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A parametrized Borsuk-Ulam theorem for a product of spheres with free $\mathbb{Z}_p$-action and free S^1-action

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In this paper, we generalize the result showed by Koikara and Mukerjee in [4] for bundles whose fibre has the same cohomology(mod p) of a product of spheres with free $\mathbb{Z}_p$-action and we estimate the size of the $\mathbb{Z}_p$-coincidence set. Further, in case of the circle group S^1, we also obtained a theorem for bundles whose fibre has rational cohomology ring isomorphic to that of $S^m \times S^n$, which extends a result proved by Nakaoka in [8]. May, 2007 ICMC-USP

Key Words: Parametrized Borsuk-Ulam theorem, characteristic polynomials, free action, product of spheres.

1. INTRODUCTION

The first parametrized version of the Borsuk-Ulam theorem was proved by Albrecht Dold in [1]. The general formulation of the parametrized Borsuk-Ulam type problem is the following:

Let $\pi : E \to B$ and $\pi' : E' \to B$ be two vector bundles over the same paracompact space B and suppose that E and E' admit an action of the cyclic group $\mathbb{Z}_p$, which preserving fibre, where p is prime. Let $f : S(E) \to E'$ be a fibre preserving equivariant map, where $S(E) \subset E$ is the sphere bundle of E. The parametrized version of the Borsuk-Ulam theorem consists of estimate the cohomological dimension of the set of points

$$Z_f = \{x \in S(E); f(x) = 0\}.$$

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The definition of the Stiefel-Whitney polynomials for vector bundles with the antipodal involution was introduced by Dold in [1] and it is an useful tool in studying parametrized Borsuk-Ulam type problem. Dold used the Stiefel-Whitney polynomials to prove that if $p = 2$ and if m and k are the dimensions of the fibres of E and E', respectively, where $m > k$ then

$$\text{cohom.dim } Z_f \geq \text{cohom.dim } (B) + m - k + 1,$$

where cohom.dim denotes the cohomological dimension.

Jaworowski in [6], maked similar constructions for vector bundles with linear periodic fibre preserving maps of a prime period p that are free on the outside of the 0-section. He defined a mod p Chern polynomial associated to E, where $E \to B$ is a vector bundle over a CW-complex B. Using the Chern polynomial, he obtained a lower bound on the size of the periodic coincidence set for fibre preserving maps $f : S(E) \to E'$ of the unit sphere bundle of E into another vector bundle in terms of the cohomology dimension. Izydorek and Rybicki in [3] also proved a parametrized version of the Borsuk-Ulam theorem for vector bundles with fibre preserving free actions of the cyclic group $\mathbb{Z}_p$.

Nakaoka in [8] obtained a result for $\mathbb{Z}_p$-actions and S^1-actions which included the cohomology version of the Dold' s theorem in presence of fixed points.

The technique introduced by Dold to solve the parametrized problem by using certain polynomials, called characteristic polynomials, is also used by Koikara and Mukerjee in [4] to show a parametrized version of the Borsuk-Ulam theorem for a product of spheres, with free $\mathbb{Z}_2$-action.

It will be goal in this paper to prove a generalization of [4] for bundles whose fibre has the same cohomology (mod p) of a product of spheres with free $\mathbb{Z}_p$-action and to estimate the size of the $\mathbb{Z}_p$-coincidence set. Further, in case of the circle group S^1, we also obtained a theorem for bundles whose fibre has rational cohomology ring isomorphic to that of $S^m \times S^n$, which extends one of the results proved by Nakaoka in [8].

2. PRELIMINARIES

We start by introducing some basics notions and notations. Let $\mathcal{U}$ a family of a space X. By the order of $\mathcal{U}$ we mean the largest number n such that $\mathcal{U}$ contains n members with non-empty intersection. The order of $\mathcal{U}$ is denoted by ord $\mathcal{U}$. We say a family $\mathcal{U}$ has *finite order* if ord $\mathcal{U} = n$ for some natural number n.

DEFINITION 2.1. [10] A space X is said to be *finitistic* [1] if every open cover of X has an open refinement with finite order.

[1]The concept of finitistic spaces was introduced by Swan [10] for working in fixed point theory (see also [11]).

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By the definition, all compact spaces and all finite dimensional (in the sense of the covering dimension dim) paracompact spaces are finitistic spaces. Specifically, the following characterization are valid.

