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Strongly damped wave problems: bootstrapping and regularity of solutions

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The aim of the article is to give a unified treatment in the Banach spaces to several problems modeled by semilinear partial differential equations of second order in time. Our results are applied to a number of examples appearing in the literature, which can be regarded as *strongly damped semilinear wave equations*. The present work greatly extends the known results on existence and regularity of solutions of such problems. In many cases, these problems have been considered previously only within the setting of Hilbert spaces with the operators appearing in the main part being self-adjoint. Here we present a more general setting involving sectorial operators and reflexive Banach spaces.
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1. INTRODUCTION

In this article we consider a class of mathematical models that can be regarded as *strongly damped semilinear wave problems*. Our aim is to give a unified treatment to several different special problems appearing in the literature and modeled by second order in time semilinear partial differential equations with structural damping (expressed by the first order in time term). Among the problems that can be included in this class we mention:

- The sine-Gordon equation

$$u_{tt} + \beta u_t - \Delta u - \alpha \Delta u_t = -\sin u + g(x),\ \ \alpha, \beta \in (0, \infty).$$

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- The perturbed wave equation of quantum mechanics

$$u_{tt} - \Delta u - \alpha \Delta u_t = -|u|^q u - \beta |u_t|^p u_t + g(x),\ \alpha \in (0, \infty).$$

- The perturbed viscous Cahn-Hilliard equation

$$\epsilon u_{tt} + u_t + \Delta^2 u - \delta \Delta u_t = \Delta(h(u)),\ \epsilon, \delta \in (0, \infty).$$

- The wave equation with structural damping

$$u_{tt} + 2\eta(-\Delta)^{\frac{1}{2}} u_t + \beta u_t - \Delta u = f(u),\ \eta \in [0, \infty), \beta \in (0, \infty).$$

Under suitable boundary conditions these problems, and many others in the class considered here, have the property that the solution operator associated to the linear part is an analytic semigroup. This fact gives a parabolic structure to these second order (wave like) semilinear problems. Our aim here is to consider such class of problems from the point of view of parabolic equations. With this in mind we give a general result on local existence, regularity and bootstrapping. We first prove that the above problems are locally well posed not only in the L^2 setting (context in which they have already been extensively studied, see [4, 5, 6, 8, 9, 10, 22, 26, 28]), but also in the L^p setting, context for which the theory has not been fully developed (see, for example, [23]). To that end we prove that the linear part of the equation generates analytic semigroup in this latter setting and study its regularization properties. Next, we prove a local existence theorem and study the regularity properties of solutions of the associated semilinear problem in the $L^p(\Omega)$ setting. We also obtain a bootstrapping argument that will lead to more regularity of solutions which, in some cases, will be classical. Lastly, we apply these results to the above mentioned examples comparing our results with the results in the literature.

To present the results proved in this paper we introduce some terminology.

DEFINITION 1.1. Let X be a reflexive Banach space, $A : D(A) \subset X \to X$ be a closed densely defined operator, $\eta > 0$ and $\theta \in [\frac{1}{2}, 1]$. The triple (η, θ, A) is said to be an *admissible triple* if

$$\|(\lambda + A)^{-1}\| \leq \frac{M}{1 + |\lambda|} \tag{1}$$

for some $M > 0$ and for all λ in the sector $\Sigma_\psi = \{\lambda \in \mathbb{C} : \psi \leq |\arg|\lambda \leq \pi$ with $\psi \in (0, \frac{\pi}{2})\} \cup \{0\}$ and either

i) $\theta \in (\frac{1}{2}, 1]$ or
ii) $\theta = \frac{1}{2}$ and $\frac{\pi}{2} > \frac{\psi}{2} + \arg(\eta + \sqrt{\eta^2 - 1})$.

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Let (η, θ, A) be an admissible triple and denote by X^α the fractional power spaces (see [1, 17]) associated to A. Consider the second order differential equation

$$\begin{cases} u_{tt} + 2\eta A^\theta u_t + Au = f(u) + g(u_t), \\ u(0) = u_0 \in X^{\frac{1}{2}},\ u_t(0) = v_0 \in X. \end{cases} \tag{2}$$

With this notation, all the above mentioned mathematical models are represented in the form of equation (2). Also in all the examples cited previously the corresponding triple is admissible.

Problems like (2) appear in the literature under the name of *strongly damped wave equations* and in most cases A must be self adjoint. Our approach allows a great generalization including much more general strongly damped wave problems then those considered so far. In particular, we allow the elliptic operators to be in non-divergence form. Besides this, the local existence results are improved significantly with respect to the nonlinear part of the equation, since other spaces can be used instead of Hilbert spaces.

Problem (2) can be written as a first order Cauchy problem in $X^{\frac{1}{2}} \times X$ in the following way

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \mathcal{A}_{(\theta)} \begin{bmatrix} u \\ v \end{bmatrix} = F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right),\ t > 0, \quad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{3}$$

where

$$\mathcal{A}_{(\theta)} = \begin{bmatrix} 0 & -I \\ A & 2\eta A^\theta \end{bmatrix} : D(\mathcal{A}_{(\theta)}) \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X \tag{4}$$

is defined by

$$\mathcal{A}_{(\theta)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ A^\theta(A^{1-\theta}\varphi + 2\eta\psi) \end{bmatrix} \tag{5}$$

for

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\mathcal{A}_{(\theta)}) = \{\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X^{\frac{3}{2}-\theta} \times X^{\frac{1}{2}};\ A^{1-\theta}\varphi + 2\eta\psi \in X^\theta\}$$

and

$$F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f(u) + g(v) \end{bmatrix}.$$

Remark 1. 1. Though the condition ii) in Definition 1.1 seems restrictive it can be verified in all interesting applications for all η in $(0, \infty)$. Unfortunately this condition cannot be avoided. If $\eta \in (0, 1)$, $\theta = \frac{1}{2}$, Δ_D is the Dirichlet Laplacian in a bounded smooth domain Ω, $p \neq 2$ and $A = -e^{i(\frac{\pi}{2} - \arg(\eta - \sqrt{\eta^2 - 1}))}\Delta_D$, then $(\eta, \frac{1}{2}, A)$ is not an admissible triple and consequently $\mathcal{A}_{(\frac{1}{2})}$ does not generate a C^0-semigroup (see [18]) and (3) is an ill posed problem.

When X is a Hilbert space and A is a positive self adjoint operator, it has been proved in [8, 9] that $\mathcal{A}_{(\theta)}$ is a sectorial operator and therefore it generates an analytic semigroup. Under the same conditions, in [10], the authors gave a characterization of the fractional power spaces associated to $\mathcal{A}_{(\theta)}$, $\theta \in [\frac{1}{2}, 1]$. In these works, the fact that X is a Hilbert space and that A is a positive self adjoint is used in an essential way. We extend the results of [8, 9, 10] to the case when the operator A is not self adjoint and more generally we consider the case when the space is a general reflexive Banach space.

The local existence for (2) in the critical exponent case have been considered in [4, 5] when A is a positive self adjoint operator and X is a Hilbert space. In [7] the case when A is the Dirichlet Laplacian in a bounded smooth domain Ω and $\theta = \frac{1}{2}$ is considered in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ for subcritical nonlinearities. We extent the results in [4, 5, 6, 7] to general $2m$-th order strongly elliptic operator, $\theta \in [\frac{1}{2}, 1]$ and critical nonlinearities.

