_{i*} have the same expressions as the ones that appear in the recognition problem for h_i, $i = 2, \ldots 8$, in [20].

THEOREM 4.2. **(Classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-reversible-equivariant bifurcations)** *If a germ $g_* = [p,q] \in \vec{\mathcal{F}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ satisfies the recognition conditions in Table 4 for the normal form h_{j_*} for some $j = 2, \ldots, 8$, then g_* is $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivalent to h_{j_*}.*

To illustrate, we detail the computations for the normal form h_{2*} and describe the bifurcation diagrams for its unfolding. Our aim here is mainly to clarify the role of L^* in the stability of solutions when we pass from the equivariant to the reversible-equivariant problem. The other cases follow the same procedure and their diagrams are obtained straightforwardly by the association between results in Table 3 given by L^*.

From [20, Theorem 4.1], if a $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant problem $g = [p,q]$ satisfies the recognition conditions

$$p_u q_v - p_v q_u = 0, \quad p_u,\ q_v,\ p_v,\ p_\lambda,\ q_\lambda,\ \rho_2,\ p_u q_\lambda - p_\lambda q_u \neq 0, \tag{17}$$

where

$$\rho_2 = q_{uv} p_u {p_v}^2 - q_{vv} p_u^2 p_v - q_{uu} p_v^3 - p_{uv} q_v p_u p_v + p_{vv} q_v p_u^2 + p_{uu} q_v p_v^2,$$

then g is equivalent to the normal form

$$h_2 = [\varepsilon_1 u + \varepsilon_4 v + \varepsilon_2 \lambda + \varepsilon_5 u^2, \varepsilon_1\varepsilon_3\varepsilon_4 u + \varepsilon_3 v + \kappa\lambda].$$

All the derivatives are calculated at the origin and $\varepsilon_i = \pm 1$, with $\kappa = |\frac{p_v}{p_\lambda q_v}| q_\lambda$, $\mathrm{sign}(\varepsilon_1) = \mathrm{sign}(p_u)$, $\mathrm{sign}(\varepsilon_2) = \mathrm{sign}(p_\lambda)$, $\mathrm{sign}(\varepsilon_3) = \mathrm{sign}(q_v)$, $\mathrm{sign}(\varepsilon_4) = \mathrm{sign}(p_v)$ and $\mathrm{sign}(\varepsilon_5) =$

sign$(\rho_2 q_v)$. The unfolding terms for h_2 are $[u,0]$, $[0,\lambda]$, $[0,1] \in \overrightarrow{\mathcal{E}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. As a consequence of Theorem 4.1, if $g_* = L^*(g) = [p,q]$ satisfies the recognition conditions (17), then g_* is equivalent to the normal form

$$h_{2*} = [\varepsilon_1 u + \varepsilon_4 v + \varepsilon_2 \lambda + \varepsilon_5 u^2, \varepsilon_1 \varepsilon_3 \varepsilon_4 u + \varepsilon_3 v + \kappa \lambda].$$

Furthermore, the unfolding terms for h_{2*} are $[u,0]$, $[0,\lambda]$, $[0,1] \in \overrightarrow{\mathcal{F}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, so that

$$H_{2*}(x,y,\lambda,\alpha,\beta,\kappa) = [(\varepsilon_1 + \alpha)u + \varepsilon_4 v + \varepsilon_2 \lambda + \varepsilon_5 u^2, \varepsilon_1 \varepsilon_3 \varepsilon_4 u + \varepsilon_3 v + \kappa \lambda + \beta]$$

is the unfolding of h_{2*}.

We now turn to the bifurcation diagrams for H_{2*}, when $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1$, $\varepsilon_2 = -1$ and $\kappa < -1$. We choose these signs in order to compare our results with the corresponding equivariant diagrams in [20, Fig. 4.].

The solution branches for $H_{2*} = [(1+\alpha)u + v - \lambda + u^2, u + v + \kappa\lambda + \beta]$ are the same as for H_2, the unfolding of h_2, but the stability on each branch shall be analysed. As mentioned above, this can be done by the existent correspondence between the eigenvalues in both contexts. A mixed-mode solution branch exists only if $\beta > 0$. On that branch, when $\det(dH_{2*})_{(x,y,\lambda)} > 0$ we have that

$$\text{sign } \operatorname{tr}(dH_{2*})_{(x,y,\lambda)} = \text{sign}(xy),$$

so an exchange of stability occurs if, and only if, $x = 0$ or $y = 0$, that is, at a secondary bifurcation point. In this case, the parts on the branch with stability "$++$" are mapped onto the mixed-mode solutions with stability "$--$", and vice-versa, by the reversing symmetries κ_1 and κ_2. This is represented in the diagrams by "$++/--$".

In Table 4 we present the complete classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant bifurcation problems. The adopted notation here forces this table to be the same as Table 2 that appears in [20, p. 1659].

