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Singularly non-autonomous semilinear parabolic problems with critical exponents and applications

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In this work we consider initial value problems of the form

$$\begin{cases} \dfrac{dx}{dt} + A(t)x = f(t,x) \\ x(\tau) = x_0, \end{cases}$$

in a Banach space X where $A(t) : D \subset X \to X$ is a linear, closed and unbounded operator which is sectorial for each t. We show local well posedness for the case when the nonlinearity f grows critically. Some applications are considered. May, 2007 ICMC-USP

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1. INTRODUCTION

Let X be a Banach space and $A(t)$, $t \in \mathbb{R}$, be a family of linear, closed and unbounded operators defined on a fixed dense subspace D of X. We consider singularly non-autonomous

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semilinear abstract parabolic problems of the form

$$\begin{cases} \dfrac{dx}{dt} + A(t)x = f(t,x), \quad t > 0 \\ x(\tau) = x_0 \in D, \end{cases} \tag{1}$$

when the nonlinearity $f : \mathbb{R} \times D \to X$ is *critical* (has the *same order* as $A(t)$). We prove, under suitable assumptions on f, that (1) is locally well posed.

Remark 1. 1. Note that $\mathbb{R}\times D \ni (t,x) \mapsto 2A(t)x \in X$ is globally Lipschitz continuous on the second variable, nonetheless (1) will not be locally well posed with $f(t,x) = -2A(t)x$. The conditions on the map f will need to carry some intrinsic regularization to overcome this difficulty.

Definition 1.1. A family $\{U(t,\tau) : \ t \geq \tau \in \mathbb{R}\} \subset L(X)$ that it satisfies

$$\begin{aligned} &1)\ U(\tau,\tau) = I, \\ &2)\ U(t,\sigma)U(\sigma,\tau) = U(t,\tau), \text{ for any } \ t \geq \sigma \geq \tau, \\ &3)\ (t,\tau) \mapsto U(t,\tau)v_0 \ \text{ is continuous for } \ t \geq \tau, \ v_0 \in X, \end{aligned}$$

it is called a *process* or *family of linear evolution operators*.

To obtain solutions for (1) we will need to obtain good properties for the solution operator associated to

$$\begin{cases} \dfrac{dx}{dt} + A(t)x = 0, \ t > \tau \\ x(\tau) = x_0 \in X. \end{cases} \tag{2}$$

We will show that, under suitable assumptions on the operators $A(t)$, $t \in \mathbb{R}$, that for each $\tau \in \mathbb{R}$ the problem (2) has a unique solution $[\tau,\infty) \ni t \mapsto x(t,\tau,x_0) \in X$ defined in $[\tau,\infty)$. Denote by $\{U(t,\tau) \in L(X) : t \geq \tau \in \mathbb{R}\}$ the operator family defined by $U(t,\tau)x_0 = x(t,\tau,x_0)$. We will see that the family of operators $\{U(t,\tau) \in L(X) : t \geq \tau \in \mathbb{R}\}$ defines a process with suitable properties.

With $\{U(t,\tau) \in L(X) : t \geq \tau \in \mathbb{R}\}$ as above, the solution of (1) will be given by

$$x(t) = U(t,\tau)x_0 + \int_\tau^t U(t,s)f(s,x(s))ds, \quad x(\tau) = x_0.$$

In the hyperbolic case, Kato [9] was the first to construct the family of evolution operators of time inhomogeneous equations, the results were later revised by himself in [10]. For the parabolic case, Sobolevskii [14] was the first to construct such family (see also[7, 13]). In the next section, the construction of the family of evolution operators is explained following some of the arguments given in [14]. We will use many of the ideas of Sobolevskii, therefore

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we will omit the proofs that can be found in [14]. Instead of a cumbersome approximation argument we use a fixed point argument to show to construct the operator family $U(t, \tau)$.

In [5], the authors show local well posedness for the problem (1) in the case where the operator $A(t)$ is time independent, that is, $A(t) = A$, A is a sectorial operator, and the nonlinearity f is critically growing. In this paper, we extend the results of [5] to the case when the problem is singularly non-autonomous; that is, the linear unbounded main part of the equation is also non-autonomous and f is critically growing.

Next we introduce, following [5], the needed terminology that give meaning to the expression *critical growth*. Roughly saying f it presents critical growth when it is of same order that A.

Denote by A_0 the operator $A(t_0)$ for some $t_0 \in \mathbb{R}$ fixed. If X_α denotes the domain of $(A_0)^\alpha$ with the graph norm, $\alpha > 0$, $X_0 := X$, denote by $\{X_\alpha : \alpha \geq 0\}$ the fractional power scale associated to A_0.

Definition 1.2. We say that $x : [0, \tau_0] \to X_1$ is an $\epsilon-$regular mild solution of (1) if $x \in C([0, \tau_0], X_1) \bigcap C((0, \tau_0], X_{1+\epsilon})$, and $x(t)$ satisfies

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s, x(s))ds.$$

Definition 1.3. For $\epsilon \geq 0$, we say that a map g is an $\epsilon-$regular map relative to the pair (X_1, X_0) if there exist $\rho > 1$, $\gamma(\epsilon)$ with $\rho\epsilon \leq \gamma(\epsilon) < 1$, and a constant c, such that $g : X_{1+\epsilon} \to X_{\gamma(\epsilon)}$ and

$$\|g(x) - g(y)\|_{X_{\gamma(\epsilon)}} \leq c\|x - y\|_{X_{1+\epsilon}}(\|x\|^{\rho-1}_{X_{1+\epsilon}} + \|y\|^{\rho-1}_{X_{1+\epsilon}} + 1) \quad \forall x, y \in X_{1+\epsilon}.$$

We observe that it is possible that g is defined in X_1 and takes values in X_0 (*it is of same order as* A) or even that g is not defined in X_1. Because of this ϵ-regularity is, at times, called critical growth condition. Our main result reads

Theorem 1.1. *Assuming that* $A(t)$ *is* uniformly sectorial (*see condition* (*a*)) *and* uniformly Hölder continuous (*see condition* (*b*)) *and that* f *is* $\epsilon-$*regular relatively to the pair* (X_1, X_0), *then* (1) *is locally well posed.*

The results presented here are a natural extension of the results that appear in the recent literature of semilinear autonomous problems (A time independent) concerning local well posedness with critically growing nonlinearities. We can say that the local well posedness for autonomous semilinear parabolic problems with critical nonlinearities has been satisfactorily answered whereas the analogous question for singularly non-autonomous problems (when the unbounded operator depend of the time) has not been considered yet. Our aim is to provide results as complete as those available for the autonomous case.

This paper is organized as follows. In Section 2, we construct a family of evolution operators, following the ideas contained in [14], in some cases presents new proofs and slightly better results with a different approach. In Section 3, we show the local well posedness of semilinear parabolic problems with critically growing nonlinearities and, in Section 4, we apply the results of the Section 3.

2. SINGULARLY NON-AUTONOMOUS LINEAR PARABOLIC EQUATIONS

Note that, for $\tau \in \mathbb{R}$ fixed, the family $\{e^{-(t-\tau)A(\tau)} : t \geq \tau\}$ is the solution operator family associated to

$$\frac{dx}{dt} + A(\tau)x = 0. \tag{3}$$

Therefore, if $\{U(t,\tau) \in L(X) : t \geq \tau \in \mathbb{R}\}$ is the linear evolution process associated to (2), the difference $\{U(t,\tau) - e^{-(t-\tau)A(\tau)} : t \geq \tau\}$ is the family of the solution operators associated to

$$\frac{dx}{dt} + A(t)x = -[A(t) - A(\tau)]e^{-(t-\tau)A(\tau)} \tag{4}$$

with trivial initial condition. Hence we should have

$$U(t,\tau) = e^{-(t-\tau)A(\tau)} + \int_\tau^t U(t,s)[A(\tau) - A(s)]e^{-(s-\tau)A(\tau)}ds. \tag{5}$$

To obtain the evolution process $\{U(t,\tau) \in L(X) : t \geq \tau \in \mathbb{R}\}$ we will use (5); that is, we show that, for each $\tau \in \mathbb{R}$, equation (5) has an unique continuous solution $\{U(t,\tau) \in L(X) : t \geq \tau\}$.

We assume that

(a) The operator $A(t) : D \subset X \to X$ is a closed densely defined operator (the domain D is fixed) and there is a constant $C > 0$ (independent of $t \in \mathbb{R}$) such that

$$\|[A(t) + \lambda I]^{-1}\|_{L(X)} \leq \frac{C}{|\lambda| + 1}; \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq 0. \tag{6}$$

To express this fact we will say that the family $A(t)$ is *uniformly sectorial.*

(b) There are constants $C > 0$ and $\epsilon > 0$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{L(X)} \leq C(t-\tau)^\epsilon, \quad (\epsilon \in (0,1]). \tag{7}$$

To express this fact we will say that the function $A(t)$ is *uniformly Hölder continuous.*

From (a), $-A(t)$ is the generator of an analytic semigroup $\{e^{-\tau A(t)} \in L(X) : \tau \geq 0\}$.

With this and of the fact that $0 \in \rho(A(t))$, follows that, for some $\delta > 0$

$$\|e^{-\tau A(t)}\|_{L(X)} \leq Ce^{-\delta\tau}, \tag{8}$$

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$$\|A(t)e^{-\tau A(t)}\|_{L(X)} \leq C\tau^{-1}e^{-\delta\tau}. \tag{9}$$

From (b) it follows that

$$\|A(t)A^{-1}(\tau)\|_{L(X)} \leq C \qquad (t,\tau) \in \mathbb{R}^2.$$

Under these assumptions we show that (5) has an unique solution $U(t,\tau)$ that is continuous for $t \geq \tau$. Fix $\tau < t_0$ and denote for $U(t)$ the operator $U(t_0, t)$. Consider the complete metric space

$$E = \{U \in C([\tau, t_0), L(X)) : \sup_{t\in[\tau,\, t_0)} \|U(t)\|_{L(X)} < \infty\}$$

with the metric

$$\|U\|_E = \sup_{t\in[\tau,\, t_0)} \|U(t)\|_{L(X)}.$$

Let S be the map defined by

$$(SU)(t) = e^{-(t_0-t)A(t)} + \int_t^{t_0} U(s)[A(t) - A(s)]e^{-(s-t)A(t)}ds.$$

If $U \in E$, then the integral in the above inequality is convergent and

$$t \mapsto \int_t^{t_0} U(s)[A(t) - A(s)]e^{-(s-t)A(t)}ds$$

is in $C([\tau, t_0), L(X))$. To show that $t \mapsto e^{-(t_0-t)A(t)}$ it belongs $C([\tau, t_0), L(X))$ we will need the following auxiliary results (see [7, 14] for a proof).

Lemma 2.1. *For $t, s, \xi \in \mathbb{R}$, the following inequalities hold*

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\|_{L(X)} \leq Ce^{-\delta\tau}(t-s)^{\epsilon}, \quad \textit{for } \tau \geq 0, \tag{10}$$

$$\|A(\xi)[e^{-\tau A(t)} - e^{-\tau A(s)}]\|_{L(X)} \leq Ce^{-\delta\tau}\tau^{-1}(t-s)^{\epsilon}, \quad \textit{for } \tau > 0. \tag{11}$$

The proof follows using the identity

$$[e^{-(t-\tau)A(\tau)} - e^{-(\theta-\tau)A(\tau)}] = -A^{-1}(\tau)\int_0^{t-\theta} e^{-\xi A(\tau)}d\xi\; A^2(\tau)\, e^{-(\theta-\tau)A(\tau)}. \tag{12}$$

Lemma 2.2. *The functions $(0,\infty)\times\mathbb{R} \ni (\tau, s) \mapsto e^{-\tau A(s)} \in L(X)$ and $(0,\infty)\times\mathbb{R}\times\mathbb{R} \ni (\tau, t, s) \mapsto A(t)e^{-\tau A(s)} \in L(X)$ are continuous in the uniform operator topology of $L(X)$.*

Corollary 2.1. *The functions*

$$(t,\tau)\mapsto e^{-(t-\tau)A(\tau)},\quad (t,\tau)\mapsto e^{-(t-\tau)A(t)},$$

$$(t,\tau)\mapsto [A(\tau)-A(t)]e^{-(t-\tau)A(\tau)} \text{ and } (t,\tau)\mapsto [A(\tau)-A(t)]e^{-(t-\tau)A(t)}$$

are continuous from $\{(t,\tau)\in\mathbb{R}^2 : t>\tau\}$ *in* $L(X)$ *with the uniform operator topology.*

Follows of the Corollary 2.1 that $t\mapsto e^{-(t_0-t)A(t)}$ is in E. In what follows, we prove that S is a contraction from E into itself and therefore has an only fixed point in E.