Recently, in [13], the following general second order partial differential equation is studied in the Hilbert setting

$$\begin{cases} Mu_{tt}(t) + Au(t) + k \cdot D(u_t(t)) = F(u(t), u_t(t)), \\ u(0) = u_0 \in D(A^{1/2}), u_t(0) = u_1 \in V = D(M^{1/2}), \end{cases} \tag{6}$$

where A is a positive self-adjoint operator, k is a positive parameter, and operators D and F satisfy suitable hypotheses. Of course, (6) covers many examples that do not fall into the class of problems considered here. Also, in [13] the authors consider not only local existence but also global existence and existence of global attractors. Nonetheless, if $M = I$ and $D = A^\theta$ ($\theta \in [\frac{1}{2}, 1]$), that is in the case of the strongly damped wave equation considered here, the space V in [13] coincides with the base Hilbert space H in which A is self-adjoint. In this situation our assumptions may lead to a more general class of nonlinearities. For example, if Δ_D denotes the Dirichlet Laplacian in a bounded smooth domain $\Omega \subset \mathbb{R}^3$, $\theta = 1$ and $D = A = -\Delta_D$, $M = I$, $H = L^2(\Omega)$, $f : \mathbb{R} \to \mathbb{R}$ and $F(u, u_t)(x) = f(u(x))$, the growth that we obtain for f is 5, whereas in [13] it is 3. We remark that in the present work we consider also operators A which are not necessarily self-adjoint as well as settings which are not restricted to Hilbert spaces. Also, we never assume that the nonlinear terms in the equation are monotone or quasi-monotone.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, L be $2m$-th order operator in Ω given by

$$L = \sum_{|\sigma| \leq 2m} a_\sigma(x) D^\sigma \tag{7}$$

and $\{B_j, j = 0, \ldots, m-1\}$ be boundary operators such that the triple $(L, \{B_j\}, \Omega)$ forms a *regular elliptic boundary value problem* (see [15]). For $p \in [2, \infty)$ define

$$\begin{aligned} &A_L : D(A_L) \subset L^p(\Omega) \to L^p(\Omega), \ D(A_L) = H^{2m}_{p,\{B_j\}}(\Omega), \\ &A_L u = Lu \quad \text{for} \quad u \in D(A_L). \end{aligned} \tag{8}$$

Then A_L is a sectorial operator in $X = L^p(\Omega)$ and $X^{\frac{1}{2}} = H^m_{p,\{B_j\}}(\Omega)$ (see [27] for the definition of the $H^s_{p,\{B_j\}}(\Omega)$ spaces). Assume throughout the paper that (η, θ, A_L) is an admissible triple and consider the problem

$$\begin{cases} u_{tt} + 2\eta A_L^{\theta} u_t + A_L u = f(u) + g(u_t), \ t > 0, \ x \in \Omega, \\ u(0) = u_0 \in X^{\frac{1}{2}}, \ u_t(0) = v_0 \in X. \end{cases} \tag{9}$$

As a particular consequence of the results of this paper we obtain:

THEOREM 1.1. *Suppose that f, g are C^1 functions such that*

i) $|f'(s)| \le c(1 + |s|^{\rho-1})$, $s \in \mathbb{R}$, where $1 < \rho \le \frac{N+mp}{N-mp} =: \bar{\rho}$ if $mp < N$, or $\rho \in (1, \infty)$ if $mp = N$,

ii) $|g'(s)| \le c(1 + |s|^{\nu-1})$, $s \in \mathbb{R}$, where $1 < \nu \le \frac{N+2mp\theta}{N} =: \bar{\nu}$.

Then, the initial boundary value problem (9) *has a unique local solution through each point $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^m_{p,\{B_j\}}(\Omega) \times L^p(\Omega)$. This solution depends continuously upon the initial condition and, for $\theta < \frac{2}{3}$, is a classical solution.*

In fact, we may consider nonlinearities depending on spatial derivatives of u. In particular, for the perturbed viscous Cahn-Hilliard equation in dimension three, the conclusion of the above theorem holds for any C^3 function h and any initial data from $H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega)$ (no growth condition is required and h need not to be a polynomial).

This paper is organized as follows. In Section 2 we show that the operator $\mathcal{A}_{(\theta)}$ generates analytic semigroup, characterize its fractional power spaces (including extrapolation spaces) and obtain an abstract local existence result. In Section 3 we consider the case when A is a $2m$-th order elliptic operator in $L^p(\Omega)$, specialize the characterization of fractional power spaces given in Section 2 to prove embedding of such spaces in suitable known spaces, obtain a general bootstrapping result and consider some particular nonlinearities that appear in the applications. Finally, in Section 4 we apply our results to some known problems including the ones mentioned in the beginning of the paper.

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2. ABSTRACT SETTING OF THE PROBLEM

In this section we study the abstract properties of the operator $\mathcal{A}_{(\theta)}$ defined by (5). We also establish local well posedness of (3) with nonlinearities f, g defined in appropriate subspaces of X. Throughout the section we assume that (η, θ, A) is an admissible triple.

2.1. Properties of the operator $\mathcal{A}_{(\theta)}$

Here we discuss the properties of $\mathcal{A}_{(\theta)}$, $\theta \in [\frac{1}{2}, 1]$, necessary to consider (3) as an abstract parabolic problem on the extrapolation space $Y_{(\theta)_{-1}}$ of Y^0 (the completion of

$(Y^0, \|\mathcal{A}_{(\theta)}^{-1}(\cdot)\|))$. To that end we first prove that $\mathcal{A}_{(\theta)} : D(\mathcal{A}_{(\theta)}) \subset Y^0 \to Y^0$ is a sectorial operator and characterize its fractional power spaces.

In the case when X is a Hilbert space and A is a positive self-adjoint operator these properties have been established in a sequence of papers [8, 9, 10]. If $-A$ is the Dirichlet Laplacian in a bounded smooth domain Ω and $X = L^p(\Omega)$, the proof that $\mathcal{A}_{(\frac{1}{2})}$ generates analytic semigroup and an approximate characterization of its fractional power spaces is given in [7].

Remark 2. 1. Note that

$$\mathcal{A}_{(\theta)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ A\varphi + 2\eta A^\theta \psi \end{bmatrix} \text{ for } \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X^1 \times X^\theta, \tag{10}$$

and that $X^1 \times X^\theta$ is a dense subset of $D(\mathcal{A}_{(\theta)})$. However the operator $\mathfrak{C} : D(\mathfrak{C}) \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$ defined by $D(\mathfrak{C}) = X^1 \times X^\theta$ and

$$\mathfrak{C} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & -I \\ A & 2\eta A^\theta \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ A\varphi + 2\eta A^\theta \psi \end{bmatrix}$$

is not a closed operator unless $\theta = \frac{1}{2}$. Note also that $D(\mathcal{A}_{(\frac{1}{2})}) = X^1 \times X^{\frac{1}{2}}$ and that, for all $\theta \in (\frac{1}{2}, 1]$, $D(\mathcal{A}_{(\theta)})$ is not a cross product of spaces.

As shown in [2, 4], the knowledge of the scale of fractional power spaces and its embedding into known spaces are the main tool to obtain local well posedness of a semilinear sectorial problem (even in the critical growth case). In this sense, we obtain here the main working tools for obtaining a general local well posedness result for (3).

PROPOSITION 2.1. *If $Y^0 = X^{\frac{1}{2}} \times X$ and $\mathcal{A}_{(\theta)} : D(\mathcal{A}_{(\theta)}) \subset Y^0 \to Y^0$ is defined by* (5), *the following holds:*

1) $\mathcal{A}_{(\theta)}$ *is closed,*
2) $0 \in \rho(\mathcal{A}_{(\theta)})$, *and*
3) *If A has compact resolvent, then $\mathcal{A}_{(\theta)}$ has compact resolvent for $\theta \in [\frac{1}{2}, 1)$.*

Proof: 1) To prove that $\mathcal{A}_{(\theta)}$ is closed we take a sequence $\left(\begin{bmatrix} \varphi_n \\ \psi_n \end{bmatrix}, \mathcal{A}_{(\theta)} \begin{bmatrix} \varphi_n \\ \psi_n \end{bmatrix} \right)$ in the graph of $\mathcal{A}_{(\theta)}$ which converges in $[X^{\frac{1}{2}} \times X] \times [X^{\frac{1}{2}} \times X]$ to $\left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix}, \begin{bmatrix} \xi \\ \nu \end{bmatrix} \right)$. From this we easily conclude that $\xi = \phi$ and

$$\begin{aligned} &\phi_n \xrightarrow{X^{\frac{1}{2}}} \phi \Rightarrow A^{1-\theta}\phi_n \xrightarrow{X} A^{1-\theta}\phi, \\ &\psi_n \xrightarrow{X} \psi \end{aligned}$$

and hence $A^{1-\theta}\phi_n + \psi_n \to A^{1-\theta}\phi + \psi$. On the other hand

$$A^{\theta}(A^{1-\theta}\phi_n + \psi_n) \to \nu.$$

From the closedness of A^θ we have that $A^{1-\theta}\phi + \psi \in D(A^\theta)$ and that

$$A^{\theta}(A^{1-\theta}\phi + \psi) = \nu.$$

Hence $\begin{bmatrix}\phi\\ \psi\end{bmatrix} \in D(\mathcal{A}_{(\theta)})$ and $\mathcal{A}_{(\theta)}\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix}\xi\\ \nu\end{bmatrix}$.