PROPOSITION 2.1. [10] *A paracompact space X is finitistic if and only if there is a compact subspace K of X such that* $\dim F < \infty$*, for every closed subspace F with $F \cap K = \emptyset$.*

Throughout this paper, we denote by $X \sim_p S^m \times S^n$ a finitistic space with mod p cohomology ring isomorphic to that spheres product $S^m \times S^n$, which admits a free action of the cyclic group $\mathbb{Z}_p$ and by $X \sim_{\mathbb{Q}} S^m \times S^n$ a finitistic space with rational cohomology ring isomorphic to that spheres product $S^m \times S^n$, which admits a free action of the circle group $G = S^1$. H^* denotes Čech cohomology, unless otherwise indicated. The symbol " $\cong$ " denotes an appropriate isomorphism between algebraic objects.

The mod p Bockstein cohomology operation associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ will be denoted by β.

3. MAIN THEOREMS

Given a topological space $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$), where $0 < m \leq n$ and m is odd, let $\pi : X \hookrightarrow E \to B$ be a fibre bundle with the fibrewise free $\mathbb{Z}_p$-action (resp. free S^1-action) such that the quotient bundle $\overline{\pi} : \overline{E} \to B$ has the cohomology extension property, in the sense of [9, chapter 5 §7], where B is a paracompact space. Let us consider $\pi' : E' \to B$ a k-dimensional vector bundle with fibrewise $\mathbb{Z}_p$-action (resp. S^1-action) over B, with k even. If $f : E \to E'$ is a fibre preserving equivariant map, denote by $Z_f = f^{-1}(0)$ and by $\overline{Z}_f$ the quotient by the action induced on Z_f. In these conditions, we proved the following results

THEOREM 3.1 (Case $G = \mathbb{Z}_p$, p an odd prime). *Suppose $q(x, y, z)$ in $H^*(B)[x, y, z]$ is a polynomial such that $q(x, y, z)|_{\overline{Z}_f} = 0$. Then there are polynomials $r_1(x, y, z), r_2(x, y, z)$ in $H^*(B)[x, y, z]$ such that*

$$q(x, y, z)|_{\overline{Z}_f} W'(x, y) = r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z),$$

in the ring $H^*(B)[x, y, z]/(x^2)$, where $W'(x, y), W_1(x, y, z)$ and $W_2(x, y, z)$ are characteristic polynomials. [2]

As a consequence, we have the following corollary, which is a parametrized version of the Borsuk-Ulam theorem.

COROLLARY 3.1. *Suppose that the fibre dimension of $E' \to B$ is equal to k. Then $q(x, y, z)|_{\overline{Z}_f} \neq 0$, for all nonzero polynomials in $H^*(B)[x, y, z]$, whose degree in x, y and z*

[2]Characteristic polynomials will be defined in Section 5.

is less than $m-k+1$. One can also say that the $H^*(B)$-homomorphism

$$\sum_{i=0}^{\frac{m-k-1}{2}} H^*(B).x.y^i \oplus \sum_{i=0}^{\frac{m-k-1}{2}} H^*(B).y^i \to H^*(\overline{Z}_f),$$

defined by $x \mapsto x|_{\overline{Z}_f}$ and $y^i \mapsto y^i|_{\overline{Z}_f}$ is a monomorphism. In particular, if $m > k$

$$\text{cohom.dim}\ \overline{Z}_f \geq \text{cohom.dim}\ (B) + m - k.$$

Theorem 3.2 (Case $G = S^1$). *Suppose that $q(y,z) \in H^*(B)[y,z]$ is a polynomial such that $q(y,z)|_{\overline{Z}_f} = 0$. Then there are polynomials $r_1(y,z)$ and $r_2(y,z)$ in $H^*(B)[y,z]$ such that*

$$q(y,z)W'(y) = r_1(y,z)W_1(y,z) + r_2(y,z)W_2(y,z),$$

where $W'(y)$, $W_1(y,z)$ and $W_2(y,z)$ are characteristic polynomials in $H^*(B)[y,z]$.

