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Regularity of the global attractor for semilinear damped wave equations

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We consider attractors $\mathbf{A}_\eta$, $\eta \in [0,1]$, corresponding to a singularly perturbed hyperbolic damped wave equation

$$u_{tt} + 2\eta A^{\frac{1}{2}} u_t + au_t + Au = f(u)$$

in $H^1_0(\Omega)\times L^2(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^3$. For dissipative nonlinearity $f \in C^2(\mathbb{R},\mathbb{R})$ satisfying $|f''(s)| \le c(1+|s|)$ with some $c > 0$, we prove that the family of attractors $\{\mathbf{A}_\eta, \eta \ge 0\}$ is upper semicontinuous at $\eta = 0$ in $H^{1+s}(\Omega)\times H^s(\Omega)$ for any $s \in (0,1)$. For dissipative $f \in C^3(\mathbb{R},\mathbb{R})$ satisfying $\lim_{|s|\to\infty} \frac{f''(s)}{s} = 0$ we prove that the attractor $\mathbf{A_0}$ for the hyperbolic damped wave equation

$$u_{tt} + au_t + Au = f(u)$$

(case $\eta = 0$) is bounded in $H^4(\Omega) \times H^3(\Omega)$ and thus is compact in the Hölder spaces $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for every $\mu \in (0, \frac{1}{2})$. As a consequence of the uniform bounds we obtain that the family of attractors $\{\mathbf{A}_\eta, \eta \in [0,1]\}$ is upper and lower semicontinuous in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for every $\mu \in (0, \frac{1}{2})$.
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1. INTRODUCTION

We study a singularly perturbed damped wave equation

$$\begin{cases} u_{tt} + 2\eta A^{\frac{1}{2}} u_t + au_t + Au = f(u),\ t > 0,\ x \in \Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega, \\ u(t,x) = 0,\ t \ge 0,\ x \in \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a smooth bounded subdomain of $\mathbb{R}^3$, A denotes the negative Laplacian in $L^2(\Omega)$ with the domain $D(A)=H^2(\Omega)\cap H^1_0(\Omega)$, a is a fixed positive number, $f:\mathbb{R}\to\mathbb{R}$ satisfies suitable regularity and growth conditions and $\eta\in[0,1]$ is a parameter.

It is known that for a dissipative $f\in C^2(\mathbb{R},\mathbb{R})$ with $\limsup_{|s|\to\infty}\frac{|f''(s)|}{1+|s|}<\infty$ there exists a C^0 semigroup $\{T_0(t)\}$ associated to the problem (1) with $\eta=0$, which has a global attractor $\mathbf{A_0}$ in $H^1_0(\Omega)\times L^2(\Omega)$ (see e.g. [2, 3, 5, 11] and references therein). It is also known that for $\eta>0$ problem (1) with dissipative $\lim_{|s|\to\infty}\frac{|f'(s)|}{1+|s|^4}=0$ defines a C^0 nonlinear semigroup $\{T_\eta(t)\}$ in $H^1_0(\Omega)\times L^2(\Omega)$, which has a global attractor $\mathbf{A}_\eta$ (see [6, 7, 8]).

In [5] it has been shown that, if a dissipative f satisfies $\lim_{|s|\to\infty}\frac{f'(s)}{s^2}=0$ ($N\geq 3$), and if

$$0\notin\sigma(-A+f'(u_0))\ \text{ whenever }\ u_0\in H^1_0(\Omega)\ \text{ and }\ u_0=A^{-1}f(u_0), \tag{2}$$

(which is known to be a generic property, see [4]) then the family of attractors for the semigroups $\{T_\eta(t)\}$ is continuous at $\eta=0$; that is

$$\sup_{a_\eta\in\boldsymbol{A}_\eta}\inf_{a_0\in\boldsymbol{A}_0}\|a_0-a_\eta\|_{H^1_0(\Omega)\times L^2(\Omega)}+\sup_{a_0\in\boldsymbol{A}_0}\inf_{a_\eta\in\boldsymbol{A}_\eta}\|a_0-a_\eta\|_{H^1_0(\Omega)\times L^2(\Omega)}\to 0\ \text{ as }\ \eta\to 0^+. \tag{3}$$

The growth of f was limited in [5] to the one needed to obtain existence of the global attractor for the problem (1) with $\eta=0$ and, simultaneously, to the one needed to ensure compactness of the Nemytskii map $f^e:H^1_0(\Omega)\to L^2(\Omega)$ associated with f, which condition was essentially used in the proof of the lower semicontinuity property.

In the present work upper semicontinuity result of [5] is proved for critically growing nonlinearities. This is based on the uniform bound on the approximating attractors in $H^2(\Omega)\times H^1(\Omega)$-norm obtained with the aid of *Alekseev's nonlinear variation of constants formula* (see [2, 7]).

The continuity of the attractors $\{\mathbf{A}_\eta,\ \eta\in[0,1]\}$ in appropriate norms serves as an important tool to transfer regularity properties from the problem (1) with $\eta>0$ (which has parabolic structure) to the problem (1) with $\eta=0$ (which has hyperbolic structure). In this procedure the study of (1) with $\eta>0$ shows a resemblance to the *Viscosity Method.* In particular, the family of problems (1) with $\eta>0$, which is mostly of mathematical interest, is considered as a useful tool allowing to achieve this goal.

Applying the above mentioned method we show in the present work that if $f\in C^2(\mathbb{R},\mathbb{R})$ is dissipative and $\lim_{|s|\to\infty}\frac{f'(s)}{s^2}=0$ then $\mathbf{A_0}$ is compact in $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$, $\mu\in(0,\frac12)$, since the approximating attractors $\mathbf{A}_\eta$, $\eta\in(0,1]$, are proved to be uniformly bounded in $H^3(\Omega)\times H^2(\Omega)$ and compact in the space of Hölder functions $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$, $\mu\in(0,\frac12)$. Additionally, if $f\in C^3(\mathbb{R},\mathbb{R})$, then in a similar manner we show that the attractor $\mathbf{A_0}$ is bounded in $H^4(\Omega)\times H^3(\Omega)$ and compact in $C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$, $\mu\in(0,\frac12)$.

In [12, Theorem 2.21] the *Galerkin method* was used to obtain sharp time and space regularity properties of $\mathbf{A_0}$. In particular it has been proved in [12] that, if $f^{(2)+\mu}$ satisfies a suitable growth assumption, then $\mathbf{A_0}$ is bounded in $H^3(\Omega)\times H^2(\Omega)$.

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In the present paper we prove that

$$\lim_{|s|\to\infty}\frac{f''(s)}{s}=0 \text{ and } f\in C^{1+j}(\mathbb{R},\mathbb{R}) \text{ imply } \mathbf{A_0}\subset H^{2+j}(\Omega)\times H^{1+j}(\Omega),\ \ j=1,2.$$

We remark that no Hölder continuity is required for f'' (or f''') to obtain within our approach the above mentioned regularity of $\mathbf{A}_0$.

The article is organized in the following way. In Section 2 we establish a suitable functional analytic framework to study nonlinear semigroups associated to (1) and properties of their attractors. In Section 3 we show that in the case when $f\in C^2(\mathbb{R},\mathbb{R})$ is dissipative and $|f''(s)|\le c(1+|s|)$ for some $c>0$, the approximating attractors are uniformly bounded in $H^2(\Omega)\times H^1(\Omega)$. For such f we conclude upper semicontinuity of $\{\mathbf{A}_\eta,\eta\in[0,1]\}$ at $\eta=0$ in $H^{1+s}(\Omega)\times H^s(\Omega)$, $s\in[0,1)$. Under the same assumptions as in Section 3, we obtain in Section 4 a uniform $H^3(\Omega)\times H^2(\Omega)-$bound for the approximating attractors and upper semicontinuity of the family of attractors in $H^{2+s}(\Omega)\times H^{1+s}(\Omega)$, $s\in[0,1)$. If, in addition, we assume that $\lim_{|s|\to\infty}\frac{f^{(2)}(s)}{s}=0$ we obtain $H^3(\Omega)\times H^2(\Omega)\cap C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)-$regularity of $\mathbf{A}_0$ ($\mu\in(0,\frac12)$) and continuity (upper and lower semicontinuity) of the family of attractors in $H^{2+s}(\Omega)\times H^{1+s}(\Omega)$, $s\in[0,1)$. In Section 5, assuming additionally that $f\in C^3(\mathbb{R},\mathbb{R})$, we establish $H^4(\Omega)\times H^3(\Omega)\cap C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)-$regularity of $\mathbf{A_0}$ ($\mu\in(0,\frac12)$) and continuity (upper and lower semicontinuity) of the family of attractors in $H^{3+s}(\Omega)\times H^{2+s}(\Omega)$, $s\in[0,1)$.

2. FUNCTIONAL ANALYTIC FRAMEWORK AND EXISTENCE OF ATTRACTORS

In this section we establish the functional analytic framework to study (1) and indicate how one can obtain the existence of attractors for the nonlinear semigroup associated to (1).

Let $X_p=L^p(\Omega)$, $p\in(1,\infty)$, and denote by $\{X_p^\alpha,\alpha\ge0\}$ the fractional power scale generated by (A,X_p). Recall that, for suitably smooth Ω, A has bounded imaginary powers (see [15]) and spaces X_p^α, $\alpha\in(0,1)$, $p>1$, coincide with $[L^p(\Omega),H^2_{p,\{I\}}(\Omega)]_\alpha=H^{2\alpha}_{p,\{I\}}(\Omega)$ (see [16]).