2) Follows immediately from the fact that

$$\mathcal{A}_{(\theta)}^{-1} = \begin{bmatrix}2\eta A^{\theta-1} & A^{-1}\\ -I & 0\end{bmatrix}.$$

3) Follows immediately from 2). ∎

As mentioned before, the proof of analyticity for $\mathcal{A}_{(\frac{1}{2})}$ and approximate characterization of the fractional power spaces $D(\mathcal{A}^{\alpha}_{(\frac{1}{2})})$, $\alpha \in [0,1]$ are given in [7] for general Banach spaces $X^{\frac{1}{2}} \times X$. Unfortunately the proofs of [7] cannot be extended to the case $\theta \in (\frac{1}{2}, 1]$ nor give an exact characterization of the fractional power spaces for $\theta = \frac{1}{2}$. Next we prove the analyticity and give an exact characterization of the domains of fractional powers $D(\mathcal{A}_{(\theta)})$ for $\theta \in [\frac{1}{2}, 1]$ in general Banach spaces $X^{\frac{1}{2}} \times X$.

We prove that $\mathcal{A}_{(\theta)}$, $\theta \in [\frac{1}{2}, 1]$, is sectorial. To that end we will follow the ideas of some unpublished notes by A. Rodriguez-Bernal [24] and Lemma 5.1 of [3].

For $\theta \in (\mathbf{\frac{1}{2}}, \mathbf{1}]$, let $\mathcal{B}_\theta : D(\mathcal{A}_{(\theta)}) \subset Y^0 \to Y^0$ be the operator defined by

$$D(\mathcal{B}_\theta) := D(\mathcal{A}_{(\theta)}) \text{ and } \mathcal{B}_\theta := \mathcal{A}_{(\theta)} + \begin{pmatrix}0 & 0\\ 0 & \frac{1}{2\eta}A^{1-\theta}\end{pmatrix}.$$

If

$$P_\theta = \begin{pmatrix} I & 0\\ \frac{1}{2\eta}A^{1-\theta} & I\end{pmatrix}, \quad P_\theta^{-1} = \begin{pmatrix} I & 0\\ -\frac{1}{2\eta}A^{1-\theta} & I\end{pmatrix}, \quad \mathcal{D}_\theta = \begin{pmatrix}\frac{1}{2\eta}A^{1-\theta} & I\\ 0 & 2\eta A^\theta\end{pmatrix},$$

then $P_\theta : D(\mathcal{A}_{(\theta)}) \to X^{\frac{3}{2}-\theta} \times X^\theta = D(\mathcal{D}_\theta)$, $P_\theta \mathcal{B}_\theta = \mathcal{D}_\theta P_\theta$ and

$$P_\theta : X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$$

are isomorphisms. The operator

$$\widetilde{\mathcal{D}}_\theta := \begin{pmatrix}\frac{1}{2\eta}A^{1-\theta} & 0\\ 0 & 2\eta A^\theta\end{pmatrix} : X^{\frac{3}{2}-\theta} \times X^\theta \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$$

is sectorial. Since

$$\left(\mathcal{D}_\theta - \widetilde{\mathcal{D}}_\theta\right)\widetilde{\mathcal{D}}_\theta^{-\gamma} : X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$$

is a bounded linear operator for $1 > \gamma > \frac{1}{2\theta}$, it follows from [15] page 177 that

$$\mathcal{D}_\theta : X^{\frac{3}{2}-\theta} \times X^\theta \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$$

is sectorial and its fractional power spaces coincide with the fractional power spaces for the operator $\widetilde{\mathcal{D}}_\theta$ with equivalent norms and therefore are given by

$$D(\mathcal{D}_\theta^\alpha) = X^{\frac{1}{2}+(1-\theta)\alpha} \times X^{\theta\alpha}.$$

For $\theta = \mathbf{\frac{1}{2}}$, $\eta > 0$, $a_\eta = \eta + \sqrt{\eta^2 - 1}$ and $\bar{a}_\eta = \eta - \sqrt{\eta^2 - 1}$ we define $\mathcal{D}_{\frac{1}{2}} : D(\mathcal{D}_{\frac{1}{2}}) := X^{\frac{1}{2}} \times X^{\frac{1}{2}} \subset X \times X \to X \times X$ by

$$\mathcal{D}_{\frac{1}{2}} = \begin{bmatrix} a_\eta A^{\frac{1}{2}} & 0 \\ 0 & \bar{a}_\eta A^{\frac{1}{2}} \end{bmatrix}.$$

If

$$P_{\frac{1}{2}} = \begin{bmatrix} \bar{a}_\eta A^{\frac{1}{2}} & I \\ a_\eta A^{\frac{1}{2}} & I \end{bmatrix}, \qquad P_{\frac{1}{2}}^{-1} = \frac{1}{\bar{a}_\eta - a_\eta} \begin{bmatrix} A^{-\frac{1}{2}} & -A^{-\frac{1}{2}} \\ -a_\eta & \bar{a}_\eta \end{bmatrix},$$

then $P_{\frac{1}{2}}\mathcal{A}_{(\frac{1}{2})} = \mathcal{D}_{\frac{1}{2}}P_{\frac{1}{2}}$ and $P_{\frac{1}{2}} : X^{\frac{1}{2}} \times X \to X \times X$ is an isomorphism.

It follows from (1) and from the results of [19] that $A^{\frac{1}{2}}$ is a sectorial operator which satisfies

$$\|(\lambda + A^{\frac{1}{2}})^{-1}\| \leq \frac{M}{1 + |\lambda|} \tag{11}$$

for any $\lambda \in \Sigma_{\frac{\psi}{2}} = \{\lambda \in \mathbb{C} : \frac{\psi}{2} \leq |\arg\lambda| \leq \pi \text{ with } \psi \in (0, \frac{\pi}{2})\}$. If $\frac{\pi}{2} > \frac{\psi}{2} + \arg a_\eta$ then $a_\eta A^{\frac{1}{2}}$ is a sectorial operator and the fractional power spaces associated to it coincide with the fractional power spaces associated to $A^{\frac{1}{2}}$ with equivalent norms. In particular $D(\mathcal{D}^\alpha_{\frac{1}{2}}) = X^{\frac{\alpha}{2}} \times X^{\frac{\alpha}{2}}$.

Remark 2. 2. When $\eta \geq 1$ both a_η and $\bar{a}_\eta$ these numbers are positive. Consequently, $\arg a_\eta = 0$ and the condition $\frac{\pi}{2} > \frac{\psi}{2} + \arg a_\eta$ is automatically satisfied.

We extend the definition of $\mathcal{B}_\theta$ to the case $\theta = \frac{1}{2}$ setting $\mathcal{B}_{\frac{1}{2}} := \mathcal{A}_{(\frac{1}{2})}$.

The following result is strongly based on [3, Lemma 5.1].

LEMMA 2.1. *If $\mathcal{B}_\theta$, P_θ and $\mathcal{D}_\theta$ are as above then,*

1) $\mathcal{B}_\theta$ *and* $\mathcal{D}_\theta$ *have the same spectrum,*
2) $\mathcal{B}_\theta$ *is sectorial,*
3) $P_\theta e^{-\mathcal{B}_\theta t} = e^{-\mathcal{D}_\theta t} P_\theta$ *for all* $t \geq 0$,
4) $P_\theta : D(\mathcal{B}_\theta^\alpha) \to D(\mathcal{D}_\theta^\alpha)$ *is an isomorphism, and*
5) *For each* $\alpha \in [0,1]$ *we have that*

$$D(\mathcal{B}_\theta^\alpha) = \left\{ \begin{bmatrix} \xi \\ \nu \end{bmatrix} : \xi \in X^{\frac{1}{2}+(1-\theta)\alpha} \text{ and } A^{1-\theta}\xi + 2\eta\nu \in X^{\theta\alpha} \right\}. \tag{12}$$

In particular, for $\alpha \in [0, \frac{1}{2}]$,

$$D(\mathcal{B}_\theta^\alpha) = X^{\frac{1}{2}+(1-\theta)\alpha} \times X^{\theta\alpha}. \tag{13}$$

Proof: 1) Follows from the equality $(\lambda - \mathcal{B}_\theta)^{-1} = P_\theta^{-1}(\lambda - \mathcal{D}_\theta)^{-1} P_\theta$.