We have the following corollary:

Corollary 3.2. *Suppose that the fibre dimension of $E' \to B$ is equal to k. Then $q(y,z)|_{\overline{Z}_f} \neq 0$, for all nonzero polynomials in $H^*(B)[y,z]$, whose degree in y and z is less than $m-k+1$. One can also say that the $H^*(B)$-homomorphism*

$$\sum_{i=0}^{\frac{m-k-1}{2}} H^*(B).y^i \to H^*(\overline{Z}_f),$$

defined by $y^i \mapsto y^i|_{\overline{Z}_f}$, is a monomorphism. In particular, if $m > k$

$$\text{cohom.dim}\ \overline{Z}_f \geq \text{cohom.dim}\ (B) + m - k - 1.$$

Remark 3. 1. Suppose that in Corollary 3.1 (resp. Corollary 3.2) B is a point. Then for any $\mathbb{Z}_p$-equivariant map (resp. S^1-equivariant map) $f : X \sim_p S^m \times S^n \to \mathbb{R}^k$, we have that cohom.dim $\overline{Z}_f \geq m - k$ (resp. cohom.dim $\overline{Z}_f \geq m - k - 1$.)

Remark 3. 2. Theorem 3.2 and Corollary 3.2 extend the result proved by Nakaoka [8, Theorem 1(ii) and its Corollary] for bundles whose fibre has rational cohomology ring isomorphic to the product of spheres, in case of free S^1-action.

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4. THE COHOMOLOGY RINGS OF THE ORBIT SPACE X/G

The results of this section was based upon work of Dotzel et al. The authors in [2], using as main tool the Leray-Serre spectral sequence, determined the possible cohomology algebra of the orbit space X/G, where X and G satisfy the properties required in Section 2. In the same conditions on X and G, we have the following lemmas, which are consequences of [2, theorems 1 and 3].

LEMMA 4.1 (Case $G = \mathbb{Z}_p$, p an odd prime). *Let $X \sim_p S^m \times S^n$ be a topological space, where $0 < m \leq n$ and m is odd. Suppose that $H^*(X;\mathbb{Z})$ is of finite type. Then $H^*(X/G;\mathbb{Z}_p)$ is a free graded module generated by the elements*

$$1, a, b, ab, \ldots, b^{\frac{m-1}{2}}, ab^{\frac{m-1}{2}}, d, ad, abd, \ldots, ab^{\frac{m-1}{2}}d,$$

subject to the relations $a^2 = 0$, $b^{\frac{m+1}{2}} = 0$ and $d^2 = 0$, where $a \in H^1(X/G;\mathbb{Z}_p)$, $b = \beta(a) \in H^2(X/G;\mathbb{Z}_p)$ and $d \in H^n(X/G;\mathbb{Z}_p)$.

Proof. By [2, Theorem 1(i)], $H^*(X/G;\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p[x,y,z]/(x^2, y^{(m+1)/2}, z^2)$ as a graded commutative algebra, where m is odd, $\deg x = 1$, $y = \beta(x)$ and $\deg z = n$. Therefore, the homomorphism

$$\mathbb{Z}_p[x,y,z]/(x^2, y^{(m+1)/2}, z^2) \to H^*(X/G;\mathbb{Z}_p) \text{ given by } (x,y,z) \mapsto (a,b,d)$$

is an isomorphism of $\mathbb{Z}_p$-algebras. ▮

LEMMA 4.2 (Case $G = S^1$). *Let $X \sim_{\mathbb{Q}} S^m \times S^n$ be a topological space, where $0 < m \leq n$ and m is odd. Then $H^*(X/G;\mathbb{Q})$ is a free graded module generated by the elements*

$$1, b, b^2, \ldots, b^{\frac{m-1}{2}}, d, db, db^2 \ldots, db^{\frac{m-1}{2}}$$

subject to the relations $b^{\frac{m+1}{2}} = 0$ and $d^2 = 0$, where $b \in H^2(X/G;\mathbb{Q})$ and $d \in H^n(X/G;\mathbb{Q})$.

Proof. By [2, Theorem 3(i)], $H^*(X/G;\mathbb{Q})$ is isomorphic to $\mathbb{Q}[y,z]/(y^{(m+1)/2}, z^2)$ as a graded commutative algebra, where m is odd, $\deg y = 2$ and $\deg z = n$. Thus, the homomorphism

$$\mathbb{Q}[y,z]/(y^{(m+1)/2}, z^2) \to H^*(X/G;\mathbb{Q}) \text{ given by } (y,z) \mapsto (b,d)$$

is an isomorphism of $\mathbb{Q}$-algebras. ▮

5. CHARACTERISTIC POLYNOMIALS

By following the technique introduced by Dold, in this section we define the characteristic polynomials associated to the fibre bundle (X, E, π, B). As in [4], we need to assume that the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$, where G is $\mathbb{Z}_p$ or S^1, has the cohomology extension property and then Leray-Hirsch theorem can be applied. There are two cases to consider, as follows.