Consider damped wave operators

$$\mathcal{A}_\eta : D(A_\eta)\subset X_2^{\frac12}\times X_2\to X_2^{\frac12}\times X_2,\quad \mathcal{A}_\eta=\begin{bmatrix}0 & -I\\ A & 2\eta A^{\frac12}+aI\end{bmatrix},\quad D(\mathcal{A}_\eta)=X_2^1\times X_2^{\frac12},\ \eta\ge0,$$

and note that

$$\mathcal{A}_\eta^{-1}: X_2^{\frac12}\times X_2\to X_2^{\frac12}\times X_2,\quad \mathcal{A}_\eta^{-1}=\begin{bmatrix}2\eta A^{-\frac12}+aA^{-1} & A^{-1}\\ -I & 0\end{bmatrix}\ \text{ for }\ \eta\ge0. \tag{4}$$

Recall from [5] that there is a certain $d>0$ such that

$$d^{-1}\Big\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\Big\|_{X_2^1\times X_2^{\frac12}}\le\Big\|\mathcal{A}_\eta\begin{bmatrix}\phi\\ \psi\end{bmatrix}\Big\|_{X_2^{\frac12}\times X_2}\le d\Big\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\Big\|_{X_2^1\times X_2^{\frac12}},\ \eta\in(0,1],\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in X_2^1\times X_2^{\frac12}. \tag{5}$$

Let $Y_2^0 := Y_2$ be the extrapolated space (see [1]); that is, the completion of the normed space $(X_2^{\frac12}\times X_2, \|\mathcal{A}_\eta^{-1}\cdot\|_{X_2^{\frac12}\times X_2})$. For some $\tilde d>0$ (independent of $\eta\in[0,1]$), we have

$$\tilde d^{-1}\|\mathcal{A}_\eta^{-1}\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{X_2^{\frac12}\times X_2}\le \|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{X_2\times X_2^{-\frac12}}\le \tilde d\|\mathcal{A}_\eta^{-1}\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{X_2^{\frac12}\times X_2},\ \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in X_2^{\frac12}\times X_2. \tag{6}$$

Thus

$$\text{the completions of } (X_2^{\frac12}\times X_2,\|\mathcal{A}_\eta^{-1}\cdot\|_{X_2^{\frac12}\times X_2}) \text{ and } (X_2^{\frac12}\times X_2,\|\cdot\|_{X_2\times X_2^{-\frac12}}) \text{ coincide} \tag{7}$$

with equivalent norms independent of $\eta\in[0,1]$, which proves that $Y_2 = X_2\times X_2^{-\frac12}$.

The operator $\mathcal{A}_\eta$ has the closed extension to Y_2 with the domain Y_2^1 and it generates (both in Y_2 and in Y_2^1) a compact C^0-semigroup of contractions $\{e^{-\mathcal{A}_\eta t}: t\ge 0\}$ which is uniformly exponentially decaying; that means, there are positive constants c and ω such that

$$\|e^{-\mathcal{A}_\eta t}\|_{L(X_2^{\frac12}\times X_2)}\leqslant ce^{-\omega t},\ t\ge 0,\ \eta\in[0,1]. \tag{8}$$

Furthermore, this semigroup is analytic for $\eta\in(0,1]$ and the fractional power scale Y_2^α, $\alpha\ge 0$, generated by $(\mathcal{A}_\eta, Y_2)$ with $\eta>0$ is characterized as (see [6, 9, 10])

$$Y_2^\alpha = X_2^{\frac{\alpha}{2}}\times X_2^{\frac{\alpha-1}{2}},\ \alpha\ge 0. \tag{9}$$

Remark 2. 1. Since the portion of the above scale corresponding to $\alpha\in[0,4]$ will play an important role in further consideration we remark that we have already proved in (5)-(7) that

$$Y_2^0 = X_2\times X_2^{-\frac12},\ Y_2^1 = X_2^{\frac12}\times X_2,\ Y_2^2 = X_2^1\times X_2^{\frac12}.$$

By definition of fractional power spaces (see [13, Definition 1.4.1]) we next have[1]

$$\begin{aligned}Y_2^3 &= \mathcal{A}_\eta^{-3}(Y_2^0)=\mathcal{A}_\eta^{-1}(Y_2^2)=\{\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y_2^0:\mathcal{A}_\eta\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y_2^2\}\\ &=\{\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y_2^0:\psi\in X_2^1,\ \phi+2\eta A^{-\frac12}\psi+aA^{-1}\psi\in A^{-1}(X_2^{\frac12})\}=X_2^{\frac32}\times X_2^1\end{aligned}$$

and, similarly,

$$\begin{aligned}Y_2^4 &= \mathcal{A}_\eta^{-4}(Y_2^0)=\mathcal{A}_\eta^{-1}(Y_2^3)=\{\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y_2^0:\mathcal{A}_\eta\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y_2^3\}\\ &=\{\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y_2^0:\psi\in X_2^{\frac32},\ \phi+2\eta A^{-\frac12}\psi+aA^{-1}\psi\in A^{-1}(X_2^1)\}=X_2^2\times X_2^{\frac32}.\end{aligned}$$

[1]Note that $A\phi+2\eta A^{\frac12}\psi+a\psi\in X_2^{\frac12}=H_0^1(\Omega)$ is equivalent to $A^{\frac12}\phi+2\eta\psi+aA^{-\frac12}\psi\in X_2^1=H^2(\Omega)\cap H_0^1(\Omega)$ and to $\phi+2\eta A^{-\frac12}\psi+aA^{-1}\psi\in X_2^{\frac32}=H^3_{I,\Delta}(\Omega)$. Note also that $A^{\frac12}\psi\in X_2^{\frac12}=H_0^1(\Omega)$ if and only if $\psi\in X_2^1=H^2(\Omega)\cap H_0^1(\Omega)$; in particular, $\psi\in X_2^1=H^2(\Omega)\cap H_0^1(\Omega)$ implies that $A^{\frac12}\psi$ has zero trace, which distinguishes $A^{\frac12}\psi$ from first order differential operators.

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As for the 'intermediate spaces', we merely remark that they are characterized with the aid of complex interpolation (see [6, Theorem 2] for details). As for the spaces $X_2^{\frac{3}{2}}$, X_2^2 appearing above we obtain firstly with the aid of [13, Definition 1.4.1] that

$$X_2^{\frac{3}{2}} = A^{-\frac{3}{2}}(X_2^0) = A^{-1}(X_2^{\frac{1}{2}}) = \{\phi \in X_2^0 : A\phi \in X_2^{\frac{1}{2}}\} = \{\phi \in X_2^{\frac{1}{2}} : A\phi \in X_2^{\frac{1}{2}}\}$$

and hence, via elliptic regularity theory,

$$X_2^{\frac{3}{2}} = \{\phi \in H_0^1(\Omega) : A\phi \in H_0^1(\Omega)\} = H^3_{\{I,\Delta\}}(\Omega).$$

In a similar manner we show that

$$X_2^2 = A^{-2}(X_2^0) = A^{-1}(X_2^1) = \{\phi \in X_2^0 : A\phi \in X_2^1\} = \{\phi \in X_2^1 : A\phi \in X_2^1\}$$

and consequently

$$X_2^2 = \{\phi \in H^2(\Omega) \cap H_0^1(\Omega) : A\phi \in H^2(\Omega) \cap H_0^1(\Omega)\} = H^4_{\{I,\Delta\}}(\Omega).$$

With this set-up we consider problem (1) in the form

$$\tfrac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right), \ t > 0, \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \tag{10}$$

where $\eta \in [0,1]$ and

$$F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 \\ f^e(u) \end{smallmatrix}\right], \ f^e(u)(x) = f(u(x)) \ \text{ for all } \ u \in X_2^{\frac{1}{2}}, \ x \in \Omega.$$

Since we are working in space dimension $N = 3$, we assume that

$$f \in C^2(\mathbb{R}, \mathbb{R}), \quad |f''(s)| \leqslant c(1 + |s|), \ s \in \mathbb{R}, \tag{11}$$

and

$$\limsup_{|s| \to \infty} \frac{f(s)}{s} < \lambda_1, \tag{12}$$

where λ_1 is the first positive eigenvalue of A.

If (11) is satisfied, then $F : Y_2^1 \to Y_2^1$ is continuously differentiable and Lipschitz continuous in bounded subsets of Y_2^1. Hence, for each $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] \in Y_2^1$, there are constants $r > 0$ and $\tau = \tau_{\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]} > 0$ such that for any $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ in the ball $B_r(\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]) \subset Y_2^1$ there is a unique mild solution

$$T_\eta(\cdot)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] := \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in C([0,\tau], Y_2^1)$$

of (10) and the map

$$B_r(\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]) \ni \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \mapsto T_\eta(\cdot)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in C([0,\tau], Y_2^1)$$

is continuously differentiable.