2) Let $\psi \in (0, \frac{\pi}{2})$ and $\Sigma_\psi = \{\lambda \in \mathbb{C} : \psi \leq |\arg(\lambda + a)| \leq \pi\}$ such that

$$\|(\lambda + \mathcal{D}_\theta)^{-1}\|_{L(X^{\frac{1}{2}} \times X)} \leq \frac{K}{|\lambda + a|}.$$

The result now follows from

$$\begin{aligned} \|(\lambda + \mathcal{B}_\theta)^{-1}\|_{L(X^{\frac{1}{2}} \times X)} &= \|P_\theta^{-1}\|_{L(X^{\frac{1}{2}} \times X)} \|(\lambda + \mathcal{D}_\theta)^{-1}\|_{L(X^{\frac{1}{2}} \times X)} \|P_\theta\|_{L(X^{\frac{1}{2}} \times X)} \\ &\leq \frac{M}{|\lambda + a|}. \end{aligned}$$

3) Follows from the fact that $P_\theta(\lambda + \mathcal{B}_\theta)^{-1} = (\lambda + \mathcal{D}_\theta)^{-1} P_\theta$ and from the formulas

$$e^{-\mathcal{B}_\theta t} = \int_\Gamma e^{\lambda t}(\lambda + \mathcal{B}_\theta)^{-1} d\lambda, \qquad e^{-\mathcal{D}_\theta t} = \int_\Gamma e^{\lambda t}(\lambda + \mathcal{D}_\theta)^{-1} d\lambda.$$

4) It follows from the formulas

$$\mathcal{B}_\theta^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\mathcal{B}_\theta t} dt, \qquad \mathcal{D}_\theta^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\mathcal{D}_\theta t} dt$$

and from 3. that $P_\theta \mathcal{B}_\theta^{-\alpha} = \mathcal{D}_\theta^{-\alpha} P_\theta$. Since $D(\mathcal{B}_\theta^\alpha) = R(\mathcal{B}_\theta^{-\alpha})$, $D(\mathcal{D}_\theta^\alpha) = R(\mathcal{D}_\theta^{-\alpha})$ and $P_\theta(X^{\frac{1}{2}} \times X) = X^{\frac{1}{2}} \times X$, we conclude that $P_\theta(D(\mathcal{B}_\theta^\alpha)) = D(\mathcal{D}_\theta^\alpha)$. Finally, to prove that $P_\theta : D(\mathcal{B}_\theta^\alpha) \to D(\mathcal{D}_\theta^\alpha)$ is bounded with bounded inverse we note that

$$\left\| P_\theta \mathcal{B}_\theta^\alpha \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\|_{X^{\frac{1}{2}} \times X} = \left\| D_\theta^\alpha P_\theta \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\|_{X^{\frac{1}{2}} \times X} = \left\| P_\theta \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\|_{D(\mathcal{D}_\theta^\alpha)}.$$

Using the fact that $P_\theta : X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$ is an isomorphism we obtain the result.

5) To characterize the fractional power spaces associated to $\mathcal{B}_\theta$ simply noting that

$$P_\theta(D(\mathcal{B}_\theta^\alpha)) = D(\mathcal{D}_\theta^\alpha) = D(\tilde{\mathcal{D}}_\theta^\alpha) = X^{\frac{1}{2}+(1-\theta)\alpha} \times X^{\theta\alpha}.$$

Hence, for $\theta \in (\frac{1}{2}, 1]$, $\begin{bmatrix}\xi\\ \nu\end{bmatrix} \in D(\mathcal{B}_\theta^\alpha)$ if and only if there exists $\begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in X^{\frac{1}{2}+(1-\theta)\alpha} \times X^{\theta\alpha}$ such that

$$P_\theta^{-1}\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix}\varphi\\ -\frac{1}{2\eta}A^{1-\theta}\varphi + \psi\end{bmatrix} = \begin{bmatrix}\xi\\ \nu\end{bmatrix},$$

which is equivalent to say that $\xi \in X^{\frac{1}{2}+(1-\theta)\alpha}$ and $\frac{1}{2\eta}A^{1-\theta}\xi + \nu \in X^{\theta\alpha}$. Thus we have (12) and it is easy to see that the spaces given by (12) coincide with the spaces given by (13) for $\alpha \in [0, \frac{1}{2}]$.

For $\theta = \frac{1}{2}$ and $\frac{\pi}{2} > \frac{\psi}{2} + \arg a_\eta$ it is easy to see that $\mathcal{B}_{\frac{1}{2}}$ is sectorial and that

$$D(\mathcal{B}^\alpha_{(\frac{1}{2})}) = X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}} \quad \text{for all} \quad \alpha \in [0, 1]. \tag{14}$$

The proof is complete. ▮

Next we prove that $\mathcal{A}_{(\theta)}$ is a sectorial operator and characterize the fractional power spaces associated to it.

THEOREM 2.1. *For each $\theta \in [\frac{1}{2}, 1]$, the operator $\mathcal{A}_{(\theta)}$ is sectorial in $X^{\frac{1}{2}} \times X$ and consequently $\{e^{-\mathcal{A}_{(\theta)}t},\ t \geq 0\}$ is analytic semigroup. The fractional power spaces associated to $\mathcal{A}_{(\theta)}$ coincide with the fractional power spaces associated to $\mathcal{B}_\theta$ with equivalent norms. Furthermore, if A has compact resolvent and $\theta \in [\frac{1}{2}, 1)$, then $\mathcal{A}_{(\theta)}$ has compact resolvent.*

Proof: Note that, for $\theta \in (\frac{1}{2}, 1]$,

$$\begin{aligned}\left\|\begin{pmatrix}0 & 0\\ 0 & \frac{1}{2\eta}A^{1-\theta}\end{pmatrix}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right\|_{X^{\frac{1}{2}}\times X} &= \frac{1}{2\eta}\|A^{1-\theta}\psi\|_X \leq C\|A^{\frac{1}{2}}\psi\|_X^{2(1-\theta)}\|\psi\|_X^{2\theta-1}\\ &\leq \tilde{C}\left\|\mathcal{B}_{(\theta)}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right\|^{2(1-\theta)}_{X^{\frac{1}{2}}\times X}\left\|\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right\|^{2\theta-1}_{X^{\frac{1}{2}}\times X}.\end{aligned}$$

From [15] page 177 we notify that

$$\begin{pmatrix}0 & 0\\ 0 & \frac{1}{2\eta}A^{1-\theta}\end{pmatrix}\mathcal{B}_{(\theta)}^{-\beta}$$

is bounded in $X^{\frac{1}{2}} \times X$ for any[1] $1 \geq \beta > 2(1-\theta)$.

[1]The restriction $1 \geq \beta > 2(1-\theta)$ excludes the case $\theta = \frac{1}{2}$ and the sectoriality of $\mathcal{A}_{(\theta)}$ as well as the characterization of its fractional powers spaces, considered above, are only valid for $\theta \in (\frac{1}{2}, 1]$. Fortunately

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Thus, for $\theta \in (\frac{1}{2}, 1]$, $\mathcal{A}_{(\theta)}$ is sectorial (see [17, Corollary 1.4.5]) and the fractional power spaces associated to it are given by (12) with equivalent norms (see [17, Theorem 1.4.8]). The case $\theta = \frac{1}{2}$ is proved in Lemma 2.1. ▌

Let $Y^0 = X^{\frac{1}{2}} \times X$ and recall (see [1]) that the extrapolation space $Y_{(\theta)_{-1}}$ of Y^0 generated by $\mathcal{A}_{(\theta)}$ is the completion of the normed space $(Y^0, \|\mathcal{A}_{(\theta)}^{-1} \cdot \|_{Y^0})$. Since $\mathcal{A}_{(\theta)}$ is a sectorial operator in Y^0, then its closed extension to $Y_{(\theta)_{-1}}$ (denoted by the same symbol) is a sectorial operator in $Y_{(\theta)_{-1}}$ with $D(\mathcal{A}_{(\theta)_{-1}}) = Y^1_{(\theta)_{-1}} = Y^0$.

The following result (adapted from [7]) provides a characterization of the extrapolation spaces $Y^{\alpha}_{(\theta)_{-1}}$ for $\alpha \in [0, \frac{1}{2}]$ in terms of the fractional power spaces X^{α} associated to A.

Lemma 2.2. *The following holds*

$$Y^{\alpha}_{(\theta)_{-1}} \supset X^{\frac{1}{2}-(1-\alpha)(1-\theta)} \times X^{-\frac{1}{2}+\alpha(1-\theta)},\ \alpha \in [0, \frac{1}{2}]. \tag{15}$$

Proof: Note that if $a_0 \in \rho(-\mathcal{A}_{(\theta)})$, then $\|\mathcal{A}_{(\theta)}^{-1}(\cdot)\|_Y$ and $\|(\mathcal{A}_{(\theta)} + a_0 I)^{-1}(\cdot)\|_Y$ give equivalent norms in Y. Without loss of generality we may assume that $\operatorname{Re}\sigma(\mathcal{A}_{(\theta)}) > 0$. Otherwise, as in [17, p. 29], we replace $\mathcal{A}_{(\theta)}$ by $\mathcal{A}_{(\theta)} + a_0 I$ with a_0 sufficiently large and the proof can be carried out in the same manner.