Recall that the *beta function* $\mathbf{B}(\cdot,\cdot):(0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ is defined by

$$\mathbf{B}(a,b)=\int_0^1(1-s)^{a-1}s^{b-1}ds.$$

Given $U,V\in E$, it follows that

$$(SU)(t)-(SV)(t)=\int_t^{t_0}[U(s)-V(s)][A(t)-A(s)]e^{-(s-t)A(t)}ds.$$

Thus

$$\begin{aligned}
&\|(SU)(t)-(SV)(t)\|_{L(X)}\\
&\le\int_t^{t_0}\|U(s)-V(s)\|_{L(X)}\,\|A(s)-A(t)\|_{L(X^1,X)}\,\|e^{-(s-t)A(t)}\|_{L(X,X^1)}ds\\
&\le\int_t^{t_0}C(s-t)^{\epsilon}C(s-t)^{-1}ds\sup_{t\le s\le t_0}\left\{\|U(s)-V(s)\|_{L(X)}\right\}\\
&\le C_1\int_t^{t_0}(s-t)^{\epsilon-1}ds\,\|U-V\|_E\le C_1(\epsilon)(t_0-t)^{\epsilon}\,\|U-V\|_E\,.
\end{aligned}$$

Following with the iterations, we get that

$$\begin{aligned}
&\|(S^nU)(t)-(S^nV)(t)\|_{L(X)}\\
&\le C(\epsilon)\mathbf{B}(1+\epsilon,\epsilon)\mathbf{B}(1+2\epsilon,\epsilon)\dots\mathbf{B}(1+(n-1)\epsilon,\epsilon)(t_0-t)^{n\epsilon}\,\|U-V\|_E\,.
\end{aligned}$$

Recalling that $\mathbf{B}(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$, where Γ is the *gamma function*, we have

$$\mathbf{B}(1+\epsilon,\epsilon)\mathbf{B}(1+2\epsilon,\epsilon)\dots\mathbf{B}(1+(n-1)\epsilon,\epsilon)=\frac{\Gamma(1+\epsilon)(\Gamma(\epsilon))^n}{\Gamma(1+n\epsilon)}.$$

That is,

$$\|(S^nU)(t)-(S^nV)(t)\|_{L(X)}\le\frac{C(\epsilon)\Gamma(1+\epsilon)[\Gamma(\epsilon)(t_0-t)^{\epsilon}]^n\,\|U-V\|_E}{\Gamma(1+n\epsilon)}.$$

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Consider the power series

$$\sum_{n=1}^{\infty} \frac{C(\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+n\epsilon)} s^n.$$

If it has infinite radius of convergence, for each $s \in \mathbb{R}$ there is a $n \in \mathbb{N}$ such that

$$\frac{C(\epsilon)\Gamma(1+\epsilon)[\Gamma(\epsilon)(t_0-t)^{\epsilon}]^n}{\Gamma(1+n\epsilon)} < 1,$$

and then by the Banach contraction principle, S^n has an unique fixed point in E and, consequently, S has an unique fixed point in E.

To verify that the radius of convergence is infinitive, we apply the ratio test. Dividing the $(n+1)^{\text{th}}$ coefficient by the n^{th} coefficient of the series we obtain

$$\Gamma(\epsilon)^{-1}\mathbf{B}(\epsilon, 1+n\epsilon).$$

Since

$$\mathbf{B}(\epsilon, 1+n\epsilon) = \int_0^1 s^{\epsilon-1}(1-s)^{n\epsilon} ds,$$

and for each $0 < s \leq 1$ the integrand is bounded by $s^{\epsilon-1}$ and pointwise converges to zero, as $n \to \infty$, it follows of the Dominated convergence theorem that $\mathbf{B}(\epsilon, 1+n\epsilon) \overset{n\to\infty}{\longrightarrow} 0$. This completes the proof that S has a unique fixed point in E.

As t_0 it is arbitrary in $[0,T]$, (5) has an unique solution $U(t,\tau)$ that is continuous in the uniform topology of the operators for all $t > \tau$.

The strong continuity of the family of evolution operators $\{U(t,\tau) : t \geq \tau\}$ in $t = \tau$ follows immediately from (5) and of the following results (see [7, 14]).

Lemma 2.3. *For $\tau \geq 0$ the function $(t, s, \tau, \xi) \mapsto A(t)e^{-\tau A(s)}A^{-1}(\xi) \in L(X)$ is continuous in the strong operator topology.*

Corollary 2.2. *For $t \geq \tau$ the functions*

$$(t,\tau) \mapsto e^{-(t-\tau)A(\tau)}, \quad (t,\tau) \mapsto e^{-(t-\tau)A(t)}$$

and

$$(t,\tau) \mapsto [A(\tau) - A(t)]e^{-(t-\tau)A(\tau)}A^{-1}(\tau)$$

are continuous in the strong operator topology.

With this, we have that the family $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defined by (5) satisfies the properties (1) and (3) in Definition 1.1. Clearly this is the process associated to (2).

Definition 2.1. We say that $[\tau, \infty) \ni t \mapsto U(t,\tau)x_0 \in X$ is a *w*eak solution of (2).

In what it is follows we show that a weak solution for (2) is in fact a solution in the classical sense.

2.1. Regularity of the family $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$.

In this section we show that $x(t) := U(t,\tau)x_0$, $t \geq \tau$, is the solution of (2) in the sense that $x(t)$ is differentiable for $t > \tau$, $x(t) \in D(A(t))$ for $t > \tau$ and it verifies (2).

To this end we give alternative expressions for $U(t,\tau)$.

2.1.1. Alternative expressions for $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$.

Let

$$\varphi_1(t,\tau) = [A(\tau) - A(t)]e^{-(t-\tau)A(\tau)}, \tag{13}$$

and assume that the integral equation

$$\Phi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \Phi(t,s)\varphi_1(s,\tau)ds \tag{14}$$

has an unique solution $\Phi(t,\tau)$, $t > \tau$. Making

$$\bar{U}(t,\tau) = e^{-(t-\tau)A(\tau)} + \int_\tau^t e^{-(t-s)A(s)}\Phi(s,\tau)ds,$$

and noting that

$$\begin{aligned}\int_\tau^t \bar{U}(t,s)\varphi_1(s,\tau)ds &= \int_\tau^t [e^{-(t-s)A(s)} + \int_s^t e^{-(t-\theta)A(\theta)}\Phi(\theta,s)d\theta]\varphi_1(s,\tau)ds\\ &= \int_\tau^t e^{-(t-s)A(s)}\varphi_1(s,\tau)ds + \int_\tau^t e^{-(t-s)A(s)}\int_\tau^s \Phi(s,\theta)\varphi_1(\theta,\tau)d\theta ds\\ &= \int_\tau^t e^{-(t-s)A(s)}\Phi(s,\tau)ds,\end{aligned}$$

we have that $\bar{U}(t,\tau) = U(t,\tau)$ for all $t \geq \tau$ and

$$U(t,\tau) = e^{-(t-\tau)A(\tau)} + \int_\tau^t e^{-(t-s)A(s)}\Phi(s,\tau)ds. \tag{15}$$

It remains to show that (14) has an unique solution to ensure that (15) holds. The expression (15) will be used to obtain regularity properties of $t \mapsto U(t,\tau)x_0$.

Note that $\varphi_1(s,\tau) = [A(\tau) - A(s)]e^{-(s-\tau)A(\tau)}$ is continuous in the uniform operator topology in all variables for $s > \tau$ (see Corollary 2.1). From (7) and (9) we have that

$$\|\varphi_1(s,\tau)\|_{L(X)} \leq Ce^{-\delta(s-\tau)}(s-\tau)^{\epsilon-1}. \tag{16}$$

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With this in mind we show that (14) has an unique solution $\Phi(t,\tau)$ that is continuous in the uniform operator topology for all (t,τ) with $t>\tau$ and which satisfies

$$\|\Phi(t,\tau)\|_{L(X)} \le C(t-\tau)^{\epsilon-1}. \tag{17}$$

The procedure is analogous to that used to obtain $U(t,\tau)$. Fix $t_0\in\mathbb{R}$, with $t_0>\tau$ and consider the complete metric space

$$Y=\{\Phi\in C([\tau,t_0),L(X)) : \sup_{t\in[\tau,\, t_0)} (t_0-t)^{1-\epsilon}\left\|\Phi(t)\right\|_{L(X)}<\infty\}$$

with the metric

$$\|\Phi\|_Y = \sup_{t\in[\tau,\, t_0)} (t_0-t)^{1-\epsilon}\left\|\Phi(t)\right\|_{L(X)}.$$

Define the map G by

$$(G\Phi)(t)=\varphi_1(t_0,t)+\int_t^{t_0}\Phi(s)\varphi_1(s,t)ds.$$

Clearly G takes Y into $C([\tau,t_0),L(X))$. Next we prove that G takes Y into itself and that, for some $n\in\mathbb{N}$, G^n is a contraction. In fact,

$$(t_0-t)^{1-\epsilon}\|(G\Phi)(t)\|_{L(X)}\le C\left[1+(t_0-t)^{\epsilon}\|\Phi\|_Y\int_0^1(1-s)^{\epsilon-1}s^{\epsilon-1}ds\right]$$

proves that $G:Y\longrightarrow Y$. Now, for $\Phi,\Psi\in Y$,

$$(G\Phi)(t)-(G\Psi)(t)=\int_t^{t_0}[\Phi(s)-\Psi(s)]\varphi_1(s,t)ds$$

and

$$(t_0-t)^{1-\epsilon}\|(G\Phi)(t)-(G\Psi)(t)\|_{L(X)}\le C\mathbf{B}(\epsilon,\epsilon)(t_0-t)^{\epsilon}\left\|\Phi-\Psi\right\|_Y.$$

Using this and iterating we obtain that

$$\|(G^n\Phi)(t)-(G^n\Psi)(t)\|_Y\le C^n\mathbf{B}(\epsilon,\epsilon)\mathbf{B}(2\epsilon,\epsilon)\dots\mathbf{B}(n\epsilon,\epsilon)(t_0-t)^{n\epsilon}\left\|\Phi-\Psi\right\|_Y.$$

That is,

$$\|(G^n\Phi)(t)-(G^n\Psi)(t)\|_{L(X)}\le\frac{C^n\Gamma(\epsilon)[\Gamma(\epsilon)(t_0-t)^{\epsilon}]^n\left\|\Phi-\Psi\right\|_Y}{\Gamma((n+1)\epsilon)}.$$

From Banach contraction principle G has an unique fixed point in $C([\tau,\ t_0),L(X))$. Concluding the proof that (14) has a unique solution.

To show that $U(t,\tau)x_0$ is a classical solution of (2), we use (14), (15) and a alternative definition of Φ given next. Define

$$\Psi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Phi(s,\tau)ds,$$

and note that

$$\begin{aligned}\int_\tau^t \Psi(t,s)\varphi_1(s,\tau)ds &= \int_\tau^t [\varphi_1(t,s) + \int_s^t \varphi_1(t,\theta)\Phi(\theta,s)d\theta]\varphi_1(s,\tau)ds \\ &= \int_\tau^t \varphi_1(t,\theta)\varphi_1(\theta,\tau)d\theta + \int_\tau^t \varphi_1(t,\theta)\int_\tau^\theta \Phi(\theta,s)\varphi_1(s,\tau)dsd\theta \\ &= \int_\tau^t \varphi_1(t,\theta)\Phi(\theta,\tau)d\theta,\end{aligned}$$

then $\Psi(t,\tau) = \Phi(t,\tau)$ and thus Φ defined by (14) also it is given by

$$\Phi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Phi(s,\tau)ds. \tag{18}$$

In the next two lemmas we study some regularity properties of the function $\Phi(t,\tau)$. These results can be found in [14] and its proofs are included here for completeness.