5.1. Case $G = \mathbb{Z}_p$, p an odd prime

Let $(X \sim_p S^m \times S^n, E, \pi, B)$ be a fibre bundle in the same hypotheses of Section 3 and let us consider the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$. It follows from Leray-Hirsh theorem that there exist generators $\mathbf{a} \in H^1(\overline{E})$, $\mathbf{b} \in H^2(\overline{E})$ and $\mathbf{d} \in H^n(\overline{E})$ such that the natural homomorphism $j^* : H^*(\overline{E}) \to H^*(X/G)$ maps $\mathbf{a}$ to a, $\mathbf{b}$ to b and $\mathbf{d}$ to d, where a, b and d are as in Lemma 4.1. Furthermore, $H^*(\overline{E})$ is a $H^*(B)$-module, via the induced homomorphism $\overline{\pi}^*$, generated by

$$1, \mathbf{a}, \mathbf{b}, \mathbf{ab}, \ldots, \mathbf{b}^{\frac{m-1}{2}}, \mathbf{ab}^{\frac{m-1}{2}}, \mathbf{d}, \mathbf{ad}, \mathbf{abd}, \ldots, \mathbf{ab}^{\frac{m-1}{2}}\mathbf{d}. \tag{1}$$

Let us first consider natural numbers m and n satisfying: $1 < m < n$. We can express the elements $\mathbf{b}^{\frac{m+1}{2}} \in H^{m+1}(\overline{E})$ and $\mathbf{d}^2 \in H^{2n}(\overline{E})$ in terms of the basis (1), that is, there exist unique elements $\omega_i, \nu_i \in H^i(B)$ such that

$$\mathbf{b}^{\frac{m+1}{2}} = \omega_{m+1} + \omega_m \mathbf{a} + \omega_{m-1}\mathbf{ab} + \cdots + \omega_2 \mathbf{b}^{\frac{m-1}{2}} + \omega_1 \mathbf{ab}^{\frac{m-1}{2}} + \alpha \mathbf{d},$$

where $\alpha \in \mathbb{Z}_p$, with $\alpha = 0$ if $n > m+1$ and

$$\mathbf{d}^2 = \nu_{2n} + \nu_{2n-1}\mathbf{a} + \cdots + \nu_{2n-m}\mathbf{ab}^{\frac{m-1}{2}} + \nu_n \mathbf{d} + \nu_{n-1}\mathbf{ad} + \cdots + \nu_{n-m}\mathbf{ab}^{\frac{m-1}{2}}\mathbf{d}.$$

Definition 5.1. The characteristic polynomials in the indeterminates x, y and z, of degrees respectively $1, 2$ and n, associated to the fibre bundle $(X \sim_p S^m \times S^n, E, \pi, B)$, are defined as follows

$$\begin{aligned} W_1(x,y,z) &= \omega_{m+1} + \omega_m x + \cdots + \omega_1 x y^{\frac{m-1}{2}} + y^{\frac{m+1}{2}} + \alpha z, \\ W_2(x,y,z) &= \nu_{2n} + \nu_{2n-1} x + \cdots + \nu_{2n-m} x y^{\frac{m-1}{2}} + \nu_n z + \cdots \\ &+ \nu_{n-m} x y^{\frac{m-1}{2}} z + z^2, \end{aligned}$$

where $\omega_i, \nu_i \in H^i(B)$ and $1 < m < n$. If we consider natural numbers m and n such that $1 < m = n$, we can express the elements $\mathbf{b}^{\frac{m+1}{2}} \in H^{m+1}(\overline{E})$ and $\mathbf{d}^2 \in H^{2m}(\overline{E})$ in terms of the basis (1), as follows

$$\begin{aligned} \mathbf{b}^{\frac{m+1}{2}} &= \omega_{m+1} + \omega_m \mathbf{a} + \cdots + \omega_1 \mathbf{ab}^{\frac{m-1}{2}} + \bar{\omega}_1 \mathbf{d} + \alpha \mathbf{ad} \qquad \text{and} \\ \mathbf{d}^2 &= \nu_{2m} + \nu_{2m-1}\mathbf{a} + \cdots + \nu_m \mathbf{ab}^{\frac{m-1}{2}} + \bar{\nu}_m \mathbf{d} + \nu_{m-1}\mathbf{ad} + \cdots + \gamma \mathbf{ab}^{\frac{m-1}{2}}\mathbf{d}, \end{aligned}$$