Proceeding as in [4] we have that

$$(\mathcal{A}_{(\theta)})^{-1+|\beta|} : Y_{(\theta)_{-1}} \overset{\substack{\text{isometric}\\ \text{isomorphism}}}{\longrightarrow} Y_{(\theta)_{-|\beta|}},\ -1 \leq \beta \leq 0. \tag{16}$$

In particular,

$$Y_{(\theta)_{-(1-\alpha)}} = Y^{\alpha}_{(\theta)_{-1}},\ 0 \leq \alpha \leq 1$$

(see [1, p. 267, Corollary 1.3.9]). From (16) and [1, p. 266, Theorem 1.3.8] we then obtain

$$\| \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y^{\alpha}_{(\theta)_{-1}}} = \| \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y_{(\theta)_{-(1-\alpha)}}} = \|\mathcal{A}_{(\theta)}^{-(1-\alpha)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y^0} = \|\mathcal{A}^{\alpha}_{(\theta)} \mathcal{A}^{-1}_{(\theta)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y^0} = \|\mathcal{A}^{-1}_{(\theta)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y^{\alpha}_{(\theta)}}.$$

Using characterization of spaces $Y^{\alpha}_{(\theta)}$ given in (13),

$$Y^{\alpha}_{(\theta)} = X^{\frac{1}{2}+\alpha(1-\theta)} \times X^{\theta\alpha},$$

results for the case $\theta = \frac{1}{2}$ can be obtained without using a perturbation argument with a simple change of variables.

we next have

$$\| \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y^{\alpha}_{(\theta)_{-1}}} = \|\mathcal{A}^{-1}_{(\theta)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{X^{\frac{1}{2}+\alpha(1-\theta)} \times X^{\theta\alpha}} = \| \begin{bmatrix} \eta A^{-(1-\theta)}\varphi + A^{-1}\psi \\ -\varphi \end{bmatrix} \|_{X^{\frac{1}{2}+\alpha(1-\theta)} \times X^{\theta\alpha}}$$

$$= \|A^{\frac{1}{2}+\alpha(1-\theta)}(\eta A^{-(1-\theta)}\varphi + A^{-1}\psi)\|_{X^0} + \|A^{\theta\alpha}\varphi\|_{X^0}$$

$$= \|\eta A^{\frac{1}{2}-(1-\alpha)(1-\theta)}\varphi + A^{-\frac{1}{2}+\alpha(1-\theta)}\psi\|_{X^0} + \|A^{\theta\alpha}\varphi\|_{X^0}, \ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in Y^0, \ \alpha \in [0, \frac{1}{2}], \ \theta \in [\frac{1}{2}, 1].$$

Consequently, since $\frac{1}{2} - (1-\alpha)(1-\theta) \geq \theta\alpha$ for $\alpha \in [0, \frac{1}{2}]$, $\theta \in [\frac{1}{2}, 1]$, we obtain that

$$\| \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{Y^{\alpha}_{(\theta)_{-1}}} \leq \max\{c+\eta, 1\} \left(\|A^{\frac{1}{2}-(1-\alpha)(1-\theta)}\varphi\|_{X^0} + \|A^{-\frac{1}{2}+\alpha(1-\theta)}\psi\|_{X^0} \right)$$

$$= \max\{c+\eta, 1\} \| \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \|_{X^{\frac{1}{2}-(1-\alpha)(1-\theta)} \times X^{-\frac{1}{2}+\alpha(1-\theta)}}, \ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in Y^0, \ \alpha \in [0, \tfrac{1}{2}], \ \theta \in [\tfrac{1}{2}, 1].$$

Since Y^0 is dense in $X^{\frac{1}{2}-(1-\alpha)(1-\theta)} \times X^{-\frac{1}{2}+\alpha(1-\theta)}$, the proof is complete. ▮

2.2. Abstract local existence result

In this subsection we establish local well posedness for the Cauchy problem (3). We would like to study this problem in the phase space $X^{\frac{1}{2}} \times X$. To pose the problem in the mentioned space we will need to consider the nonlinear term F as a map with values in the extrapolated space $Y_{(\theta)_{-1}}$ associated to $\mathcal{A}_{(\theta)}$ in Y^0. Whenever F maps Y^0 into $Y^{\alpha}_{(\theta)_{-1}}$ for certain $\alpha \in (0, 1)$, then local well posedness result is known (see [17]). If F does not possess this property we will use the concept of ε-regular maps and ε-regular solutions to (3) (see [2]).

Definition 2.1. F is an ε-regular map relative to the pair $(Y^0, Y_{(\theta)_{-1}})$ with constants $C > 0$, $\rho > 1$, $\varepsilon \in (0, \frac{1}{\rho})$, and $1 > \gamma \geq \rho\varepsilon$ if and only if

$$\|F(\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}) - F(\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix})\|_{Y^{\gamma}_{(\theta)_{-1}}} \leq C\| \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \|_{Y^{1+\varepsilon}_{(\theta)_{-1}}} \left(1 + \| \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \|^{\rho-1}_{Y^{1+\varepsilon}_{(\theta)_{-1}}} + \| \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \|^{\rho-1}_{Y^{1+\varepsilon}_{(\theta)_{-1}}} \right), \ \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \in Y^{1+\varepsilon}_{(\theta)_{-1}}. \tag{17}$$

If the above relation holds with $\varepsilon = 0$ and $\gamma \in (0, 1)$ then we say that F is subcritical relative to $(Y^0, Y_{(\theta)_{-1}})$.

Definition 2.2. Let $\varepsilon > 0$, $\tau > 0$, $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^0$ and $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} : [0, \tau] \to Y^0$. We say that $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix}$ is an ε-regular solution to (3) on $[0, \tau]$ if and only if

- $\left[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}\right] \in C([0,\tau], Y^{1}_{(\theta)_{-1}}) \cap C((0,\tau], Y^{1+\varepsilon}_{(\theta)_{-1}})$,
- $\left[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}\right]$ satisfies the Cauchy integral formula:

$$\left[\begin{smallmatrix} u(t) \\ v(t) \end{smallmatrix}\right] = e^{\mathcal{A}t} \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{\mathcal{A}(t-s)} F\left(\left[\begin{smallmatrix} u(s) \\ v(s) \end{smallmatrix}\right]\right) ds,\ t \in [0,\tau]. \tag{18}$$

Proposition 2.2. ([4, Proposition 6]) *If F is an ε-regular map relative to $(Y^0, Y_{(\theta)_{-1}})$, then for each $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^0$ there exists a unique ε-regular solution to* (3) *on certain interval $[0,\tau]$. When $F = \sum_{i=1}^n F_i$ and F_i are ε_i-regular maps with constants ρ_i, γ_i, C_i and*

$$\min\{\gamma_i(\varepsilon_i);\ 1 \le i \le n\} =: \tilde{\gamma} > \tilde{\varepsilon} := \max\{\varepsilon_i;\ 1 \le i \le n\}, \tag{19}$$

then there exists a unique $\tilde{\varepsilon}$-regular solution to (3)*. In addition, we have*[2]

$$\left[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}\right] \in C((0,\tau], Y^{1+\tilde{\gamma}}_{(\theta)_{-1}}) \cap C^1((0,\tau], Y^{1+\tilde{\gamma}^-}_{(\theta)_{-1}}).$$

3. THE CASE WHEN A IS A $2M$-TH ORDER ELLIPTIC OPERATOR IN $L^P(\Omega)$

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, L be $2m$-th order operator in Ω given by (7). Then, A_L given in (8) is sectorial in $X = L^p(\Omega)$, $p \in (1,\infty)$ (see [15]). Assume that $\operatorname{Re}\sigma(A_L) > 0$ and that A_L has bounded imaginary powers.

Remark 3. 1. Note that, if A_L is a maximal accretive operator in a Hilbert space, then it has bounded imaginary powers (see [20]). For conditions on $(L, \{B_j\}, \Omega)$ ensuring that A_L has this property in the L^p-setting we refer to [14].

Throughout this section we assume that (η, θ, A_L) is an admissible triple and denote by X^α, $\alpha \in \mathbb{R}$, the fractional power spaces associated to A_L. From the results in [27] we have that $X^\alpha = H^{2m\alpha}_{p,\{B_j\}}(\Omega)$, $\alpha \in [0,1]$; in particular, $X^{\frac{1}{2}} = H^m_{p,\{B_j\}}(\Omega)$. We denote $Y^0_p = H^m_{p,\{B_j\}}(\Omega) \times L^p(\Omega)$.

Lemma 3.1. *For $\alpha \in [0,\frac{1}{2}]$ and $\theta \in [\frac{1}{2},1]$ we have*

$$Y^\alpha_{(\theta)_{-1}} \supset X^{\frac{1}{2}-(1-\alpha)(1-\theta)} \times L^q(\Omega)\ \ for\ q \ge \frac{Np}{N+mp(1-2\alpha(1-\theta))},\ N > mp,\ p \ge 2, \tag{20}$$

[2]Throughout the paper s^- denotes any number smaller than $s \in \mathbb{R}$.