Lemma 2.4. *For each $0 \le \eta \le \epsilon$, $\tau < \theta \le t$*

$$\|\varphi_1(t,\tau) - \varphi_1(\theta,\tau)\|_{L(X)} \le C\,(t-\theta)^{\epsilon-\eta}\,(\theta-\tau)^{\eta-1}. \tag{19}$$

Proof: Note that

$$\begin{aligned}\varphi_1(t,\tau) - \varphi_1(\theta,\tau) = &- [A(t) - A(\theta)]e^{-(t-\tau)A(\tau)} \\ &- [A(\theta) - A(\tau)][e^{-(t-\tau)A(\tau)} - e^{-(\theta-\tau)A(\tau)}].\end{aligned} \tag{20}$$

From this and (7), it follows immediately that

$$\|[A(t) - A(\theta)]e^{-(t-\tau)A(\tau)}\|_{L(X)} \le C(t-\theta)^{\epsilon}(t-\tau)^{-1} \tag{21}$$

and that

$$\|[A(\theta) - A(\tau)][e^{-(t-\tau)A(\tau)} - e^{-(\theta-\tau)A(\tau)}]\|_{L(X)} \le C(\theta-\tau)^{\epsilon-1}. \tag{22}$$

From (12) it follows that

$$\|[A(\theta)-A(\tau)][e^{-(t-\tau)A(\tau)} - e^{-(\theta-\tau)A(\tau)}]\|_{L(X)} \le C\,(t-\theta)(\theta-\tau)^{\epsilon-2}. \tag{23}$$

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Interpolating with exponents $1-\epsilon$ and ϵ, from (22) and (23), we have

$$\|[A(\theta)-A(\tau)][e^{-(t-\tau)A(\tau)}-e^{-(\theta-\tau)A(\tau)}]\|_{L(X)} \leq C\,(t-\theta)^{\epsilon}\,(\theta-\tau)^{-1}. \tag{24}$$

Hence, from (21) and (24) we have

$$\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{L(X)} \leq C\,(t-\theta)^{\epsilon}\,(\theta-\tau)^{-1}.$$

Using this and (16) the result follows by interpolation with exponents $\frac{\epsilon-\eta}{\epsilon}$ e $\frac{\eta}{\epsilon}$. □

LEMMA 2.5. *For each* $0<\eta\leq\epsilon$, $\tau<\theta\leq t$,

$$\|\Phi(t,\tau)-\Phi(\theta,\tau)\|_{L(X)} \leq C\,(t-\theta)^{\epsilon-\eta}(\theta-\tau)^{\eta-1}. \tag{25}$$

Proof: From (14), we have that

$$\begin{aligned}\Phi(t,\tau)-\Phi(\theta,\tau) &= \varphi_1(t,\tau)-\varphi_1(\theta,\tau)\\ &\quad+\int_\theta^t \varphi_1(t,s)\Phi(s,\tau)ds+\int_\tau^\theta[\varphi_1(t,s)-\varphi_1(\theta,s)]\Phi(s,\tau)ds.\end{aligned}$$

From this and (19) and from the fact that $\Phi\in Y$ the result follows. □

2.1.2. Classical solutions

For each $v_0\in X$, we show (following [14]) that the function $(\tau,\infty)\ni t\mapsto v(t):=U(t,\tau)v_0\in X$ is continuously differentiable, $v(t)\in D$ for all $t\in(\tau,\infty)$ and satisfies (2).

Fix $v_0\in X$ and let $t-\tau\geq\gamma>0$, $0<\rho<\gamma$. Define $v_\rho:(\tau+\gamma,\infty)\to X$ by

$$v_\rho(t):=U_\rho(t,\tau)v_0=e^{-(t-\tau)A(\tau)}v_0+\int_\tau^{t-\rho}e^{-(t-s)A(s)}\Phi(s,\tau)v_0ds.$$

To show that $v(\cdot):(\tau,\infty)\to X$ is continuously differentiable we will proceed in the following manner. For each $\gamma>0$ and $t_0>\tau+\gamma$, we show that, for all $0<\rho<\gamma$, $v_\rho(\cdot)$ and $\frac{d}{dt}v_\rho(\cdot)$ converge $v(\cdot)$ and $A(\cdot)v(\cdot)$ in $C([\tau+\gamma,t_0],X)$. In particular, this will show that $\dot{v}(t)=A(t)v(t)$.

For fixed $\tau\in\mathbb{R}$, $t_0>\tau$, $0<\gamma<t_0-\tau$, and arbitrary $\rho\in(0,\gamma)$ note that

$$\sup_{t\in[\tau+\gamma,t_0]}\|U_\rho(t,\tau)v_0-U(t,\tau)v_0\|_X\leq C\epsilon^{-1}[(t-\tau)^{\epsilon}-(t-\rho-\tau)^{\epsilon}]\|v_0\|_X\overset{\rho\to0}{\longrightarrow}0.$$

In addition, $[\tau+\gamma,t_0]\ni t\mapsto U_\rho(t,\tau)v_0\in X$ is continuously differentiable. Using (13), (15) and (18) we have that

$$\begin{aligned}\frac{\partial U_\rho(t,\tau)}{\partial t}v_0 = & -A(t)e^{-(t-\tau)A(\tau)}v_0 - A(t)\int_\tau^{t-\rho} e^{-(t-s)A(s)}\Phi(s,\tau)v_0ds\\ & + \int_{t-\rho}^t \varphi_1(t,s)\Phi(s,\tau)v_0ds + e^{-\rho A(t-\rho)}[\Phi(t-\rho,\tau)-\Phi(t,\tau)]v_0\\ & + [e^{-\rho A(t-\rho)}-I]\Phi(t,\tau)v_0.\end{aligned} \tag{26}$$

Let us show that the last three terms in the above expression converges to zero as $\rho\to 0$, uniformly in $[\tau+\gamma,t_0]$ and that the first two converge uniformly in $[\tau+\gamma,t_0]$ to $A(\cdot)v(\cdot)$.

From (16) and from the fact that $\Phi\in Y$ we have that

$$\sup_{t\in[\tau+\gamma,t_0]}\|\int_{t-\rho}^t\varphi_1(t,s)\Phi(s,\tau)ds\|_{L(X)}\le C\epsilon^{-1}\rho^\epsilon(\gamma-\rho)^{\epsilon-1}\overset{\rho\to 0}{\longrightarrow}0. \tag{27}$$

Note that the functions $[\tau+\gamma,t_0]\ni t\mapsto\Phi(t,\tau)v_0\in X$ is continuous and that $[e^{-\rho A(t-\rho)}-I]\to 0$ as $\rho\to 0$ in the strong topology of operators uniformly for $t\in[\tau+\gamma,t_0]$. It follows from the compactness of $\{\Phi(t,\tau)v_0 : t\in[\tau+\gamma,t_0]\}\subset X$ that

$$\sup_{t\in[\tau+\gamma,t_0]}[e^{-\rho A(t-\rho)}-I]\Phi(t,\tau)v_0\overset{\rho\to 0}{\longrightarrow}0.$$

It follows from (25) that, for any $\eta\in(0,\epsilon)$,

$$\sup_{t\in[\tau+\gamma,t_0]}\|e^{-\rho A(t-\rho)}[\Phi(t-\rho,\tau)-\Phi(t,\tau)]v_0\|_X\le C\rho^{\epsilon-\eta}(\gamma-\rho)^{\eta-1}\|v_0\|_X\overset{\rho\to 0}{\longrightarrow}0.$$

It remains to prove that $A(t)\int_\tau^{t-\rho}e^{-(t-s)A(s)}\Phi(s,\tau)v_0ds$ converges, uniformly in $[\tau+\gamma,t_0]$, to $A(t)\int_\tau^t e^{-(t-s)A(s)}\Phi(s,\tau)v_0ds$. Note that

$$\begin{aligned}A(t)\int_\tau^{t-\rho}e^{-(t-s)A(s)}\Phi(s,\tau)v_0ds = & A(t)\int_\tau^{t-\rho}e^{-(t-s)A(s)}[\Phi(s,\tau)-\Phi(t,\tau)]v_0ds\\ & + A(t)\int_\tau^{t-\rho}e^{-(t-s)A(s)}\Phi(t,\tau)v_0ds.\end{aligned}$$

From (25), for $0<\rho_2<\rho_1<\gamma$ and $0<\eta<\epsilon$, we have that

$$\sup_{t\in[\tau+\gamma,t_0]}\left\|A(t)\int_{t-\rho_1}^{t-\rho_2}e^{-(t-s)A(s)}[\Phi(s,\tau)-\Phi(t,\tau)]ds\right\|_{L(X)}\le C\frac{(\gamma-\rho_1)^{\eta-1}}{\epsilon-\eta}[\rho_2^{\epsilon-\eta}-\rho_1^{\epsilon-\eta}].$$

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Showing that

$$A(t)\int_\tau^{t-\rho} e^{-(t-s)A(s)}[\Phi(s,\tau)-\Phi(t,\tau)]ds,$$

converges, as $\rho\to 0$, in $C([\tau+\gamma,t_0],L(X))$, to

$$A(t)\int_\tau^{t} e^{-(t-s)A(s)}[\Phi(s,\tau)-\Phi(t,\tau)]ds.$$

To prove the convergence of

$$A(t)\int_\tau^{t-\rho} e^{-(t-s)A(s)}\Phi(t,\tau)v_0ds,$$

we note that $\Phi(t,\tau)v_0$ is continuous in t for $t\in[\tau+\gamma,t_0]$ and hence, $\{\Phi(t,\tau)v_0 : t\in[\tau+\gamma,t_0]\}$ is compact. So, it is enough to show that

$$A(t)\int_\tau^{t-\rho} e^{-(t-s)A(s)}ds,$$

defines an operator in $L(X)$ which is bounded uniformly for $t\in[\tau+\gamma,t_0]$ and to use that

$$\sup_{t\in[\tau+\gamma,t_0]}\left\|A(t)\int_\tau^t e^{-(t-s)A(s)}wds - A(t)\int_\tau^{t-\rho} e^{-(t-s)A(s)}wds\right\|_X \stackrel{\rho\to 0}{\longrightarrow} 0 \tag{28}$$

for all $w\in D$, $\|w\|_X\le 1$ (which clearly holds).