4.2.1. Stability preserving and $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivalence

In this subsection we discuss occurrence of stable solutions and their persistence under equivalence. Stable hyperbolic solutions are always expected in standard equivariant bifurcations, since the action of the group of symmetries is generically absolutely irreducible. Also, if a solution looses stability at a bifurcation point, it is also expected that another stable solution takes place, in the so-called symmetry-breaking phenomenon. On the other hand, in the presence of a reversibility linear stability can not occur. In fact, we have ([19, Lemma 1.1]):

Lemma 4.3. *If ξ is an eigenvalue of a reversible linear mapping, then so is $-\xi$ and its complex conjugate $\bar{\xi}$.*

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TABLE 4.
Classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-reversible-equivariant singularities. All derivatives are calculated at the origin, and $\epsilon_i = \pm 1$ with sign given by the corresponding derivative.

$$h_{2*} = [\epsilon_1 u + \epsilon_4 v + \epsilon_2 \lambda + \epsilon_5 u^2, \epsilon_1\epsilon_3\epsilon_4 u + \epsilon_3 v + \kappa\lambda]$$
$$h_{3*} = [\epsilon_1 u + \mu v + \epsilon_2 \lambda, \epsilon_1\epsilon_2\epsilon_4 u + \epsilon_3 v + \epsilon_4\lambda + \epsilon_5\lambda^2]$$
$$h_{4*} = [\epsilon_1 u + \epsilon_2\epsilon_3\epsilon_4 v + \epsilon_2 \lambda, \eta u + \epsilon_3 v + \epsilon_4\lambda + \epsilon_5\lambda^2]$$
$$h_{5*} = [\epsilon_1 u^2 + \mu v + \epsilon_2 \lambda, \epsilon_5 u + \epsilon_3 v + \epsilon_4\lambda]$$
$$h_{6*} = [\epsilon_1 u + \epsilon_5 v + \epsilon_2 \lambda, \eta u + \epsilon_3 v^2 + \epsilon_4\lambda]$$
$$h_{7*} = [\epsilon_1 u + \epsilon_5 v + \epsilon_2 \lambda^2, \eta u + \epsilon_3 v + \epsilon_4\lambda]$$
$$h_{8*} = [\epsilon_1 u + \mu v + \epsilon_2 \lambda, \epsilon_5 u + \epsilon_3 v + \epsilon_4\lambda^2]$$

Normal Form	Recognition	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	Modal Parameter	Unfolding Terms in $\vec{\mathcal{F}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$
h_{2*}	$p_u q_v - p_v q_u = 0$ $p_u, q_v, p_v, p_\lambda, q_\lambda, \rho_2,$ $p_u q_\lambda - p_\lambda q_u \neq 0$	p_u	p_λ	q_v	p_v	$\rho_2 q_v$	$\kappa = \left\lvert \frac{p_v}{p_\lambda q_v} \right\rvert q_\lambda$	$[u, 0]$ $[0, \lambda]$ $[0, 1]$
h_{3*}	$p_\lambda q_u - p_u q_\lambda = 0$ $p_u, q_v, p_\lambda, q_\lambda, \rho_3,$ $p_u q_v - p_v q_u \neq 0$	p_u	p_λ	q_v	q_λ	$\rho_3 p_\lambda$	$\mu = \left\lvert \frac{q_\lambda}{p_\lambda q_v} \right\rvert p_v$	$[v, 0]$ $[0, u]$ $[0, 1]$
h_{4*}	$p_v q_\lambda - p_\lambda q_v = 0$ $p_u, q_v, p_\lambda, q_\lambda, \rho_4,$ $p_u q_v - p_v q_u \neq 0$	p_u	p_λ	q_v	q_λ	$\rho_4 p_\lambda$	$\eta = \left\lvert \frac{p_\lambda}{p_u q_\lambda} \right\rvert q_u$	$[v, 0]$ $[0, u]$ $[0, 1]$
h_{5*}	$p_u = 0$ $p_\lambda, q_v, q_\lambda, p_{uu},$ $p_v q_\lambda - p_\lambda q_v \neq 0$	p_{uu}	p_λ	q_v	q_λ	q_u	$\mu = \left\lvert \frac{q_\lambda}{p_\lambda q_v} \right\rvert p_v$	$[v, 0]$ $[u, 0]$ $[0, 1]$
h_{6*}	$q_v = 0$ $p_u, p_\lambda, q_\lambda, p_v, q_u, q_{vv},$ $p_u q_\lambda - p_\lambda q_u \neq 0$	p_u	p_λ	q_{vv}	q_λ	p_v	$\eta = \left\lvert \frac{p_\lambda}{p_u q_\lambda} \right\rvert q_u$	$[0, v]$ $[0, u]$ $[1, 0]$
h_{7*}	$p_\lambda = 0$ $p_u, p_v, q_v, q_\lambda, p_{\lambda\lambda},$ $p_u q_v - p_v q_u \neq 0$	p_u	$p_{\lambda\lambda}$	q_v	q_λ	p_v	$\eta = \left\lvert \frac{p_v}{p_u q_v} \right\rvert q_u$	$[\lambda, 0]$ $[0, u]$ $[1, 0]$
h_{8*}	$q_\lambda = 0$ $p_u, p_\lambda, q_u, q_v, q_{\lambda\lambda}$ $p_u q_v - p_v q_u \neq 0$	p_u	p_λ	q_v	$q_{\lambda\lambda},$	q_u	$\mu = \left\lvert \frac{q_u}{p_u q_v} \right\rvert p_v$	$[0, \lambda]$ $[v, 0]$ $[1, 0]$