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and

$$Y^{1+\alpha}_{(\theta)_{-1}} = Y^{\alpha}_{(\theta)} = H^{m+2m\alpha(1-\theta)}_{p,\{B_j\}}(\Omega) \times H^{2m\alpha\theta}_{p,\{B_j\}}(\Omega). \tag{21}$$

Proof: Denote by p' the conjugate of $p \in (1,\infty)$. Note from [1] that

$$\left(X^{-\frac{1}{2}+\alpha(1-\theta)}\right)^* = (X^*)^{\frac{1}{2}-\alpha(1-\theta)} = H^{2m(\frac{1}{2}-\alpha(1-\theta))}_{p',\{B_j\}}(\Omega) \subset L^r(\Omega)$$

for $1 \le r \le \dfrac{Np}{N(p-1) - mp(1-2\alpha(1-\theta))}$, which completes the proof. ▮

3.1. Bootstrapping

In the particular situation of this section we employ a bootstrapping argument, similar to that used for parabolic equations, to obtain additional regularity of the solutions of (3).

THEOREM 3.1. *If F an ε_r-regular map with respect to the pair of spaces $(Y^0_r, Y_{(\theta)_{-1}})$ for every $r \ge p$ with $\gamma(\varepsilon_r) = \frac{1}{2}$, then ε_r-regular solution to* (3) *on* $[0,\tau]$ *satisfies for each* $r \in [p,\infty)$ *the condition*

$$\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \in C^1((0,\tau], H^{(m+m(1-\theta))^-}_{r,\{B_j\}}(\Omega) \times H^{(m\theta)^-}_{r,\{B_j\}}(\Omega)). \tag{22}$$

Consequently, we have

$$\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \in C^1((0,\tau], C^{(m+m(1-\theta))^-}(\bar\Omega) \times C^{(m\theta)^-}(\bar\Omega)).$$

If $\theta < \frac{2}{3}$ and F takes $C^{(m+m(1-\theta))^-}(\bar\Omega) \times C^{(m+m(1-\theta))^-}(\bar\Omega)$ into $X \times C^{\gamma}(\bar\Omega)$ for some $\gamma > 0$, then the solution to (9) *is in fact classical, that is*

$$\begin{bmatrix} u(\cdot) \\ L^{\theta} u_t(\cdot) \\ u_{tt}(\cdot) \end{bmatrix} \in C((0,\tau_{u_0,v_0}), C^{2m}(\bar\Omega) \times C(\bar\Omega) \times C(\bar\Omega)). \tag{23}$$

Furthermore, if $\frac{1}{2} > \rho\varepsilon_r$, then for $B \subset Y^0_p$ bounded the local solutions $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])$ to (3) *with given $\theta \in (\frac{1}{2},1)$, $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$ exist until certain $\tau = \tau(B,\theta) > 0$ and $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\,;\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$, $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t (t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\,;\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$ are bounded subsets of the product space $H^{(m+m(1-\theta))^-}_{r,\{B_j\}}(\Omega) \times H^{(m\theta)^-}_{r,\{B_j\}}(\Omega)$ for each $t \in (0,\tau)$, $\theta \in [\frac{1}{2},1)$ and $r \in [p,\infty)$.*

Proof: First let us consider regularizing properties of the solutions uniform with respect to the initial condition varying in bounded sets. Then condition (22) follows.

It is known (see [2, 17] and [7, Theorem 5]) that there is $\tau_0 = \tau_0(B,\theta) > 0$ such that the sets

$$B_\zeta = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \theta)\,;\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B,\ t \in [\zeta,\tau_0]\},$$
$$\dot B_\zeta = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t (t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \theta)\,; \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B,\ t \in [\zeta,\tau_0]\},$$

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are bounded in $H^{(m+m(1-\theta))^-}_{p,\{B_j\}}(\Omega)\times H^{(m\theta)^-}_{p,\{B_j\}}(\Omega)$ for each $\zeta\in(0,\tau_0)$. We next have that:

i) if $N>mp(1-\theta)$, then by the embedding properties these sets are also bounded in $H^m_{q_1{}^-,\{B_j\}}(\Omega)\times L^{q_1{}^-}(\Omega)$ with $q_1=\frac{Nq_0}{N-mq_0(1-\theta)}$ and $q_0:=p$. In addition, we have that $\frac{q_1}{q_0}=\frac{N}{N-mp(1-\theta)}=:\bar\mu>1$ and following the regularization argument with $Y^0_{q_1^-}:=H^m_{q_1^-,\{B_j\}}(\Omega)\times L^{q_1^-}(\Omega)$ we get boundedness of the sets B_ζ, $\dot B_\zeta$ in $H^{(m+m(1-\theta))^-}_{q_1^-,\{B_j\}}(\Omega)\times H^{(m\theta)^-}_{q_1^-,\{B_j\}}(\Omega)$.

If $N>mq_1(1-\theta)$, we return to the beginning of i) and the procedure can be repeated. Also, $\frac{q_2}{q_1}$ will remain bounded from below by $\bar\mu$.

What was said above proves the existence of a finite number k, such that after k repetitions we reach certain $q_k\ge 2$, for which $N\le mq_k(1-\theta)$ and the sets B_ζ, $\dot B_\zeta$ are bounded in $H^{(m+m(1-\theta))^-}_{q_k^-,\{B_j\}}(\Omega)\times H^{(m\theta)^-}_{q_k^-,\{B_j\}}(\Omega)$. Since finally obtained space is embedded in $H^m_{r,\{B_j\}}(\Omega)\times L^r(\Omega)$ for every $r\in[p,\infty)$, the proof is complete.

For $\theta<\frac{2}{3}$ we have $2m\theta<m+m(1-\theta)$. Rewriting (9) in the form $A_Lu=f(u)+g(u_t)-u_{tt}-A^\theta_Lu_t$ and using Schauder type estimate for elliptic equations, (23) follows. ▮

3.2. Local existence for particular nonlinearities

The following local existence results follows from Proposition 2.2 and Lemma 3.1.

Corollary 3.1. *For $p\ge 2$ assume that*
i) $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz continuous function if $mp>N$,
ii) f satisfies the condition

$$|f(s_1)-f(s_2)|\leqslant c|s_1-s_2|(1+|s_1|^{\rho-1}+|s_2|^{\rho-1}),\ \ s_1,s_2\in\mathbb{R}^1, \tag{24}$$

with certain $1<\rho\le\frac{N+mp}{N-mp}=:\bar\rho$ and $c>0$ if $mp<N$,
iii) condition (24) holds with arbitrarily large but fixed $\rho\in(1,\infty)$ if $mp=N$.

Then, the map $F\left(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\right):=\left[\begin{smallmatrix}0\\f^e(w_1)\end{smallmatrix}\right]$ takes Y^0_p into $Y^{\alpha^*_p}_{(\theta)_{-1}}$ and is Lipschitz continuous on bounded subsets of Y^0_p, where $\alpha^*_p=\frac{1}{2}$ in the case i) or iii) or when $\theta=1$, and $\alpha^*_p=\min\{\frac12,\frac{N+mp-\rho(N-mp)}{2mp(1-\theta)}\}$, $\rho<\bar\rho$, in the case ii).