The operator $F(t,\tau)=A(t)\displaystyle\int_\tau^t e^{-(t-s)A(s)}ds$ is clearly defined in D. Observe that

$$F(t,\tau)w = A(t)\int_\tau^t e^{-(t-s)A(t)}wds + A(t)\int_\tau^t\left[\int_s^t e^{-(t-\xi)A(t)}[A(t)-A(s)]e^{-(\xi-s)A(s)}wd\xi\right]ds,$$

where we have used that

$$[e^{-\tau A(t)}-e^{-\tau A(s)}]w = \int_0^\tau e^{-(\tau-\xi)A(t)}[A(s)-A(t)]e^{-\xi A(s)}wd\xi. \tag{29}$$

Changing the order or integration, it follows that

$$\begin{aligned} F(t,\tau)w &= [I-e^{-(t-\tau)A(t)}]\left[I-\int_\tau^t \varphi_1(t,s)ds\right]w \\ &+ A(t)\int_\tau^t e^{-(t-\xi)A(t)}\left[\int_\xi^t \varphi_1(t,s)ds + \int_\tau^\xi[\varphi_1(t,s)-\varphi_1(\xi,s)]ds\right]d\xi\ w \\ &+ A(t)\int_\tau^t e^{-(t-\xi)A(t)}[A(t)-A(\xi)]A^{-1}(\xi)F(\xi,\tau)wd\xi. \end{aligned} \tag{30}$$

From (30), using (7), (8), (9), (16) and (19) we obtain, for $0<\eta<\epsilon$, that

$$\|F(t,\tau)w\|_X \le C\|w\|_X + C\int_\tau^t |t-\xi|^{\epsilon-1}\|F(\xi,\tau)w\|_X d\xi.$$

It follows from the Generalized Gronwall Lemma (see [8]) that

$$\|F(t,\tau)w\|_X \le C\|w\|_X.$$

This shows that $F(t,\tau)$ extends to a bounded linear operator in X (which we again denote by $F(t,\tau)$) which is bounded uniformly for $t\in[\tau+\gamma,t_0]$. The characterization of $F(t,\tau)w$ for $w\in X$ is obtained in the following way: If $D \ni w_n \stackrel{n\to\infty}{\longrightarrow} w$, then

$$z_n = \int_\tau^{t-\rho} e^{-(t-s)A(s)} w_n ds \stackrel{n\to\infty}{\longrightarrow} z = \int_\tau^t e^{-(t-s)A(s)} w ds$$

and since $F(t,\tau)$ is a bounded linear operator

$$F(t,\tau)w_n = A(t)z_n \stackrel{n\to\infty}{\longrightarrow} y.$$

From the fact that $A(t)$ is closed, we have that $F(t,\tau)w = A(t)z$; that is,

$$F(t,\tau)w = A(t)\int_\tau^t e^{-(t-s)A(s)} w ds, \ \forall w \in X. \tag{31}$$

Consequently

$$\sup_{t\in[\tau+\gamma,t_0]} \left\| A(t)\int_\tau^t e^{-(t-s)A(s)} ds \right\|_{L(X)} \le C. \tag{32}$$

Hence,

$$\sup_{t\in[\tau+\gamma,t_0]} \{\|U_\rho(t,\tau)v_0 - U(t,\tau)v_0\|_X + \|\frac{d}{dt}U_\rho(t,\tau)v_0 - A(t)U(t,\tau)v_0\|_X\} \stackrel{\rho\to 0}{\longrightarrow} 0$$

and $v(t) = U(t,\tau)v_0$ satisfies (2).

Corollary 2.3. *For all $t>\tau\in\mathbb{R}$*

$$\left\|A(t)U(t,\tau)A(\tau)^{-1}\right\|_{L(X)} \le C. \tag{33}$$

Also,

$$\|A(t)U(t,\tau)\|_{L(X)} \le C(t-\tau)^{-1}. \tag{34}$$

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Proof: This result follows from (9), (17), from [14, (1.37),(1.42)] and of the identity

$$A(t)U(t,\tau) = A(t)e^{-(t-\tau)A(\tau)} + A(t)\int_\tau^t e^{-(t-s)A(s)}[\Phi(s,\tau) - \Phi(t,\tau)]ds$$
$$+ A(t)\int_\tau^t e^{-(t-s)A(s)}\Phi(t,\tau)ds.$$

□

The estimate (33) it will be improved in the next section, however it plays a fundamental role in this process. The estimate (34) is already in its best form.

COROLLARY 2.4. *For $s \le \tau \le t$, we have*

$$U(t,s) = U(t,\tau)U(\tau,s). \tag{35}$$

Proof: Given $w_0 \in X$ and $v_0 = U(\tau,s)w_0$, the functions $U(t,s)w_0$ and $U(t,\tau)U(\tau,s)w_0$ are both solutions of (2) which are continuous for $t \ge \tau$, continuously differentiable for $t > \tau$. In view of the uniqueness they must coincide. Since w_0 is arbitrary, (35) follows. □

2.2. Fractional powers

In this section, we study the relationship between the fractional powers associated to the operator $A(t)$ and the family of evolution operators $\{U(t,\tau) : t \ge \tau \in \mathbb{R}\}$. Many of the results presented here can be found in [14] and therefore many of the proofs are omitted or presented in a abridged version.

The conditions imposed on the operators $A(t)$ allow us to define the fractional power operators $A(t)^\alpha$, $\alpha \in \mathbb{R}$ in the following manner

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\tau A(t)}\tau^{\alpha-1}d\tau$$

if $\alpha > 0$ and $A(t)^\alpha = (A(t)^{-\alpha})^{-1}$ if $\alpha > 0$. An exhaustive treatment of fractional powers of operators of the positive type can be found in [2].

In the results that follow, we seek for relationships between the powers of $A(t)$ and the family of evolution operators $\{U(t,\tau) : t \ge \tau \in \mathbb{R}\}$ that are independent of t. Because of this, some classical proofs are repeated.

The next lemma is fundamental to obtain estimates for $U(t,\tau)$ in $L(X,X^\alpha)$.

LEMMA 2.6. *Assume that $\{A(t) : t \in \mathbb{R}\}$ satisfy (a). For $\alpha < \beta < \gamma$, there exist $C > 0$, independent of t, such that*

$$\|A(t)^\beta x\|_X \le C\|A(t)^\gamma x\|_X^{\frac{\beta-\alpha}{\gamma-\alpha}}\|A(t)^\alpha x\|_X^{\frac{\gamma-\beta}{\gamma-\alpha}}, \qquad (x \in D(A^\gamma(t))). \tag{36}$$

There are constants $C > 0$ and $\delta > 0$, independent of β and t, such that

$$\|A(t)^{\beta}e^{-\tau A(t)}\|_{L(X)} \leq Ce^{-\delta\tau}\tau^{-\beta}, \quad \text{for } \beta \geq 0 \tag{37}$$

$$\|[e^{-\tau A(t)} - I]A(t)^{-\beta}\|_{L(X)} \leq C\tau^{\beta}, \quad \text{for } 0 \leq \beta \leq 1. \tag{38}$$

With this, we can prove the following Corollary.

COROLLARY 2.5. *For any $\tau, r, t, \xi \in \mathbb{R}$, $t \leq r$, $\tau > 0$ and $0 \leq \alpha \leq 1$ we have*

$$\|A(\xi)^{\alpha}[e^{-\tau A(r)} - e^{-\tau A(t)}]\|_{L(X)} \leq C\tau^{-\alpha}(r-t)^{\epsilon}, \tag{39}$$

and

$$\|A(\xi)^{\alpha}[A(r)e^{-\tau A(r)} - A(t)e^{-\tau A(t)}]\|_{L(X)} \leq C\tau^{-\alpha-1}(r-t)^{\epsilon(1-\alpha)}.$$

Proof: Following of the (10), (11) and (36) that

$$\begin{aligned}\|A(\xi)^{\alpha}[e^{-\tau A(r)} - e^{-\tau A(t)}]\|_{L(X)} &\leq \|A(\xi)[e^{-\tau A(r)} - e^{-\tau A(t)}]\|^{\alpha}\|[e^{-\tau A(r)} - e^{-\tau A(t)}]\|^{1-\alpha} \\ &\leq C\tau^{-\alpha}(r-t)^{\epsilon}.\end{aligned}$$

From the identities

$$A(r)e^{-\tau A(r)} - A(t)e^{-\tau A(t)} = A(r)[e^{-\tau A(r)} - e^{-\tau A(t)}] + [A(r) - A(t)]A(t)^{-1}A(t)e^{-\tau A(t)},$$

$$A(\xi)[A(r)e^{-\tau A(r)} - A(t)e^{-\tau A(t)}] = A(\xi)A(r)^{-1}\Big[A(r)e^{-\frac{\tau}{2}A(r)}\Big]^2 - A(\xi)A(t)^{-1}\Big[A(t)e^{-\frac{\tau}{2}A(t)}\Big]^2$$

and using (7), (9) and (11), it follows from (36) that

$$\|A(\xi)^{\alpha}[A(r)e^{-\tau A(r)} - A(t)e^{-\tau A(t)}]\|_{L(X)} \leq C\tau^{-\alpha-1}(r-t)^{\epsilon(1-\alpha)}.$$

□

The next Lemma was taken from [14] (see also [7]).

LEMMA 2.7. *For $0 \leq \beta \leq \alpha$ and $\beta \leq 1 + \epsilon$,*

$$\|A(t)^{\alpha}e^{-(t-\tau)A(t)}A(\tau)^{-\beta}\|_{L(X)} \leq C(\alpha,\beta)(t-\tau)^{\beta-\alpha}e^{-\delta(t-\tau)}. \tag{40}$$

With this we have the following result.

Corollary 2.6. *For any $\alpha \in [0,1]$, $\theta \in [0,\infty)$, $0 \le \beta \le \theta$ and $\xi, r, \tau \in \mathbb{R}$ there exist a constant C such that*

$$\|A(\xi)^{\alpha} A(r)^{\theta} e^{-sA(r)} A(\tau)^{-\beta}\|_{L(X)} \le C s^{-\alpha-\theta+\beta}, \quad \forall s > 0. \tag{41}$$

Proof: From (36) and (40) immediately follows that

$$\begin{aligned} &\|A(\xi)^{\alpha} A(r)^{\theta} e^{-sA(r)} A(\tau)^{-\beta}\| \\ &\qquad \le \|A(\xi)A(r)^{-1}\|^{\alpha} \|A(r)^{\theta+1} e^{-sA(r)} A(\tau)^{-\beta}\|^{\alpha} \|A(r)^{\theta} e^{-sA(r)} A(\tau)^{-\beta}\|^{1-\alpha} \\ &\qquad \le C s^{-\alpha-\theta+\beta}. \end{aligned}$$

This completes the proof. □

Lemma 2.8. *For any $t \ge \tau \in \mathbb{R}$*

$$U(t,\tau)A(\tau)^{-1} = e^{-(t-\tau)A(t)} A(\tau)^{-1} - \int_{\tau}^{t} e^{-(t-s)A(t)} [A(s) - A(t)] U(s,\tau) A(\tau)^{-1} ds.$$

Proof: The operator $\varphi(s) = -e^{-(t-s)A(t)} U(s,\tau) A(\tau)^{-1}$ $(\tau \le s \le t)$ is differentiable with respect to s and

$$\varphi'(s) = e^{-(t-s)A(t)} [A(s) - A(t)] U(s,\tau) A(\tau)^{-1}.$$

Integrating with respect to s from τ to t the result follows. □

The next result is of fundamental importance. In it, the fact that we can consider $\alpha > 0$, is what allow us to treat the problem 1 in the case of critical nonlinearities.

Lemma 2.9. *For $0 \le \beta \le \alpha < 1 + \epsilon$, we have*

$$\|A(t)^{\alpha} U(t,\tau) A(\tau)^{-\beta}\|_{L(X)} \le C(\alpha,\beta)(t-\tau)^{\beta-\alpha}. \tag{42}$$

See [7, 14] for a proof.

As a simple consequence of Lemma 2.9 we have that.