$$\rho_2 = q_{uv} p_u p_v^2 - q_{vv} p_u^2 p_v - q_{uu} p_v^3 - p_{uv} q_v p_u p_v + p_{vv} q_v p_u^2 + p_{uu} q_v p_v^2$$
$$\rho_3 = q_{uu} p_\lambda^3 - p_{uu} p_\lambda^2 q_\lambda - q_{u\lambda} p_\lambda^2 p_u + q_{\lambda\lambda} p_\lambda p_u^2 + p_{u\lambda} p_\lambda q_\lambda p_u - p_{\lambda\lambda} p_u^2 q_\lambda$$
$$\rho_4 = q_{\lambda\lambda} p_\lambda q_v^2 - p_{vv} p_\lambda q_\lambda^2 - q_{v\lambda} p_\lambda q_\lambda q_v - p_{vv} q_\lambda^3 - p_{\lambda\lambda} q_v^2 q_\lambda + p_{v\lambda} q_\lambda^2 q_v$$

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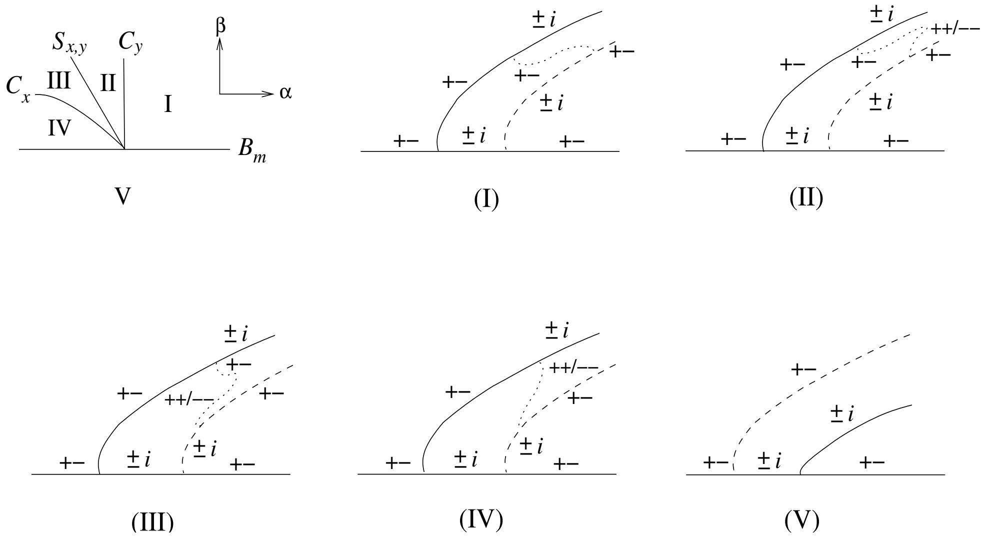

FIG. 1. Transition varieties and bifurcation diagrams for H_{2*}

Hence, it is a general fact that a stable solution may occur in reversible systems only if the reversing symmetries do not belong to its isotropy subgroup. Under the present action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, a steady-state (x, y, λ) is stable if, and only if, it is a mixed mode saddle in the associated $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant problem and such that the trace of the jacobian $(dg_*)_{(x,y,\lambda)}$ is positive. The next proposition gives the nature of each equilibrium of the bifurcation problem g_* from the stability (or instability) of the same point as an equilibrium of g given by [20]. The main point is the role of L^* in the exchange in nature of the steady states for one case from the other.

Proposition 4.1. *Let g_* be a $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant bifurcation problem under the self-dual representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{R}^2$. Let (x, y, λ) be a steady-state of g_*. Then we have the following:*

1. *If (x, y, λ) is a trivial or a pure-mode solution of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant problem where the eigenvalues are either both positive or both negative, then (x, y, λ) is a saddle type solution in the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant problem, denoted in the diagrams by "$+-$";*

2. *If (x, y, λ) is a trivial or a pure-mode of saddle type of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivariant problem, then (x, y, λ) is a non-hyperbolic solution of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant problem with purely imaginary eigenvalues, denoted in the diagrams by "$\pm i$";*

3. *If* (x, y, λ) *is a mixed-mode solution of the* $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$*equivariant problem where the eigenvalues have nonzero real part with same sign, then* (x, y, λ) *is a saddle in the* $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$*reversible-equivariant problem, denoted in the diagrams by* "$+-$";

4. *If* (x, y, λ) *is a mixed-mode of saddle type of the* $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$*equivariant problem, then* (x, y, λ) *is a solution of the* $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$*reversible-equivariant problem where the eigenvalues have positive real part (if* $\mathrm{tr}(dg_*) > 0$*) or negative real part (if* $\mathrm{tr}(dg_*) < 0$*), denoted in the diagrams by* "$++$" *or* "$--$", *respectively. Also, to every mixed-mode solution with* "$++$" *stability corresponds another (symmetric) mixed-mode solution with* "$--$" *stability, and vice-versa.*

Proof: It is a direct consequence of the results in Table 3. □

An exchange of stability along the mixed-mode branch of type *4.* accurs precisely where $\mathrm{tr}(dh_{i*})_{(x,y,\lambda)}$ changes sign. It is easy to see that in all the cases, $\mathrm{tr}(dh_{i*})_{(x,y,\lambda)} = 0$ if and only if $x = 0$ or $y = 0$. That is, the trace vanishes at a secondary bifurcation point. We note that in the corresponding equivariant context this exchange of stability on the mixed-mode branch occurs only for the unfolding H_5, with $\mu < 0$, out of a secondary bifurcation, yielding a Hopf bifurcation point (see [20, Fig. 12, p. 1665]). For the corresponding reversible-equivariant H_{5*}, such a point is not encountered, since the eigenvalues on that branch are of opposite signs.

