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Characterization of non-autonomous attractors

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In this paper we give a characterization of pullback attractors for non-autonomous problems that are perturbations of autonomous problems when the attractor for the latter is the union of unstable manifolds of hyperbolic equilibria. Using this we study asymptotically autonomous problems (as $t \to -\infty$ and as $t \to +\infty$) and prove that if the autonomous limits are gradient and their attractors are the union of the unstable manifolds of hyperbolic equilibria (which may be completely different at $-\infty$ and $+\infty$) then the pullback attractors can be characterized as the union of the unstable manifolds of the backwards limit whereas the forward point attractors are the hyperbolic orbits which converge to the equilibria. A forward attractor may be defined which is compact, invariant and attracts bounded sets. October, 2006 ICMC-USP

1. INTRODUCTION

Infinite-dimensional dissipative dynamical systems are models for many problems from Mathematical Physics, Chemistry or Biology and studying the global attractors associated to them has been the subject of the work of many researchers throughout the last thirty five years. In these models it is natural to consider situations for which the dynamical systems are non-autonomous. In contrast to the autonomous case, for which the concept of a global

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attractor is settled and for which (in some particular class) a characterization of global attractor is available, the concept of a global attractor for non-autonomous dynamical systems is not yet settled (in our view) and a characterization (in any particular class) is not available.

It is our aim with this paper to contribute to the evolution of the concept of global attractors for non-autonomous dynamical systems and provide a characterization of these attractors for some cases which include certain asymptotically autonomous problems (backward and forward). We also intend to study non-autonomous dynamics in such a way that autonomous dynamics can be viewed as a particular case.

The main result proved in this paper (after many pages of notation and natural hypotheses) can be summarized as: *The pullback attractor for a non-autonomous dynamical system that is a perturbation of an autonomous gradient dynamical system for which all the equilibrium points are hyperbolic is the union of the unstable manifolds of hyperbolic global solutions* (*i.e. the linearization around these trajectories have exponential dichotomies*).

In order to describe properly the results in this paper we will need to introduce some terminology. Let $\mathcal{Z}$ be a Banach space, and $\eta \in (0,1]$ a parameter, and consider the semilinear non-autonomous problems

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_\eta(t,y) \\ &y(\tau) = y_0 \in \mathcal{Z}, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_0(y) \\ &y(\tau) = y_0 \in \mathcal{Z}. \end{aligned} \tag{1.2}$$

where $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of a C^0-semigroup of bounded linear operators and, for $\eta \in [0,1]$, $f_\eta(t,\cdot)$ is a differentiable function which is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ with Lipschitz constant independent of η and t. Assume that, for each $\tau \in \mathbb{R}$ and $y_0 \in \mathcal{Z}$ the unique solutions of (1.1) and (1.2) exist for all $t \geq \tau$.

Denote by $t \mapsto T_\eta(t,\tau)y_0$ the solution for (1.1). Then,

$$T_\eta(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s,\tau)y_0)\, ds. \tag{1.3}$$

On the other hand, if $t \mapsto T_0(t-\tau)y_0$ denotes the solution for (1.2). Then

$$T_0(t-\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_0(T_0(s-\tau)y_0)\, ds. \tag{1.4}$$

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Definition 1.1. Denote by $\mathcal{C}(\mathcal{Z})$ the space of continuous (nonlinear) operators $\mathcal{T}$: $\mathcal{Z} \to \mathcal{Z}$. A family $\{S(t,\tau) : \ t \geq \tau \in \mathbb{R}\} \subset C(\mathcal{Z})$ that satisfies

$$\begin{aligned}
&1)\ S(\tau,\tau) = I,\\
&2)\ S(t,\sigma)S(\sigma,\tau) = S(t,\tau), \ \text{for each } t \geq \sigma \geq \tau,\\
&3)\ (t,\tau) \mapsto S(t,\tau)z_0 \text{ is continuous for } t \geq \tau, \ z_0 \in \mathcal{Z}.
\end{aligned}$$

is called a nonlinear process. In the particular case when each $S(t,\tau) \in L(\mathcal{Z})$ (the space of bounded linear operators from $\mathcal{Z}$ into itself) we say that $\{S(t,\tau) : \ t \geq \tau \in \mathbb{R}\}$ is a linear process.

If $S(t,\tau)$ is a nonlinear (resp. linear) process such that for $t \geq \tau$ $S(t,\tau)$ depends only on the elapsed time $t-\tau$ then defining $S_0(t-\tau) = S(t,\tau)$ gives rise to the family $\{S_0(t) : t \geq 0\}$ which is called a nonlinear (resp. linear) semigroup.

It is clear that $\{T_\eta(t,\tau) \in C(\mathcal{Z}), \ t \geq \tau \in \mathbb{R}\}$, $\eta > 0$, and $\{T_0(t-\tau) \in C(\mathcal{Z}), \ t \geq \tau \in \mathbb{R}\}$ are nonlinear processes and therefore $\{T_0(t) : t \geq 0\}$ is a nonlinear semigroup.

When $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is a nonlinear process with the property that $S(t,\tau) = S_0(t-\tau)$ for all $t \geq \tau \in \mathbb{R}$, the *forward* dynamics as $t \to \infty$ for the dynamical system $\{S_0(t) : t \geq 0\}$ and the *pullback* one as $\tau \to -\infty$ for $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ are the same. However, one of the things we wish to emphasise here is that these 'dynamical limits' are totally unrelated for general processes $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$, in which entirely different qualitative properties of systems can arise depending whether we analyze the *forward* or the *pullback* long time behavior (see [7, 15]). In Section 4 we give a contribution characterizing the *pullback* dynamics and the *forward* dynamics making evident that they are completely unrelated even when they can be characterized.

Let us recall the definition of a global attractor and related concepts (see [9]) for a nonlinear semigroup $\{S(t) : \ t \geq 0\}$. For each $z \in \mathcal{Z}$, the positive orbit $\gamma^+(z)$ through z is defined as $\gamma^+(z) = \{S(t)z : t \geqslant 0\}$. A backward solution through z is a continuous function $\xi : (-\infty, 0] \to \mathcal{Z}$ such that $\xi(0) = z$ and, for any $s \leqslant 0$, $S(t)\xi(s) = \xi(t+s)$ for $0 \leqslant t \leqslant -s$. A global solution through z is a function $\xi : \mathbb{R} \to \mathcal{Z}$ such that $\xi(0) = z$, $s \in \mathbb{R}$ and $t \geqslant 0$ we have that $S(t)\xi(s) = \xi(t+s)$.

When $\mathcal{Z}$ is infinite-dimensional, in general backward or global solutions do not exist: the existence of such a solution will depend on the choice of the initial condition z. Furthermore, when a backward solution exists, it may not be unique. Let the negative orbit through z be defined as

$$\gamma^-(z) = \cup_{t \geqslant 0} H(t,z),$$

where $H(t,z) = \{y \in \mathcal{Z} :$ there is a backward solution through z defined by $\xi : (-\infty, 0] \to \mathcal{Z}$ with $\xi(0) = z$ and $\xi(-t) = y\}$.

The complete orbit $\gamma(z)$ through z is defined as $\gamma(z) = \gamma^-(z) \cup \gamma^+(z)$. If B and C are subsets of $\mathcal{Z}$, we say that the set B *attracts the set* C *under* $S(t)$ if $\mathrm{dist}(S(t)C, B) \to 0$ as $t \to \infty$. A set $A \subset \mathcal{Z}$ is said to be *invariant* under $\{S(t) : t \geq 0\}$ if, for any $z \in A$, there is a complete orbit $\gamma(z)$ through z such that $\gamma(z) \subset A$ or equivalently if $S(t)A = A$ for any $t \geqslant 0$.

Definition 1.2. A set $\mathcal{A} \subseteq \mathcal{Z}$ is said to be the global attractor for the semigroup $\{S(t) : t \geqslant 0\}$ if it is compact, invariant and attracts bounded subsets of $\mathcal{Z}$.

We assume that the nonlinear semigroup $\{T_0(s) : s \geq 0\}$ has a global attractor A_0.

The notion of a global attractor for a nonlinear process $\{S(t,\tau) : \ t \geq \tau \in \mathbb{R}\}$ requires much more care. Since any fixed set A will not, in general, be invariant in the above sense for a non-autonomous process, we define *invariance* in this context as follows:

- A family $\{A(t) \subset \mathcal{Z} : t \in [\sigma,\infty)\}$ is invariant under $\{S(t,\tau) : \ t \geq \tau \in \mathbb{R}\}$ if $S(t,\tau)A(\tau) = A(t)$ for all $t \geq \tau \geq \sigma$.

With this in mind one would immediately think that a non-autonomous attractor should be defined as it follows

- A family $\{A(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$ with $A(t)$ compact for all $t \in \mathbb{R}$ is a non-autonomous attractor if it is invariant under $\{S(t,\tau) : \ t \geq \tau \in \mathbb{R}\}$ and attracts bounded sets; that is, for each bounded set $B \subset \mathcal{Z}$ and $\tau \in \mathbb{R}$ we have that

$$\lim_{t\to\infty} \mathrm{dist}(S(t,\tau)B, A(t)) = 0.$$

Unfortunately such definition is likely to be satisfied only for some very specific and restrictive situations (e.g. if $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is uniformly asymptotically compact in the sense of [8]). Some very simple examples of systems that we would expect to possess a 'global non-autonomous attractor' will not have an attractor in the sense of this definition (see Remark 4.2). This is essentially due to the fact that some of the forward dynamics may be associated with solutions that blow up in finite backward time (see Remark 4.2).

At this point it becomes clear that we may not neglect the invariance which ensures that all globally defined bounded solutions lie within the attractor. Indeed, in the autonomous case the attractor is exactly the union of all globally defined bounded orbits [19],

$$\mathcal{A} = \{\xi : S(t)\xi \text{ exists for all } t \in \mathbb{R} \text{ and is bounded}\}, \tag{1.5}$$

and the properties of invariance and attraction define the global attractor. It is also clear that the non-autonomous asymptotic dynamics is not completely described by the globally defined bounded solutions. Hence we must account for both.

As part of this effort, we now introduce pullback attractors (see [8], where the sets $A(t)$ below are referred to as *kernel sections*, and [11, 18]).

Definition 1.3. A family of $\{A(t) : \ t \in \mathbb{R}\}$ of subsets of $\mathcal{Z}$ *pullback attracts* a bounded set $B \subset \mathcal{Z}$ under $\{S(t,\tau) : \ t \geq \tau \in \mathbb{R}\}$ if

$$\lim_{\tau\to-\infty} \mathrm{dist}(S(t,\tau)B, A(t)) = 0, \ \forall t \in \mathbb{R}.$$

A family of compact sets $\{A(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$ with $\overline{\cup_{t\in\mathbb{R}} A(t)}$ compact is a *pullback attractor* for $\{S(t,\tau): \ t \geq \tau \in \mathbb{R}\}$ if it is invariant and pullback attracts all bounded subsets of $\mathcal{Z}$ under $\{S(t,\tau): \ t \geq \tau \in \mathbb{R}\}$.