If, in addition to (11), the condition (12) is satisfied, the solutions of (10) are defined for all $t \geq 0$ and are bounded uniformly in bounded subsets of Y_2^1, which is a consequence of the properties of the functional $\mathcal{L} : Y_2^1 \to \mathbb{R}$,

$$\mathcal{L}([\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}]) = \frac{1}{2}\|w_1\|^2_{X_2^{\frac{1}{2}}} + \frac{1}{2}\|w_2\|^2_{X_2} - \int_\Omega \int_0^{w_1} f(s)dsdx,\ [\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}] \in Y_2^1. \tag{13}$$

Namely, for some $\varepsilon > 0$ and $\mathcal{C}_\varepsilon > 0$, $\mathcal{L}$ satisfies under the mentioned assumptions

$$\mathcal{L}([\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}]) \geq \frac{\varepsilon}{2\lambda_1}\|w_1\|^2_{X_2^{\frac{1}{2}}} + \frac{1}{2}\|w_2\|^2_{X_2} - \mathcal{C}_\varepsilon,\ [\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}] \in Y_2^1, \tag{14}$$

is decreasing along solutions $\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] = [\begin{smallmatrix} u \\ v \end{smallmatrix}]\,(t, [\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}], \eta)$ of (10) and

$$\|\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_{Y_2^1} \leqslant c_{\mathcal{L}}\sqrt{1 + \mathcal{L}([\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}])} \tag{15}$$

with a constant $c_{\mathcal{L}}$ independent of $\eta \in [0, 1]$. Consequently, all solutions of (10) are globally defined and bounded uniformly for $t \geq 0$ and $[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}]$ in bounded subsets of Y_2^1.

Hence we have a nonlinear semigroup $\{T_\eta(t) : t \geq 0\}$ associated to (10) and next we outline the proof that this semigroup has a global attractor. In fact, to obtain the existence of attractor for $\{T_\eta(t) : t \geq 0\}$, we only need to prove that the set of equilibria for (10) is bounded and that $\{T_\eta(t) : t \geq 0\}$ is asymptotically compact.

The set of equilibria $\mathcal{E} = \{[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}] \in Y_2^1 :\ u_0 = A^{-1}f^e(u_0),\ v_0 = 0\}$ is bounded in Y_2^2. Indeed, from (12) we have

$$\|A^{\frac{1}{2}}u_0\|_{X_2} \leqslant M_0^{\mathcal{E}},\ [\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}] \in \mathcal{E} \tag{16}$$

and, since (11) ensures that f^e takes bounded subsets of $X_2^{\frac{1}{2}}$ into bounded subsets of X_2,

$$\|Au_0\|_{X_2} = \|f^e(u_0)\|_{X_2} \leqslant \sup_{\|\phi\|_{X_2^{\frac{1}{2}}} \leqslant M_0^{\mathcal{E}}} \|f^e(\phi)\|_{X_2} = M_1^{\mathcal{E}}. \tag{17}$$

The asymptotic compactness property of $\{T_\eta(t) : t \geq 0\}$ is proved in [2] for the case $\eta = 0$ and is an easy consequence of compactness of the linear semigroup $\{e^{-\mathcal{A}_\eta t} : t \geq 0\}$ in the case $\eta > 0$ (see [7, 8] for even faster growth).

Therefore, if $\eta \in [0, 1]$ and (11), (12) hold, then there exists a C^1-semigroup $\{T_\eta(t)\}$ of global solutions of (10) in Y_2^1, which has a global attractor $\mathbf{A}_\eta$.

3. UPPER SEMICONTINUITY RESULT FOR $F(U)$ GROWING LIKE $|U|^3$

In this section we obtain uniform Y_2^2−bound on the attractors when f satisfies (11)-(12) and prove (under the same assumptions on f) upper semicontinuity of $\{\mathbf{A}_\eta,\ \eta \geq 0\}$ at $\eta = 0$.

The proof of the Y_2^2−bound for $\cup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ will be divided into several lemmas.

Lemma 3.1. *If* (11) *and* (12) *are satisfied, then*

$$\sup_{\eta\in[0,1]} \sup_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \mathcal{L}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leq \sup_{\left[\begin{smallmatrix} u_{0} \\ 0 \end{smallmatrix}\right]\in\mathcal{E}} \mathcal{L}(\left[\begin{smallmatrix} u_{0} \\ 0 \end{smallmatrix}\right]). \tag{18}$$

Consequently, $\bigcup_{\eta\in[0,1]} \boldsymbol{A}_\eta$ *is bounded in* Y_2^1-*norm.*

Proof: For $\eta\in[0,1]$ and $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\in \boldsymbol{A}_\eta$ there exists a precompact complete orbit through $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$. Denoting it by $\gamma_\eta=\{T_\eta(t)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\ t\in\mathbb{R}\}$ we observe that

$$\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] = T_\eta(k)T_\eta(-k)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$$

for arbitrary $k\in\mathbb{N}$. The sequence $\{T_\eta(-k)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ has a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ convergent strongly in Y_2^1 to certain element of the α-limit set $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ of $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$. Since $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])\subset\mathcal{E}$, from (15) and (16) we have that

$$\mathcal{L}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leqslant \mathcal{L}(T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]),$$

$$\mathcal{L}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leqslant \lim_{k_l\to\infty} \mathcal{L}(T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leqslant \sup_{\left[\begin{smallmatrix} u_{0} \\ 0 \end{smallmatrix}\right]\in\mathcal{E}} \mathcal{L}(\left[\begin{smallmatrix} u_{0} \\ 0 \end{smallmatrix}\right]).$$

From this and (14) we obtain

$$\sup_{\eta\in(0,1]} \sup_{\left[\begin{smallmatrix} \phi_{\eta} \\ \psi_{\eta} \end{smallmatrix}\right]\in \mathbf{A}_\eta} \| \left[\begin{smallmatrix} \phi_{\eta} \\ \psi_{\eta} \end{smallmatrix}\right] \|_{Y_2^1} \leqslant c_{\mathcal{L}} \sup_{\left[\begin{smallmatrix} u_{0} \\ 0 \end{smallmatrix}\right]\in\mathcal{E}} \sqrt{1+\mathcal{L}(\left[\begin{smallmatrix} u_{0} \\ 0 \end{smallmatrix}\right])} =: \zeta \tag{19}$$

and the result is proved. ▮

From (12), there exists $K_f>\max\{0,f'(0)\}$ such that

$$s^{-1}(f(s)-f(0)) < K_f \quad \text{for each } 0\neq s\in\mathbb{R}. \tag{20}$$

Decompose f as $f(s)=\tilde{f}(s)+\hat{f}(s)$ where

$$\tilde{f}(s)=f(s)-f(0)-K_f s,\quad \hat{f}(s)=f(0)+K_f s,\quad s\in\mathbb{R},$$

and

$$s\tilde{f}(s)\leqslant 0 \text{ for } s\in\mathbb{R}. \tag{21}$$

Denote by $\tilde{T}_\eta(t)\left[\begin{smallmatrix} u_{0} \\ v_{0} \end{smallmatrix}\right]$ the solution of

$$\tfrac{d}{dt}\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] + A_\eta\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \tilde{f}^e(\tilde{u}) \end{smallmatrix}\right] =: \tilde{F}(\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]),\ t>0,\quad \left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_{0} \\ v_{0} \end{smallmatrix}\right]. \tag{22}$$

With this decomposition we write Alekseev's nonlinear variation of constants formula (see [7, Lemma 7]; also [2]),

$$T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] = \tilde{T}_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] + \int_0^t (\partial\tilde{T}_\eta)(t-s, T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\hat{F}(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])ds, \quad t>0, \tag{23}$$

where $\partial\tilde{T}_\eta$ is partial derivative of the map $(0,\infty)\times Y_2^1 \ni (t,V_0) \to \tilde{T}_\eta(t)V_0 \in Y_2^1$ with respect to V_0 and

$$\hat{F}(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\K_f w_1 + f(0)\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right] \in Y_2^1.$$

We recall from [13, Theorem 3.4.4] that $(\partial\tilde{T}_\eta)(\cdot,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\hat{F}(\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])$ is a solution of

$$\frac{d}{dt}W + A_\eta W = \tilde{F}'(\tilde{T}_\eta(t)V_0)W,\ t>0, \quad W(0) = \hat{F}(V_0). \tag{24}$$

In what follows we will derive suitable estimates for the solutions of (22) and (24).

Lemma 3.2. *If $\tilde{f}$ is as before, then for each $r>0$ there are positive constants c_r and ω_r such that*

$$\sup_{\eta\in(0,1]} \sup_{\|\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_{Y_2^1}\le r} \|\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\eta)\|_{Y_2^1} \le c_r e^{-\omega_r t}, \quad t\ge 0, \tag{25}$$

where $\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\eta)$ is the solution of (22).