We end this subsection giving sufficient conditions for linear stability of hyperbolic solutions to $g_* = 0$ be invariant by $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$equivalence. This is the analogous of [9, X, Proposition 3.2] for equivariant bifurcations:

PROPOSITION 4.2. *Let* g_* *be a* $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$*reversible-equivariant bifurcation problem in the plane and let* (x, y, λ) *be a hyperbolic solution of* $g_* = 0$. *Then the stability of* (x, y, λ) *is invariant under* $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$*equivalence if any of the following hold:*

(a) (x, y, λ) *is a trivial or a pure mode solution;*

(b) (x, y, λ) *is a mixed-mode solution and* $\det(dg_*)_{(x,y,\lambda)} < 0$;

(c) (x, y, λ) *is a mixed-mode solution and* $\det(dg_*)_{(x,y,\lambda)} > 0$, *with* $p_u \cdot q_v \geq 0$ *at the origin.*

Proof: This is the same as the proof of [9, X, Proposition 3.2], since the group of equivalences in $\overrightarrow{\mathcal{F}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is the same as in $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, and here we use the anti-diagonal form of $(dg_*)_{(x,y,\lambda)}$ in place of the diagonal form of $(dg)_{(x,y,\lambda)}$ to prove part (a). □

4.3. Bifurcation problems with reversing symmetry group $\mathbf{D_4}$

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In [11], the authors classify degenerate $\mathbf{D}_4-$equivariant bifurcation problems on the plane up to topological codimension 2, presenting the nondegeneracy conditions, universal unfoldings and their bifurcation diagrams. In this subsection we use those results and Theorem 4.1 to obtain the classification of $\mathbf{D}_4-$reversible-equivariant bifurcations on the plane when the rotating generator $\xi = \pi/2$ acts as a reversing symmetry. As mentioned previously, $\mathbf{D}_4$ is the only dihedral group of even order n for which $\xi = 2\pi/n$ acting as a reversibility leads to a self-dual representation. As we shall see, in the same way as done in the previous subsection for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, the notation here is chosen so that the expressions of normal forms, their recognition conditions and unfolding terms shall coincide with the equivariant classification as it appears in [11, Table II].

We consider the action of the dihedral group $\mathbf{D}_4$ on $\mathbb{C}$ generated by

$$\kappa : z \mapsto \bar{z} \quad \text{and} \quad \xi : z \mapsto e^{i\frac{\pi}{2}} z,$$

where the flip κ is a symmetry and the rotation ξ is a reversing symmetry, that is, $\sigma(\kappa) = 1$ and $\sigma(\xi) = -1$, which corresponds to a self-dual representation of $\mathbf{D}_4$ on the plane. In [2] we deduce the general form of one-parameter $\mathbf{D}_4-$reversible-equivariant germs on $\mathbb{C}$, which is given by

$$g_*(z, \lambda) = s(u, v, \lambda) z^3 + q(u, v, \lambda)\bar{z}, \tag{18}$$

where $u = z\bar{z}$ and $v = z^4 + \bar{z}^4$ are the generators of the $\mathbf{D}_4-$invariants.

In [11] the authors give the general form of one-parameter $\mathbf{D}_4-$equivariant germs in real coordinates:

$$g(x, y, \lambda) = p(N, \Delta, \lambda) \begin{pmatrix} x \\ y \end{pmatrix} + r(N, \Delta, \lambda)\delta \begin{pmatrix} x \\ -y \end{pmatrix}, \tag{19}$$

where $\delta = y^2 - x^2$ and p, r are elements of the ring $\mathcal{E}(\mathbf{D}_4)$ of smooth $\mathbf{D}_4-$invariant function germs generated by $N = x^2 + y^2$ and $\Delta = \delta^2$. In real coordinates, we have that

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{20}$$

is the matrix of a $\mathbf{D}_4-$reversible-equivariant linear isomorphism. Via the pullback L^* (14), we use (19) to rewrite a general one-parameter $\mathbf{D}_4-$reversible-equivariant germ:

$$g_*(x, y, \lambda) = r(N, \Delta, \lambda)\delta \begin{pmatrix} x \\ y \end{pmatrix} + p(N, \Delta, \lambda) \begin{pmatrix} x \\ -y \end{pmatrix}. \tag{21}$$

Now, we shall denote g in (19) and $g_* = L^*(g)$ in (21) in the invariant coordinates as $g = pg_1 + rg_2 \equiv [p, r]$ and $g_* = pg_{1*} + rg_{2*} \equiv [p, r]$, respectively, where $g_1(x, y, \lambda) = (x, y)$ and $g_2(x, y, \lambda) = \delta(x, -y)$ are the generators of the $\mathbf{D}_4-$equivariants and $g_{1*}(x, y, \lambda) = L^*(g_1)(x, y, \lambda) = (x, -y)$ and $g_{2*}(x, y, \lambda) = L^*(g_2)(x, y, \lambda) = \delta(x, y)$ are the generators of the $\mathbf{D}_4-$reversible-equivariants.