For autonomous problems, this concept of a pullback attractor coincides with the standard definition of the attractor, and if there exists a pullback attractor $\{A(t) : t \in \mathbb{R}\}$ for $\{S(t,\tau): \ t \geq \tau \in \mathbb{R}\}$ then (see [8])

$$\{A(t) : t \in \mathbb{R}\} = \{\xi(t) : \xi(\cdot) : \mathbb{R} \to \mathcal{Z} \text{ is bounded and } S(t,\tau)\xi(\tau) = \xi(t)\}$$

(cf. (1.5)). This fact shows that this concept is the natural generalization to a non-autonomous framework of the notion of global attractor for autonomous dynamical systems. It is also clear that the pullback attractor will not necessarily enjoy any kind of forward attraction; this is demonstrated by the example presented in Remark 4.2. Except in very specific situations, for example when the non-autonomous nonlinear process is asymptotically autonomous backwards and forward to the same nonlinear semigroup, the pullback behavior and the forward behavior will not be related (see Theorems 4.1 and 4.2, and [7, 17, 13] for other specific cases).

Note that the assumption that $A(t)$ be uniformly bounded is not a standard one in the literature, and without it the definition still reduces to the familiar one in the autonomous case. Indeed, there are examples in which allowing $A(t)$ to be unbounded, particular as $t \to +\infty$, is a useful weakening of the definition. Nevertheless, the uniformity imposed here occurs in most interesting applications, and allows for stronger results, while ruling out some potentially pathological behavior (e.g. unstable sets that do not belong to the attractor, see [12]).

If $\{T_0(t) : t \geq 0\}$ is gradient (Definition 2.2), has a global attractor A_0 and (1.2) possesses a finite number of stationary solutions y_i^*, $1 \leq i \leq n$, all of them hyperbolic, then A_0 is the union of the unstable manifolds $W_0^u(y_i^*)$ of these hyperbolic equilibria, i.e., the structure of A_0 is completely understood. Actually, this is essentially the class of equations in Banach spaces for which a detailed knowledge of the structure of the attractor is known.

We show in [5] that, if (1.1) is a small perturbation of (1.2), to each hyperbolic equilibrium point of (1.2) there corresponds a hyperbolic global solution (see also Section 2) and the corresponding stable and unstable manifolds behave continuously as $\eta \to 0$. These facts lead to upper and lower semicontinuity results for the pullback attractors $A_\eta(t)$ with respect to A_0 (see [14, 5]) (without characterizing the pullback attractors $A(t)$).

We give a characterization of the pullback attractor $\{A_\eta(t) : t \in \mathbb{R}\}$ for the process $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ arising from (1.1) when f_η is a small perturbation of f_0. Ideally, this characterization should maintain some resemblance with the characterization of the autonomous attractor A_0. We also want to characterize the forward attractor (when possible), insisting on the requirement that this be an invariant set (giving up invariance properties to have forward attraction, as with the 'uniform attractor' of [8], is a direction that we do not wish to pursue). On the other hand, it is fundamental to recognize that forward attraction is in general not related to pullback attraction, and we also aim to highlight this.

In order to advance the existing theory of attractors from the autonomous to non-autonomous domain, we will study perturbations of gradient autonomous dynamical systems for which we have a good characterization of the attractor. In this case, if all the equilibrium points of the limiting autonomous problems are hyperbolic, we can give a good characterization of the pullback attractor (Section 2 and the proof in Section 3) as well as define and characterize a forward attractor with invariance properties (Section 4). In particular, in Section 3 we will prove the following results:

Consider the initial value problem (1.1) as a non-autonomous perturbation of (1.2), i.e., for any $r > 0$,

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\sup_{y\in B(0,r)}\{\|f_\eta(t,y)-f_0(y)\|_{\mathcal{Z}}+\|(f_\eta)_y(t,y)-(f_0)_y(y)\|_{\mathcal{Z}}\}=0.$$

Assume that (1.2) is gradient and that all solutions of (2.3) are hyperbolic equilibrium solutions for (1.2) and that (1.2) has a global attractor A_0 which is given by $A_0 = \cup_{i=1}^n W_0^u(y_i^*)$, where $W_0^u(y_i^*)$ is the unstable manifold associated to y_i^*. Then

- *Structure of pullback attractors*

$$A_\eta(t)=\bigcup_{i=1}^n W_\eta^u(\xi_{i,\eta}^*)(t),$$

for all $t\in\mathbb{R}$, where $W_\eta^u(\xi_{i,\eta}^*)$ denotes the unstable manifold associated the global hyperbolic solutions $\xi_{i,\eta}^*$ (which we prove to exist near each of the hyperbolic equilibrium points y_i^*).

- *Backward and forward limits of complete trajectories*

For every bounded complete trajectory of (1.1), $\xi_\eta(t)$, there are $1\le j\le n$ and $1\le k\le n$ such that

$$\lim_{t\to\infty}\|\xi_\eta(t)-\xi_{j,\eta}^*(t)\|_{\mathcal{Z}}=0 \text{ and } \lim_{t\to-\infty}\|\xi_\eta(t)-\xi_{k,\eta}^*(t)\|_{\mathcal{Z}}=0.$$

- *Forward point attractor*

For each $(\tau,y_0)\in\mathbb{R}\times\mathcal{Z}$ there is a $1\le j\le n$ such that

$$\lim_{t\to\infty}\|T_\eta(t,\tau)y_0-\xi_{j,\eta}^*(t)\|_{\mathcal{Z}}=0.$$

In the second part (Section 4) of this paper we take advantage of the above results and apply them to asymptotically autonomous dynamical systems. We allow the backwards and forward dynamics to be different, so that, in particular, we can show how pullback and forward dynamics are unrelated. Because asymptotically autonomous systems can be reduced to non-autonomous perturbations of an autonomous equation, we can also describe the structure of the attractors. Moreover, we identify hyperbolic trajectories for the non-autonomous problem that are the true time-dependent attracting structures

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for every bounded complete trajectory; this goes further than merely identifying their asymptotic limits (equilibria of the autonomous equations), which is the usual approach ([2], [3]). Finally, we remark that we can also obtain a forward time-dependent attractor in this framework.

2. ON THE STRUCTURE OF PULLBACK ATTRACTORS

We start by describing some of the results in [5] where we obtain the continuity of the unstable manifolds of hyperbolic equilibria for (1.2) under non-autonomous perturbation.

It is easy to see that the following result holds

THEOREM 2.1. *Assume that*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\sup_{z\in B(0,r)}\|f_\eta(t,z)-f_0(z)\|_{\mathcal{Z}}=0,\quad \textit{for each } r>0. \tag{2.1}$$

Then, for each $r>0$ *and* $T>0$*,*

$$\lim_{\eta\to 0}\sup\{\|T_\eta(t,\tau)z-T_0(t-\tau)z\|,\tau\in\mathbb{R},\ t\in[\tau,\tau+T]\ \textit{and}\ \|z\|\le r\}\to 0. \tag{2.2}$$

A solution of (1.2) is an equilibrium solution if it satisfies

$$\mathfrak{B}y+f_0(y)=0. \tag{2.3}$$

Suppose that y_0^* is solution of (1.2). It follows that, if $\mathcal{A}=\mathfrak{B}+f_0'(y_0^*)$, then $\mathcal{A}$ generates a strongly continuous semigroup $\{e^{\mathcal{A}t}:t\geqslant 0\}$ of bounded linear operators.

DEFINITION 2.1. An equilibrium solution y_0^* to (1.2) is said to be hyperbolic if the following are satisfied:

1. The spectrum of $\mathcal{A}$ does not intersect the imaginary axis and the set $\sigma^+=\{\lambda\in\sigma(\mathcal{A}):\mathrm{Re}\lambda>0\}$ is compact.
This allow us to choose a smooth closed simple curve γ in $\rho(\mathcal{A})\cap\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>0\}$ oriented counterclockwise and enclosing σ^+ and define the projection

$$\mathcal{Q}=\mathcal{Q}(\sigma^+)=\frac{1}{2\pi\mathrm{i}}\int_\gamma(\lambda I-\mathcal{A})^{-1}d\lambda. \tag{2.4}$$

If $\mathcal{Z}^+=\mathcal{Q}(\mathcal{Z})$, $\mathcal{Z}^-=(I-\mathcal{Q})(\mathcal{Z})$, and $\mathcal{A}^\pm=\mathcal{A}_{|_{\mathcal{Z}^\pm}}$, then $\mathcal{Z}=\mathcal{Z}^+\oplus\mathcal{Z}^-$, $\mathcal{A}^-$ generates a strongly continuous semigroup on $\mathcal{Z}^-$ and $\mathcal{A}^+\in L(\mathcal{Z}^+)$.

2. There are constants $\bar{M}\ge 1$ and $\beta>0$ such that

$$\begin{aligned}\|e^{\mathcal{A}^+t}\|_{L(\mathcal{Z}^+)}&\leqslant\bar{M}e^{\beta t},\quad t\leqslant 0,\\ \|e^{\mathcal{A}^-t}\|_{L(\mathcal{Z}^-)}&\leqslant\bar{M}e^{-\beta t},\quad t\geqslant 0.\end{aligned} \tag{2.5}$$

Next we recall the definition of a gradient nonlinear semigroup

Definition 2.2. We say that a nonlinear semigroup $\{S(t) : t \geq 0\}$ is *gradient* if $\{S(t)z : t \geq 0\}$ is relatively compact for each $z \in \mathcal{Z}$ and there exists a continuous function $V : \mathcal{Z} \to \mathbb{R}$ such that

- $t \mapsto V(T_0(t)z) : [0, \infty) \to \mathcal{Z}$ is nonincreasing for each $z \in \mathcal{Z}$.
- If $z \in \mathcal{Z}$ is such that there is a global solution $\xi(\cdot) : \mathbb{R} \to \mathcal{Z}$ through $\xi(0) = z$ and $V(\xi(t)) = V(z)$ for all $t \in \mathbb{R}$, then z is a solution of (2.3).

The function $V : \mathcal{Z} \to \mathbb{R}$ is called a Liapunov function for $\{S(t) : t \geq 0\}$.