Proof: As in [2, 7] we consider for $\delta\ge 0$ a suitable Lyapunov type functional

$$\tilde{\mathcal{L}}_\delta(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]) = \frac{1}{2}\|A^{\frac{1}{2}}w_1\|_{X_2}^2 + \frac{1}{2}\|w_2\|_{X_2}^2 - \int_\Omega\int_0^{w_1}\tilde{f}(s)dsdx + \delta\int_\Omega w_1w_2dx, \tag{26}$$

$\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right] \in Y_2^1$. The integral $\int_\Omega\int_0^{w_1}\tilde{f}(s)dsdx$ is nonpositive and from the Poincaré's inequality $\lambda_1\|w_1\|_{X_2}^2 \le \|A^{\frac{1}{2}}w_1\|_{X_2}^2$ we have

$$\frac{1}{4}(\|A^{\frac{1}{2}}w_1\|_{X_2}^2 + \|w_2\|_{X_2}^2) \le \tilde{\mathcal{L}}_\delta(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]), \quad 0<\delta\le\delta_0 := \frac{1}{2}\min\{\lambda_1,1\},\ \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1. \tag{27}$$

Estimating in a standard way and using (21) we obtain, for a suitable choice of $\delta\in(0,\delta_0)$,

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{L}}_\delta(\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right]) &= -a\|\tilde{v}\|_{X_2}^2 - 2\eta\|A^{\frac{1}{4}}\tilde{v}\|_{X_2}^2 + \delta\|\tilde{v}\|_{X_2}^2 + \delta\int_\Omega \tilde{u}\left(-A\tilde{u} - a\tilde{v} - 2\eta A^{\frac{1}{2}}\tilde{v} + \tilde{f}(\tilde{u})\right)dx\\ &\le -\frac{\delta}{2}(\|A^{\frac{1}{2}}\tilde{u}\|_{X_2}^2 + \|\tilde{v}\|_{X_2}^2), \quad \eta\in(0,1].\end{aligned} \tag{28}$$

In particular, for $\delta=0$ we get

$$\|A^{\frac{1}{2}}w_1\|_{X_2}^2 + \|w_2\|_{X_2}^2 \le 2\tilde{\mathcal{L}}_0(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]), \quad \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1, \tag{29}$$

and

$$\frac{d}{dt}\tilde{\mathcal{L}}_0([\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}]) \leq 0. \tag{30}$$

Since $|\tilde{f}(s)| \leq (c+K_f)|s|(1+|s|^2)$, we have

$$|\int_\Omega \int_0^{\tilde{u}} \tilde{f}(s)dsdx| \leq (c+K_f)(\|\tilde{u}\|_{X_2}^2 + \|\tilde{u}\|_{X_4}^4).$$

Therefore, from (29), (30) and Sobolev embedding we infer that for some constant $c_0 > 1$

$$|\int_\Omega \int_0^{\tilde{u}} \tilde{f}(s)dsdx| \leq c_0 \left(1 + \tilde{\mathcal{L}}_0([\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}])\right) \|A^{\frac{1}{2}}\tilde{u}\|_{X_2}^2. \tag{31}$$

Following (26), (28), (31) we conclude that for each $r > 0$ there is a constant $\rho_r > 0$ such that

$$\frac{d}{dt}\tilde{\mathcal{L}}_\delta(t, [\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}](u_0, v_0, \eta) \leq -\rho_r\, \tilde{\mathcal{L}}_\delta(t, [\begin{smallmatrix}\tilde{u}\\ \tilde{v}\end{smallmatrix}](u_0, v_0, \eta)), \quad t \geq 0,\ \eta \in [0,1],\ \|[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}]\|_{Y_2^1} \leq r.$$

With the estimate (27) condition (25) now follows easily. ▮

We next estimate solutions of (24) uniformly for $\eta \in (0,1]$ and $V_0 \in \bigcup_{\eta\in(0,1]} \boldsymbol{A}_\eta$.

Lemma 3.3. *If* $\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix} \in \boldsymbol{A}_\eta$, $\eta \in (0,1]$ *and* $W = [\begin{smallmatrix}w\\ \chi\end{smallmatrix}]$ *is the solution of* (24) *with* $W(0) = \hat{F}(\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}) = \begin{bmatrix}0\\ K_f\phi_\eta + f(0)\end{bmatrix}$, *then there are positive constants* ϵ *and* L_ϵ *such that*

$$\|[\begin{smallmatrix}w\\ \chi\end{smallmatrix}]\|_{Y_2^1} \leqslant L_\epsilon e^{-\epsilon t}, \quad t \geq 0. \tag{32}$$

Proof: As a consequence of (19) we have

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\in \boldsymbol{A}_\eta} \|\hat{F}(\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix})\|_{Y_2^1} \leq \tilde{\zeta}. \tag{33}$$

We also remark that the equation for w can be rewritten as

$$w_{tt} + 2\eta A^{\frac{1}{2}} w_t + a w_t + Aw = (f'(\tilde{u}_\eta) - f'(0))w + (f'(0) - K_f)w \tag{34}$$

with

$$f'(0) - K_f < 0. \tag{35}$$

For $\delta \geq 0$ and $[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}] \in Y_2^1$ we define

$$\hat{\mathcal{L}}_\delta([\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}]) = \frac{1}{2}\|w_2\|_{X_2}^2 + \frac{1}{2}(K_f - f'(0))\|w_1\|_{X_2}^2 + \frac{1}{2}\|A^{\frac{1}{2}} w_1\|_{X_2}^2 + \delta \int_\Omega w_1 w_2 dx. \tag{36}$$

Differentiating $\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ w_t \end{smallmatrix}\right]\right)$ and using (34) we get

$$\frac{d}{dt}\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right) = -\epsilon\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right) + R_{\delta,\epsilon}, \tag{37}$$

where

$$\begin{aligned} R_{\delta,\epsilon} &= \frac{\epsilon}{2}\|w_t\|^2_{X_2} + \frac{\epsilon}{2}(K_f - f'(0))\|w\|^2_{X_2} + \frac{\epsilon}{2}\|A^{\frac{1}{2}}w\|^2_{X_2} + \epsilon\delta\int_\Omega w w_t dx \\ &+ \int_\Omega (f'(\tilde{u}_\eta) - f'(0)) w w_t dx + \delta\int_\Omega (f'(\tilde{u}_\eta) - f'(0)) w^2 dx \\ &+ \delta\int_\Omega (f'(0) - K_f) w^2 dx - \delta\|A^{\frac{1}{2}}w\|^2_{X_2} - a\|w_t\|^2_{X_2} - a\delta\int_\Omega w w_t dx \\ &- 2\eta\|A^{\frac{1}{4}}w_t\|^2_{X_2} - 2\delta\eta\int_\Omega w_t A^{\frac{1}{2}} w dx + \delta\|w_t\|^2_{X_2}. \end{aligned} \tag{38}$$

Applying (11) we obtain from (38)

$$\begin{aligned} R_{\delta,\epsilon} \leqslant & \frac{\epsilon}{2}\|w_t\|^2_{X_2} + \frac{\epsilon}{2}(K_f - f'(0))\|w\|^2_{X_2} + \frac{\epsilon}{2}\|A^{\frac{1}{2}}w\|^2_{X_2} + \frac{\epsilon\delta}{2}\|w\|^2_{X_2} + \frac{\epsilon\delta}{2}\|w_t\|^2_{X_2} \\ &+ \frac{a}{2}\|w_t\|^2_{X_2} + \frac{c^2}{2a}\|w\|^2_{X_6}\|\tilde{u}_\eta\|^2_{X_6}(1 + \|\tilde{u}_\eta\|_{X_6})^2 \\ &+ \delta c^2\|w\|^2_{X_6}\|\tilde{u}_\eta\|_{X_6}(1 + \|\tilde{u}_\eta\|_{X_6}) + \delta(f'(0) - K_f)\|w\|^2_{X_2} - \delta\|A^{\frac{1}{2}}w\|^2_{X_2} \\ &- a\|w_t\|^2_{X_2} + \frac{\lambda_1\delta}{2}\|w\|^2_{X_2} + \frac{a^2\delta}{2\lambda_1}\|w_t\|^2_{X_2} + 4\delta\eta\|w_t\|^2_{X_2} + \frac{\delta\eta}{4}\|A^{\frac{1}{2}}w\|^2_{X_2} + \delta\|w_t\|^2_{X_2}. \end{aligned}$$

From Lemma 3.1 and Lemma 3.2, there are thus positive constants r, $\bar{c}_r$ and ω_r such that

$$\begin{aligned} R_{\delta,\epsilon} \leqslant & -\left(\delta - \frac{\epsilon}{2}\right)(K_f - f'(0))\|w\|^2_{X_2} - \left(\delta - \frac{\epsilon}{2} - \frac{\epsilon\delta}{2\lambda_1} - \frac{\delta}{2} - \frac{\delta\eta}{4}\right)\|A^{\frac{1}{2}}w\|^2_{X_2} \\ &- \left(\frac{a}{2} - \frac{\epsilon}{2} - \delta - \frac{\epsilon\delta}{2} - \frac{\delta a^2}{2\lambda_1} - 4\delta\eta\right)\|w_t\|^2_{X_2} + \bar{c}_r e^{-\omega_r t}\|A^{\frac{1}{2}}w\|^2_{X_2},\ \delta \in (0,1),\ \epsilon > 0. \end{aligned} \tag{39}$$

From (35), (37), (39) one can choose $\epsilon > 0$, $\delta > 0$ to obtain with the aid of (25),

$$\frac{d}{dt}\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right) \leqslant -\epsilon\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right) + \bar{c}_r e^{-\omega_r t}\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right), \quad t \geq 0,\ \eta \in (0,1]. \tag{40}$$

Solving differential inequality (40) and using (33), (36) we get the result. ∎

We will next strengthen (32) to $X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}$ −estimate with $\alpha \in (0, \frac{1}{2})$. First, let us show that there is a constant $\hat{d} \geq 1$ such that

$$\hat{d}^{-1}\|\cdot\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}} \leq \|\mathcal{A}^\alpha_\eta \cdot\|_{Y_2^1} \leq \hat{d}\|\cdot\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}},\ \forall \eta \in (0,1]. \tag{41}$$

For this we recall from [6, Proposition 1 (vi)] that

$$\|\mathcal{A}_\eta^{\mathrm{i}t}\|_{L(Y_2^1)} \le e^{\frac{\pi}{2}|t|},\ t\in\mathbb{R},\ \eta\in(0,1]. \tag{42}$$