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TABLE 5.
Steady-state branches and stability for $\mathbf{D}_4$-equivariant and $\mathbf{D}_4$-reversible-equivariant bifurcations

Solution Type	Equation	Equivariant stability (signs of eigenvalues)	Reversible-equivariant stability (signs of eigenvalues)
Trivial	$x = 0$ $y = 0$	p (twice)	p $-p$
R	$y = 0$ $p - x^2 r = 0$	$p_N - r + x^2(2p_\Delta - r_N) - 2x^4 r_\Delta$ r	$p_N - r + x^2(2p_\Delta - r_N) - 2x^4 r_\Delta$ $-r$
S	$x = y$ $p = 0$	p_N $-r$	$\sqrt{-p_N r}$ $-\sqrt{-p_N r}$
T	$p = 0$ $r = 0$	sign $\det(dg) =$ sign $(p_\Delta r_N - p_N r_\Delta)$ sign $\mathrm{tr}(dg) =$ sign $\big(N(p_N - 2\Delta r_\Delta)$ $+ \Delta(2p_\Delta - r_N)\big)$	sign $\det(dg_*) =$ sign $(p_N r_\Delta - p_\Delta r_N)$ sign $\mathrm{tr}(dg_*) =$ sign $\big(-\delta(p_N - 2\Delta r_\Delta)$ $-\delta N(2p_\Delta - r_N)\big)$

Regarding the bifurcation diagrams, the symmetries of a solution branch are defined by its isotropy subgroup. The lattice of isotropy subgroups of $\mathbf{D}_4$ is given by $\mathbf{D}_4$, $\mathbb{Z}_2(\kappa)$, $\mathbb{Z}_2(\xi\kappa)$ and $\mathbf{1}$. Following the notation of [11], the solution branches with isotropy subgroups $\mathbb{Z}_2(\kappa)$, $\mathbb{Z}_2(\xi\kappa)$ and $\mathbf{1}$ are denoted by R, S and T, respectively.

The solution branches of $g_* = 0$ coincide with the solution branches of $g = 0$; their equations and expressions to study stability of solutions are given in Table 5. For the present case, the solutions with isotropies containing reversing symmetries lies on the trivial and S branches; the trivial solutions are of saddle type, for all $\lambda \neq 0$, and on the $S-$branch the eigenvalues are either a pair of real eigenvalues of opposite signs or a pair of purely imaginary eigenvalues. As a result, hyperbolic solutions on such branches are never stable. On the other hand, stability can occur on the R and T solution branches, once these have isotropy generated only by symmetries. Along the $T-$branch, there are cases where the two eigenvalues of $(dg_*)_{(x,y,\lambda)}$ change from both positive to both negative (or vice-versa), this change occurring at a secondary bifurcation point.

Analogously as for the $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$reversible-equivariant example, the transition varieties coincide with the transition varieties for the $\mathbf{D}_4-$equivariant bifurcation problem g (see Remark 4.1).

In Table 6 we present all the universal unfoldings up to topological codimension 2, with a choice of $+1$ or -1 in some of the coefficients $\varepsilon_i = \pm 1$ that appear in the normal forms. This choice follows [11], since our aim here is to explicit similarities and differences between both contexts. We use m and n for modal parameters and α and β for simple unfolding parameters. The unfolding I corresponds to the nondegenerate normal form. Normal forms

TABLE 6.

Universal Unfoldings for $\mathbf{D}_4$-reversible-equivariant bifurcations

	Universal Unfolding $[p, r]$	Topological Codimension
I	$[-\lambda + mN, 1]$, $m \neq 0, 1$	0
II	$[-\lambda + N^2 + \alpha N, \varepsilon_2]$	1
III	$[-\lambda + (1+\alpha)N + \varepsilon_2 \Delta, 1]$	1
IV	$[-\lambda + \varepsilon_1 N + m\Delta, N + \alpha]$, $m \neq 0$	1
V	$[\lambda^2 + mN + \lambda N + \alpha, \varepsilon_2]$, $m \neq 0, \varepsilon_2$	1
VI	$[-\lambda + N^3 + \beta N^2 + \alpha N, \varepsilon_2]$	2
VII	$[-\lambda + N + \varepsilon_2 N\Delta + \alpha N + \beta\Delta, 1]$	2
VIII	$[-\lambda + N + m\Delta^2 + \beta\Delta, \varepsilon_2 N + \alpha]$, $m \neq 0$	2
IX	$[-\lambda + N, \varepsilon_2\Delta + m\lambda^2 + \beta N + \alpha]$, $m \neq 0$	2
X	$[-\lambda + mN^2 + n\Delta + \alpha N, N + \varepsilon_2\Delta + \beta]$, $mn \neq 0$, $m + n \neq \frac{1}{2}$	2
XI	$[\lambda^2 + \varepsilon_1 N^2 + m\lambda N + \alpha + \beta N, \varepsilon_2]$, $m \geq 0$, $m^2 \neq 4\varepsilon_1$	2
XII	$[\lambda^2 + \varepsilon_1 N + \varepsilon_2\Delta + m\lambda N, \varepsilon_1]$, $m \geq 0$, $m^2 \neq 4\varepsilon_2$	2
XIII	$[\lambda^2 + \varepsilon_1 N + \alpha, \lambda + m\Delta + \beta]$, $m \neq 0$	2
XIV	$[-\lambda^3 + mN + \varepsilon_1\lambda N + \alpha + \beta\lambda, 1]$, $m \neq 0, 1$	2
XV	$[\lambda^2 + mN + \alpha + \beta\lambda N, \varepsilon_1]$, $m \neq 0, \varepsilon_1$	2

II - V are germs of topological codimension 1 and normal forms VI - XV are of topological codimension 2.