Corollary 2.7. *If $0 \le \alpha < \epsilon$, $0 \le \beta \le 1$, then*

$$\|A(\xi)^{\alpha} A(t) U(t,\tau) A(\tau)^{-\beta}\|_{L(X)} \le C(\alpha,\beta)(t-\tau)^{\beta-\alpha-1}. \tag{43}$$

LEMMA 2.10. *If $\beta > \alpha$ and $0 < \beta - \alpha \leq 1$, then*

$$\|A(t)^\alpha[U(t,\tau) - I]A(\tau)^{-\beta}\|_{L(X)} \leq C(\alpha,\beta)(t-\tau)^{\beta-\alpha}. \tag{44}$$

Proof: Observe that

$$U(t,\tau) - I = e^{-(t-\tau)A(t)} - I + \int_\tau^t e^{-(t-s)A(t)}[A(s) - A(t)]U(s,\tau)ds.$$

From (7), (40), (42) and Corollary 2.6 it follows that

$$\begin{aligned}\|A(t)^\alpha[U(t,\tau) - I]A(\tau)^{-\beta}\| &\leq \| \int_0^{t-\tau} A(t)^\alpha \frac{d}{ds} e^{-sA(t)} A(\tau)^{-\beta} ds\| \\ &+ \int_\tau^t \|A(t)^\alpha e^{-(t-s)A(t)}\| \, \|[A(s) - A(t)]A(s)^{-1}\| \, \|A(s)U(s,\tau)A(\tau)^{-\beta}\| ds \\ &\leq C(\alpha,\beta)(t-\tau)^{\beta-\alpha}.\end{aligned}$$

□

LEMMA 2.11. *For any $\alpha, \beta \in [0,1]$,*

$$\|A(\xi)^\alpha[U(t,\tau) - e^{-(t-\tau)A(t)}]A(\tau)^{-\beta}\|_{L(X)} \leq C(\alpha,\beta)(t-\tau)^{\beta-\alpha+\epsilon}. \tag{45}$$

LEMMA 2.12. *If $0 \leq \beta \leq 1$ e $0 \leq \alpha < \epsilon$, we have*

$$\|A(\xi)^\alpha A(t)[U(t,\tau) - e^{-(t-\tau)A(t)}]A(\tau)^{-\beta}\|_{L(X)} \leq C(\alpha,\beta)(t-\tau)^{\beta-\alpha-1+\epsilon}. \tag{46}$$

Note that, for $\beta > 0$,

$$A(\tau)^{-\beta} - A(t)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty [e^{-sA(\tau)} - e^{-sA(t)}]s^{\beta-1} ds. \tag{47}$$

Now, for $\gamma \in [0,1]$, using (10) and (11), we obtain from (36) that

$$\|A(t)^\gamma[e^{-sA(t)} - e^{-sA(\tau)}]\|_{L(X)} \leq Ce^{-\delta s}s^{-\gamma}(t-\tau)^\epsilon. \tag{48}$$

From (47) and (48), for $0 < \beta < 1$ and $\gamma < \beta$,

$$\|A(t)^\gamma[A(t)^{-\beta} - A(\tau)^{-\beta}]\|_{L(X)} \leq C(\gamma,\beta)(t-\tau)^\epsilon. \tag{49}$$

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LEMMA 2.13. *For $0 \leq \alpha \leq 1$, $0 \leq \beta \leq \nu < 1+\epsilon$, $0 < \nu - \alpha \leq 1$, and $\tau < t \leq r$, we have*

$$\|A(\xi)^\alpha [U(r,\tau) - U(t,\tau)]A(\tau)^{-\beta}\|_{L(X)} \leq C(\alpha,\beta,\gamma)(r-t)^{\nu-\alpha}(t-\tau)^{\beta-\nu}. \tag{50}$$

Proof: From the identity

$$\begin{aligned}
&A(\xi)^\alpha [U(r,\tau) - U(t,\tau)]A(\tau)^{-\beta} \\
&\quad = \Big\{ A(\xi)^\alpha [U(r,t) - e^{-(r-t)A(r)}]A(t)^{-\nu+\epsilon}A(t)^{-\epsilon} - A(\xi)^\alpha \int_0^{r-t} A(r)^{1-\nu} e^{-sA(r)} ds \\
&\quad + A(\xi)^\alpha \int_0^{r-t} A(r)^{1-\gamma} e^{-sA(r)} A(r)^\gamma [A(r)^{-\nu} - A(t)^{-\nu}] ds \Big\} A(t)^\nu U(t,\tau)A(\tau)^{-\beta},
\end{aligned}$$

applying (49), (41), (42) and (45) for $\alpha < \gamma < \nu < \gamma + \epsilon$ and $\alpha < 1$ (or $\alpha = \gamma = 1 < \nu < 1+\epsilon$) we get the result. □

COROLLARY 2.8. *For any $0 \leq \beta \leq \nu < 1+\epsilon$ and $\nu > 1$, we have*

$$\|[A(r)U(r,\tau) - A(t)U(t,\tau)]A(\tau)^{-\beta}\|_{L(X)} \leq C(r-t)^{\nu-1}(t-\tau)^{\beta-\nu}. \tag{51}$$

Proof: Since $\nu \in (1,2)$ we have that

$$\begin{aligned}
\|[A(r)U(r,\tau) - A(t)U(t,\tau)]A(\tau)^{-\beta}\| &\leq \|A(r)[U(r,\tau) - U(t,\tau)]A(\tau)^{-\beta}\| \\
&+ \|[A(r) - A(t)]A(t)^{-1}\| \, \|A(t)^{-(\nu-1)}\| \, \|A(t)^\nu U(t,\tau)A(\tau)^{-\beta}\| \\
&\leq C(r-t)^{\nu-1}(t-\tau)^{\beta-\nu} + C(r-t)^\epsilon (t-\tau)^{\beta-\nu} \leq C(r-t)^{\nu-1}(t-\tau)^{\beta-\nu},
\end{aligned}$$

and (51) is proved. □

COROLLARY 2.9. *For all $0 \leq \theta < \epsilon$, $(\tau,\infty) \ni t \mapsto A(\xi)^\theta A(t)U(t,\tau)A(\tau)^{-\beta} \in L(X)$ is Hölder continuous with exponent $\eta < \epsilon(1 - \frac{\theta}{\epsilon})$.*

Proof: From (36), (43) and (51) with $1+\epsilon > \nu = 1+\alpha > 1$, it follows that, for $\theta \in [0,\alpha]$,

$$\|A^\theta(\xi)[A(r)U(r,\tau) - A(t)U(t,\tau))]A(\tau)^{-\beta}\| \leq C(\alpha,\beta,\gamma)(t-\tau)^{\beta-\alpha-1}(r-t)^{\alpha)1-\frac{\theta}{\alpha})}. \tag{52}$$

Which completes the proof. □

2.3. Nonhomogeneous Linear equations

For $A(t)$ satisfying the conditions of de Section 2, $f : \mathbb{R} \to X$ continuous, $\tau \in \mathbb{R}$ e $v_0 \in X$, we considerer the problem

$$\frac{dv}{dt} + A(t)v = f(t), \; t > \tau \tag{53}$$

$$v(\tau) = v_0. \tag{54}$$

For $t \geq \tau$, define

$$v(t) = U(t,\tau)v_0 + \int_\tau^t U(t,s)f(s)ds. \tag{55}$$

Is clear that $v(\cdot)$ satisfies (54). Assume that there are constants positive C, ρ such that

$$\|f(t) - f(s)\| \leq C(t-s)^\rho, \quad \forall\ t, s \in \mathbb{R}. \tag{56}$$

In this case, we will show that $v(t)$ defined by (55) is continuously differentiable for $t > \tau$ and satisfies the equation (53).

The differentiability of $U(t,\tau)v_0$ has been studied in Section 2.1. To see that

$$z(t) := \int_\tau^t U(t,s)f(s)ds \tag{57}$$

is continuously differentiable with derivative

$$w(t) := -A(t)\int_\tau^t U(t,s)f(s)ds + f(t),$$

first we will show that $z(t)$ is right differentiable and that

$$\frac{d^+}{dt}z(t) = w(t),\ t \geq \tau. \tag{58}$$

After that we will show that $w(t)$ is continuous for $t \geq \tau$ to conclude the result.

Proposition 2.1. *The function $z(t)$ defined for* (57) *is right differentiable and satisfies* (58) *for all $t \geq \tau$.*

Proof: Let $t < r$, using (35), (43) and (46), we obtain

$$\begin{aligned}
&\frac{1}{r-t}\left[\int_\tau^r U(r,s)f(s)\,ds - \int_\tau^t U(t,s)f(s)\,ds\right] \\
&\quad= \frac{1}{r-t}\int_t^r U(r,s)f(s)\,ds + \frac{1}{r-t}[U(r,t)-I]\int_\tau^t U(t,s)f(s)\,ds \\
&\quad= \frac{1}{r-t}\int_t^r U(r,s)f(s)\,ds + \frac{1}{r-t}[U(r,t)-I]A(t)^{-1}\Big\{[I - e^{-(t-\tau)A(t)}]f(t) \\
&\quad+ A(t)\int_\tau^t \Big[U(t,s) - e^{-(t-s)A(t)}\Big]\,ds\,f(t) + A(t)\int_\tau^t U(t,s)[f(s)-f(t)]ds\Big\}.
\end{aligned}$$

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We will show that

$$\begin{aligned}\frac{d^+}{dt}\int_\tau^t U(t,s)f(s)\,ds &= f(t) - A(t)\int_\tau^t U(t,s)[f(s)-f(t)]\,ds \\ &\quad - A(t)\int_\tau^t [U(t,s)-e^{-(t-s)A(t)}]\,ds\ f(t) - [I-e^{-(t-\tau)A(t)}]f(t),\end{aligned} \tag{59}$$

and that the right hand side of the above expression is continuous for all $t \geq \tau$. For this, we will need some auxiliary lemmas.

LEMMA 2.14. *For $0 \leq \alpha < \min\{\epsilon, \rho\}$, the functions*

i) $$t \mapsto A(\xi)^\alpha A(t)\int_\tau^t U(t,s)[f(s)-f(t)]ds,$$

ii) $$t \mapsto A(\xi)^\alpha A(t)\int_\tau^t [U(t,s)-e^{-(t-s)A(t)}]ds\ f(t),$$

iii) $$t \mapsto [I-e^{-(t-\tau)A(t)}]f(t),$$

are continuous in t for $t \geq \tau$ and

iv) $$A(\xi)^\alpha e^{-(t-\tau)A(t)}f(t),$$

is continuous for $t > \tau$.

Proof: To prove $i)$ note that, for $r > t$,

$$\begin{aligned}&A(r)\int_\tau^r U(r,s)[f(s)-f(r)]ds - A(t)\int_\tau^t U(t,s)[f(s)-f(t)]ds \\ &= A(r)\int_t^r U(r,s)[f(s)-f(r)]ds + \int_\tau^t [A(r)U(r,s)-A(t)U(t,s)][f(s)-f(t)]ds \\ &+ A(r)\int_\tau^t U(r,s)[f(t)-f(r)]ds.\end{aligned}$$

From (52) with $1 < \nu < 1+\rho-\alpha$ and from (43), it follows that

$$\begin{aligned}&\left\|A(\xi)^\alpha A(r)\int_\tau^r U(r,s)[f(s)-f(r)]ds - A(\xi)^\alpha A(t)\int_\tau^t U(t,s)[f(s)-f(t)]ds\right\| \\ &\quad \leq C(r-t)^{\rho-\alpha} + C(r-t)^{\nu-1}(t-\tau)^{-\nu+1+\rho-\alpha} + C(r-t)^\rho[(r-\tau)^{-\alpha}-(r-t)^{-\alpha}].\end{aligned}$$

Showing that the function defined by (1) is continuous.

To prove $ii)$ observe that (46) implies

$$\left\|A(\xi)^\alpha A(t)\int_{t-\rho}^t [U(t,s)-e^{-(t-s)A(t)}]ds\right\|_{L(X)} \leq C\rho^{\epsilon-\alpha}. \tag{60}$$

After that note that, for $0 < 2\rho < t-\tau$

$$A(r)\int_\tau^{r-\rho}[U(r,s)-e^{-(r-s)A(r)}]ds\ f(r)-A(t)\int_\tau^{t-\rho}[U(t,s)-e^{-(t-s)A(t)}]ds\ f(t)$$
$$=A(r)\int_{t-\rho}^{r-\rho}[U(r,s)-e^{-(r-s)A(r)}]ds\ f(r)+\int_\tau^{t-\rho}[A(r)U(r,s)-A(t)U(t,s)]ds\, f(r)$$
$$-\int_\tau^{t-\rho}[A(r)e^{-(r-s)A(r)}-A(t)e^{-(t-s)A(t)}]dsf(r)$$
$$+A(t)\int_\tau^{t-\rho}[U(t,s)-e^{-(t-s)A(t)}]ds\,[f(r)-f(t)].$$

From (45) we have that

$$\left\|A(\xi)^\alpha A(r)\int_{t-\rho}^{r-\rho}[U(r,s)-e^{-(r-s)A(r)}]ds\right\|\le\frac{C}{\epsilon-\alpha}[(r-t+\rho)^{\epsilon-\alpha}-\rho^{\epsilon-\alpha}],$$

and that

$$\left\|A(\xi)^\alpha A(t)\int_\tau^{t-\rho}[U(t,s)-e^{-(t-s)A(t)}]ds[f(r)-f(t)]\right\|\le C[(t-\tau)^{\epsilon-\alpha}-\rho^{\epsilon-\alpha}](r-t)^\rho.$$

For $\gamma>0$ we have from (37), (38), (10) and (11) that

$$\left\|\int_\tau^{t-\rho}A(\xi)^\alpha[A(r)e^{-(r-s)A(r)}-A(t)e^{-(t-s)A(t)}]ds\right\|$$
$$\le\left\|\int_\tau^{t-\rho}A(\xi)^\alpha A(r)^{1+\gamma}e^{-(t-s)A(r)}[e^{-(r-t)A(r)}-I]A(r)^{-\gamma}ds\right\|$$
$$+\left\|\int_\tau^{t-\rho}A(\xi)^\alpha[A(r)e^{-(t-s)A(r)}-A(t)e^{-(t-s)A(t)}]ds\right\|$$
$$\le C(r-t)^{\min\{\gamma,\,\epsilon(1-\alpha)\}}[(t-\tau)^{-\alpha-\gamma}+(t-\tau)^{-\alpha}].$$

After that, if $\alpha<\theta<\epsilon$, it follows from (52) that

$$\left\|\int_\tau^{t-\rho}A(\xi)^\alpha[A(r)U(r,s)-A(t)U(t,s)]ds\right\|\le\frac{C}{\theta}(r-t)^{\theta(1-\frac{\alpha}{\theta})}[(t-\tau)^{-\theta}-\rho^{-\theta}].$$

Showing that

$$t\mapsto A(\xi)^\alpha A(t)\int_\tau^{t-\rho}[U(t,s)-e^{-(t-s)A(t)}]ds\ f(t) \tag{61}$$

is continuous in compact time intervals of (τ,∞), for any suitably small $\rho>0$. From (60) we obtain that the function above converges, uniformly on compact time intervals, to function in *ii*) as $\rho\to0$. Hence, the function in *ii*) is continuous for $t>\tau$.