If $\{T_0(t) : t \geq 0\}$ is gradient, has a global attractor A_0 and (2.3) has a finite number of solutions y_i^*, $1 \leq i \leq n$, all of them hyperbolic, then A_0 is the union of the unstable manifolds $W_0^u(y_i^*)$ of these hyperbolic equilibria; that is,

$$A_0 = \bigcup_{i=1}^{n} W_0^u(y_i^*). \tag{2.6}$$

where

$$W_0^u(y_i^*) = \{\, z \in \mathcal{Z} : \text{ there is a backward solution } y(t) \text{ of } (1.2) \\ \text{satisfying } y(\tau) = z \text{ and such that } \lim_{t \to -\infty} \|y(t) - y_i^*\| = 0\}.$$

In this case $\omega(z)$, the ω-limit set of a point $z \in \mathcal{Z}$, must be a connected set contained in the set of equilibria $\mathcal{E} = \{y_1^*, \cdots, y_n^*\}$ and this ensures that it is exactly one equilibrium point. Now we can define $\mathfrak{N} = V(\mathcal{E}) := \{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ with $\mathfrak{n}_i < \mathfrak{n}_j$, $\leq i < j \leq p \leq n$ and $\mathcal{E}_k = \{y_i^* \in \mathcal{E} : V(y_i^*) = \mathfrak{n}_k\}$

With this notation we have the following result (see Hale [9])

Theorem 2.2. *If* $\{T_0(t) : t \geq 0\}$ *is gradient which has a global attractor* A_0, $V : \mathcal{Z} \to \mathbb{R}$ *is its Liapunov function and* (2.3) *has a finite number of solutions* y_i^*, $1 \leq i \leq n$, *all of them hyperbolic, then* A_0 *is given by* (2.6) *and if* $y(\cdot) : \mathbb{R} \to \mathcal{Z}$ *is a global solution for* (1.2), *then there are* $1 \leq k_1 < k_2 \leq p$, $y_i^* \in \mathcal{E}_{k_1}$ *and* $y_j^* \in \mathcal{E}_{k_2}$, *such that*

$$\lim_{t \to -\infty} y(t) = y_j^* \text{ and } \lim_{t \to +\infty} y(t) = y_i^*$$

This is the case in many examples that have gradient structure (at least generically).

A characterization of the pullback attractor (the analog of the autonomous attractor) as in (2.6) is not available in the literature for any class of infinite-dimensional problems. The finite-dimensional case is proved in Langa et al. [14].

Our first task will be to find the substitutes for the hyperbolic equilibrium points. In [5] it is shown that near each of the hyperbolic equilibrium y_i^* there is a unique global solution $\xi_{i,\eta}^*$ which enjoy a hyperbolic structure. In order to be more precise we need the notion of exponential dichotomy, which we now introduce.

Definition 2.3. We say that a linear process $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy with exponent β and constant M if there exists a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset L(\mathcal{Z})$ such that

1. $Q(t)U(t,s) = U(t,s)Q(s)$, for all $t \geq s$;
2. The restriction $U(t,s)_{|_{R(Q(s))}}$, $t \geq s$ is an isomorphism from $R(Q(s))$ into $R(Q(t))$ and its inverse is denoted by $U(s,t) : R(Q(t)) \to R(Q(s))$.
3. There are constants $\omega > 0$ and $M \geq 1$ such that

$$\begin{aligned} &\|U(t,s)(I - Q(s))\| \leq Me^{-\omega(t-s)} \quad t \geq s \\ &\|U(t,s)Q(s)\| \leq Me^{\omega(t-s)}, \quad t \leq s \end{aligned} \tag{2.7}$$

Now we will define the analog of a hyperbolic equilibrium for non-autonomous problems (1.1). But first we need to introduce some more terminology. Consider the problem

$$\begin{aligned} &\dot{z} = \mathcal{A}z + B_\eta(t)z \\ &z(\tau) = z_0 \in \mathcal{Z}, \end{aligned} \tag{2.8}$$

where $\mathbb{R} \ni t \mapsto B_\eta(t) \in L(\mathcal{Z})$ is strongly continuous. It is well known that the problem (2.8) has a unique mild solution $U_\eta(t,\tau)z_0$ for each $z_0 \in \mathcal{Z}$ which satisfies

$$U_\eta(t,\tau)z_0 = e^{\mathcal{A}(t-\tau)}z_0 + \int_\tau^t e^{\mathcal{A}(t-s)}B_\eta(s)U_\eta(s,\tau)z_0 ds. \tag{2.9}$$

The family $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is a linear process as in Definition 1.1. We say that (2.8) has exponential dichotomy if $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has exponential dichotomy.

Now we are ready to define the analog of a hyperbolic equilibrium for non-autonomous systems.

Definition 2.4. Let $\xi : \mathbb{R} \to \mathcal{Z}$ be a global solution of (1.1). We say that ξ is hyperbolic if

$$\begin{aligned} &\dot{z} = \mathfrak{B}z + (f_\eta)_y(t,\xi(t))z \\ &z(\tau) = z_0 \in \mathcal{Z} \end{aligned}$$

has exponential dichotomy. A global solution which has exponential dichotomy will be called a *global hyperbolic solution.*

The following result can be adapted from Theorem 7.6.11 in [10].

THEOREM 2.3. *Suppose that* $\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|B_\eta(t)\|_{L(\mathcal{Z})}=0$ *and that* $\mathcal{A}$ *is the generator of a* C^0*-semigroup such that* (2.5) *is satisfied for some* $\beta>0$ *and* $\bar{M}\geq 1$*. Then, for each* $M>M_1$ *and* $\omega<\beta$*, there is a* $\eta_0>0$ *such that, for all* $\eta\leq\eta_0$, (2.8) *has exponential dichotomy with exponent* ω *and constant* M*.*

Hereafter we assume that (1.2) has a global attractor A_0, (1.2) has a finite number of solutions y_i^*, $1\leq i\leq n$, all of them hyperbolic and that A_0 satisfies (2.6). Setting $z=y-y_i^*$, we rewrite the equation (1.2) as

$$\begin{aligned}\dot{z}&=\mathcal{A}^i z+h_i(z)\\ z(\tau)&=z_0=y_0-y_i^*\end{aligned}\tag{2.10}$$

where $\mathcal{A}^i=\mathfrak{B}+f_0'(y_i^*)$, $h_i(z)=f_0(y_i^*+z)-f_0(y_i^*)-f_0'(y_i^*)z$. Hence, 0 is an equilibrium solution for (2.10) and $h_i(0)=0$, $h_i'(0)=0\in L(\mathcal{Z})$.

In order to characterize the non-autonomous attractor for (1.1) we first note that, for suitably small η, associated to each hyperbolic equilibrium y_0^* of (1.2) there is a unique global solution ξ_0^* which stays near y_0^* for all $t\in\mathbb{R}$. This is proved in the following result of [5]

PROPOSITION 2.1. *Assume that, for any* $r>0$*,*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\sup_{y\in B(0,r)}\{\|f_\eta(t,y)-f_0(y)\|_{\mathcal{Z}}+\|(f_\eta)_y(t,y)-(f_0)_y(y)\|_{\mathcal{Z}}\}=0.\tag{2.11}$$

Then, there exists $\eta_0>0$ *such that, for each* $0<\eta\leq\eta_0$ *and* $1\leq i\leq n$*, there are solutions* $\xi_{i,\eta}^*:\mathbb{R}\to\mathcal{Z}$ *such that*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\max_{1\leq i\leq n}\|\xi_{i,\eta}^*(t)-y_i^*\|_{\mathcal{Z}}=0$$

and

$$\dot{y}=\mathcal{A}^i y+B_\eta^i(t)y\tag{2.12}$$

has exponential dichotomy, where $B_\eta^i(t)=(f_\eta)_y(\xi_{i,\eta}^*(t))-f_0'(y_i^*)$ *.*

The exponential dichotomy for (2.12) follows easily from (2.5) and Theorem 2.3.

Proceeding as in the autonomous case we change variables $z(t)=y-\xi_{i,\eta}^*(t)$ in (1.1) and rewrite (1.1) as

$$\begin{aligned}\dot{z}&=(\mathcal{A}+B_\eta^i(t))z+h_\eta^i(t,z)\\ z(\tau)&=z_0\end{aligned}\tag{2.13}$$

where $h_\eta^i(t,z)=f_\eta(\xi_{i,\eta}^*(t)+z)-f_\eta(\xi_{i,\eta}^*(t))-(f_\eta)_y(\xi_{i,\eta}^*(t))z$. Hence, 0 is a globally defined bounded solution for (2.13) and $h_\eta^i(t,0)=0$, $(h_\eta^i)_y(t,0)=0\in L(\mathcal{Z})$.

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Definition 2.5. The unstable manifold of a hyperbolic solution $\xi^*_{i,\eta}$ to (2.13) is the set

$$W^u_\eta(\xi^*_{i,\eta}) = \{ (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \text{ there is a backward solution } z(t,\tau,\zeta) \text{ of } (1.1) \\ \text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to-\infty} \|z(t,\tau,\zeta)-\xi^*_{i,\eta}(t)\| = 0\}.$$

The stable manifold of a hyperbolic solution $\xi^*_{i,\eta}$ to (2.13) is the set

$$W^s_\eta(\xi^*_{i,\eta}) = \{ (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \text{ there is a forward solution } z(t,\tau,\zeta) \text{ of } (1.1) \\ \text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to+\infty} \|z(t,\tau,\zeta)-\xi^*_{i,\eta}(t)\| = 0\}.$$

The intersection of the unstable (stable) manifold with a neighborhood of the curve $(\cdot,\xi(\cdot))$ in $\mathbb{R}\times\mathcal{Z}$ is called a local unstable (stable) manifold and is denoted by $W^u_{\eta,\text{loc}}$ ($W^s_{\eta,\text{loc}}$).

The following proposition summarizes the main results proved in [5].