As a consequence

$$D(\mathcal{A}_\eta^\alpha) = [Y_2^1, D(\mathcal{A}_\eta)]_\alpha,\ \alpha\in(0,1),\ \eta\in(0,1],$$

and

$$\hat{d}^{-1}\|\mathcal{A}_\eta^\alpha\cdot\|_{Y_2^1} \le \|\cdot\|_{[Y_2^1,D(\mathcal{A}_\eta)]_\alpha} \le \hat{d}\|\mathcal{A}_\eta^\alpha\cdot\|_{Y_2^1},\ \alpha\in(0,1),\ \eta\in(0,1], \tag{43}$$

where $\hat{d}$ is independent of $\eta\in(0,1]$ (see [1, §I.2.9 (2.9.9)]). Since (5) ensures that $\|\mathcal{A}_\eta\cdot\|_{Y_2^1}$ and $\|\cdot\|_{Y_2^2}$ are equivalent norms uniformly for $\eta\in(0,1]$, we infer that $\|\cdot\|_{[Y_2^1,D(\mathcal{A}_\eta)]_\alpha}$ and $\|\cdot\|_{[Y_2^1,Y_2^2]_\alpha}$ are also equivalent uniformly for $\eta\in(0,1]$. Recalling that $[Y_2^1,Y_2^2]_\alpha = X^{\frac{1+\alpha}{2}}\times X^{\frac{\alpha}{2}} := Y_2^{1+\alpha}$, $\alpha\in(0,1)$, (see [6, Proposition 3]), we get (41).

Lemma 3.4. *If* $\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in\boldsymbol{A}_\eta$, $\eta\in(0,1]$, $\alpha\in(0,\frac12)$ *and* $W=\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right]$ *is the solution of* (24) *with* $W(0)=\hat{F}(\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right])$*, then there are positive constants* d'' *and* ω'' *such that*

$$\|\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right]\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}} \leqslant d''e^{-\omega''t},\quad t\geqslant 0. \tag{44}$$

Proof: For $\alpha\in(0,\frac12)$ and the solution $W=\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right]$ of (24) with $W(0)=\hat{F}(\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right])$ we obtain

$$\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right] = e^{-\mathcal{A}_\eta t}\mathcal{A}_\eta^\alpha\hat{F}(\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]) + \int_0^t e^{-\mathcal{A}_\eta(t-s)}\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}0\\ f'(\tilde{u}_\eta)w-K_fw\end{smallmatrix}\right]ds,\ t\geqslant 0$$

and from (8) we get

$$\|\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right]\|_{Y_2^1} \leqslant ce^{-\omega t}\|\mathcal{A}_\eta^\alpha\hat{F}(\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right])\|_{Y_2^1} + c\int_0^t e^{-\omega(t-s)}\|\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}0\\ f'(\tilde{u}_\eta)w-K_fw\end{smallmatrix}\right]\|_{Y_2^1}ds,\ t\geqslant 0. \tag{45}$$

As a consequence of (19) and (41), for each $\alpha\in(0,\frac12)$ there exists $\zeta_\alpha>0$ such that

$$\sup_{\eta\in(0,1]}\ \sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in\boldsymbol{A}_\eta}\|A_\eta^\alpha\hat{F}(\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right])\|_{Y_2^1} \leqslant \hat{d}\|K_f\phi_\eta+f(0)\|_{X_2^{\frac{\alpha}{2}}} \leqslant \zeta_\alpha \tag{46}$$

and

$$\begin{aligned}\|\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}0\\ f'(\tilde{u}_\eta)w-K_fw\end{smallmatrix}\right]\|_{Y_2^1} &\leqslant \hat{d}\|A^{\frac{\alpha}{2}}(f'(\tilde{u}_\eta)w-K_fw)\|_{X_2}\\ &= \hat{d}\|A^{\frac{\alpha-1}{2}}A^{\frac12}(f'(\tilde{u}_\eta)w-K_fw)\|_{X_2}.\end{aligned} \tag{47}$$

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From [1, Theorem 1.3.8 and 1.4.12], we have

$$\begin{aligned}
&\|A^{\frac{\alpha-1}{2}}A^{\frac{1}{2}}(f'(\tilde u_\eta)w-K_fw)\|_{X_2}=\|A^{\frac{1}{2}}(f'(\tilde u_\eta)w-K_fw)\|_{X_2^{\frac{\alpha-1}{2}}}\\
&=\|A^{\frac{1}{2}}(f'(\tilde u_\eta)w-K_fw)\|_{\left(X_2^{\frac{1-\alpha}{2}}\right)^*}\leqslant K_\alpha\|A^{\frac{1}{2}}(f'(\tilde u_\eta)w-K_fw)\|_{X_{\frac{6}{5-2\alpha}}}.
\end{aligned}\tag{48}$$

As a consequence of (47), (48) we thus get

$$\begin{aligned}
&\|\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}0\\ f'(\tilde u_\eta)w-K_fw\end{smallmatrix}\right]\|_{Y_2^1}\leqslant\tilde K_\alpha\|\nabla(f'(\tilde u_\eta)w-K_fw)\|_{X_{\frac{6}{5-2\alpha}}}\\
&\leqslant\tilde K_\alpha\|f''(\tilde u_\eta)\nabla\tilde u_\eta w+f'(\tilde u_\eta)\nabla w-K_f\nabla w)\|_{X_{\frac{6}{5-2\alpha}}}.
\end{aligned}\tag{49}$$

Since $\alpha\in(0,\frac{1}{2})$, from (11) we have

$$\begin{aligned}
\|f'(\tilde u_\eta)\nabla w-K_f\nabla w\|_{X_{\frac{6}{5-2\alpha}}}&\leqslant\|c(1+|\tilde u_\eta|^2)\nabla w\|_{X_{\frac{6}{5-2\alpha}}}+K_f\|\nabla w\|_{X_{\frac{6}{5-2\alpha}}}\\
&\leqslant(c+K_f)\|\nabla w\|_{X_{\frac{6}{5-2\alpha}}}+c\|\tilde u_\eta\|_{X_6}^2\|\nabla w\|_{X_{\frac{6}{3-2\alpha}}}\\
&\leqslant K'\left(\|A^{\frac{1}{2}}w\|_{X_2}+\|A^{\frac{1}{2}}\tilde u_\eta\|_{X_2}^2\|A^{\frac{1+\alpha}{2}}w\|_{X_2}\right)
\end{aligned}\tag{50}$$

and

$$\begin{aligned}
\|f''(\tilde u_\eta)\nabla\tilde u_\eta w\|_{X_{\frac{6}{5-2\alpha}}}&\leqslant\|c(1+|\tilde u_\eta|)\|_{X_6}\|\nabla\tilde u_\eta\|_{X_2}\|w\|_{X_{\frac{6}{1-2\alpha}}}\\
&\leqslant K''\left(1+\|A^{\frac{1}{2}}\tilde u_\eta\|_{X_2}\right)\|A^{\frac{1}{2}}\tilde u_\eta\|_{X_2}\|A^{\frac{1+\alpha}{2}}w\|_{X_2}.
\end{aligned}\tag{51}$$

By (25), (32) and (49)-(51) there are positive numbers d' and w' such that

$$\|\mathcal{A}_\eta^\alpha\left[\begin{smallmatrix}0\\ f'(\tilde u_\eta)w-K_fw\end{smallmatrix}\right]\|_{Y_2^1}\leqslant d'e^{-\omega' t}\left(1+\|w\|_{X_2^{\frac{1+\alpha}{2}}}\right),\ t\geqslant 0.\tag{52}$$

Inserting (46), (52) into the right hand side of (45) and using (41) we then have

$$\|\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right]\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}\leqslant c\hat d^2e^{-\omega t}\zeta_\alpha+c\hat d d'\int_0^t e^{-\omega(t-s)}e^{-\omega' s}\left(1+\|\left[\begin{smallmatrix}w\\ \chi\end{smallmatrix}\right]\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}\right)ds,\ t\geqslant 0$$

and therefore (44) follows from Gronwall's inequality. ∎

As a consequence of Lemma 3.4 we obtain the following result

Lemma 3.5. *For every $\alpha\in(0,\frac{1}{2})$, $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is a bounded subset of $X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}$.*

Proof: Following the proof of [5, Lemma 3.5] we have that any complete orbit $\gamma_\eta(\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right])$, $\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right]\in\boldsymbol{A}_\eta$, is precompact in Y_2^2. Hence, a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right]\}$ exists, convergent

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in Y_2^2 to an element of the α-limit set $\alpha_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \subset \mathcal{E}$. Then, letting $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] =: \left[\begin{smallmatrix} u_{\eta} \\ v_{\eta} \end{smallmatrix}\right](t, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta)$, we have from (22)

$$\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right](t, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta) = e^{-\mathcal{A}_\eta t}T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_\eta(t-s)}\tilde{F}(\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right](s, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta))ds,$$

and (since $\mathcal{E}$ is bounded in Y_2^2; see (17)) choosing $\eta > 0$ we find certain $N_\eta \in \mathbb{N}$ such that for all $k_l \geqslant N_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$

$$\|T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_{Y_2^2} \leqslant M_1^{\mathcal{E}} + 1. \tag{53}$$