Here we present the bifurcation diagrams for the universal unfoldings I, IV and VIII (Figures 2, 3 and 4), including the transition varieties in the α, β-plane for the last case. For the remaining cases, the diagrams have the same shape as for the corresponding equivariants, with stability given by Proposition 4.3.

We then consider the normal form I. We have $p(N, \Delta, \lambda) = -\lambda + mN$ and $r(N, \Delta, \lambda) = 1$, $m \neq 0, 1$. The R and S branches are given by

$$R : \begin{cases} y = 0 \\ \lambda = (m-1)x^2 \end{cases} \quad \text{and} \quad S : \begin{cases} x = y \\ \lambda = 2mx^2 \end{cases}$$

The eigenvalues on the R and S branches are given by $\nu_1 = m - 1$, $\nu_2 = -1 < 0$ and $\eta_1 = \sqrt{-m}$, $\eta_2 = -\sqrt{-m}$, respectively. Examining the signs of these eigenvalues, we obtain three different types of bifurcation diagrams. Note that, for the normal form I, hyperbolic solutions are never stable (see Figure 2).

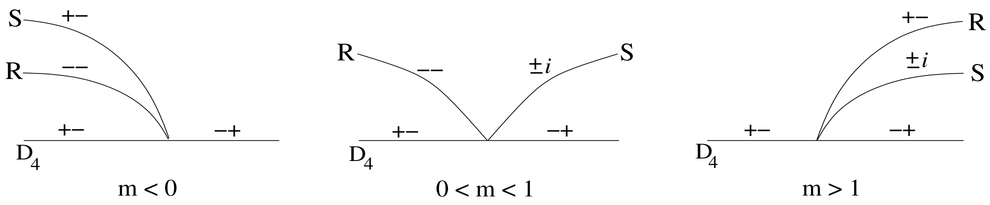

FIG. 2. Persistent bifurcation diagrams for the normal form I: $[-\lambda + mN, 1]$, $m \neq 0, 1$.

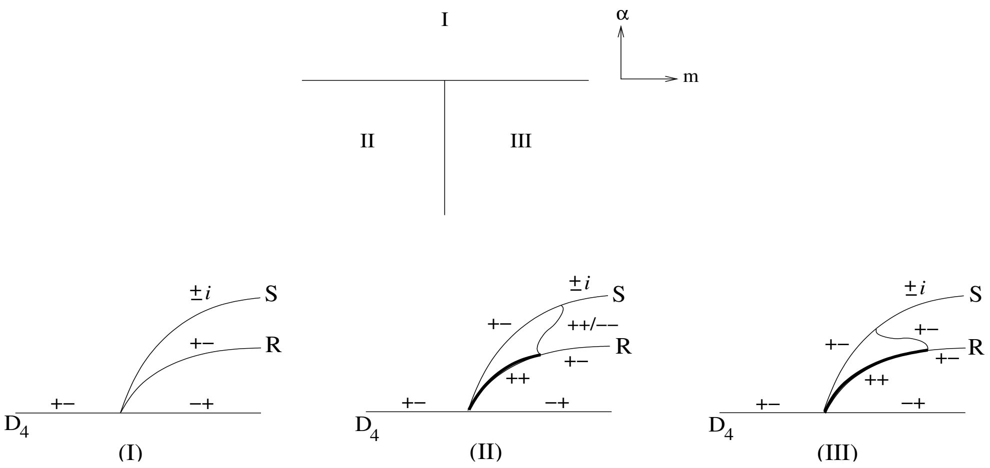

FIG. 3. Persistent bifurcation diagrams for the unfolding of normal form IV: $[-\lambda + \varepsilon_1 N + m\Delta, N + \alpha]$, $m \neq 0$ and $\varepsilon_1 = 1$.

Let us now consider the normal form IV. We have $p(N, \Delta, \lambda) = -\lambda + \varepsilon_1 N + m\Delta$ and $r(N, \Delta, \lambda) = N + \alpha$, $m \neq 0$. The R, S and T branches are given by

$$R : \left\{ \begin{array}{lcl} y & = & 0 \\ \lambda & = & (\varepsilon_1 - \alpha)x^2 + (m-1)x^4 \end{array} \right. , \qquad S : \left\{ \begin{array}{lcl} x & = & y \\ \lambda & = & 2\varepsilon_1 x^2 \end{array} \right.$$

$$\text{and} \quad T : \left\{ \begin{array}{lcl} N & = & -\alpha \\ \lambda & = & \varepsilon_1 N + m\Delta. \end{array} \right.$$

A $T-$branch exists if $\alpha < 0$, and $(dg_*)_{(x,y,\lambda)}$ has positive determinant when $m < 0$. In this case, we look at its trace, which is $-\delta\varepsilon_1 = -(y^2 - x^2)\varepsilon_1$. Therefore, an exchange of stability of type "++" to "−−" is forced to occur along the T-branch. This occurs at a secondary bifurcation, namely at the intersection of the S and T branches (when $y = x$), and is represented in the diagrams by "$++/--$".