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for unique elements $\omega_i, \bar{\omega}_i, \nu_i, \bar{\nu}_i \in H^i(B)$ and $\alpha, \gamma \in \mathbb{Z}_p$. In this case, the characteristic polynomials are given by

$$\begin{aligned} W_1(x,y,z) &= \omega_{m+1} + \omega_m x + \cdots + \omega_1 x y^{\frac{m-1}{2}} + y^{\frac{m+1}{2}} + \bar{\omega}_1 z + \alpha x z, \\ W_2(x,y,z) &= \nu_{2m} + \nu_{2m-1} x + \cdots + \nu_m x y^{\frac{m-1}{2}} + \bar{\nu}_m z + \nu_{m-1} x z + \cdots \\ &\cdots + \gamma x y^{\frac{m-1}{2}} z + z^2, \text{ for } 1 < m = n. \end{aligned}$$

We can substitute these elements for the indeterminates x, y and z respectively and we obtain the homomorphism of $H^*(B)$-algebras,

$$\sigma : H^*(B)[x,y,z] \to H^*(\overline{E}) \text{ given by } (x,y,z) \mapsto (\mathbf{a}, \mathbf{b}, \mathbf{d}). \tag{2}$$

We have that Ker (σ) is an ideal generated by the characteristic polynomials x^2, $W_1(x,y,z)$ and $W_2(x,y,z)$ and consequently,

$$\frac{H^*(B)[x,y,z]}{(x^2, W_1(x,y,z), W_2(x,y,z))} \cong H^*(\overline{E}). \tag{3}$$

Now, given a polynomial $q(x,y,z) \in H^*(B)[x,y,z]$ we will denote by $q(x,y,z)|_{\overline{E}}$ the image of $q(x,y,z)$ by the map σ defined in (2) and $q(x,y,z)|_{\overline{Z}_f}$ the image of $q(x,y,z)$ by the following composite

$$H^*(B)[x,y,z] \to H^*(\overline{E}) \to H^*(\overline{Z}_f)$$

given by $(x,y,z) \mapsto (\mathbf{a}, \mathbf{b}, \mathbf{d}) \mapsto (i^*\mathbf{a}, i^*\mathbf{b}, i^*\mathbf{d})$, where i^* denotes the induced by the natural inclusion.

Next, we need to define the characteristic polynomials associated to the k-dimensional vector bundle $E' \to B$ over B, with k even. For this, let us denote by SE' the sphere bundle of $E' \to B$. Then, one has that $\overline{SE'}$ is a bundle of $(k-1)$-dimensional lens space L_p^{k-1}. Since $H^*(L_p^{k-1}; \mathbb{Z}_p)$ is generated by the elements $a' \in H^1(L_p^{k-1}; \mathbb{Z}_p)$, $b' = \beta(a') \in H^2(L_p^{k-1}; \mathbb{Z}_p)$ with the relations $(a')^2 = 0$ and $(b')^{\frac{k}{2}} = 0$, it follows from Leray-Hirsh theorem that there exist generators $\mathbf{a'} \in H^1(\overline{SE'})$ and $\mathbf{b'} \in H^2(\overline{SE'})$ such that the homomorphism $(j')^* : H^*(\overline{SE'}) \to H^*(L_p^{k-1})$ maps $\mathbf{a'}$ to a' and $\mathbf{b'}$ to b'. Furthermore, $H^*(\overline{SE'})$ is a $H^*(B)$-module, via the induced homomorphism $(\overline{\pi'})^*$, generated by the elements

$$1, \mathbf{a}', \ \mathbf{b}', \ \mathbf{a}' \, \mathbf{b}', \ldots, \mathbf{a}' \, (\mathbf{b}')^{\frac{k-2}{2}}.$$

We can express $(\mathbf{b}')^{\frac{k}{2}} \in H^k(\overline{SE'})$ as follows

$$(\mathbf{b}')^{\frac{k}{2}} = \omega'_k + \omega'_{k-1}\mathbf{a}' + \cdots + \omega'_1 \mathbf{a}' (\mathbf{b}')^{\frac{k-2}{2}},$$

for unique elements $\omega'_i \in H^i(B)$.