Furthermore, there exists a unique mild solution $\left[\begin{smallmatrix}u(\cdot)\\v(\cdot)\end{smallmatrix}\right]\in C([0,\tau_{u_0,v_0}),Y^0_p)$ to (3) through $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y^0_p$ and for α^*_p as above we have that

$$\begin{aligned}&\left[\begin{smallmatrix}u(\cdot)\\v(\cdot)\end{smallmatrix}\right]\in C((0,\tau_{u_0,v_0}),H^{m+2m\alpha^*_p(1-\theta)}_{p,\{B_j\}}(\Omega)\times H^{2m\alpha^*_p\theta}_{p,\{B_j\}}(\Omega)),\\&\left[\begin{smallmatrix}u(\cdot)\\v(\cdot)\end{smallmatrix}\right]\in C^1((0,\tau_{u_0,v_0}),H^{(m+2m\alpha^*_p(1-\theta))^-}_{p,\{B_j\}}(\Omega)\times H^{(2m\alpha^*_p\theta)^-}_{p,\{B_j\}}(\Omega)).\end{aligned}\tag{25}$$

If ii) holds with $\rho = \bar{\rho}$, then

$$\begin{aligned}
\|F([\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}]) - F([\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}])\|_{Y^{\gamma(\varepsilon)}_{(\theta)_{-1}}} &\leq C\|f^e(u_1) - f^e(u_2)\|_{L^{\frac{Np}{N+mp(1-2\gamma(\varepsilon)(1-\theta))}}(\Omega)} \\
&\leq c\|u_1 - u_2\|_{H^{m+2m\varepsilon\theta}_{p,\{B_j\}}(\Omega)} \left(1 + \|u_1\|^{\bar{\rho}-1}_{H^{m+2m\varepsilon\theta}_{p,\{B_j\}}(\Omega)} + \|u_2\|^{\bar{\rho}-1}_{H^{m+2m\varepsilon\theta}_{p,\{B_j\}}(\Omega)}\right) \\
&= c\|[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}] - [\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}]\|_{Y^{1+\varepsilon}_{(\theta)_{-1}}} \left(1 + \|[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}]\|^{\bar{\rho}-1}_{Y^{1+\varepsilon}_{(\theta)_{-1}}} + \|[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}]\|^{\bar{\rho}-1}_{Y^{1+\varepsilon}_{(\theta)_{-1}}}\right),
\end{aligned} \tag{26}$$

where $\varepsilon \in [0, \frac{1}{2\bar{\rho}}]$, $\gamma(\varepsilon) = \bar{\rho}\varepsilon$ and $[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}], [\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}] \in H^{m+2m\varepsilon\theta}_{p,\{B_j\}}(\Omega) \times H^{2m\varepsilon\theta}_{p,\{B_j\}}(\Omega)$.

Therefore, for each $\varepsilon \in (0, \frac{1}{2\bar{\rho}}]$, the nonlinear map $F([\begin{smallmatrix} u \\ v \end{smallmatrix}]) := [\begin{smallmatrix} 0 \\ f^e(u) \end{smallmatrix}]$ is ε-regular relative to $(Y^0_p, Y_{(\theta)_{-1}})$ with constant $\gamma(\varepsilon)$ and there exists a unique ε-regular solution to (3) through $[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}] \in Y^0_p$ satisfying both relations in (25) with $\alpha^*_p = \gamma(\varepsilon) \in (0, \frac{1}{2}]$.

Proof: The result follows from Lemma 3.1, Sobolev embedding properties of spaces $H^s_p(\Omega)$ into Lebesgue spaces, Hölder inequality and Proposition 2.2. ∎

In a similar manner as in the case of Corollary 3.1 we obtain

Corollary 3.2. *For $p \geq 2$ assume that*

$$|g(s_1) - g(s_2)| \leq c|s_1 - s_2|(1 + |s_1|^{\nu-1} + |s_2|^{\nu-1}),\ \ s_1, s_2 \in \mathbb{R}, \tag{27}$$

where

$$1 < \nu \leq \frac{N + 2mp\theta}{N} =: \bar{\nu}. \tag{28}$$

Define $\varepsilon_\nu = \max\{0, \frac{N(\nu-1)-mp\theta}{2mp\theta\nu}\}$, take $\gamma(\varepsilon_\nu) = \frac{1}{2}$ and denote by g^e the Nemytskiĭ operator associated with $g : \mathbb{R} \to \mathbb{R}$. Then, we have

$$\begin{aligned}
\|F([\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}]) - F([\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}])\|_{Y^{\gamma(\varepsilon_\nu)}_{(\theta)_{-1}}} &\leq C\|g^e(v_1) - g^e(v_2)\|_{L^{\frac{Np}{N+mp(1-2\gamma(\varepsilon_\nu)(1-\theta))}}(\Omega)} \\
&= \|g^e(v_1) - g^e(v_2)\|_{L^{\frac{Np}{N+mp\theta}}(\Omega)} \\
&\leq c\|v_1 - v_2\|_{H^{2m\varepsilon_\nu\theta}_{p,\{B_j\}}(\Omega)} \left(1 + \|v_1\|^{\nu-1}_{H^{2m\varepsilon_\nu\theta}_{p,\{B_j\}}(\Omega)} + \|v_2\|^{\nu-1}_{H^{2m\varepsilon_\nu\theta}_{p,\{B_j\}}(\Omega)}\right) \\
&= c\|[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}] - [\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}]\|_{Y^{1+\varepsilon_\nu}_{(\theta)_{-1}}} \left(1 + \|[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}]\|^{\nu-1}_{Y^{1+\varepsilon_\nu}_{(\theta)_{-1}}} + \|[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}]\|^{\nu-1}_{Y^{1+\varepsilon_\nu}_{(\theta)_{-1}}}\right),
\end{aligned}$$

where $[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}], [\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}] \in H^{m+2m\varepsilon_\nu\theta}_{p,\{B_j\}}(\Omega) \times H^{2m\varepsilon_\nu\theta}_{p,\{B_j\}}(\Omega)$. Therefore the nonlinear map $F([\begin{smallmatrix} u \\ v \end{smallmatrix}]) := [\begin{smallmatrix} 0 \\ g^e(v) \end{smallmatrix}]$ is ε_ν-regular relative to $(Y^0_p, Y_{(\theta)_{-1}})$ with $\gamma(\varepsilon_\nu) = \frac{1}{2}$.

Furthermore, there exists a unique ε_ν-regular solution $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \in C([0, \tau_{u_0,v_0}), Y^0_p)$ to (3) through $[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}] \in Y^0_p$ and (25) holds for $\alpha^*_p = \frac{1}{2}$.

Corollary 3.3. *Suppose that the assumptions of Corollaries 3.1, 3.2 hold and let* $F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = F_1\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) + F_2\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right)$ *where* $F_1\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) := \left[\begin{smallmatrix} 0 \\ f^e(u) \end{smallmatrix}\right]$, $F_2\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) := \left[\begin{smallmatrix} 0 \\ g^e(v) \end{smallmatrix}\right]$. *Then Proposition 2.2 applies with* $\tilde{\gamma} = \frac{1}{2}$ *and certain* $\tilde{\varepsilon} = \max\{\frac{1}{2\tilde{\rho}}, \frac{1}{\varepsilon_\nu}\}$. *Thus, there exists a unique* ε_ν*-regular solution* $\left[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}\right] \in C([0,\tau_{u_0,v_0}), Y^0_p)$ *to* (3) *through* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^0_p$ *and* (25) *holds for* $\alpha^*_p = \frac{1}{2}$.

The approach developed above works as well for nonlinearities depending on the space derivatives of u. The corollary below gives a sample result of that kind.

Corollary 3.4. *For* $2p > N$ *assume that* $h \in C^{2+Lipschitz}(\mathbb{R},\mathbb{R})$. *For* $2p \leq N$ *assume that*

$$|h''(s_1) - h''(s_2)| \leqslant c|s_1 - s_2|(1 + |s_1|^{\rho-3} + |s_2|^{\rho-3}),\ s_1, s_2 \in \mathbb{R}^1, \tag{29}$$

with

i) $\rho \in (1,\infty)$ *if* $N = 2p$, *or*
ii) $\rho \in (1, \frac{N}{N-2p}]$ *if* $3p \geq N > 2p$, *or*
iii) $\rho \in (1, \frac{N}{N-2p}]$ *if* $N > 2p$ *and* h *is a polynomial.*

Then $F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) := \left[\begin{smallmatrix} 0 \\ h''^e(u)|\nabla u|^2 + h'^e(u)\Delta u \end{smallmatrix}\right]$ *is* $\frac{1}{2\rho}$*-regular with respect to the pair of spaces* $(Y^0_p, Y_{(\frac{1}{2})_{-1}})$ *with* $\gamma = \frac{1}{2}$. *Therefore, there exists a unique* $\frac{1}{2\rho}$*-regular solution* $\left[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}\right] \in C([0,\tau_{u_0,v_0}), Y^0_p)$ *to* (3) *through* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^0_p$ *and* (25) *holds for* $\alpha^*_p = \frac{1}{2}$, $m = 2$, $\theta = \frac{1}{2}$.