Therefore, we get

$$\begin{aligned} &\|\mathcal{A}_\eta\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right](t, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta)\|_{Y_2^1} \leqslant \|e^{-\mathcal{A}_\eta t}\|_{L(Y_2^1)}\|A_\eta T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_{Y_2^1} \\ &+ \int_0^t \|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_2^1)}\|\mathcal{A}_\eta\left(\tilde{F}(\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right](s, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta)\right)\|_{Y_2^1}ds \\ &\le ce^{-\omega t}(M_1^{\mathcal{E}} + 1) + \int_0^t ce^{-\omega(t-s)}\|\mathcal{A}_\eta\left(\tilde{F}(\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right](s, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta)\right)\|_{Y_2^1}ds. \end{aligned} \tag{54}$$

From (11), (5) and (25) we have, for $\eta \in (0, 1]$,

$$\begin{aligned} &\|\mathcal{A}_\eta\tilde{F}\left(\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right]\right)\|_{Y_2^1} = \|\mathcal{A}_\eta\left[\begin{smallmatrix} 0 \\ \tilde{f}(\tilde{u}_\eta) \end{smallmatrix}\right]\|_{Y_2^1} \leqslant (1 + 2\eta + \frac{a}{\lambda_1})\|A^{\frac{1}{2}}(f(\tilde{u}_\eta) - f(0) - K_f\tilde{u}_\eta)\|_{X_2} \\ &\leqslant C(\|\left(1 + |\tilde{u}_\eta|^2\right)\nabla\tilde{u}_\eta\|_{X_2} + \|\nabla\tilde{u}_\eta\|_{X_2}) \leqslant (C+1)\|\nabla\tilde{u}_\eta\|_{X_2} + C\||\tilde{u}_\eta|^2\nabla\tilde{u}_\eta\|_{X_2} \\ &\leqslant (C+1)c_\zeta e^{-\omega_\zeta t} + C\|\tilde{u}_\eta\|_{X_6}^2\|\nabla\tilde{u}_\eta\|_{X_6} \leqslant (C+1)c_\zeta e^{-\omega_\zeta t} + \tilde{C}\|A^{\frac{1}{2}}\tilde{u}_\eta\|_{X_2}^2\|A\tilde{u}_\eta\|_{X_2} \\ &\leqslant (C+1)c_\zeta e^{-\omega_\zeta t} + d\tilde{C}c_\zeta^2 e^{-2\omega_\zeta t}\|\mathcal{A}_\eta\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right]\|_{Y_2^1}. \end{aligned} \tag{55}$$

Joining (54), (55), solving the resulting integral inequality and using (5) we conclude that a positive constant $\hat{c}$ exists such that

$$\forall_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \in \mathbf{A}_\eta} \forall_{k_l \ge N_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])} \|\left[\begin{smallmatrix} \tilde{u}_{\eta} \\ \tilde{v}_{\eta} \end{smallmatrix}\right](t, T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \eta)\|_{Y_2^2} \leqslant \hat{c},\ t \ge 0,\ \eta \in (0, 1]. \tag{56}$$

We can now use (23) to conclude the proof. First note that, from Lemma 3.4, we have

$$\begin{aligned} &\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\in\mathbf{A}_\eta}\int_0^t \|(\partial\tilde{T}_\eta)(t-s, T_\eta(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\hat{F}(T_\eta(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}ds \\ &\leqslant \int_0^t \sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right]\in\mathbf{A}_\eta}\|(\partial\tilde{T}_\eta)(t-s, \left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])\hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}ds \\ &\leqslant \frac{d''}{\omega''}\left(1 - e^{-\omega'' t}\right) \leqslant \frac{d''}{\omega''},\ t \ge 0. \end{aligned} \tag{57}$$

Connecting (57), (56) and (23) we get the result. ▌

Using Lemma 3.5 we derive the uniform bound on the attractors in Y_2^2−norm.

Theorem 3.1. *If (11) and (12) hold, then $\bigcup_{\eta\in(0,1]} \boldsymbol{A}_\eta$ is a bounded subset of Y_2^2; that is, there is a positive constant $\mathbf{C}$ such that*

$$\sup_{\eta\in(0,1]} \sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in\boldsymbol{A}_\eta} \| \left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right] \|_{Y_2^2} \leq \mathbf{C}. \tag{58}$$

Proof: Since a complete orbit $\gamma_\eta(\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right])$, $\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right] \in \boldsymbol{A}_\eta$, is precompact in Y_2^2, a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right]\}$ exists, convergent in Y_2^2 to an element of the α-limit set $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right])$ $\subset \mathcal{E}$. Recall that $\mathcal{E}$ is bounded in Y_2^2, let $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right] =: \left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right]$ and write the integral formula

$$\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right] = e^{-\mathcal{A}_\eta t}T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_\eta(t-s)}F(\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right])ds,\ t\geq 0,$$

to obtain, with the aid of (8) and (53),

$$\begin{aligned}\|\mathcal{A}_\eta\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right]\|_{Y_2^1} &\leqslant \|e^{-\mathcal{A}_\eta t}\|_{L(Y_2^1)}\|\mathcal{A}_\eta T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\ v_{0\eta}\end{smallmatrix}\right]\|_{Y_2^1} + \|\left(I-e^{-\mathcal{A}_\eta t}\right)F(\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right])\|_{Y_2^1}\\ &\quad+\int_0^t \|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_2^1)}\|\mathcal{A}_\eta\left(F(\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right])\right)\|_{Y_2^1}ds\\ &\leq ce^{-\omega t}(M_1^{\mathcal{E}}+1)+(1+ce^{-\omega t})\|F(\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right])\|_{Y_2^1}\\ &\quad+\int_0^t ce^{-\omega(t-s)}\|A_\eta\left(F(\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right])\right)\|_{Y_2^1}ds.\end{aligned} \tag{59}$$

Then, for each $\eta\in(0,1]$, we have

$$\begin{aligned}&\|\mathcal{A}_\eta\big(F\left(\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\right)\big)\|_{Y_2^1} = \|\mathcal{A}_\eta\left[\begin{smallmatrix}0\\ f(u_\eta)-f(0)\end{smallmatrix}\right]\|_{Y_2^1}\\ &\leqslant C\|\nabla(f(u_\eta)-f(0))\|_{X_2}+a\|f(u_\eta)-f(0)\|_{X_2} = C\|f'(u_\eta)\nabla u_\eta\|_{X_2}+a\|f(u_\eta)-f(0)\|_{X_2}\end{aligned}$$

so that, using (11) and Lemma 3.5 with $\alpha\in[\frac{1}{3},\frac{1}{2})$, we get

$$\begin{aligned}\|\mathcal{A}_\eta\big(F\left(\left[\begin{smallmatrix}u_\eta\\ v_\eta\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\right)\big)\|_{Y_2^1} &\leqslant \tilde{C}(\|(1+|u_\eta|^2)\nabla u_\eta\|_{X_2}+1)\\ &\leqslant \hat{C}(\||u_\eta|^2\nabla u_\eta\|_{X_2}+1) \leqslant \hat{C}(\|u_\eta\|^2_{X_{\frac{6}{\alpha}}}\|\nabla u_\eta\|_{X_{\frac{6}{3-2\alpha}}}+1)\\ &\leqslant \hat{C}'(\|u_\eta\|^3_{X_2^{\frac{1+\alpha}{2}}}+1)\leqslant \bar{C}.\end{aligned} \tag{60}$$

From (59), (60) and (5) we obtain (58). The proof is thus complete. ▌

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Theorem 3.2. *Under the assumptions of Theorem 3.1 the family of attractors* $\{\mathbf{A}_\eta,\ \eta \in (0,1]\}$ *is upper semicontinuous at* $\eta = 0$ *in* Y_2^1*; that is*

$$\sup_{a_\eta \in \boldsymbol{A}_\eta} \inf_{a_0 \in \boldsymbol{A}_0} \|a_0 - a_\eta\|_{Y_2^1} \to 0 \quad as \quad \eta \to 0^+.$$

Proof. If $\eta_n \to 0^+$ and $\left[\begin{smallmatrix} u_{0\eta_n} \\ v_{0\eta_n} \end{smallmatrix}\right] \in \boldsymbol{A}_{\eta_n}$, then as a consequence of Theorem 3.1 and compactness of the embedding $Y_2^2 \hookrightarrow Y_2^1$ a subsequence $\{\left[\begin{smallmatrix} u_{0\eta_{n_k}} \\ v_{0\eta_{n_k}} \end{smallmatrix}\right]\}$ exists, convergent in Y_2^1 to a certain $\left[\begin{smallmatrix} \hat{u}_0 \\ \hat{v}_0 \end{smallmatrix}\right] \in Y_2^1$. Whenever $\left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]$ in Y_2^1, from [5, Lemma 3.1], for every $\tau > 0$ we have

$$\sup_{t \in [0,\tau]} \|T_{\eta_n}(t) \left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right] - T_0(t) \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y_2^1} \to 0 \text{ as } \eta_n \to 0^+.$$

Therefore, it is possible to construct a complete bounded orbit of the limit semigroup $\{T_0(t)\}$ through $\left[\begin{smallmatrix} \hat{u}_0 \\ \hat{v}_0 \end{smallmatrix}\right]$. Since such orbit is contained in $\mathbf{A_0}$, the proof is complete. ▌

As a consequence of the compact embedding $Y_2^2 \hookrightarrow Y_2^{1+\alpha}$, $\alpha \in [0,1)$, we have that

Corollary 3.1. *The family of attractors* $\{\mathbf{A}_\eta,\ \eta \in (0,1]\}$ *is upper semicontinuous at* $\eta = 0$ *in* $X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}$ *for each* $\alpha \in [0,1)$*; that is,* $\sup_{a_\eta \in \boldsymbol{A}_\eta} \inf_{a_0 \in \boldsymbol{A}_0} \|a_0 - a_\eta\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}} \to 0$ *as* $\eta \to 0^+$*, for each* $\alpha \in [0,1)$*.*

4. $(H^3 \times H^2) \cap (C^{1+\mu} \times C^\mu)$–REGULARITY OF $\mathbf{A_0}$ WITH $\mu \in (0, \frac{1}{2})$

In this section uniform $H^3(\Omega) \times H^2(\Omega)$–bound for the attractors is established. We also obtain $H^3(\Omega) \times H^2(\Omega) \cap C^{1+\mu}(\bar{\Omega}) \times C^\mu(\bar{\Omega})$–regularity of the limit attractor $\mathbf{A}_0$ with $\mu \in (0, \frac{1}{2})$.