Proposition 2.2. *Let $\eta\in(0,1]$, $f_\eta : \mathbb{R}\times\mathcal{Z}\to\mathcal{Z}$ differentiable. Consider the initial value problem* (1.1)*. Assume that all solutions of* (2.3) *are hyperbolic equilibrium solutions for* (1.2) *and that* (1.2) *has a global attractor A_0 which is given by* (2.6)*. Assume that* (2.11) *is satisfied for any $r>0$. Under these assumptions the following hold:*

1. *If $A_i = \mathfrak{B} + f_0'(y^*_i)$, the spectrum of $\mathcal{A}_i$ does not intersect the imaginary axis and the set $\sigma^+_i = \{\lambda\in\sigma(\mathcal{A}_i) : \text{Re}\lambda > 0\}$ is compact. If γ_i is a smooth curve in $\rho(\mathcal{A}_i)\cap\{\lambda\in\mathbb{C} : \text{Re}\lambda>0\}$ oriented counterclockwise and enclosing σ^+_i and*

$$\mathcal{Q}_i = \mathcal{Q}_i(\sigma^+_i) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma_i} (\lambda I - \mathcal{A}_i)^{-1} d\lambda \tag{2.14}$$

then there are constants $\bar M_i \ge 1$ and $\beta_i>0$ such that

$$\|e^{\mathcal{A}^+_i t}\|_{L(\mathcal{Z}^+_i)} \leqslant \bar M_i e^{\beta_i t}, \quad t\leqslant 0,$$
$$\|e^{\mathcal{A}^-_i t}\|_{L(\mathcal{Z}^-_i)} \leqslant \bar M_i e^{-\beta_i t}, \quad t\geqslant 0,$$

where $\mathcal{Z}^+_i = R(\mathcal{Q}_i)$, $\mathcal{Z}^-_i = R(I-\mathcal{Q}_i)$, $A^\pm_i = A|_{\mathcal{Z}^\pm_i}$, $A^+_i \in L(Z^+_i)$.

2. *For each suitably small η, there are globally defined solutions of* (2.13) *$\xi^*_{i,\eta} : \mathbb{R}\to\mathcal{Z}$ with $\lim_{\eta\to0}\sup_{t\in\mathbb{R}}\|\xi^*_{i,\eta}(t) - y^*_i\|_{L(\mathcal{Z})} = 0$ and such that*

$$\dot z = \mathfrak{B}z + (f_\eta)_y(t,\xi^*_{i,\eta}(t))z \tag{2.15}$$

has an exponential dichotomy; that is, there is a family of projections $\{Q^i_\eta(t) :\ t\in\mathbb{R}\}$ such that the conditions in Definition 2.4 are satisfied, where $U^i_\eta(t,\tau)$ is the solution operator associated to (2.15)*.*

3. *For any* $\tau \in \mathbb{R}$

$$\lim_{\eta\to 0}\sup_{t\geq\tau}\|U_\eta^i(t,\tau)(I-Q_\eta^i(\tau))-e^{\mathcal{A}_i(t-\tau)}(I-\mathcal{Q}_i)\|_{L(\mathcal{Z})}\to 0,$$

$$\lim_{\eta\to 0}\sup_{t\leq\tau}\|U_\eta^i(t,\tau)Q_\eta^i(\tau)-e^{\mathcal{A}_i(t-\tau)}\mathcal{Q}_i\|_{L(\mathcal{Z})}\to 0.$$

Furthermore, for any $T>0$,

$$\lim_{\eta\to 0}\sup_{|t-\tau|\leq T}\|U_\eta^i(t,\tau)-e^{\mathcal{A}_i(t-\tau)}\|_{L(\mathcal{Z})}\to 0.$$

4. *The projections* $Q_\eta^i(t)$ *and* $\mathcal{Q}_i$ *satisfy*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|Q_\eta^i(t)-\mathcal{Q}_i\|_{L(\mathcal{Z})}=0.$$

5. *For suitably small* $\epsilon>0$ *there are functions*

$$\mathbb{R}\times B(0,\epsilon)\ni(\tau,z)\mapsto \Sigma_{i,\eta}^{*,u}(\tau,Q_\eta^i(\tau)w)\in(I-Q_\eta^i(\tau))\mathcal{Z}$$

$$\mathbb{R}\times B(0,\epsilon)\ni(\tau,z)\mapsto \Sigma_{i,\eta}^{*,s}(\tau,(I-Q_\eta^i(\tau))w)\in Q_\eta^i(\tau)\mathcal{Z}$$

such that

$$W^u_{\eta,\mathrm{loc}}(\xi^*_{i,\eta})=\{(\tau,\xi^*_{i,\eta}+w):w=(Q_\eta^i(\tau)w,\Sigma_{i,\eta}^{*,u}(\tau,Q_\eta^i(\tau)w)),\tau\in\mathbb{R},\|w\|_{\mathcal{Z}}\leq\epsilon\}$$
$$W^s_{\eta,\mathrm{loc}}((\xi^*_{i,\eta}))=\{(\tau,\xi^*_{i,\eta}+w):w=(\Sigma_{i,\eta}^{*,s}(\tau,(I-Q_\eta^i(\tau))w),(I-Q_\eta^i(\tau))w),\tau\in\mathbb{R},\|w\|_{\mathcal{Z}}\leq\epsilon\}.$$

6. *Finally, the unstable and stable manifolds behave continuously at* $\eta=0$ *in the sense that*

$$\sup_{t\leq\tau}\sup_{\|w\|_{\mathcal{Z}}\leq\epsilon}\left\{\|Q_\eta^i(t)w-\mathcal{Q}_iw\|_{\mathcal{Z}}+\|\Sigma_{i,\eta}^{*,u}(t,Q_\eta^i(t)w)-\Sigma_0^{*,u}(\mathcal{Q}_iw)\|_{\mathcal{Z}}\right\}\overset{\eta\to 0}{\longrightarrow}0.$$

$$\sup_{t\geq\tau}\sup_{\|w\|_{\mathcal{Z}}\leq\epsilon}\left\{\|Q_\eta^i(t)w-\mathcal{Q}_iw\|_{\mathcal{Z}}+\|\Sigma_{i,\eta}^{*,s}(t,(I-Q_\eta^i(t))w)-\Sigma_0^{*,s}((I-\mathcal{Q}_i)w)\|_{\mathcal{Z}}\right\}\overset{\eta\to 0}{\longrightarrow}0.$$

In what follows we suppose the following:

i) For each $\eta\in(0,1]$ there exists a pullback attractor $\{A_\eta(t)\}_{t\in\mathbb{R}}$ associated to (1.1).

ii) There exist η_0 and a compact attracting set $K \subset \mathcal{Z}$ such that, for all $B \subset \mathcal{Z}$ bounded

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{\eta\leq\eta_0} dist(T_\eta(t,\tau)B, K) = 0. \tag{2.16}$$

In particular, this implies (see [8]) that

$$\bigcup_{\eta\leq\eta_0} \bigcup_{t\in\mathbb{R}} A_\eta(t) \subset K. \tag{2.17}$$

Now we are ready to state our main theorem

THEOREM 2.4. *Let $\eta \in [0,1]$, $f_\eta : \mathbb{R}\times\mathcal{Z} \to \mathcal{Z}$ be differentiable. Consider the initial value problem* (1.1)*. Assume that* (1.2) *is gradient and that all solutions of* (2.3) *are hyperbolic equilibrium solutions for* (1.2) *and that* (1.2) *has a global attractor A_0 which is given by* (2.6)*. Assume that* (2.11) *is satisfied for any $r > 0$.*

1.If we denote by $W^u_\eta(\xi^*_{i,\eta})(\tau) = \{\, \zeta \in \mathcal{Z} :$ there is a backward solution $z(t,\tau,\zeta)$ of (1.1) *satisfying $z(\tau,\tau,\zeta) = \zeta$ and such that $\lim_{t\to-\infty} \|z(t,\tau,\zeta) - \xi^*_{i,\eta}(t)\| = 0\}$, then the attractor $\{A_\eta(\tau) : \tau \in \mathbb{R}\}$ of* (1.1) *is such that*

$$A_\eta(\tau) = \bigcup_{i=1}^{n} W^u_\eta(\xi^*_{i,\eta})(\tau).$$

2.For each globally defined bounded solution $\xi_\eta(\cdot)$ of (1.1) *and $\eta \leq \eta_0$, there are $1 \leq j \leq n$ and $1 \leq k \leq n$ such that*

$$\lim_{t\to\infty} \|\xi_\eta(t) - \xi^*_{j,\eta}(t)\|_{\mathcal{Z}} = 0 \text{ and } \lim_{t\to-\infty} \|\xi_\eta(t) - \xi^*_{k,\eta}(t)\|_{\mathcal{Z}} = 0. \tag{2.18}$$

3.For each $(\tau, y_0) \in \mathbb{R}\times\mathcal{Z}$ there is a $1 \leq j \leq n$ such that

$$\lim_{t\to\infty} \|T_\eta(t,\tau)y_0 - \xi^*_{j,\eta}(t)\|_{\mathcal{Z}} = 0.$$

Using this theorem we give in Section 4.1 a characterization of the pullback attractors in the case of asymptotically autonomous problems at $-\infty$ and in Section 4.2 we define forward invariant attractors and give a characterization of them for the case of asymptotically autonomous problems at $+\infty$.

3. PROOF OF THEOREM 2.4

Before we can start the proof of Theorem 2.4 we need the following very important lemma.

LEMMA 3.1. *Let η_k be a sequence of positive numbers such that $\eta_k \to 0$ as $n \to \infty$. Assume that there is a sequence $\xi_{\eta_k}(\cdot) : \mathbb{R} \to \mathcal{Z}$ of solutions* (1.1) *such that* $\overline{\cup_{k\in\mathbb{N}} \cup_{t\in\mathbb{R}} \xi_{\eta_k}(t)}$ *is compact. Then, for any sequence $\{t_k\}$ in $\mathbb{R}$, there is a subsequence which we again denote by ξ_{η_k} and a globally defined bounded solution $y(\cdot)$ of* (1.2) *such that*

$$\lim_{k\to\infty} \xi_{\eta_k}(t+t_k) \to y(t) \tag{3.1}$$

uniformly for t in compact subsets of $\mathbb{R}$.