Definition 5.2. The characteristic polynomial in the indeterminates x and y of degrees respectively 1 and 2, associated to the vector bundle $E' \to B$ is defined as follows

$$W'(x,y) = \omega'_k + \omega'_{k-1}x + \cdots + \omega'_1 x y^{\frac{k-2}{2}} + y^{\frac{k}{2}},$$

where $\omega'_i \in H^i(B)$.

From similar arguments to those used above, we have the following isomorphism of $H^*(B)$-algebra

$$\frac{H^*(B)[x,y]}{(x^2, W'(x,y))} \cong H^*(\overline{SE'}) \text{ defined by } x \mapsto \mathbf{a}' \text{ and } y \mapsto \mathbf{b}'.$$

5.2. Case $G = S^1$

Let $(X \sim_{\mathbb{Q}} S^m \times S^n, E, \pi, B)$ be a fibre bundle as in Section 3 and consider the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$. It follows from Leray-Hirsh theorem and Lemma 4.2 that

$$\frac{H^*(B)[y,z]}{(W_1(y,z), W_2(y,z))} \cong H^*(\overline{E}) \tag{4}$$

where, if $1 < m < n$

$$\begin{aligned} W_1(y,z) &= \omega_{m+1}1 + \omega_{m-1}y + \cdots + \omega_2 y^{\frac{m-1}{2}} + y^{\frac{m+1}{2}} + \alpha z \quad \text{and} \\ W_2(y,z) &= \nu_{2n} + \nu_{2n-2}y + \cdots + \nu_{2n-(m-1)}y^{\frac{m-1}{2}} + \nu_n z + \nu_{n-2}yz + \cdots \\ &\cdots + \nu_{n-(m-1)}y^{\frac{m-1}{2}}z + z^2, \end{aligned}$$

are the characteristic polynomials associated to $(X \sim_{\mathbb{Q}} S^m \times S^n, E, \pi, B)$, where $\omega_i, \nu_i \in H^i(B)$, $\alpha \in \mathbb{Q}$ with $\alpha = 0$, if $n > m+1$.

In case that $1 < m = n$, we have the following characteristic polynomials

$$\begin{aligned} W_1(y,z) &= \omega_{m+1} + \omega_{m-1}y + \cdots + \omega_2 y^{\frac{m-1}{2}} + y^{\frac{m+1}{2}} + \bar{\omega}_1 z \quad \text{and} \\ W_2(y,z) &= \nu_{2m} + \nu_{2m-2}y + \cdots + \nu_{2m-(m-1)}y^{\frac{m-1}{2}} + \nu_m z + \nu_{m-2}yz + \cdots \\ &\cdots + \nu_1 y^{\frac{m-1}{2}}z + \gamma z, \end{aligned}$$

where $\omega_i, \bar{\omega}_i \in H^i(B)$ and $\gamma \in \mathbb{Q}$.

Following the same argument of the previous case and observing that $\overline{SE'}$ is a bundle of $(k-2)$-dimensional complex projective space $S^{k-1}/S^1 = P_{\frac{(k-2)}{2}}(\mathbb{C})$, we have that

$$\frac{H^*(B)[y]}{(W'(y))} \cong H^*(\overline{SE'}), \text{ where } W'(y) = \omega'_{m+1}1 + \omega'_{m-1}y + \cdots + \omega'_2 y^{\frac{k-2}{2}} + y^{\frac{k}{2}}$$

is the characteristic polynomial associated to vector bundle $E' \to B$.

6. PROOF OF THE RESULTS

Proof (Proof of Theorem 3.1). Let $q(x,y,z)$ be a polynomial in $H^*(B)[x,y,z]$ such that $q(x,y,z)|_{\overline{Z}_f} = 0$. It follows from the continuity of the cohomology theory, that there is an open subset V in $\overline{E}$, with $V \supset \overline{Z}_f$ such that $q(x,y,z)|_V = 0$. One has that from the exact sequence

$$\cdots \longrightarrow H^*(\overline{E}, V) \xrightarrow{j_1^*} H^*(\overline{E}) \longrightarrow H^*(V) \longrightarrow \cdots$$

there exists $\mu \in H^*(\overline{E}, V)$ such that $j_1^*(\mu) = q(x,y,z)|_{\overline{E}}$, where the map $j_1 : \overline{E} \to (\overline{E}, V)$ denotes the natural inclusion. One can take the map of the orbit spaces $\overline{f} : \overline{E} - \overline{Z}_f \to \overline{E'} - \{0\}$ induced by the equivariant map $f : E \to E'$. Since the induced cohomology homomorphism $\overline{f}^*$ is a $H^*(B)$-homomorphism and $W'(\mathbf{a}', \mathbf{b}') = 0$, we have that

$$W'(x,y)|_{\overline{E}-\overline{Z}_f} = W'(\mathbf{a}, \mathbf{b}) = W'(\overline{f}^*(\mathbf{a}'), \overline{f}^*(\mathbf{b}')) = \overline{f}^*(W'(\mathbf{a}', \mathbf{b}')) = 0.$$