Consider

$$\tfrac{d}{dt}\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right],\ t > 0, \quad \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{smallmatrix}\right], \tag{61}$$

and the solution $\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])$ of

$$\tfrac{d}{dt}\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ f(u_\eta) \end{smallmatrix}\right],\ t > 0, \quad \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_2^1. \tag{62}$$

Define

$$\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t, u_\eta) := \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] - e^{-\mathcal{A}_\eta t} \left[\begin{smallmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} A^{-1}f(0) \\ 0 \end{smallmatrix}\right]. \tag{63}$$

Then $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t, u_\eta)$ solves

$$\tfrac{d}{dt}\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ f(u_\eta) - f(0) \end{smallmatrix}\right],\ t > 0, \quad \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \tag{64}$$

and fulfils the integral equation

$$\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t,u_\eta) = \int_0^t e^{-\mathcal{A}_\eta(t-s)} \left[\begin{smallmatrix} 0 \\ f(u_\eta(s))-f(0) \end{smallmatrix}\right] ds,\ t \geq 0. \tag{65}$$

From (8) we also have for all $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_2^1$

$$\|e^{-\mathcal{A}_\eta t} \begin{bmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{bmatrix} \|_{Y_2^1} \leq ce^{-\omega t} \| \begin{bmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{bmatrix} \|_{Y_2^1},\ t \geq 0,\ \eta \in (0,1]. \tag{66}$$

It is easy to see that there is a constant $d_1 > 0$, independent of $\eta \in (0,1]$, such that

$$d_1^{-1} \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y_2^3} \leq \|\mathcal{A}_\eta^2 \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y_2^1} \leq d_1 \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y_2^3},\ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y_2^3. \tag{67}$$

Lemma 4.1. *If* (11) *and* (12) *hold, then* $\tilde{\mathbf{A}} := \begin{bmatrix} -A^{-1}f(0) \\ 0 \end{bmatrix} + \bigcup_{\eta \in (0,1]} \boldsymbol{A}_\eta$ *is a bounded subset of* Y_2^3 *and the union of attractors* $\bigcup_{\eta \in (0,1]} \boldsymbol{A}_\eta$ *is bounded in* $H^3(\Omega) \times H^2(\Omega)$*−norm.*

Proof: If $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, we infer from Theorem 3.1 that $f(\phi_\eta) - f(0) \in X_2^1 = H^2(\Omega) \times H_0^1(\Omega)$. Following the characterization of fractional power spaces in Remark 2.1 we then conclude that $\left[\begin{smallmatrix} 0 \\ f(\phi_\eta)-f(0) \end{smallmatrix}\right] \in Y_2^3$. Next, recalling (67) we have

$$\begin{aligned} &\|{\mathcal{A}_\eta}^2 \left[\begin{smallmatrix} 0 \\ f(\phi_\eta)-f(0) \end{smallmatrix}\right] \|_{Y_2^1} \leq d_1 \| \left[\begin{smallmatrix} 0 \\ f(\phi_\eta)-f(0) \end{smallmatrix}\right] \|_{Y_2^3} = d_1 \| \left[\begin{smallmatrix} 0 \\ f(\phi_\eta)-f(0) \end{smallmatrix}\right] \|_{X_2^{\frac{3}{2}} \times X_2^1} \\ &= d_1 \|A(f(\phi_\eta) - f(0))\|_{X_2} = d_1 \|f''(\phi_\eta)|\nabla \phi_\eta|^2 + f'(\phi_\eta)\Delta\phi_\eta\|_{X_2}, \end{aligned}$$

and hence (via (58))

$$\sup_{\eta \in (0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \boldsymbol{A}_\eta} \|{\mathcal{A}_\eta}^2 \left[\begin{smallmatrix} 0 \\ f(\phi_\eta)-f(0) \end{smallmatrix}\right] \|_{Y_2^1} \leq \tilde{\mathbf{C}}. \tag{68}$$

If $\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]$ is a solution of (62) with $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, then for $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t,u_\eta)$ in (65) we have[2]

$$\begin{aligned} &\sup_{\eta \in (0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \boldsymbol{A}_\eta} \|\mathcal{A}_\eta^2 \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t,u_\eta)\|_{Y_2^1} \\ &\leq \sup_{\eta \in (0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \boldsymbol{A}_\eta} \int_0^t \|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_2^1)} \|\mathcal{A}_\eta^2 (\left[\begin{smallmatrix} 0 \\ f(u_\eta(s))-f(0) \end{smallmatrix}\right])\|_{Y_2^1} ds \leq \frac{c}{\omega}\tilde{\mathbf{C}}(1 - e^{-\omega t}),\ t \geq 0. \end{aligned}$$

This, together with (67), implies

$$\sup_{t \geq 0} \sup_{\eta \in (0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \boldsymbol{A}_\eta} \| \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t,u_\eta)\|_{Y_2^3} \leq \frac{cd_1}{\omega}\tilde{\mathbf{C}}. \tag{69}$$

[2]We remark that $\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right](t, \left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right]) \in \boldsymbol{A}_\eta$ for all $t \geq 0$, $\eta \in (0,1]$.

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Fix $\eta \in (0,1]$, $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta$ and let $\gamma_\eta = \{T_\eta(t)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, t \in \mathbb{R}\}$ be the double-sided orbit through $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$. If $k_n \to \infty$ and $\begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix} = T_\eta(-k_n)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$, then (63) reads

$$\begin{aligned}\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} &= T_\eta(k_n)\begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix} = \begin{bmatrix} u^n_\eta \\ v^n_\eta \end{bmatrix}(k_n) \\ &= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(k_n, u^n_\eta) + e^{-\mathcal{A}_\eta k_n}\begin{bmatrix} u_{0n\eta} - A^{-1}f(0) \\ v_{0n\eta} \end{bmatrix} + \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix}, \ n \in \mathbb{N},\end{aligned}$$

where $\begin{bmatrix} u^n_\eta \\ v^n_\eta \end{bmatrix}$ is a solution of (62) with $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix}$. Since $\begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix} = T_\eta(-k_n)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, from (19) and (66), taking the limit in Y^1_2 we obtain

$$\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} - \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix} = \lim_{k_n \to \infty} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(k_n, u_\eta). \tag{70}$$

With the aid of (69) we infer that $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} - \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix}$ is simultaneously a limit of a sequence convergent weakly in Y^3_2, which translates into the bound

$$\sup_{\eta \in (0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta} \| \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} - \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix} \|_{Y^3_2} \leq \frac{cd_1}{\omega}\tilde{\mathbf{C}}. \tag{71}$$

This in the light of (9) and the elliptic regularity theory completes the proof.[3] ▮

Corollary 4.1. *Under the assumptions of Lemma 4.1 the union* $\bigcup_{\eta \in (0,1]} \boldsymbol{A}_\eta$ *of attractors is precompact in* $C^{1+\mu}(\bar{\Omega}) \times C^\mu(\bar{\Omega})$ *for each* $\mu \in (0, \frac{1}{2})$.

Below we address the regularity properties of the limit attractor $\mathbf{A_0}$.

Theorem 4.1. *Suppose that* (2), (11), (12) *hold and that* $\lim_{|s|\to\infty} \frac{f'(s)}{s^2} = 0$. *Then the attractor* $\boldsymbol{A}_0$ *for the semigroup* $\{T_0(t)\}$ *associated to* (10) *with* $\eta = 0$ *in* $H^1_0(\Omega) \times L^2(\Omega)$ *is bounded in* $H^3(\Omega) \times H^2(\Omega)$ *and compact in* $C^{1+\mu}(\bar{\Omega}) \times C^\mu(\bar{\Omega})$, $\mu \in (0, \frac{1}{2})$.

Proof: As shown in [5], the family of attractors $\boldsymbol{A}_\eta$ is lower semicontinuous at $\eta = 0$ with respect to the Hausdorff semidistance in Y^1_2. Therefore, if $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} \in \boldsymbol{A}_0$ then $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}$ is a limit in Y^1_2 of a sequence $\{\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}\}$ as $\eta_n \to 0^+$, where $\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \in \boldsymbol{A}_{\eta_n}$. As a consequence of Lemma 4.1 the sequence $\{\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}\}$ is bounded in $H^3(\Omega) \times H^2(\Omega)-$norm by certain positive constant $D > 0$ uniformly with respect to $n \in \mathbb{N}$. The latter ensures that $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}$ is a weak limit of a subsequence bounded in $H^3(\Omega) \times H^2(\Omega)-$norm and thus is bounded by the same

[3]The idea of decomposing the solution as in this proof comes back to [14, Theorem 2.4].