Proof: Since $\overline{\cup_{k\in\mathbb{N}} \cup_{t\in\mathbb{R}} \xi_{\eta_k}(t)}$ is compact, there is a subsequence which we again denote by η_k and $y_0 \in \mathcal{Z}$ such that $\xi_{\eta_k}(t_k) \to y_0$. Let $y(\cdot) : [0,\infty)$ be the solution of (1.2) such that $y(0) = y_0$. Of course, this solution is bounded. If $t > t_k$ it follows from the continuity of the nonlinear process $T_{\eta_k}(t+t_k, t_k)$, uniformly for $t_k \in \mathbb{R}$ and for t in compact subsets of $[0,\infty)$ (equation (2.2)), that

$$\xi_{\eta_k}(t+t_k) = T_{\eta_k}(t+t_k, t_k)\xi_{\eta_k}(t_k) \to T_0(t-t_k)y_0 = y(t)$$

uniformly for t in compact subsets of $[0,\infty)$. Proceeding similarly, $\xi_{\eta_k}(t_k - 1)$ has a convergent subsequence $\xi_{\eta_k^1}(t_k-1)$ with limit y_{-1}. Defining $y(t) := T_0(t+1)y_{-1}$, $t \in [-1,\infty)$, we have that $y(0) = T_0(t_k-(t_k-1))y_{-1} = \lim_{k\to\infty} T_{\eta_k^1}(t_k, t_k-1)\xi_{\eta_k^1}(t_k-1) = \lim_{k\to\infty} \xi_{\eta_k^1}(t_k) = y_0$ and $y(t) = T_0(t+1)y_{-1}$ for all $t \geq -1$. From this we have that $y(\cdot) : [-1,\infty) \to \mathcal{Z}$ is a solution of (1.2) with $y(-1) = y_{-1}$, $y(0) = y_0$ and

$$\xi_{\eta_k^1}(t+t_k) = T_{\eta_k}(t+t_k, t_k-1)\xi_{\eta_k^1}(t_k-1) \to T_0(t+1)y_{-1} = y(t)$$

uniformly for t in compact subsets of $[-1,\infty)$. Suppose that:

- We have obtained subsequences $\{\xi_{\eta_k^i}\}_{n=1}^{\infty}$ for $1 \leq i \leq m-1$ with the property that $\{\eta_k^i\}_{n=1}^{\infty}$ is a subsequence of $\{\eta_k^{i-1}\}_{n=1}^{\infty}$ and such that $\xi_{\eta_k^i}(t_k - i) \to y_{-i}$, $1 \leq i \leq m-1$.
- We have defined $y(t)$ by $\lim_{n\to\infty} \xi_{\eta_k^i}(t+t_k) = y(t)$, in $[-i, -i+1]$ and, consequently, $y(t) : [-i,\infty) \to \mathcal{Z}$ is a solution of (1.2) with $y(-j) = y_{-j}$, $0 \leq j \leq i$ and $\xi_{\eta_k^i}(t+t_k)$ converges to $y(t)$ uniformly for t in compact subsets of $[-i,\infty)$, $1 \leq i \leq m-1$.

Now construct $\{\eta_k^m\}_{n=1}^{\infty}$ a subsequence of $\{\eta_k^{m-1}\}_{n=1}^{\infty}$ such that $\xi_{\eta_k^m}(t_k - m)$ is convergent. Let y_{-m} be its limit and define $y(t) = T_0(t+m)y_{-m}$ for $t \in [-m, -m+1]$. Then $y(t)$ is a solution of (1.2) with $y(-i) = y_{-i}$, $0 \leq i \leq m$ and $\xi_{\eta_k^m}(t+t_k)$ converges to $y(t)$ uniformly for t in compact subsets of $[-m,\infty)$.

With this we have constructed a sequence $\{\xi_{\eta_k^k}\}_{k=1}^{\infty}$ and a solution $y(\cdot) : \mathbb{R} \to \mathcal{Z}$ of (1.2) with $y(-i) = y_{-i}$ for all $i \in \mathbb{N}$ and such that $\xi_{\eta_k^k}(t+t_k) \to y(t)$ uniformly for t in compact subsets of $\mathbb{R}$. This concludes the proof.□

The following lemma is taken from [9] and its proof is added here for completeness only.

LEMMA 3.2. *Assume that $\{T_0(t) : t \geq 0\}$ is a gradient nonlinear semigroup which has a global attractor A_0 and such that* (2.3) *has a finite number of solutions $\mathcal{E} = \{y_i^* : 1 \leq$*

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$i \leq n\}$, all of them hyperbolic. Let $V : \mathcal{Z} \to \mathbb{R}$ be the Liapunov function associated with $\{T_0(t) : t \geq 0\}$ and $V(\mathcal{E}) = \{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ with $\mathfrak{n}_i < \mathfrak{n}_{i+1}$, $1 \leq i \leq p-1$.

If $1 \leq j \leq p$ and $\mathfrak{n}_{j-1} < r < \mathfrak{n}_j$, then $\mathcal{Z}_r^j = \{z \in \mathcal{Z} : V(z) \leq r\}$ is positively invariant under $\{T_0(t) : t \geq 0\}$ and $\{T_{0,r}^j(t) : t \geq 0\}$, the restriction of $\{T_0(t) : t \geq 0\}$ to $\mathcal{Z}_r^j$, has a global attractor A_0^j given by

$$A_0^j = \cup\{W^u(y_\ell^*) : V(y_\ell^*) \leq \mathfrak{n}_{j-1}\}.$$

In particular, $V(z) \leq \mathfrak{n}_{j-1}$ for all $z \in A_0^j$.

Proof: It is clear from the definition of the Liapunov function that $\mathcal{Z}_r^j$ is invariant under $\{T_0(t) : t \geq 0\}$. To prove the existence of an attractor for $\{T_{0,r}^j(t) : t \geq 0\}$ we note that it inherits from $\{T_0(t) : t \geq 0\}$ the properties required to obtain the existence of a global attractor; namely, orbits of bounded subsets of $\mathcal{Z}_r$ are bounded, $\{T_{0,r}^j(t) : t \geq 0\}$ is bounded dissipative and $\{T_{0,r}^j(t) : t \geq 0\}$ is asymptotically compact. Hence, $\{T_{0,r}^j(t) : t \geq 0\}$ has a global attractor A_0^j. The restriction V_r^j of V to $\mathcal{Z}_r^j$ is a Liapunov function for $\{T_{0,r}^j(t) : t \geq 0\}$ and the characterization of A_0^j follows. The last statement is an immediate consequence of the characterization of A_0^j.□

Now we are ready to prove the main result of this paper

Proof of Theorem 2.4: We prove (1.) and (2.) by contradiction. Let $\xi_{\eta_k}(\cdot)$ be a globally defined bounded solution of (1.1) (with $\eta = \eta_k$) that is not in the unstable (stable) manifold of any ξ_{i,η_k}^*, $1 \leq i \leq n$.

Our first task is to rule out the possibility that there is a sequence of solutions that does not end backward (resp. forward) near one of the equilibrium points $y_i^* \in \mathcal{E}$. To prove this we use Lemma 3.1 and the fact that (1.2) is gradient.

Before giving all the details, we briefly summarize the argument.

From the hypothesis on the sequence of solutions $\{\xi_{\eta_k}(\cdot)\}$, each $\xi_{\eta_k}(\cdot)$ must leave backward (resp. forward) any neighborhood $\mathcal{O}$ of the set of equilibria $\mathcal{E}$, because otherwise it would be in the unstable (resp. stable) manifold of one of the hyperbolic solutions $\{\xi_{i,\eta_k}^*\}$. So we can certainly find a sequence $\{t_k\}$ such that $\xi_{\eta_k}(t_k) \notin \mathcal{O}$.

Now, the uniform convergence of $T_\eta(t,s)$ to $T_0(t-s)$ (via Lemma 3.1) together with the fact that every solution in A_0 must end backward (resp. forward) in a single point of $\mathcal{E}$ guarantees that there is a finite time T^* such that for some $0 \leq s_k \leq T^*$, $\xi_{\eta_k}(t_k - s_k) \in \mathcal{O}$ (resp. $\xi(\eta_k(t+t_k)) \in \mathcal{O}$), i.e. the solution $\xi_{\eta_k}(\cdot)$ reenters this neighborhood of the set of equilibria in a finite time. However, our hypothesis implies that $\xi_{\eta_k}(\cdot)$ must leave this neighborhood once more as time runs backwards (resp. forwards).

The fact that this procedure must end will give rise to a contradiction, and follows from the fact that the semigroup $\{T_0(t) : t \geq 0\}$ is gradient, since its Liapunov function must be decreasing along solutions and in particular this will not allow the solution $\xi_{\eta_k}(\cdot)$ to revisit the neighborhood of a given equilibria.

Now we formalize this procedure. Let $V : \mathcal{Z} \to \mathbb{R}$ be the Liapunov function associated with the nonlinear semigroup $\{T_0(t) : t \geq 0\}$, let $\mathcal{E} = \{y_1^*, \cdots, y_n^*\}$, $\mathfrak{N} = V(\mathcal{E}) := \{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ with $\mathfrak{n}_i < \mathfrak{n}_j$, $\leq i < j \leq p \leq n$ and $\mathcal{E}_k = \{y_i^* \in \mathcal{E} : V(y_i^*) = \mathfrak{n}_k\}$, $1 \leq k \leq p$.

If $0 < \nu < \epsilon_0 = \frac{1}{2}\min\{\mathfrak{n}_k - \mathfrak{n}_{k-1} : 1 \leq k \leq p\}$ define

$$\mathcal{O}_\nu(y_j^*) = V^{-1}(\mathfrak{n}_j - \nu, \mathfrak{n}_j - \nu) \cap B(y_j, \nu).$$

Note that for some $\eta_\nu > 0$, $\mathcal{O}_\nu(y_j^*) \supseteq B(y_j, \eta_\nu)$. We write

$$\mathcal{O}_\nu = \bigcup_{j=1}^{n} \mathcal{O}_\nu(y_j^*).$$

It is clear that we can choose $\nu_0 \leq \epsilon_0$ such that, for all $\nu < \nu_0$ we have that

- $\mathcal{O}_\nu(y_i^*) \cap \mathcal{O}_\nu(y_j^*) = \emptyset$, for all $1 \leq i < j \leq n$.
- Each $\mathcal{O}_\nu(y_j^*)$ contains exactly one equilibrium solution y_j^* Proposition 2.2, (6).
- There is an $\eta_0 > 0$ such that in each $\mathcal{O}_\nu(y_j^*)$ there is a unique globally defined hyperbolic solution $\xi_{j,\eta}^*$.

Recall that all solutions that stay in the neighborhood $\mathcal{O}_\nu$ of $\mathcal{E}$ for all t in an interval of the form $(-\infty, \tau]$ ($[\tau, \infty)$) must be in the unstable (resp. stable) manifold of one of the $\xi_{i,\eta}^*$.

Now choose ϵ' with $\nu_0 > \epsilon' > \epsilon$. The first stage of the argument is to use our hypothesis to choose a sequence $\{t_k^1\}$ of real numbers such that $\xi_{\eta_k}(t_k^1)$ lies outside $\mathcal{O}_\epsilon$. It follows from Lemma 3.1 that there is a subsequence, which we again denote by ξ_{η_k} such that (3.1) holds; that is,

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k^1) \to y^1(t)$$

uniformly for t in compact subsets of $\mathbb{R}$.