On the other hand, from the exact sequence

$$\cdots \longrightarrow H^*(\overline{E}, \overline{E} - \overline{Z}_f) \xrightarrow{j_2^*} H^*(\overline{E}) \longrightarrow H^*(\overline{E} - \overline{Z}_f) \longrightarrow \cdots$$

there exists $\theta \in H^*(\overline{E}, \overline{E} - \overline{Z}_f)$ satisfying the condition $j_2^*(\theta) = W'(x,y)|_{\overline{E}}$, where $j_2 : \overline{E} \to (\overline{E}, \overline{E} - \overline{Z}_f)$ is the natural inclusion. Hence,

$$q(x,y,z)W'(x,y)|_{\overline{E}} = j_1^*(\mu) j_2^*(\theta) = j^*(\mu \smile \theta)$$

by naturality of cup product. Let us observe that

$$\mu \smile \theta \in H^*(\overline{E}, V \cup (\overline{E} - \overline{Z}_f)) = H^*(\overline{E}, \overline{E})$$

which implies $\mu \smile \theta = 0$. Thus, $q(x,y,z)W'(x,y)|_{\overline{E}} = 0$ and by (3) we conclude that there exist polynomials $r_1(x,y,z)$ and $r_2(x,y,z)$ in $H^*(B)[x,y,z]$ such that

$$q(x,y,z)W'(x,y) = r_1(x,y,z)W_1(x,y,z) + r_2(x,y,z)W_2(x,y,z)$$

in the ring $H^*(B)[x,y,z]/(x^2)$. This completes the proof. ▌

Proof (Proof of Corollary 3.1). Let $q(x,y,z) \in H^*(B)[x,y,z]$ be a nonzero polynomial satisfying the condition $\deg q(x,y,z) < m - k + 1$ and suppose by contradiction that $q(x,y,z)|_{\overline{Z_f}} = 0$. One then has, by Theorem 3.1 that

$$q(x,y,z)W'(x,y) = r_1(x,y,z)W_1(x,y,z) + r_2(x,y,z)W_2(x,y,z),$$

in $H^*(B)[x,y,z]/(x^2)$.

Note that $\deg W'(x,y) = k$, $\deg W_1(x,y,z) = m+1$ and $\deg W_2(x,y,z) = 2n$, implies $\deg\ q(x,y,z) \geq m+1-k$, which evidently is impossible. ▌

Proof (Proof of Theorem 3.2). Let $q(y,z)$ be a polynomial in $H^*(B)[y,z]$ such that $q(y,z)|_{\overline{Z}_f} = 0$. By similar arguments used in the proof of Theorem 3.1, we conclude that $q(y,z)W'(y)|_{\overline{E}} = 0$. Therefore, by (4) we have that there are polynomials $r_1(y,z)$ and $r_2(y,z)$ in $H^*(B)[y,z]$ such that

$$q(y,z)W'(y) = r_1(y,z)W_1(y,z) + r_2(y,z)W_2(y,z).$$

The proof is complete. ▌

Proof (Proof of Corollary 3.2). Let $q(y,z) \in H^*(B)[y,z]$ be a nonzero polynomial such that $\deg q(y,z) < m-k+1$. If $q(y,z)|_{\overline{Z}_f} = 0$, by Theorem 3.2

$$q(y,z)W'(y) = r_1(y,z)W_1(y,z) + r_2(y,z)W_2(y,z),$$

where $\deg W'(y) = k$, $\deg W_1(y,z) = m+1$ and $\deg W_2(y,z) = 2n$. Thus, we conclude that $\deg q(y,z) \geq m+1-k$, which is a contradiction. ▌