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To prove the continuity of the $iii)$ we only need to prove that $t \mapsto e^{-(t-\tau)A(t)}$ is continuous for $t > \tau$. This follows from

$$\begin{aligned} e^{-(r-\tau)A(r)}f(r) - e^{-(t-\tau)A(t)}f(t) &= [e^{-(r-\tau)A(t)} - e^{-(t-\tau)A(t)}]f(t) \\ &+ [e^{-(r-\tau)A(r)} - e^{-(r-\tau)A(t)}]f(t) + e^{-(r-\tau)A(r)}[f(r) - f(t)], \end{aligned} \tag{62}$$

(10), (56) and from the continuity of $r \mapsto e^{-(r-\tau)A(t)}$.

The proof of $iv)$ follows immediately from (39). This concludes the proof of Lemma. □

The proof of the following lemma can be found in [7].

LEMMA 2.15. *For any $x \in X$, $t \in \mathbb{R}$, and $r > t$*

$$\frac{1}{r-t}[U(r,t) - I]A(t)^{-1}x \longrightarrow -x, \quad \textit{quando} \quad r \to t^+.$$

As the expression of the right hand side of (59) is continuous for $t > \tau$, it follows $t \mapsto \int_\tau^t U(t,s)f(s)ds$ is continuously differentiable for $t > \tau$ and from Lemma 2.1, we have

$$\frac{d}{dt}\int_\tau^t U(t,s)f(s)ds = f(t) - A(t)\int_\tau^t U(t,s)f(s)ds.$$

Consequently, if $v(t)$ is defined by (55), then $v(t)$ is differentiable for $t > \tau$ and satisfies (53),(54). Furthermore, it is the unique solution of (53),(54). This concludes the proof of Proposition 2.1. □

3. SEMILINEAR PARABOLIC PROBLEMS

In this section we aim to show a result on existence, uniqueness and continuity with respect to initial data for (1) when f has critical growth. We borrow the notations of [5].

3.1. Local existence of solutions

Throughout this section we assume that $\{A(t) : t \in \mathbb{R}\}$ satisfies (6), (7). In this section we prove that if f is an $\epsilon-$regular map (see Definition 1.3) the problem (1) is locally well posed.

For $\epsilon, \rho, \gamma(\epsilon)$ and c positive, $0 \le \epsilon < 1, \rho > 1, \rho\epsilon \le \gamma(\epsilon) < 1$, and $\nu(t)$ with $0 \le \nu(t) \le \delta$, $\lim_{t\to 0^+}\nu(t) = 0$, define $\mathcal{F} := \mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, \nu(\cdot))$ as the family of functions f such that, for $t > \tau$, $f(t, \cdot)$ is an $\epsilon-$regular map relative to the pair (X_1, X_0), that is, $f(t,\cdot) : X_{1+\epsilon} \longrightarrow$

$X_{\gamma(\epsilon)}$ and

$$\|f(t,v)-f(t,w)\|_{X_{\gamma(\epsilon)}} \leq c\|v-w\|_{X_{1+\epsilon}}(\|v\|^{\rho-1}_{X_{1+\epsilon}}+\|w\|^{\rho-1}_{X_{1+\epsilon}}+\nu(t)t^{-\gamma(\epsilon)+\epsilon}), \tag{63}$$

$$\|f(t,v)\|_{X_{\gamma(\epsilon)}} \leq c(\|v\|^{\rho}_{X_{1+\epsilon}}+\nu(t)t^{-\gamma(\epsilon)}), \tag{64}$$

for all $v,w\in X_{1+\epsilon}$.

Without loss of generality we can assume that the function $\nu(\cdot)$ is non-decreasing. In most cases in the argument below we will fix the parameters $\epsilon,\rho,\gamma(\epsilon)$ and c, and we will denote the class $\mathcal{F}$ defined above by $\mathcal{F}(\nu(\cdot))$.

If (6) and (7) they are satisfied, we saw in Section 2, that for each t, $A(t)$ is the infinitesimal generator of an analytic semigroup in X_0, that we will denote for $\{e^{-\tau A(t)}:\tau\geq 0\}$ and there exist a family of evolution operators $\{U(t,\tau):t\geq\tau\in\mathbb{R}\}$ associate the $A(t)$. Let M a positive real constant, such that

$$\|U(t,\tau)e\|_{X_{1+\alpha}} \leq M|t-\tau|^{-1+\beta-\alpha}\|e\|_{X_\beta}, \qquad 0\leq\beta\leq 1+\alpha\leq 2, \tag{65}$$

(see Lemma 2.9).

For each $\tau\in[0,T]$, the map

$$[\tau,T]\in t\longmapsto U(t,\tau)u_0\in X_{\alpha+1}$$

is continuous, for all $u_0\in X_\beta$ and $t>\tau$, with τ fixed. In fact, fix t_2, with $t_1>t_2>\tau$. Given $u_0\in X_\beta$, for any $\gamma>\alpha$, $\gamma-\alpha\leq 1$, follows from (44) that

$$\begin{aligned}
&\|[U(t_1,\tau)-U(t_2,\tau)]u_0\|_{X_{\alpha+1}} \leq \|[U(t_1,t_2)-I]U(t_2,\tau)u_0\|\\
&\leq \|U(t_1,t_2)-I\|_{L(X_{\gamma+1},X_{\alpha+1})}\|U(t_2,\tau)u_0\|_{X_{\gamma+1}}\\
&\leq C(\alpha,\gamma)(t_1-t_2)^{\gamma-\alpha}(t_2-\tau)^{-1+\beta-\gamma}\|u_0\|_{X_\beta} \overset{t_1\to t_2}{\longrightarrow} 0.
\end{aligned}$$

We can now state the main result of this section.

THEOREM 3.1. *Let $f\in\mathcal{F}(\epsilon,\rho,\gamma(\epsilon),c,\nu(\cdot))$, $0<\epsilon<\rho_0$. If $w_0\in X_1$, there exist $r>0$ and $\tau_0>0$, with the property that for any $v_0\in B_{X_1}(w_0,r)$ there exists a continuous function $v(\cdot,v_0):[0,\tau_0]\to X_1$, with $v(0)=v_0$, which is the unique $\epsilon-$regular mild solution starting at v_0 of the problem*

$$\begin{aligned}
&\frac{dv}{dt}+A(t)v=f(t,v), \quad t>0,\\
&v(0)=v_0.
\end{aligned} \tag{66}$$

This solution satisfies

$$v\in C((0,\tau_0],X_{1+\epsilon}),$$

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$$t^\theta \|v(t,v_0)\|_{X_{1+\theta}} \overset{t\to 0^+}{\longrightarrow} 0, \quad 0<\theta<\gamma(\epsilon).$$

Moreover, if $v_0, z_0 \in B_{X_1}(w_0, r)$*, then*

$$t^\theta \|v(t,v_0)-v(t,z_0)\|_{X_{1+\theta}} \le C\|v_0-z_0\|_{X_1}, \quad \forall t\in[0,\tau_0], 0\le\theta\le\theta_0<\gamma(\epsilon).$$

The constants above depend on the following: $\tau_0=\tau_0(w_0,A(\cdot),\nu(\cdot),\epsilon,\rho,\gamma(\epsilon),c,M)$*,* $r=r(w_0,\epsilon,\rho,\gamma(\epsilon),c,M)$ *and* $C=C(\theta_0,\epsilon,\gamma(\epsilon),M)$*.*

Before we prove above theorem we need some lemmas.

Lemma 3.1. *The operators* $(t-\tau)^\alpha U(t,\tau): X_1 \longrightarrow X_{1+\alpha}$*,* $t>\tau$*, are bounded linear operators, satisfying*

$$\|(t-\tau)^\alpha U(t,\tau)\|_{L(X_1,X_{1+\alpha})} \le M, \quad \text{with } M \text{ independent of } t \text{ and } \tau,\ t>\tau.$$

Moreover, given a compact subset J *of* X_1*, we have*

$$\lim_{t\to\tau^+}\sup_{x\in J}\|(t-\tau)^\alpha U(t,\tau)x\|_{X_{1+\alpha}}=0.$$

Proof: The first part follow of (65) with $\beta=1$.

The operators $(t-\tau)^\alpha U(t,\tau): X_1 \longrightarrow X_{1+\alpha}$ are bounded, uniformly in $t\ge\tau$. Taking $x\in J$, since $X_{1+\alpha}\overset{d}{\hookrightarrow} X_1$, there exist $x_n\in X_{1+\alpha}$ such that $x_n \overset{X_1}{\to} x$. Thus

$$\|(t-\tau)^\alpha U(t,\tau)x\|_{X_{1+\alpha}} \le \|(t-\tau)^\alpha U(t,\tau)x_n\|_{X_{1+\alpha}} + M\|x_n-x\|_{X_1}.$$

Thus

$$0\le \liminf_{t\to\tau^+}\|t^\alpha U(t,\tau)x\|_{X_{1+\alpha}} \le \limsup_{t\to\tau^+}\|(t-\tau)^\alpha U(t,\tau)x\|_{X_{1+\alpha}} \le M\|x_n-x\|_{X_1} \overset{n\to\infty}{\longrightarrow} 0.$$

Given $\epsilon>0$, for each $x\in J$ there exist $\sigma_x>\tau$, such that

$$\|(t-\tau)^\alpha U(t,\tau)x\|_{X_{1+\alpha}} < \epsilon/2, \quad t\in[\tau,\sigma_x].$$

For $y\in B_{X_1}(x,\frac{\epsilon}{2M})$, we obtain

$$\|(t-\tau)^\alpha U(t,\tau)y\|_{X_{1+\alpha}} < \epsilon,\ t\in[\tau,\sigma_x].$$

Since J is compact in X_1, there exist $x_1,\cdots,x_n$ such that $J\subset\cup_{i=1}^n B_{X_1}(x_i,\frac{\epsilon}{2M})$. Choose $\sigma_0=\min_{i=1\dots n}\{\sigma_{x_i}\}$, then

$$\sup_{x\in J}\|(t-\tau)^\alpha U(t,\tau)x\|_{X_{1+\alpha}} < \epsilon, \quad t\in[\tau,\sigma_0].$$

and

$$\lim_{t\to\tau^+}\sup_{x\in J}\|(t-\tau)^\alpha U(t,\tau)x\|_{X_{1+\alpha}}=0.$$

□

Define, for $0\le\theta\le\gamma(\epsilon)$,

$$\mathbf{B}_\epsilon^\theta=\max_{0\le\xi\le\theta}\{\mathbf{B}(\gamma(\epsilon)-\xi,1-\gamma(\epsilon)),\mathbf{B}(\gamma(\epsilon)-\xi,1-\rho\epsilon)\}.$$

Next the two Lemmas are similar Lemmas 2 and 3 of [5], the proofs are exactly the same and will be omitted.