Since $y^1(\cdot)$ must enter backward (resp. forward) $\mathcal{O}_\epsilon$ we have that there is a fixed negative (resp. positive) number T_1 such that $\xi_{\eta_k}(T_1 + t_k^1) \in \mathcal{O}_\epsilon(y_j^*)$ for some $1 \leq j \leq p$. However, since by assumption $\xi_{\eta_k}(\cdot)$ is not in the unstable manifold of y_j^*, it must leave the η_ϵ neighbourhood of y_j^*, and so there is a sequence $t_k^2 < T_1 + t_k^1$ (resp. $t_k^2 > T_1 + t_k^1$) such that

$$\xi_{\eta_k}(t_k^2) \in \overline{\mathcal{O}_{\epsilon'}(y_j^*)} \backslash B(y_j^*, \eta_\epsilon). \tag{3.2}$$

BACKWARD CASE

We now embark on a case-by-case analysis. In each case we show that we can restart the entire procedure around a different equilibrium with a higher value of the Lyapunov function. Given that there only a finite number of distinct values of the Lyapunov function at the equilibria, this produces a contradiction.

Note that since $\xi_{\eta_k}(t_k^2) \in \mathcal{O}_{\epsilon'}(y_j^*)$ we must have

$$|\xi_{\eta_k} - y_j^*| < \epsilon' \qquad \text{and} \qquad V(\xi_{\eta_k}(t_k^2)) \in (\mathfrak{n}_j - \epsilon', \mathfrak{n}_j + \epsilon'),$$

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while the fact that $\xi_{\eta_k}(t_k^2) \notin B(y_j^*, \eta_\epsilon)$ means that

$$|\xi_{\eta_k}(t_k^2) - y_j^*| \geq \eta_\epsilon.$$

Case (a). For some $\delta > 0$, for infinitely many k we have $V(\xi_{\eta_k}(t_k^2)) \geq \mathfrak{n}_j + \delta$.

We can use Lemma 3.1 to obtain a subsequence, which we again denote by ξ_{η_k}, such that

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k^2) \to y^2(t)$$

uniformly for t in compact subsets of $\mathbb{R}$. It follows that $V(y^2(0)) \geq \mathfrak{n}_j + \delta$, and since $y^2(\cdot)$ must enter $\mathcal{O}_\epsilon$ as t decreases, there is a fixed negative number T_2 such that $\xi_{\eta_k}(T_2 + t_k^2) \in \mathcal{O}_\epsilon(y_\ell^*)$ for some $1 \leq \ell \leq n$ such that $V(y_\ell^*) > V(y_j^*)$.

We can now follow the initial argument (before our case-by-case analysis) and find a sequence of times $t_k^3 < t_k^2$ such that

$$\xi_{\eta_k}(t_k^3) \in \overline{\mathcal{O}_{\epsilon'}(y_l^*)} \backslash B(y_l^*, \eta_\epsilon) \qquad \text{where} \qquad V(y_l^*) > V(y_j^*),$$

i.e. we can restart the argument from (3.2), but now with y_j^* replaced by some y_l^* for which $V(y_l^*) > V(y_j^*)$.

It is only possible to fall into case (a) a finite number of times.

Case (b). For some subsequence, $V(\xi_{\eta_k}(t_k^2)) \to \mathfrak{n}_j$.

Since $\eta_\epsilon \leq |\xi_{\eta_k}(t_k^2) - y_j^*| \leq \epsilon'$, if we use Lemma 3.1 to find a convergent subsequence with

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k^2) \to y^2(t)$$

uniformly for t in compact subsets of $\mathbb{R}$, it follows that $y^2(0) = \mathfrak{n}_j$ but $\eta_\epsilon \leq |y^2(0) - y_j^*| \leq \epsilon'$. In particular, $y^2(0)$ is not any of the equilibria, and the trajectory $y^2(\cdot)$ must leave a neighbourhood of y_j^* and tend to some y_l^* with $V(y_l^*) > V(y_j^*)$. We now follow the argument in case (a), so once again it is only possible to fall into case (b) a finite number of times.

Case (c). There exists a $\gamma > 0$ and a k_0 such that for all $k \geq k_0$,

$$\mathfrak{n}_j - \epsilon' < V(\xi_{\eta_k}(t_k^2)) < \mathfrak{n}_j - \gamma.$$

We note immediately we may assume that for any $\delta > 0$ there exists a k'_δ such that

$$V(\xi_{\eta_k}(t)) \leq \mathfrak{n}_j + \delta \qquad \forall\ t \leq t_k^2,\ k \geq k'_\delta \tag{3.3}$$

with $k'_\delta \geq k_0$, for otherwise, using the continuity of V, we must have

$$V(\xi_{\eta_k}(t_k^3)) \geq \mathfrak{n}_j + \delta/2$$

for some $t_k^3 < t_k^2$, for infinitely many k: hence we can return to case (a) with t_k^2 replaced by t_k^3. Since case (a) can only be repeated a finite number of times, we eventually find ourselves in case (c) with (3.3) valid.

In this situation we choose ϵ'' with $\nu_0 > \epsilon'' > \epsilon' > \epsilon$ and take any δ with $0 < \delta < \epsilon$. Set

$$B_\delta = \{\xi_{\eta_k}(\theta) : V(\xi_{\eta_k}(\theta)) \le \mathfrak{n}_j - \delta, \text{ for some } \theta \in \mathbb{R} \text{ and some } k \ge k_0\}.$$

Using Lemma 3.2, we can find $t_\delta^* > 0$ such that

$$\sup\{V(T_0(t)B)\} < \mathfrak{n}_j - \epsilon'' \qquad \forall\, t \ge t_\delta^*. \tag{3.4}$$

Now use the continuity of V and the uniform convergence of T_η to T_0 to choose $k_\delta \ge k'_\delta$ sufficiently large that

$$\sup\{V(T_{\eta_k}(t, t-s)B)\} < \mathfrak{n}_j - \epsilon' \qquad \forall\, s \in [t_\delta^*, 2t_\delta^*] \qquad \forall\, k \ge k_\delta. \tag{3.5}$$

It follows by induction that if $\xi_{\eta_k}(t_k^2 - t) \in B$ then $\xi_{\eta_k}(t_k^2 - t + \tau) \in B$ for all $\tau \ge t_\delta^*$.

We now claim that for all $k \ge k_\delta$ we must have

$$V(\xi_{\eta_k}(t_k^2 - t)) > \mathfrak{n}_j - \delta \qquad \forall\, t \ge t_\delta^*.$$

Indeed, suppose not. Then it follows that for some $t \ge t^*$ that

$$V(\xi_{\eta_k}(t_k^2 - t)) \le \mathfrak{n}_j - \delta,$$

i.e. $\xi_{\eta_k}(t_k^2 - t) \in B$. But then (3.5) shows that $V(\xi_{\eta_k}(t_k^2)) < \mathfrak{n}_j - \epsilon'$, a contradiction.

It follows that for any $\delta > 0$, there exist k_δ and t_δ^* such that

$$\mathfrak{n}_j - \delta < V(\xi_{\eta_k}(t_k^2 - t)) < \mathfrak{n}_j + \delta \qquad \forall\, k \ge k_\delta,\ t \ge t_\delta^*.$$

Now suppose that it is not true that for k sufficiently large, $\xi_{\eta_k}(\cdot)$ lies asymptotically (as $t \to -\infty$) in a ν_0 neighbourhood of one of the equilibria. Then, relabelling k to choose an appropriate subsequence, there must exist a sequence of times $T_k(n)$ with $T_k(n) \to \infty$ such that

$$|\xi_{\eta_k}(t_k^2 - T_k(n)) - y_k^*| \ge \nu_0 \qquad \text{for all } k = 1, \ldots, n.$$

Choosing n_k such that $T_k(n_k) \ge t_{1/k}^*$ it follows with $t_k^3 = t_k^2 - T_k(n_k)$ that we have a sequence $\xi_{\eta_k}(t_k^3)$ with

$$|\xi_{\eta_k}(t_k^3) - y_j^*| \ge \nu_0 \qquad \text{and} \quad V(\xi_{\eta_k}(t_k^3)) \to \mathfrak{n}_j \text{ as } k \to \infty.$$

However, this is essentially the situation in case (b), which can occur only a finite number of times.

FORWARD CASE:

There are two possibilities

a) Infinitely many $\xi_{\eta_k}(t_k^2)$ satisfy $\mathfrak{n}_j - \epsilon' \le V(\xi_{\eta_k}(t_k^2)) \le \mathfrak{n}_j - \epsilon$

b) There exists a k_0 such that $\xi_{\eta_k}(t_k^2)$ satisfy $\mathfrak{n}_j + \epsilon \le V(\xi_{\eta_k}(t_k^2)) \le \mathfrak{n}_j + \epsilon'$, for all $k \ge k_0$.

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Let us first deal with the case b). For $s \geq 0$ *let*

$$T_0(t_k^2 + s - t_k^2)\xi_{\eta_k}(t_k^2) = w_k(s)$$
$$T_\eta\,(\,t_k^2\,+\,s, t_k^2\,)\xi_{\eta_k}(t_k^2) = \xi_{\eta_k}(t_k^2 + s) = z_k(s)$$

Note that $V(w_k(s)) \leq V(\xi_{\eta_k}(t_k^2)) \leq \mathfrak{n}_j + \epsilon'$ *for all* $s \geq 0$.

Choose $\nu_0 > \epsilon'' > \epsilon' > \epsilon > \hat{\epsilon}$ *and* $t^* > 0$ *such that, if*

$$B = \{\xi_{\eta_k}(\theta) : \mathfrak{n}_j + \hat{\epsilon} \leq V(\xi_{\eta_k}(\theta)) \leq \mathfrak{n}_j + \epsilon'',\ \theta \in \mathbb{R},\ k \geq k_0\},$$

then,

$$\sup\{V(T_0(t^*)B)\} < \mathfrak{n}_j + \hat{\epsilon}. \tag{3.6}$$

Now we choose $\eta_1 > 0$ *such that, if* $\eta < \eta_1$ *then*

$$V(w_k(s)) \leq V(\xi_{\eta_k}(t_k^2)) \leq \mathfrak{n}_j + \epsilon',\ \forall\, s \in [0, t^*]\ \textit{and}$$
$$\sup\{V(T_\eta(t_k^2 + t^*, t_k^2)B)\} < \mathfrak{n}_j + \epsilon.$$

From (3.6) *we obtain that* $V(T_0(t^*)\xi_{\eta_k}(t_k^2)) = V(w_k(t^*)) < \mathfrak{n}_j + \hat{\epsilon}$ *and consequently* $V(\xi_{\eta_k}(t_k^2 + t^*)) < \mathfrak{n}_j + \epsilon$. *Now we are back at the starting point and the procedure can be repeated. If in every step we fall in case b) and we must take* k_0 *larger at each step the situation a) also happens and if not we prove that all solutions* ξ_{η_k}, *for suitably large* k, *must end forward in a neighborhood of* y_j^* *and we have a contradiction.*

If a) *happens we obtain from Lemma 3.1 that there is a subsequence, which we again denote by* ξ_{η_k} *such that* (3.1) *holds; that is,*

$$\lim_{k\to\infty} \xi_{\eta_k}(t) \to y^2(t - t_k^2)$$

uniformly for $t - t_k^2$ *in compact subsets of* $\mathbb{R}$. *Since* $\mathfrak{n}_j - \epsilon \leq V(y^2(0)) \leq \mathfrak{n}_j - \epsilon'$ *and since* $y^2(\cdot)$ *must enter forward* $\mathcal{O}_\epsilon$ *we have that there is a fixed positive number* T_2 *such that* $\xi_{\eta_k}(T_2 + t_k^2) \in \mathcal{O}_\epsilon(y_\ell^*)$ *for some* $1 \leq \ell \leq n$ *such that* $V(y_\ell^*) < V(y_j^*)$ *and must again leave this neighborhood.*

Repeating this procedure a finite number of times we obtain that this procedure must end proving that, for suitably large k, the solution ξ_{η_k} must end backward (forward) in the neighborhood $\mathcal{O}_\epsilon$. Since every solution ξ_{η_k} that stays in this neighborhood for all t in a interval of the form $(-\infty, t_k]$ ($[t_k, \infty)$) must be in the unstable (stable) manifold of ξ_{j,η_k}^*, we have a contradiction and that concludes the proof of (1.) (2.).