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The three distinct regimes for unfoldings of the normal form IV for $\varepsilon_1 = 1$ are presented in Figure 3.

Finally, consider the normal form VIII so that $p(N, \Delta, \lambda) = -\lambda + N + m\Delta^2 + \beta\Delta$ and $r(N, \Delta, \lambda) = N+\alpha$, with $m \neq 0$. In this case, the transition varieties determine five regimes presented in Figure 4. The R, S and T branches are given by

$$R : \begin{cases} y &= 0 \\ \lambda &= (1-\alpha)x^2 + (\beta-1)x^4 + mx^8 \end{cases}, \qquad S : \begin{cases} x &= y \\ \lambda &= 2x^2 \end{cases}$$

$$\text{and} \quad T : \begin{cases} N &= -\alpha \\ \lambda &= N + m\Delta^2 + \beta\Delta. \end{cases}$$

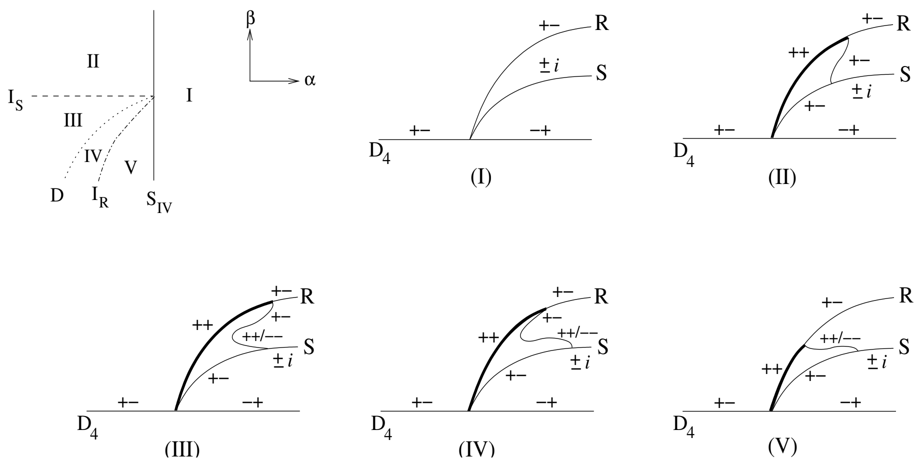

FIG. 4. Transition varieties and persistent bifurcation diagrams for the unfolding of normal form VIII: $[-\lambda + N + m\Delta^2 + \beta\Delta, N + \alpha]$, $m \neq 0$.

The $T-$branch exists only if $\alpha < 0$. As we shall see in Proposition 4.3, $T-$solutions of unfolding for all normal forms are hyperbolic solutions, and we give here the details of how eigenvalues "$++$" and "$--$" occur for case VIII. For that, we investigate the sign of the trace of $(dg_*)_{(x,y,\lambda)}$ when the determinant is positive, i.e., when $m < 0$:

$$\text{tr}(dg_*)_{(x,y,\lambda)} = \delta\big(N - 1 - 2N(2m\Delta + \beta)\big),$$

so

$$\text{sign} \;\; \text{tr}(dg_*)_{(x,y,\lambda)} = -\text{sign}(\delta) = -\text{sign}(y^2 - x^2).$$

Hence, an exchange of stability from "$++$" to "$--$" is forced to occur along the T-branch when it meets the S-branch $(y = x)$, represented in the diagrams by "$++/--$".

Remark 4. 2. $T-$solution branches can only occur for the unfoldings IV, VIII, IX, X and XIII. In this case, solutions with stability "$++$" are mapped onto solutions with stability "$--$", and vice-versa, by the reversing symmetries of $\mathbf{D}_4$. In the diagrams, this is represented by "$++/--$" on the $T-$branch.

4.3.1. **Stability preserving and $\mathbf{D}_4-$equivalence**

We already know that a stable hyperbolic solution in the reversible context can be found only if the reversing symmetries do not belong to its isotropy subgroup. For the present representation of $\mathbf{D}_4$ on $\mathbb{C}$, hyperbolic solutions on the trivial and S branches are never stable, and stability can occur on the R and T branches. In the next proposition we obtain the nature of each equilibrium for all the unfoldings of Table 6 from the stability (or instability) of the same point as equilibrium of the associated equivariant unfolding given by [11].

PROPOSITION 4.3. *Let g_* be a reversible-equivariant bifurcation problem under the present self-dual representation of $\mathbf{D}_4$ on $\mathbb{C}$ and let (x, y, λ) be a steady-state. We then have the following:*