Lemma 3.2. *Let $f\in\mathcal{F}(\epsilon,\rho,\gamma(\epsilon),c,\nu(\cdot))$. If $v\in C((0,\tau_0],X_{1+\epsilon})$, then, for all $0\le\theta<\gamma(\epsilon)$,*

$$t^\theta\|\int_0^t U(t,s)f(s,v(s))ds\|_{X_{1+\theta}}\le cM\mathbf{B}_\epsilon^\theta(\nu(t)+t^{\gamma(\epsilon)-\rho\epsilon}[\lambda(t)]^\rho),\quad 0<t\le\tau_0,$$

where $\lambda(t):=\sup_{s\in(0,t]}\{s^\epsilon\|v(s)\|_{X_{1+\epsilon}}\}$.

Lemma 3.3. *Let $f\in\mathcal{F}(\epsilon,\rho,\gamma(\epsilon),c,\nu(\cdot))$. If $v,w\in C((0,\tau_0],X_{1+\epsilon})$ satisfy $t^\epsilon\|v(t)\|_{X_{1+\epsilon}}\le\mu$ and $t^\epsilon\|w(t)\|_{X_{1+\epsilon}}\le\mu$, for some $\mu>0$ then, for all $0\le\theta<\gamma(\epsilon)$, we have*

$$t^\theta\|\int_0^t U(t,s)[f(s,v(s))-f(s,w(s))]ds\|_{X_{1+\theta}}\le\Gamma_\theta(t)\sup_{s\in[0,\tau_0]}\{s^\epsilon\|v(s)-w(s)\|_{X_{1+\epsilon}}\},$$

where

$$\Gamma_\theta(t)=cM\mathbf{B}_\epsilon^\theta[\nu(t)+t^{\gamma(\epsilon)-\rho\epsilon}2\mu^{\rho-1}].$$

From (7) and (11), it follows easily that

$$\|A(\zeta)[e^{-\tau A(t)}-e^{-\tau A(s)}]A(\eta)^{-1}\|_{L(X_0)}\le Ce^{-\delta\tau}|t-s|^\epsilon.\tag{67}$$

From this, we can show the following lemma.

Lemma 3.4. *If $v_0\in X_1$, we have that*

$$\|U(t,0)v_0-v_0\|_{X_1}\stackrel{t\to0^+}{\longrightarrow}0.$$

Proof: In fact, follows of (67) and of (45) with $\alpha=\beta=1$ that

$$\begin{aligned}\|U(t,0)v_0-v_0\|_{X_1}&\le\|[U(t,0)-e^{-tA(t)}]\|_{L(X_1,X_1)}\|v_0\|_{X_1}\\&\quad+\|[e^{-tA(t)}-e^{-tA(0)}]\|_{L(X_1,X_1)}\|v_0\|_{X_1}+\|e^{-tA(0)}v_0-v_0\|_{X_1}\\&\le Ct^\epsilon\|v_0\|_{X_1}+\|e^{-tA(0)}v_0-v_0\|_{X_1}\stackrel{t\to0^+}{\longrightarrow}0.\qquad\square\end{aligned}$$

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Proof of Theorem 3.1. Let μ, r and δ be the positive numbers defined by

$$8cM\mathbf{B}_\epsilon^\epsilon \mu^{\rho-1} = 1, \tag{68}$$

$$r = \frac{\mu}{4M} = \frac{1}{4M(8cM\mathbf{B}_\epsilon^\epsilon)^{\frac{1}{\rho-1}}}. \tag{69}$$

$$cM\delta\mathbf{B}_\epsilon^\epsilon = \min\{\frac{\mu}{8}, \frac{1}{4}\} \tag{70}$$

Also, for fixed $w_0 \in X_1$, choose $\tau_0 \in (0,1]$ such that

$$\begin{aligned} &\nu(t) < \delta, , \qquad 0 \le t \le \tau_0 \text{ and} \\ &\|t^\epsilon U(t,0)w_0\|_{X_{1+\epsilon}} \le \frac{\mu}{2}, \qquad 0 \le t \le \tau_0. \end{aligned} \tag{71}$$

Notice that these choices imply that $\Gamma_\epsilon(t) \le \frac{1}{2}$ for $t \in [0, \tau_0]$.

We search for solutions in

$$K(\tau_0) = \{v \in C((0,\tau_0], X_{1+\epsilon}) : \sup_{t\in(0,\tau_0]} \{t^\epsilon \|v(t)\|_{X_{1+\epsilon}}\} \le \mu\}.$$

Consider in $K(\tau_0)$ the metric

$$d_{K(\tau_0)}(v_1, v_2) = \sup_{t\in(0,\tau_0]} \{t^\epsilon \|v_1(t) - v_2(t)\|_{X_{1+\epsilon}}\}.$$

The space $K(\tau_0)$ with the metric $d_{K(\tau_0)}$ it is a complete metric space. Assume that $v_0 \in X_1$ with $\|v_0 - w_0\|_{X_1} < r$ and on $K(\tau_0)$ define the map

$$(Tv)(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s,v(s))ds. \tag{72}$$

Let us first prove that, for any $v_0 \in B_{X_1}(w_0, r)$, $Tv \in K(\tau_0)$; that is, T takes $K(\tau_0)$ into itself. After that we will show that $T : K(\tau_0) \to K(\tau_0)$ is a strict contraction.

To see that T defines a map from $K(\tau_0)$ into itself, we will begin showing that

$$\text{if } v \in K(\tau_0), \text{ then } Tv \in C((0,\tau_0], X_{1+\epsilon}). \tag{73}$$

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Fix $t_2 \in (0, \tau_0]$ and let $\tau_0 \geq t_1 > t_2$, we have

$$\begin{aligned}
\|(Tv)(t_1) - (Tv)(t_2)\|_{X_{1+\epsilon}} &\leq \|[U(t_1,0) - U(t_2,0)]v_0\|_{X_{1+\epsilon}} \\
&+ \| \int_0^{t_1} U(t_1,s)f(s,v(s))ds - \int_0^{t_2} U(t_2,s)f(s,v(s))ds\|_{X_{1+\epsilon}} \\
&\leq \|[U(t_1,t_2) - I]U(t_2,0)v_0\|_{X_{1+\epsilon}} + \|[U(t_1,t_2) - I] \int_0^{t_2} U(t_2,s)f(s,v(s))ds\|_{X_{1+\epsilon}} \\
&+ \| \int_{t_2}^{t_1} U(t_1,s)f(s,v(s))ds\|_{X_{1+\epsilon}}.
\end{aligned} \tag{74}$$

First note that, for $\eta > \epsilon$ and $0 < \eta - \epsilon \leq 1$, Lemma 2.10 implies that

$$\|[U(t_1,t_2) - I]\|_{L(X_{1+\eta}, X_{1+\epsilon})} \leq C(\eta, \epsilon)(t_1 - t_2)^{\eta - \epsilon},$$

that is,

$$\|[U(t_1,t_2) - I]\|_{L(X_{1+\eta}, X_{1+\epsilon})} \overset{t_1 \to t_2^+}{\longrightarrow} 0.$$

Now, from (42), we have

$$\|U(t_2,0)v_0\|_{X_{1+\eta}} \leq C(\eta) t_2^{-\eta} \|v_0\|_{X_1}.$$

Also, if $\eta > \epsilon$, $0 < \eta - \epsilon \leq 1$, from of (42), (44) and (64) it follows that

$$\begin{aligned}
\| \int_0^{t_2} U(t_2,s)f(s,v(s))ds\|_{X_{1+\eta}} &\leq \int_0^{t_2} \|U(t_2,s)\|_{L(X_{\gamma(\epsilon)}, X_{1+\eta})} \|f(s,v(s))\|_{X_{\gamma(\epsilon)}} ds \\
&\leq \int_0^{t_2} M(t_2 - s)^{-1+\gamma(\epsilon)-\eta} (\|v(s)\|^{\rho}_{X_{1+\epsilon}} + \nu(s)s^{-\gamma(\epsilon)})ds \\
&\leq M\mu^{\rho} \int_0^{t_2} (t_2 - s)^{-1+\gamma(\epsilon)-\eta} s^{-\rho\epsilon} ds + M\delta \int_0^{t_2} (t_2 - s)^{-1+\gamma(\epsilon)-\eta} s^{-\gamma(\epsilon)} ds \\
&\leq M\nu^{\rho} \mathbf{B}(\gamma(\epsilon) - \eta, 1 - \rho\epsilon) t_2^{\gamma(\epsilon)-\eta-\rho\epsilon} + M\delta \mathbf{B}(\gamma(\epsilon) - \eta, 1 - \gamma(\epsilon)) t_2^{-\eta}].
\end{aligned}$$

Hence, the two first terms on the right hand side of (74) tends to zero as $t_1 \to t_2^+$.

Finally, for the third term of the (74), observe that

$$\begin{aligned}
\| \int_{t_2}^{t_1} U(t_1,s)f(s,v(s))ds\|_{X_{1+\epsilon}} &\leq cM \int_{t_2}^{t_1} (t_1 - s)^{-1+\gamma(\epsilon)-\epsilon} (\nu(s)s^{-\gamma(\epsilon)} + \|v(s)\|^{\rho}_{X_{1+\epsilon}})ds \\
&\leq cM\delta \int_{t_2}^{t_1} (t_1 - s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\gamma(\epsilon)} ds + cM \int_{t_2}^{t_1} (t_1 - s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\rho\epsilon} (s^{\epsilon} \|v(s)\|_{X_{1+\epsilon}})^{\rho} ds \\
&\leq cM\delta t_1^{-\epsilon} \int_{t_2/t_1}^{1} (1 - s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\gamma(\epsilon)} ds + cM\mu^{\rho} t_1^{\gamma(\epsilon)-\epsilon-\rho\epsilon} \int_{t_2/t_1}^{1} (1 - s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\rho\epsilon} ds,
\end{aligned}$$

which goes to zero as $t_1 \to t_2^+$. The case $t_1 < t_2$ is similar.

Let us now show that, if $v \in K(\tau_0)$, then $t^\epsilon \|(Tv)(t)\|_{X_{1+\epsilon}} \leq \mu$, for all $t \in (0, \tau_0]$. In fact, from (64), (68), (69), (70) and (71) we have

$$\begin{aligned}
t^\epsilon \|(Tv)(t)\|_{X_{1+\epsilon}} &\leq \|t^\epsilon U(t,0)v_0\|_{X_{1+\epsilon}} + cMt^\epsilon \int_0^t (t-s)^{-1+\gamma(\epsilon)-\epsilon} (\|v(s)\|^\rho_{X_{1+\epsilon}} + \nu(s) s^{-\gamma(\epsilon)}) ds \\
&\leq \|t^\epsilon U(t,0)v_0\|_{X_{1+\epsilon}} + cM\delta t^\epsilon \int_0^t (t-s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\gamma(\epsilon)} ds \\
&+ cMt^\epsilon \int_0^t (t-s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\rho\epsilon} [s^\epsilon \|v(s)\|_{X_{1+\epsilon}}]^\rho ds \\
&\leq \|t^\epsilon U(t,0)[v_0 - w_0]\|_{X_{1+\epsilon}} + \|t^\epsilon U(t,0)w_0\|_{X_{1+\epsilon}} + cM\delta \mathbf{B}^\epsilon_\epsilon + cM\mu^\rho \mathbf{B}^\epsilon_\epsilon \\
&\leq Mr + \|t^\epsilon U(t,0)w_0\|_{X_{1+\epsilon}} + cM\delta \mathbf{B}^\epsilon_\epsilon + cM\mu^\rho \mathbf{B}^\epsilon_\epsilon \leq \mu.
\end{aligned}$$

This shows that T takes $K(\tau_0)$ into itself. The next step is to prove that the map T is a contraction from $K(\tau_0)$ into itself. From (72), we see that

$$t^\epsilon \|(Tv_1)(t) - (Tv_2)(t)\|_{X_{1+\epsilon}} \leq t^\epsilon \| \int_0^t U(t,s)[f(s, v_1(s)) - f(s, v_2(s))] ds \|_{X_{1+\epsilon}}.$$

Taking $\theta = \epsilon$ in the Lemma 3.3 and remembering that $\Gamma_\epsilon(t) \leq \frac{1}{2}$ for $t \in [0, \tau_0]$, it follows that T is a strict contraction in $K(\tau_0)$ and

$$\|T(v_1) - T(v_2)\|_{K(\tau_0)} \leq \frac{1}{2} \|v_1 - v_2\|_{K(\tau_0)}.$$

By the Banach contraction principle we have that T has an unique fixed point in $K(\tau_0)$. We will denote this fixed point by $X(t, v_0)$, for each $\|v_0 - w_0\|_{X_1} < r$, $0 \leq t \leq \tau_0$. Note that, from (73), $X(\cdot, v_0) \in C((0, \tau_0], X_{1+\epsilon})$.