Finally, (3.) is an easy consequence of the proof of (2.) and (2.16). Indeed, given (τ, y_0), define $\xi_\eta(t) = T_\eta(t, \tau)y_0$. Now, following as the in the proof of (2.), take a sequence $\{t_k^1\}$ in $[\tau, \infty)$ such that $\xi_{\eta_k}(t_k^1)$ is outside the $\mathcal{O}_\epsilon$. To follow exactly the argument (2.), we only have to assure that $\xi_{\eta_k}(t_k^1)$ has a convergent subsequence. If $\{t_k^1\}$ is bounded this is a consequence of (2.2). Now, if $\{t_k^1\}$ is unbounded, given $\rho > 0$, there exists k_0 and a subsequence of $\xi_{\eta_k}(t_k^1)$ (labelled again by ξ_{η_k}) such that $\xi_{\eta_k}(t_k^1) \in \mathcal{O}_\rho(K)$ for all $k \geq k_0$. This implies that ξ_{η_k} is totally bounded, and then sequencially compact. Note that all the argument to get (2.) is made forward in time, so it can be repeated in the $[\tau, \infty)$ interval in order to get (3.). ∎

4. ASYMPTOTICALLY AUTONOMOUS PROBLEMS

As in Section 1 we consider a Banach space $\mathcal{Z}$ and the semilinear problem

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f(t,y) \\ y(\tau) &= y_0, \end{aligned} \tag{4.1}$$

where $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of a C^0-semigroup of bounded linear operators and $f(t,\cdot)$ is a differentiable function which is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ with Lipschitz constant independent of t. Denote by $t \mapsto T(t,\tau)y_0$ the solution for (1.1). Then, $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defines a nonlinear process. Assume that the problem (4.1) has a non-autonomous attractor $\{A(t) : t \in \mathbb{R}\}$.

4.1. Asymptotically Autonomous Problems at $-\infty$

Assume that

$$\lim_{t\to-\infty} \sup_{z\in B(0,r)} \|f(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})} = 0, \qquad \text{for each } r > 0, \tag{4.2}$$

and that (1.2) has an autonomous attractor A_0.

Suppose that (2.3) has a finite number of solutions, all of them hyperbolic and that A_0 is given by (2.6). We now now prove the following:

PROPOSITION 4.1. *Assume that* (4.2) *holds and that all solutions of* (2.3) *are hyperbolic. Then, there are solutions* $\xi^*_{i_-} : \mathbb{R} \to \mathcal{Z}$, $1 \leq i_- \leq n^-$, *such that*

$$\lim_{t\to-\infty} \max_{1\leq i_-\leq n^-} \|\xi^*_{i_-}(t) - y^*_{i_-}\|_{\mathcal{Z}} = 0. \tag{4.3}$$

Furthermore, there is a $\tau \in \mathbb{R}$ *such that*

$$\dot{y} = \mathcal{A}^{i_-}y + B^{i_-}(t)y \tag{4.4}$$

has exponential dichotomy in $(-\infty, \tau]$, *where* $A^{i_-} = \mathfrak{B} + f_0'(y^\star_{i_-})$ *and* $B^{i_-}(t) = f_y(t, \xi^*_{i_-}(t)) - f_0'(y^*_{i_-})$.

Proof: The proof of this result reduces to the proof of (2) in Proposition 2.2, cutting the nonlinearities f and f_0 around $y^*_{i_-}$ in such a way that the fixed point argument works. To be more specific, we fix $1 \leq i_- \leq n^-$ and consider the change of variables $z = y - y^*_{i_-}$ in

(4.1). In this new variable (4.1) becomes

$$\dot{z} = \mathcal{A}_{i_-} z + \tilde{g}_{i_-}(t, z) \tag{4.5}$$

where $\tilde{g}_{i_-}(t,z) = f(t, z + y^*_{i_-}) - f_0(y^*_{i_-}) - f'_0(y^*_{i_-})z$. Cut $\tilde{g}_{i_-}$ outside a small neighborhood of $z = 0$ and suitably large negative times $t \le \tau$ in such a way that it becomes globally Lipschitz and bounded with very small Lipschitz constant and bound. Denote by g_{i_-} the new nonlinearity and consider, for $t \le \tau$,

$$z(t) = e^{\mathcal{A}_{i_-}(t-\tau)} z(\tau) + \int_\tau^t e^{\mathcal{A}_{i_-}(t-s)} g_{i_-}(s, (z(s)))ds.$$

Hence

$$\mathcal{Q}_{i_-} z(t) = \int_\infty^t e^{\mathcal{A}_{i_-}(t-s)} \mathcal{Q} g_{i_-}(s, (z(s)))ds$$

and

$$(I - \mathcal{Q}_{i_-})z(t) = \int_{-\infty}^t e^{\mathcal{A}_{i_-}(t-s)} (I - \mathcal{Q}) g_{i_-}(s, (z(s)))ds.$$

Consequently, there exists in a small neighborhood of $z = 0$ a globally defined solution of (4.1) if and only if

$$T_{i_-}(z)(t) = \int_\infty^t e^{\mathcal{A}_{i_-}(t-s)} \mathcal{Q}_{i_-} g_{i_-}(s, (z(s)))ds + \int_{-\infty}^t e^{\mathcal{A}_{i_-}(t-s)} (I - \mathcal{Q}_{i_-}) g_{i_-}(s, (z(s)))ds$$

has a unique fixed point in the set

$$\{z : \mathbb{R} \to \mathcal{Z} : \sup_{t \in \mathbb{R}} \|z(t)\|_{\mathcal{Z}} \le \epsilon\}$$

with suitably small ϵ. That follows assuming that, for $z, z_1, z_2 \in B(0, \epsilon)$, $\|g_{i_-}(t,z)\|_{\mathcal{Z}} \le \delta$ and that $\|g_{i_-}(t, z_1) - g_{i_-}(t, z_2)\|_{\mathcal{Z}} \le \delta \|z_1 - z_2\|_{\mathcal{Z}}$, with $\delta > 0$ suitably small. As a consequence of this it follows that $\xi^*_{i_-}(\cdot)$ is uniformly close to $y^*_{i_-}$. This solution is hyperbolic on $\mathbb{R}$. Hence, $\xi^*_{i_-}$ is a hyperbolic solution of (4.5) for all t large and negative. Hence, $y^*_{i_-} + \xi^*_{i_-}$ is a hyperbolic solution of (4.1) in $(-\infty, \tau]$ with $-\tau > 0$ suitably large. This ensures that (1.) below holds. Thus, the following result holds

Theorem 4.1. *Let $f : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ be a differentiable function which satisfies* (4.2). *Consider the initial value problem* (4.1). *Assume that all solutions of* (2.3) *are hyperbolic equilibrium solutions for* (1.2) *and that* (1.2) *has a global attractor A_0 which is given by* (2.6).

1.If we denote by $W^u(\xi^*_{i,\eta})(\tau) = \{ (\tau, \zeta) \in \mathbb{R} \times \mathcal{Z} :$ there is a backward solution $z(t, \tau, \zeta)$ of (1.1)*satisfying $z(\tau, \tau, \zeta) = \zeta$ and such that $\lim_{t \to -\infty} \|z(t, \tau, \zeta) - \xi^*_{i,\eta}(t)\| = 0\}$, then the attractor $\{A(\tau) : \tau \in \mathbb{R}\}$ of* (4.1) *is such that $A(\tau) = \cup_{i=1}^n W^u_\eta(\xi^*_{i,\eta})(\tau)$.*

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2.For each globally defined bounded solution $\xi(\cdot)$ of (4.1) *we have that there is* $1 \leq i_- \leq n^-$ *such that*

$$\lim_{t \to -\infty} \|\xi(t) - \xi^*_{i_-}(t)\|_{\mathcal{Z}} = 0. \tag{4.6}$$

Proof: The proof of (1) is a consequence of Proposition 4.1 and Theorem 2.4. Indeed, since we may write (4.1) as a small non-autonomous perturbation of (1.2) considering

$$f_\nu(t,y) = \begin{cases} f(t,y), & \text{if } t \leq -\nu \\ f(\nu,y), & \text{if } t > -\nu. \end{cases}$$

From Theorem 2.4, for suitably large ν, there exists a pullback attractor $\{A_\nu(s) : s \in \mathbb{R}\}$ given by $A(s) = \cup_{i=1}^n W^u_\eta(\xi^*_{i,\nu})(s)$ for

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_\nu(t,y) \\ &y(\tau) = y_0, \end{aligned} \tag{4.7}$$

To obtain the pullback attractor for (4.1) we first note that (4.7) and (4.1) coincide for $t \leq \tau \leq \nu$. Hence $A(t) = A_\nu(t)$ for $t \leq \nu$. To recover $A(t)$ for $t \geq \nu$ we only have take advantage of the invariance to see that $A(t) = T(t,\tau)A(\nu)$, for all $t \geq \nu$.

Now, (2) is also essentially proved since, by (4.3), every complete trajectory approaches one of the equilibria $y^*_{i_-}$ as $t \to -\infty$, so that, in particular, (2) holds.□

Remark 4. 1. It is clear from the above proof that in order to characterize the pullback attractor $\{A(t) : t \geq 0\}$ it is not necessary that $A(t)$ remains bounded as $t \to \infty$. This accounts for many cases in the existing literature where the pullback attractors do not remain bounded as $t \to \infty$ (see [11, 18]).