1. *If $(x, y, \lambda) = (0, 0, \lambda)$, then it is a saddle point in the $\mathbf{D}_4-$reversible-equivariant problem, denoted in the diagrams by "$+-$";*
2. *If (x, y, λ) is an $R-$solution of saddle type of the $\mathbf{D}_4-$equivariant problem, then (x, y, λ) is a hyperbolic solution in the $\mathbf{D}_4-$reversible-equivariant problem with either positive or negative eigenvalues, denoted in the diagrams by "$++$" or "$--$", respectively;*
3. *If (x, y, λ) is an $R-$ or an $S-$solution of the $\mathbf{D}_4-$equivariant problem, where the eigenvalues are either both positive or both negative, then (x, y, λ) is a saddle of the $\mathbf{D}_4-$reversible-equivariant problem, denoted in the diagrams by "$+-$";*
4. *If (x, y, λ) is an $S-$solution of saddle type of the $\mathbf{D}_4-$equivariant problem, then (x, y, λ) is a non-hyperbolic solution of the $\mathbf{D}_4-$reversible-equivariant problem with purely imaginary eigenvalues, denoted in the diagrams by "$\pm i$";*
5. *If (x, y, λ) is a $T-$solution of the $\mathbf{D}_4-$equivariant problem where the eigenvalues have nonzero real part with same sign, then (x, y, λ) is a saddle in the $\mathbf{D}_4-$reversible-equivariant problem, denoted in the diagrams by "$+-$";*
6. *If (x, y, λ) is a $T-$solution of saddle type in the $\mathbf{D}_4-$equivariant problem, then (x, y, λ) is a solution in the $\mathbf{D}_4-$reversible-equivariant problem with eigenvalues with positive real parts (if $\mathrm{tr}(dg_*) > 0$) or negative real parts (if $\mathrm{tr}(dg_*) < 0$), denoted in the diagrams by "$++$" or "$--$", respectively. Also, to every mixed-mode solution with "$++$" stability corresponds another (symmetric) mixed-mode solution with "$--$" stability, and vice-versa.*

Proof: It is a direct consequence of the results in Table 5. □

We finally give sufficient conditions for linear stability of hyperbolic equilibria of g_* be invariant by $\mathbf{D}_4-$equivalence:

PROPOSITION 4.4. *Let g_* be a $\mathbf{D}_4-$reversible-equivariant bifurcation problem on the plane and let (x, y, λ) be a hyperbolic steady state. Then the stability of (x, y, λ) is invariant under $\mathbf{D}_4-$equivalence if any of the following holds:*

(a) (x, y, λ) *is the trivial solution;*
(b) (x, y, λ) *is of type R or S;*
(c) (x, y, λ) *is of type T and* $\det(dg_*)_{(x,y,\lambda)} < 0$;
(d) (x, y, λ) *is of type T,* $\det(dg_*)_{(x,y,\lambda)} > 0$, $p_N \neq 0$ *with* $p_N \cdot p_\Delta \geq 0$, $p_N \cdot r_N \leq 0$ *e* $p_N \cdot r_\Delta \leq 0$ *at the origin.*

Proof: The proof of the cases $(a)-(c)$ follows the steps of the proof of [11, Proposition 3.2], taking into account the form of the matrix of $(dg_*)_{(x,y,\lambda)}$ on each branch. We now consider case (d). First, we notice that stability is preserved by $\mathbf{D}_4-$equivalence if, and only if, it is preserved by changes of coordinates $S(x, \lambda)$ in the target. Since $\det S(x, y, \lambda) > 0$, we have

$$\mathrm{sign}\big(\det(dg_*)_{(x,y,\lambda)}\big) = \mathrm{sign}\big(\det S(x, y, \lambda)(dg_*)_{(x,y,\lambda)}\big),$$

which are positive by hypothesis. So to check stability preserving we compare $\mathrm{tr}(dg_*)_{(x,y,\lambda)}$ and $\mathrm{tr}\big(S(x, y, \lambda)(dg_*)_{(x,y,\lambda)}\big)$: According to (21), the jacobian matrix of g_* at a mixed-mode solution is

$$(dg_*)_{(x,y,\lambda)} = \begin{pmatrix} (p + r\delta)_x x & (p + r\delta)_y x \\ (-p + r\delta)_x y & (-p + r\delta)_y y \end{pmatrix}. \tag{22}$$

Thus, we obtain

$$\mathrm{tr}(dg_*)_{(x,y,\lambda)} = -2\delta(p_N + 2Np_\Delta - Nr_N - 2\Delta r_\Delta). \tag{23}$$

The $\mathbf{D}_4-$equivariance implies that

$$S(x, y, \lambda) = \begin{pmatrix} a + bx^2 & cxy + dx^3y \\ cxy + dxy^3 & a + by^2 \end{pmatrix},$$

where a, b, c and d are $\mathbf{D}_4-$invariants. So we have

$$\begin{aligned} \mathrm{tr}\big(S(x, y, \lambda)(dg_*)_{(x,y,\lambda)}\big) \;=\;& -2\delta\big[(ap_N + bNp_N - dx^2y^2p_N \\ &- 4(b + c)x^2y^2p_\Delta + 2(b - c)x^2y^2r_N) \\ &+ 2N(ap_\Delta + bNp_\Delta - dx^2y^2p_\Delta) - N(ar_N + bNr_N + dx^2y^2r_N) \\ &- 2\Delta(ar_\Delta + bNr_\Delta + dx^2y^2r_\Delta)\big]. \end{aligned}$$

Now, since $a(0,0,0)$ is positive and $p_N \neq 0$, with $p_N \cdot p_\Delta \geq 0$, $p_N \cdot r_N \leq 0$ and $p_N \cdot r_\Delta \leq 0$ at the origin, we have

$$\text{sign } \mathrm{tr}(dg_*)_{(x,y,\lambda)} = \text{sign } \mathrm{tr}\big(S(x,y,\lambda)(dg_*)_{(x,y,\lambda)}\big).$$

□

The proof is complete.

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