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constant D. As a consequence, we get

$$\sup_{\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]\in \boldsymbol{A}_0} \| \begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix} \|_{H^3(\Omega)\times H^2(\Omega)} < \infty,$$

which completes the proof. ▮

COROLLARY 4.2. *Under the assumptions of Theorem 4.1, the family* $\{\mathbf{A}_\eta,\ \eta\in(0,1]\}$ *of attractors is upper and lower semicontinuous at* $\eta=0$ *in* $C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)$ *for each* $\mu\in(0,\frac12)$*; that is*

$$\sup_{a_\eta\in\boldsymbol{A}_\eta}\inf_{a_0\in\boldsymbol{A}_0}\|a_0-a_\eta\|_{C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)}+\sup_{a_0\in\boldsymbol{A}_0}\inf_{a_\eta\in\boldsymbol{A}_\eta}\|a_0-a_\eta\|_{C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)}\to 0 \quad as \quad \eta\to 0^+,$$

for each $\mu\in(0,\frac12)$.

5. $H^4(\Omega)\times H^3(\Omega)\cap C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$−REGULARITY OF $\mathbf{A_0}$ WITH $\mu\in(0,\frac12)$

In this section uniform $H^4(\Omega)\times H^3(\Omega)$−bound for approximating attractors is derived and $H^4(\Omega)\times H^3(\Omega)\cap C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$−regularity of the limit attractor $\mathbf{A_0}$ is concluded with $\mu\in(0,\frac12)$.

LEMMA 5.1. *Suppose that the assumptions of Lemma 4.1 hold and, in addition,* $f\in C^3(\mathbb{R},\mathbb{R})$*. Then, the union of attractors* $\bigcup_{\eta\in(0,1]}\mathbf{A}_\eta$ *is bounded in* $H^4(\Omega)\times H^3(\Omega)$.

Proof: We differentiate (64) and substitute $w_3=\frac{d}{dt}w_2$ to get

$$\tfrac{d}{dt}\left[\begin{smallmatrix}w_2\\ w_3\end{smallmatrix}\right]+\mathcal{A}_\eta\left[\begin{smallmatrix}w_2\\ w_3\end{smallmatrix}\right]=\begin{bmatrix}0\\ f'(u_\eta)v_\eta\end{bmatrix},\ t>0,\quad \left[\begin{smallmatrix}w_2\\ w_3\end{smallmatrix}\right]_{t=0}=\left[\begin{smallmatrix}0\\ f(u_0)-f(0)\end{smallmatrix}\right]. \tag{72}$$

We then have

$$\left[\begin{smallmatrix}w_2\\ w_3\end{smallmatrix}\right](t,u_\eta)=e^{-\mathcal{A}_\eta t}\left[\begin{smallmatrix}0\\ f(u_0)-f(0)\end{smallmatrix}\right]+\int_0^t e^{-\mathcal{A}_\eta(t-s)}\begin{bmatrix}0\\ f'(u_\eta(s))v_\eta(s)\end{bmatrix}ds,\ t\ge 0, \tag{73}$$

and from (8) (whenever an appropriate smoothness can be provided)

$$\begin{aligned}\|\mathcal{A}_\eta^2\left[\begin{smallmatrix}w_2\\ w_3\end{smallmatrix}\right](t,u_\eta)\|_{Y_2^1}&\le ce^{-\omega t}\|\mathcal{A}_\eta^2\left[\begin{smallmatrix}0\\ f(u_0)-f(0)\end{smallmatrix}\right]\|_{Y_2^1}\\ &\quad+\int_0^t ce^{-\omega(t-s)}\|\mathcal{A}_\eta^2\begin{bmatrix}0\\ f'(u_\eta(s))v_\eta(s)\end{bmatrix}\|_{Y_2^1}ds,\ t\ge 0.\end{aligned} \tag{74}$$

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For $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta$ condition (71) ensures that $\psi_\eta \in X_2^1$ and

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta} \|\psi_\eta\|_{X_2^1} \leq \frac{cd_1}{\omega}\tilde{\mathbf{C}}, \tag{75}$$

whereas Lemma 4.1 implies that $\phi_\eta \in X_6^1 \cap C^{1+\mu}(\bar{\Omega})$ for $\mu \in (0, \frac{1}{2})$ and

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta} \|\phi_\eta\|_{X_6^1 \cap C^{1+\mu}(\bar{\Omega})} < \infty, \ \mu \in (0, \frac{1}{2}). \tag{76}$$

Since norms $\|\mathcal{A}_\eta^2 \cdot \|_{Y_2^1}$ and $\|\cdot\|_{X_2^{\frac{3}{2}} \times X_2^1}$ are equivalent uniformly for $\eta \in (0,1]$ (see (67)) and

$$\|f'(\phi_\eta)\psi_\eta\|_{X_2^1} = \|\Delta(f'(\phi_\eta)\psi_\eta)\|_{X_2} = \|f'''(\phi_\eta)|\nabla\phi_\eta|^2\psi_\eta + f''(\phi_\eta)\Delta\phi_\eta\psi_\eta$$
$$+ (f''(\phi_\eta) + f'(\phi_\eta))\nabla\phi_\eta\nabla\psi_\eta + f'(\phi_\eta)\Delta\psi_\eta\|_{X_2} \quad \text{for} \quad \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta,$$

then (75)-(76) lead to the estimate

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta} \|f'(\phi_\eta)\psi_\eta\|_{X_2^1} \leq \mathbf{G}. \tag{77}$$

If $\begin{bmatrix} u_\eta \\ v_\eta \end{bmatrix}$ is a solution of (62) through $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, then (68), (74) imply

$$\|\mathcal{A}_\eta^2 \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}(t, u_\eta)\|_{Y_2^1} \leq c\tilde{\mathbf{C}} + \frac{c}{\omega} \sup_{s\geq 0} \|\mathcal{A}_\eta^2 \begin{bmatrix} 0 \\ f'(u_\eta(s))v_\eta(s) \end{bmatrix}\|_{Y_2^1}, \ t \geq 0,$$

which via (67), (9) and (77) leads to

$$\sup_{t\geq 0} \sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta} \|\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}(t, u_\eta)\|_{X_2^{\frac{3}{2}} \times X_2^1} \leq cd_1 \left(\tilde{\mathbf{C}} + \frac{d_1}{\omega}\mathbf{G}\right) =: \mathbf{D}. \tag{78}$$

For any fixed $t \geq 0$, $\eta \in (0,1]$), function $z := w_1(t, u_\eta)$ can be viewed as the solution of

$$Az = -w_3(t, u_\eta) - aw_2(t, u_\eta) - A^{\frac{1}{2}}w_2(t, u_\eta) + (f(u_\eta) - f(0)) =: \mathbf{H}(t, u_\eta). \tag{79}$$

As a consequence of (76), (78),

$$\sup_{t\geq 0} \sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta} \|\mathbf{H}(t, u_\eta)\|_{X_2^1} < \infty.$$

Hence

$$\sup_{t\geq 0}\ \sup_{\eta\in(0,1]}\ \sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \|w_1(t,u_\eta)\|_{X_2^2} \leq \tilde{\mathbf{D}},$$

which together with (78) proves that

$$\sup_{t\geq 0}\ \sup_{\eta\in(0,1]}\ \sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \| \left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right](t,u_\eta)\|_{Y_2^4} \leq \hat{\mathbf{D}}. \tag{80}$$

As in the proof of Lemma 4.1 for $\eta\in(0,1]$ and $\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta$ let $\gamma_\eta=\{T_\eta(t)\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right], t\in\mathbb{R}\}$ be the orbit through $\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]$. If $k_n\to\infty$ and $\left[\begin{smallmatrix}u_{0n\eta}\\ v_{0n\eta}\end{smallmatrix}\right]=T_\eta(-k_n)\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]$, then (63) leads to (70) with the limit taken in Y_2^1 (for details see the proof of Lemma 4.1). Thanks to (80) and weak convergence property we then obtain

$$\sup_{\eta\in(0,1]}\ \sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \| \left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]-\left[\begin{smallmatrix}A^{-1}f(0)\\ 0\end{smallmatrix}\right]\|_{Y_2^4}\leq \hat{\mathbf{D}}, \tag{81}$$

which completes the proof. ▌

Theorem 5.1. *If the assumptions of Theorem 4.1 hold and, in addition, $f\in C^3(\mathbb{R},\mathbb{R})$, then $\mathbf{A_0}$ is bounded in $H^4(\Omega)\times H^3(\Omega)$ and compact in $C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$ for each $\mu\in(0,\frac{1}{2})$.*

Proof: The result follows from property (3), Lemma 5.1 and Sobolev type embedding. ▌

Corollary 5.1. *Under the assumptions of Theorem 5.1, the family $\{\mathbf{A}_\eta,\ \eta\in(0,1]\}$ of attractors is upper and lower semicontinuous at $\eta=0$ in $C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$ for each $\mu\in(0,\frac{1}{2})$; that is*

$$\sup_{a_\eta\in \boldsymbol{A}_\eta}\inf_{a_0\in \boldsymbol{A}_0}\|a_0-a_\eta\|_{C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)}+\sup_{a_0\in \boldsymbol{A}_0}\inf_{a_\eta\in \boldsymbol{A}_\eta}\|a_0-a_\eta\|_{C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)}\to 0 \ \text{as}\ \eta\to 0^+,$$

for each $\mu\in(0,\frac{1}{2})$.

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