4.2. Asymptotically Autonomous Problems at $+\infty$

Assume that

$$\lim_{t \to +\infty} \sup_{z \in B(0,r)} \|f(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})} = 0, \quad \text{for each } r > 0, \tag{4.8}$$

and that (1.2) has an autonomous attractor A_0. We note that the nonlinearity f_0 in this subsection may be different from that in the previous subsection and consequently the attractor A_0 in this subsection may be different from that in the previous one.

Assume that (2.3) has a finite number of solutions, all of them hyperbolic and that A_0 is given by (2.6).

Definition 4.1. We say that a family $\{A^+(t) \subset \mathcal{Z} : t \geq t_0\}$ is a *time dependent forward attractor* for (4.1) if:

• $A^+(t)$ is compact for each $t \geq t_0$,

• $\{A^+(t) : t \geq \tau\}$ is invariant in the sense that $T(t,\tau)A^+(\tau) = A^+(t)$ for all $t \geq \tau \geq t_0$, and

• $\text{dist}(T(t,\tau)B, A^+(t)) \to 0$ as $t \to \infty$ for each bounded set $B \subset \mathcal{Z}$ and for any $\tau \geq t_0$.

Remark 4. 2. The concept of a global attractor for non-autonomous problems is yet something that deserves deep reflection. One certainly would desire that any definition of global attractor would include all globally defined bounded solutions. That is sufficient to define the global attractor for autonomous problems. If we define

$$A(t) = \{\xi(t) : \xi : \mathbb{R} \to \mathbb{R} \text{ is a bounded solution}\},$$

then $\{A(t) : t \in \mathbb{R}\}$ is the pullback attractor for our problem. Thus, the pullback attractor is certainly an important part of what should be called a global attractor. In particular, in the autonomous case, there is nothing else. In order that we can see that the pullback attractors will not be able to describe everything on the asymptotic behavior of a truly non-autonomous problem we consider the following example

$$\dot{x} = \lambda(t)x - x^3 \tag{4.9}$$

with $\lambda : \mathbb{R} \to \mathbb{R}$ being a smooth function with the property that $0 \leq \lambda(t) \leq 1$, $\lambda(t) = 0$ for all $t \leq 0$ and $\lambda(t) = 1$ for all $t \geq 1$.

We note that the pullback attractor for (4.9) is such that $A(t) = \{0\}$ for all $t \in \mathbb{R}$. Nonetheless, for $t \geq 1$ (4.9) has three stationary solutions $x_0 = 0$, $x_- = -1$ and $x_+ = 1$ with the equilibrium $x_0 = 0$ being unstable. If we look at the solution that in time $t_0 = 1$ is $x(t_0) = 1$ and solve the equation for $t \leq t_0$ we see that $x(0, t_0, 1) = x_1 > 0$ and therefore $x(t, t_0, 1) = \frac{1}{\sqrt{2t + x_1^2}}$, for $-\frac{x_1^2}{2} < t \leq 0$ and it blows up as $t \to -\left(\frac{x_1^2}{2}\right)^+$. We also see that the set $\{-1, 0, 1\}$ attracts points of $\mathbb{R}$ and that $[-1, 1]$ attracts forward bounded subsets of $\mathbb{R}$.

In the study of the asymptotic autonomous problems we would like to say that the pullback attractor is described by the limit as $t \to -\infty$ whereas the forward attractor is described by the limit as $t \to +\infty$. The properties of attraction when $t \to +\infty$ are very important but to have a good knowledge of the set of all trajectories that are globally defined and bounded (the pullback attractors) is also very important. We accomplish both.

We can define for each $t > 1$ $A^+(t) = [-1, 1]$ with the property that $T(t,\tau)A(\tau) = A(t)$ for all $t \geq \tau \geq 1$ and one can easily see that $d(T(t,\tau)B, A(t)) \to 0$ as $t \to \infty$. One would like to define such sets in general and obtain its attraction properties as $t \to \infty$. Of course we are only able to define them in such a simple way because of the very specific form of our non-autonomous equation. Next we consider more general and very interesting situations for which that can still be done.

Assume that (4.1) gives rises to a nonlinear process $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ for which there is an absorbing ball $B(0, r_0)$. Consider $f_k(t,z)$ the function which coincides with f in $[k,\infty) \times \mathcal{Z}$ and which is equal to $f(k,z)$ for all $t < k$ and $z \in \mathcal{Z}$. Then

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \sup_{z\in B(0,r_0)} \|f_k(t,z) - f_0(z)\|_{\mathcal{Z}} + \|(f_k)_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})} = 0. \tag{4.10}$$

We have proved in [5] that the family of attractors for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_k(t,y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.11}$$

behaves upper and lower semicontinuously as $k \to \infty$ with the limit attractor being the attractor for (1.2). If we denote by $\{A_k(t) : t \in \mathbb{R}\}$ the attractor for (4.11), we have that

$$A_j(t) = A_k(t), \quad \text{for all } j > k \text{ and } t \in \mathbb{R}.$$

Let k_0 be such that for $k \geq k_0$ the pullback attractor of (4.11) coincides with the union of the unstable manifold of $\{\xi^*_{i,k}\}$ with $\sup_{t\in\mathbb{R}} \|\xi^*_{i,k}(t) - y^*_i\|_{\mathcal{Z}} \to 0$ as $k \to \infty$. Define $A^+(t) = A_{k_0}(t)$ for $t \geq k_0$.

If we define $T_\infty(t,\tau) = T_0(t-\tau)$, it follows from the fact that

$$\sup_{t\geq\tau} \|T_k(t,\tau)B - T_\infty(t,\tau)B\|_{\mathcal{Z}} \to 0, \text{ as } k \to \infty$$

and from the lower semicontinuity of attractors that, given $\epsilon > 0$ there is a $T_\epsilon > 0$ such that, for all $t \geq T_\epsilon$, $T_\infty(t,\tau)B \subset O_\epsilon(A_0)$ and an $N \in \mathbb{N}$ such that $T_k(t,\tau)B \subset O_\epsilon(T_\infty(t,\tau)B) \subset O_{2\epsilon}(A_0) \subset O_{3\epsilon}(A^+(t))$ for all $t \geq k \geq N$. This proves the following result

Theorem 4.2. *There is a $t_0 \in \mathbb{R}$ and a time dependent forward attractor $\{A^+(t) : t \geq t_0\}$ for* (4.1).

The following results state that in this case there is a finite number of hyperbolic trajectories that attract all other trajectories as $t \to \infty$.

Proposition 4.2. *Assume that* (4.8) *holds. Then, there are solutions $\xi^*_{j+} : \mathbb{R} \to \mathcal{Z}$, $1 \leq j \leq n^+$, such that*

$$\lim_{t\to+\infty} \max_{1\leq j\leq n} \|\xi^*_{j+}(t) - y^*_{j+}\|_{\mathcal{Z}} = 0. \tag{4.12}$$

Furthermore, there is a $t_0 \in \mathbb{R}$ such that

$$\dot{y} = \mathcal{A}^{j+}y + B^{j+}(t)y \tag{4.13}$$

has an exponential dichotomy in $[t_0, +\infty)$, where $A^{i_+} = \mathfrak{B} + f_0'(y^\star_{i_+})$ and $B^{j+}(t) = f_y(t, \xi^*_{j+}(t)) - f_0'(y^*_{j+})$.

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Proof: Again, the proof of this result reduces to the proof of (2) in Proposition 2.2, cutting the nonlinearities f in the same way as before to make (4.10) hold. To be more specific, we fix $1 \leq j \leq n^+$ and consider the change variables $z = y - y^*_{j+}$ in (4.1). In this new variable (4.1) becomes

$$\dot{z} = \mathcal{A}^{j+} z + \tilde{g}_{j+}(t, z) \tag{4.14}$$

where $\tilde{g}_{j+}(t,z) = f(t, z + y^*_{j+}) - f_0(y^*_{j+}) - f_0'(y^*_{j+})z$. Cut $\tilde{g}_{j+}$ outside a small neighborhood of $z = 0$ and suitably large times in such a way that it becomes globally Lipschitz and bounded with very small Lipschitz constant and bound. Let g_{j+} the new nonlinearity and proceed exactly as in the previous section (asymptotically autonomous in $-\infty$) to obtain the existence of a global hyperbolic solution $\xi^*_{j+}(\cdot)$ for the changed equation which is uniformly close to y^*_{j+}. Now, ξ^*_{j+} is a solution of (4.14) for all t large enough. Hence, $y^*_{j+} + \xi^*_{j+}$ is a solution of (4.1) in $[\tau, \infty)$ with $\tau > 0$ suitably large. This solution is hyperbolic on $\mathbb{R}$. Hence, $\xi^*_{i_+}$ is a hyperbolic solution of (4.5) for all t large enough. Hence, $y^*_{i_+} + \xi^*_{i_+}$ is a hyperbolic solution of (4.1) in $[\tau, +\infty)$ with $\tau > 0$ suitably large. □

Ball and Peletier [3] (see also [2]) prove that, in fact a little more that (4.12) is true, namely that, given each $(\tau, y_0) \in \mathbb{R} \times \mathcal{Z}$, there exists $1 \leq j+ \leq n$ such that

$$\lim_{t \to \infty} \|T(t, \tau)y_0 - y^*_{j+}\|_{\mathcal{Z}} = 0.$$

Thus, finally the following results holds

THEOREM 4.3. *Let $f : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ be a differentiable function which satisfies* (4.8). *Consider the initial value problem* (4.1). *Assume that all solutions of* (2.3) *are hyperbolic equilibrium solutions for* (1.2) *and that* (1.2) *has a global attractor A_0 which is given by* (2.6).

Then, for each $(\tau, y_0) \in \mathbb{R} \times \mathcal{Z}$, there exists $1 \leq j+ \leq n$ such that

$$\lim_{t \to \infty} \|T(t, \tau)y_0 - \xi^*_{j+}(t)\|_{\mathcal{Z}} = 0. \tag{4.15}$$

In particular, for each globally defined bounded solution $\xi(\cdot)$ of (4.1) *there is $1 \leq j+ \leq n$ such that*

$$\lim_{t \to \infty} \|\xi(t) - \xi^*_{j+}(t)\|_{\mathcal{Z}} = 0. \tag{4.16}$$

Note that usually results on asymptotically autonomous systems in the literature show that the forward asymptotic behavior of the equations tends to limiting structures within the limit attractor, for instance equilibria of the limit equations, which, in general, are not solutions of the non-autonomous system. (Although there are non-gradient examples showing that the limiting behavior can differ from that of the limt system, e.g. [16, 20].) Theorem 4.3 goes a little further, since it describes the forward long time dynamics by means of hyperbolic solutions of the non-autonomous equations.

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