7. ESTIMATING THE SIZE OF THE $\mathbb{Z}_p$-COINCIDENCE SET

Let $(X \sim_p S^m \times S^n, E, \pi, B)$ be a fibre bundle as in Section 3. Now consider $E'' \to B$ a vector bundle of dimension l and let $f : E \to E''$ be a fibre preserving map (here, we do not assume that E'' has a $\mathbb{Z}_p$-action). Suppose that $T : E \to E$ is a generator of the free $\mathbb{Z}_p$-action in E. The $\mathbb{Z}_p$-coincidence set $A(f)$ is the set of points x in E such that f maps the entire $\mathbb{Z}_p$-orbit of x to a single point, that is,

$$A(f) = \{x \in E; f(T^i(x)) = f(x),\ \forall i = 1, \dots, p-1\}.$$

In the above conditions, one has the following

Theorem 7.1. Cohom.dim $A(f) \geq$ Cohom.dim$(B) + m - (p-1)l$.

Proof. Let us consider a vector bundle $M \to B$, which is the Whitney sum of p copies of the l-dimensional vector bundle $E'' \to B$. One then has that $M = E'' \oplus \dots \oplus E''$ admits an action of the cyclic group $\mathbb{Z}_p$, generated by a periodic homeomorphism $t_M : M \to M$ of period p given by

$$t_M(m_1, \dots, m_{p-1}, m_p) = (m_p, m_1, \dots, m_{p-1}),$$

for each $(m_1, \dots, m_p) \in M$.

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Let us denote by Δ the subspace of M consisting of the all points $(m_1, \dots, m_p)$ in M such that $m_1 = \dots = m_p$. Therefore $\Delta \to B$ is a subbundle of $M \to B$, which is called diagonal bundle. Each fibre M_b of M can be represented as a direct sum $\Delta_b \oplus \Delta_b^{\perp}$, where $\Delta_b^{\perp}$ is the orthogonal complement of Δ_b. The bundle $M \to B$ is the Whitney sum of the bundles $\Delta \to B$ and $\Delta^{\perp} \to B$. Let us observe that $\Delta^{\perp}$ is a $\mathbb{Z}_p$-subspace of M and $\mathbb{Z}_p$ acts freely on the sphere bundle $S\Delta^{\perp} \subset \Delta^{\perp}$. Since $\Delta \to B$ is a l-dimensional bundle, the fibre dimension of $\Delta^{\perp} \to B$ is equal to $k = (p-1)l$, which is even. Consider the fibre preserving $\mathbb{Z}_p$-equivariant map $F : E \to M$ defined by

$$F(x) = (f(x), f(Tx), \dots, f(T^{p-1}x)).$$

The linear projection along of the diagonal defines an equivariant fibre preserving map $r : (M, M - \Delta) \to (\Delta^{\perp}, \Delta^{\perp} - 0)$, where 0 is the zero section of $\Delta^{\perp}$. Let $h = F \circ r$ be the composition given by

$$(E, E - A(f)) \to (M, M - \Delta) \to (\Delta^{\perp}, \Delta^{\perp} - 0)$$

with $Z_h = h^{-1}(0) = (F \circ r)^{-1}(0) = F^{-1}(\Delta) = A(f)$. Since $h : E \to \Delta^{\perp}$ is an equivariant fibre preserving map, it follows from Corollary 3.1 that

$$\text{cohom.dim}\ \overline{Z}_h = \text{cohom.dim}\ \overline{A(f)} \geq \text{cohom.dim}\ (B) + m - (p-1)l,$$

which completes the proof. ∎

Remark 7. 1. In particular, if B is a point and $f : X \sim_p S^m \times S^n \to \mathbb{R}^l$ is a continuous map, where $m > (p-1)l$, one has that $A(f) \neq \emptyset$. Moreover,

$$\text{cohom.dim} A(f) \geq m - (p-1)l.$$

This result extends to spaces which has the same cohomology(mod p) of a product of spheres the classical version of the Borsuk-Ulam theorem for $\mathbb{Z}_p$-actions proved by Munkholm in [7].

Remark 7. 2. Koikara and Mukerjee in [5] obtained an estimative of the size of the $\mathbb{Z}_p$-coincidence set $A(f)$ for maps of fibre bundles with closed orientable differentiable manifolds as fibres under certain conditions. Let us observe that the estimative determined by Theorem 7.1 and Remark 7.1 can not be obtained from [5], since in Theorem 7.1 the fibres of the fibre bundles are respectively $(X \sim_p S^m \times S^n)$, which is not necessary a differentiable manifold and $\mathbb{R}^l$, which is an open manifold.

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