Following [5] we can prove that $t^\theta \|X(t, v_0)\|_{X_{1+\theta}} \to 0$ as $t \to 0$ for all $0 < \theta < \gamma(\epsilon)$.

Let us prove now that

$$\lim_{t \to 0^+} \|X(t, v_0) - v_0\|_{X_1} = 0.$$

In fact, from Lemmas 3.2 and 3.4 we have

$$\begin{aligned}
\|X(t, v_0) - v_0\|_{X_1} &\leq \|U(t,0)v_0 - v_0\|_{X_1} + \int_0^t \|U(t,s) f(s, v(s))\|_{X_1} ds \\
&\leq \|U(t,0)v_0 - v_0\|_{X_1} + cM\mathbf{B}_\epsilon [\nu(t) + t^{\gamma(\epsilon)-\rho\epsilon} [\sup_{0<s\leq t} \{s^\epsilon \|X(s, v_0)\|_{1+\epsilon}\}]^\rho] \overset{t \to 0^+}{\longrightarrow} 0.
\end{aligned}$$

From all this we see that $X(t, v_0)$ is an $\epsilon-$regular solution starting at v_0 and it is the unique $\epsilon-$regular solution starting at v_0 in the set $K(\tau_0)$. We will hereafter call it the K-solution starting at v_0.

The proofs of uniqueness of $\epsilon-$regular solutions and the continuity with respect to initial data follows exactly as in Theorem 1 of [5] and therefore we omit it. This concludes the proof of Theorem 3.1.

4. APPLICATIONS

Consider the problem

$$\begin{cases} u_t = a(t,x)\Delta u + f(u), \quad x \in \Omega,\ t > 0 \\ u(t,x) = 0, \quad x \in \partial\Omega,\ t \geq 0 \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{75}$$

By a solution of this problem we mean a solution $u(t)$ of the Cauchy problem for the ordinary differential equation of the first order

$$\frac{du}{dt} + A(t)u = f(u), \quad u(0) = u_0, \tag{76}$$

in the Banach space $L^2(\Omega)$, where the operator

$$A(t)u = a(t,x)\Delta u,$$

with domain of definition $D = H^2(\Omega) \cap H_0^1(\Omega)$. Here Ω is a bounded domain of $\mathbb{R}^N$, with regular boundary and $N \geq 3$ and we assume that $a(t,x)$ be continuously differentiable in x and that there are positive constants c_0 and c_1 such that $0 < c_0 \leq a(t,x) \leq c_1$.

Also, suppose that the functions $a(t,x)$ and $b(t,x) := \nabla_x a(t,x)$ it belongs $L^\infty(\Omega)$ and $L^\infty(\Omega^N)$ respectively, for any t. Furthermore, assume that $t \mapsto a(t,\cdot) \in L^\infty(\Omega)$, $t \mapsto b(t,\cdot) \in L^\infty(\Omega^N)$ are Hölder continuous functions with exponent $\epsilon > 0$ and constant C.

As for the nonlinearities f, we assume that $f \in C^1(\mathbb{R},\mathbb{R})$ and satisfies de growth condition

$$|f'(u)| \leq c(1 + |u|^{\rho-1}), \quad \rho = \frac{N+2}{N-2} > 1. \tag{77}$$

Lemma 4.1. *The operator $A(t)$ is the infinitesimal generator of the analytic semigroup of contractions in $L^2(\Omega)$.*

Proof: See [12, Theorem 3.6, p. 215]. □

The scale of fractional powers spaces $\{E_q^\alpha\}_{\alpha\in\mathbb{R}}$ associated to $A(t)$ satisfy

$$E_q^\alpha \hookrightarrow H_q^{2\alpha}(\Omega), \quad \alpha \geq 0,$$
$$E_q^{-\alpha} = (E_{q'}^\alpha)', \quad \alpha \geq 0,$$

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where $H^\alpha_q(\Omega)$ are the potential Bessel spaces (see [3, 6]). We will denote E^α_2 simply for E_α.

From the Sobolev Embedding Theorem, we obtain

$$\begin{cases} E_\alpha \hookrightarrow L^r(\Omega) & \text{for } r \leq \frac{2N}{N-4\alpha}, 0 \leq \alpha < \frac{N}{4} \\ E_0 = L^2(\Omega) & \\ E_\alpha \hookleftarrow L^s(\Omega) & \text{for } s \geq \frac{2N}{N-4\alpha}, -\frac{N}{4} < \alpha \leq 0, \end{cases} \tag{78}$$

with continuous embeddings. Moreover, we have $E_{-1/2} = (H^1_0(\Omega))' =: H^{-1}(\Omega)$. Notice that the realization $A_\alpha(t)$ of $A(t)$ in E_α is a isometry from $E_{\alpha+1}$ into E_α and $A_\alpha(t) : D(A_\alpha(t)) \subset E_{\alpha+1} \to E_\alpha$ is a sectorial operator. Furthermore, $D(A_\alpha(t)^\beta) = E_{\alpha+\beta}$.

Denote $X_\alpha := E_{\alpha-\frac{1}{2}}$, $\alpha \geq -\frac{1}{2}$ and $\mathcal{A}(t) : X_1 \subset X_0 \to X_0$ the $X_{-\frac{1}{2}}$-realization of $A(t)$. The fractional powers spaces associated to $\mathcal{A}(t)$ satisfy

$$\begin{cases} X_\alpha \hookrightarrow L^r(\Omega) & \text{for } r \leq \frac{2N}{N+2-4\alpha}, \frac{1}{2} \leq \alpha < \frac{1}{2} + \frac{N}{4} \\ X_{\frac{1}{2}} = L^2(\Omega) & \\ X_\alpha \hookleftarrow L^s(\Omega) & \text{for } s \geq \frac{2N}{N+2-4\alpha}, \frac{1}{2} - \frac{N}{4} < \alpha \leq \frac{1}{2}. \end{cases} \tag{79}$$

Lemma 4.2. *With the notation above, the operator $\mathcal{A}(t) : X_1 \subset X_0 \longrightarrow X_0$ is the infinitesimal generator of the an analytic semigroup in X_0 and satisfies the conditions* (6) *and* (7).

Proof: For $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\varphi \in H^1_0(\Omega)$, using the divergence theorem, we have

$$\begin{aligned} \int_\Omega [a(t,x) - a(s,x)]\Delta u(x)\varphi(x)dx &= \int_\Omega [a(t,x) - a(s,x)]\varphi(x)\Delta u(x)dx \\ &= -\int_\Omega \nabla_x([a(t,x) - a(s,x)]\varphi(x))\nabla u(x)dx. \end{aligned}$$

From this, we obtain

$$\begin{aligned} &\left|\int_\Omega [a(t,x) - a(s,x)]\Delta u(x)\varphi(x)dx\right| \\ &= \left|(t-s)^\epsilon \int_\Omega \frac{\nabla_x([a(t,x) - a(s,x)]\varphi(x))}{(t-s)^\epsilon}\nabla u(x)dx\right| \\ &\leq \left|(t-s)^\epsilon \int_\Omega \frac{\nabla_x[a(t,x) - a(s,x)]}{(t-s)^\epsilon}\varphi(x)\nabla u(x)dx\right| \\ &\quad + \left|(t-s)^\epsilon \int_\Omega \frac{[a(t,x) - a(s,x)]}{(t-s)^\epsilon}\nabla\varphi(x)\nabla u(x)dx\right| \\ &\leq |t-s|^\epsilon \| \frac{\nabla_x[a(t,x) - a(s,x)]}{(t-s)^\epsilon}\varphi\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\quad + |t-s|^\epsilon \| \frac{[a(t,x) - a(s,x)]}{(t-s)^\epsilon}\nabla\varphi\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\leq C|t-s|^\epsilon\|\varphi\|_{H^1_0(\Omega)}\|u\|_{H^1_0(\Omega)}. \end{aligned}$$

Now, taking the supremum for $\varphi \in H_0^1(\Omega)$, with $\|\varphi\|_{H_0^1(\Omega)} \le 1$, we get

$$\|[\mathcal{A}(t) - \mathcal{A}(s)]u\|_{H^{-1}(\Omega)} \le C|t-s|^{\epsilon}\|u\|_{H^1(\Omega)}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega).$$

It follows from the definition of $\mathcal{A}(t)$ and from the fact that $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$ that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{L(X_1,X_0)} \le C|t-s|^{\epsilon}.$$

□

Lemma 4.3. *Let $\rho > 1$, $f \in C^1(\mathbb{R},\mathbb{R})$ satisfying the condition (77). Then*

$$|f(u) - f(v)| \le c|u-v|(1 + |u|^{\rho-1} + |v|^{\rho-1}), \tag{80}$$

$$|f(u)| \le c(1 + |u|^{\rho}). \tag{81}$$

Proof: The proof of (80) follows from Mean Value Theorem and (81) follows from (80), with $v = 0$ and from triangular inequality. □

For $\epsilon \in (0, \frac{1}{2\rho})$, it follows from (79) and (81) that

$$X_{1+\epsilon} \hookrightarrow L^{\frac{2N}{N+2-4(1+\epsilon)}}(\Omega) \xrightarrow{f} L^{\frac{2N}{[N+2-4(1+\epsilon)]\rho}}(\Omega) = L^{\frac{2N}{N+2-4\rho\epsilon}}(\Omega) \hookrightarrow X_{\rho\epsilon}$$

From (79), (80) and Hölder's inequality with exponents $r = \frac{2N}{N+2-4(1+\epsilon)}$, $r' = \frac{2N}{4-4(\rho-1)\epsilon}$,

$$\begin{aligned}
&\|f(u(\cdot)) - f(v(\cdot))\|_{X_{\rho\epsilon}} \le c_1 \|f(u(\cdot)) - f(v(\cdot))\|_{L^{\frac{2N}{N+2-4\rho\epsilon}}(\Omega)} \\
&\le c_1 \|c|u-v|(1+|u|^{\rho-1}) + |v|^{\rho-1}\|_{L^{\frac{2N}{N+2-4\rho\epsilon}}(\Omega)} \\
&\le cc_1 \|u-v\|_{L^{\frac{2N}{N+2-4(1+\epsilon)}}(\Omega)} \|1 + |u|^{\rho-1} + |v|^{\rho-1}\|_{L^{\frac{2N}{4-4(\rho-1)\epsilon}}(\Omega)} \\
&\le c_2 \|u-v\|_{L^{\frac{2N}{N+2-4(1+\epsilon)}}(\Omega)} \left(1 + \|u\|^{\rho-1}_{L^{\frac{2N(\rho-1)}{4-4(\rho-1)\epsilon}}(\Omega)} + \|v\|^{\rho-1}_{L^{\frac{2N(\rho-1)}{4-4(\rho-1)\epsilon}}(\Omega)}\right).
\end{aligned}$$

To conclude, note that $\rho - 1 = \frac{4}{N-2}$ and $\frac{2N(\rho-1)}{4-4(\rho-1)\epsilon} = \frac{2N}{N+2-4(1+\epsilon)}$.

This proves that f is an $\epsilon-$regular map relative to the pair (X_0, X_1).

Follows of the Theorem 3.1 that the problem (75) has an unique $\epsilon-$regular solution v starting $v_0 \in H_0^1(\Omega)$.

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