Proof: In the cases ii) and iii), using Hölder inequality and Sobolev type embedding, we have that

$$\begin{aligned}
&\|(h''(u_1) - h''(u_2))|\nabla u_1|^2\|_{L^q(\Omega)} \leq C\|u_1 - u_2\|_{H^s_p(\Omega)}(1 + \|u_1\|^{\rho-3}_{H^s_p(\Omega)} + \|u_2\|^{\rho-3}_{H^s_p(\Omega)})\|u_1\|^2_{H^s_p(\Omega)},\\
&\|h''(u_2)(|\nabla u_1|^2 - |\nabla u_2|^2)\|_{L^q(\Omega)} \leq C\|u_1 - u_2\|_{H^s_p(\Omega)}(1 + \|u_2\|^{\rho-2}_{H^s_p(\Omega)})\|u_1 + u_2\|_{H^s_p(\Omega)},\\
&\|(h'(u_1) - h'(u_2))\Delta u_1\|_{L^q(\Omega)} \leq C\|u_1 - u_2\|_{H^s_p(\Omega)}(1 + \|u_1\|^{\rho-2}_{H^s_p(\Omega)} + \|u_2\|^{\rho-2}_{H^s_p(\Omega)})\|u_1\|_{H^s_p(\Omega)},\\
&\|h'(u_2)\Delta(u_1 - u_2)\|_{L^q(\Omega)} \leq C\|u_1 - u_2\|_{H^s_p(\Omega)}(1 + \|u_2\|^{\rho-1}_{H^s_p(\Omega)}),
\end{aligned}$$

where $s = 2 + \frac{1}{\rho}$, $q = \frac{Np}{N+p}$ is given by (20) with $\theta = \alpha = \frac{1}{2}$ and $m = 2$. Hence

$$\|F\left(\left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right]\right) - F\left(\left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right]\right)\|_{Y^{\frac{1}{2}}_{(\frac{1}{2})_{-1}}} \leq C\|\left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right]\|_{Y^{1+\frac{1}{2\rho}}_{(\frac{1}{2})_{-1}}}\left(1 + \|\left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right]\|^{\rho-1}_{Y^{1+\frac{1}{2\rho}}_{(\frac{1}{2})_{-1}}} + \|\left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right]\|^{\rho-1}_{Y^{1+\frac{1}{2\rho}}_{(\frac{1}{2})_{-1}}}\right)$$

Consequently the assumptions of Proposition 2.2 are satisfied with $n = 1$, $\gamma = \frac{1}{2}$ and $\epsilon = \frac{1}{2\rho}$ and the result follows easily. ∎

4. EXAMPLES

In the following examples we illustrate the applicability of our results showing that they considerably generalize the results about local well posedness appearing in the literature.

Example 4.1. *(Generalized wave equation of quantum mechanics)* The Dirichlet problem for equation

$$u_{tt} - 2\eta\Delta u_t - \Delta u = f(u) + g(u_t) \tag{30}$$

in bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, has been studied in [16] within the Hilbert setting with the growth restrictions

$$\begin{aligned} &|f(s)| \leq c(1+|s|^\rho) \text{ with } \rho < \frac{N+2}{N-2} \text{ for } N \geq 3, \\ &|g(s)| \leq c(1+|s|^\nu) \text{ with } \nu < \frac{N+4}{N}. \end{aligned} \tag{31}$$

In [4, 5] also the critical growth $\bar{\rho} = \frac{N+2}{N-2}$ was allowed. In the present paper we extend these studies to the Banach spaces and operators in non-divergence form. We thus consider the problem

$$\begin{cases} u_{tt} + 2\eta L u_t + Lu = f(u) + g(u_t), \ t > 0, \ x \in \Omega, \\ B_0 u = \ldots = B_{m-1} u = 0, \ t > 0, \ x \in \partial\Omega, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ x \in \Omega. \end{cases} \tag{32}$$

where L is as in (7). If A_L is the operator given by (8) the above problem can be written in the form (9). We note that $(\eta, 1, A_L)$ is an admissible triple for each $\eta > 0$.

Based on Corollary 3.3 we allow that the nonlinearities f and g satisfy the following growth restrictions

$$\begin{aligned} &|f(s)| \leq c(1+|s|^\rho) \text{ with } \rho \leq \frac{N+mp}{N-mp} \text{ for } N > mp, \ p \in [2,\infty), \\ &|g(s)| \leq c(1+|s|^\nu) \text{ with } \nu \leq \frac{N+2mp}{N}, \end{aligned} \tag{33}$$

which in the Hilbert setting coincide with (31) in the subcritical case and with [4, 5] in the critical case. From Corollary 3.3 we infer that the solution of (32) satisfies the condition

$$\begin{bmatrix} u(\cdot) \\ u_t(\cdot) \\ u_{tt}(\cdot) \end{bmatrix} \in C((0,\tau_{u_0,v_0}), H^m_{p,\{B_j\}}(\Omega) \times H^{m^-}_{p,\{B_j\}}(\Omega) \times H^{m^-}_{p,\{B_j\}}(\Omega)). \tag{34}$$

It is clear that the sine-Gordon equation and the perturbed wave equation of quantum mechanics can be considered as particular cases of (32).

Example 4.2. Here we consider a particular equation of the form (9)

$$u_{tt} + (-\Delta_D)^3 u_t + (-\Delta_D)^4 u = f(u) + g(u_t), \ t > 0, \ x \in \Omega, \tag{35}$$

where Δ_D is Laplacian with zero Dirichlet boundary condition in $L^p(\Omega)$. We refer to [11] for the proof that $L = \Delta^4$ with Dirichlet boundary conditions defines a uniformly strongly elliptic operator. For the proof that $[(-\Delta_D)^4]^{\frac{3}{4}} = (-\Delta_D)^3$ see [21]. If A_L is the operator given by (8), the above problem can be written in the form (9). We note that $(\eta, \frac{3}{4}, A_L)$ is an admissible triple for each $\eta > 0$.

The local well posedness of (35) follows from Corollary 3.3 if f and g satisfy the following growth conditions

$$\begin{aligned} &|f(s)| \le c(1 + |s|^\rho) \ \text{ with } \ \rho \le \frac{N+4p}{N-4p} \ \text{ for } \ N > 4p, \ p \in [2, \infty), \\ &|g(s)| \le c(1 + |s|^\nu) \ \text{ with } \ \nu \le \frac{N+6p}{N}. \end{aligned} \tag{36}$$

Furthermore, as a consequence of Theorem 3.1, we have

$$\begin{bmatrix} u(\cdot) \\ u_t(\cdot) \\ u_{tt}(\cdot) \end{bmatrix} \in C((0, \tau_{u_0,v_0}), H^{\frac{5}{2}}_{r,\{B_j\}}(\Omega) \times H^{\frac{5}{2}^-}_{r,\{B_j\}}(\Omega) \times H^{\frac{3}{2}^-}_{r,\{B_j\}}(\Omega)), \ r \ge p. \tag{37}$$

Example 4.3. In a bounded domain $\Omega \subset R^N$ with $C^{4+\mu_0}$ boundary consider a perturb viscous Cahn-Hilliard equation:

$$\begin{cases} u_{tt} + u_t + \Delta^2 u - \delta \Delta u_t = \Delta(h(u)), \ x \in \Omega, \ t > 0, \\ u = \Delta u = 0, \ x \in \partial\Omega, \ t > 0, \\ u(0, x) = u_0(x), \ u_t(0, x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{38}$$

(see [28] with $\varepsilon = 1$) where $\delta > 0$ is a constant and $h \in C^3(\mathbb{R}, \mathbb{R})$. With $A = \Delta_D^2$ in $L^p(\Omega)$, $p \ge 2$, considered on the domain $D(\Delta_D^2) = \{\phi \in H^4_p(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\}$, $\eta = \frac{\delta}{2}$ and $\theta = \frac{1}{2}$ this problem falls into the abstract formulation (9) with $f(u) = h''^e(u)|\nabla u|^2 + h'^e(u)\Delta u$ and $g(u_t) = -u_t$. Since $A^{\frac{1}{2}} = -\Delta_D$ we can choose the angle ψ for the sector $\Sigma_{\frac{\psi}{2}}$ in (11) as small as needed (see [25]) and $(\eta, \frac{1}{2}, A)$ is an admissible triple for any $\eta > 0$.

There is an extensive literature devoted to the classical parabolic Cahn-Hilliard equation (see [12, 22, 26] and references therein). Recently, problem (38) has been studied in [28]. In what follows we describe the local well posedness for (38) in the L^p-setting. We show that at most some (rather general) growth conditions are needed for local existence; in particular for dimension $N = 3$ and for $p = 2$ we require that the nonlinearity is C^3 with no additional restrictions.

If the conditions of Corollary 3.4 hold, then Theorem 3.1 applies and the resulting $\frac{1}{2\rho}$-regular solution $\left[\begin{smallmatrix} u \\ u_t \end{smallmatrix}\right] \in C([0, \tau_{u_0,v_0}), H^2_{p,\{B_j\}}(\Omega) \times L^p(\Omega))$ of the abstract counterpart of (38) satisfies

$$\begin{bmatrix} u(\cdot) \\ u_t(\cdot) \\ u_{tt}(\cdot) \end{bmatrix} \in C((0, \tau_{u_0,v_0}), H^3_{r,\{B_j\}}(\Omega) \times H^{3^-}_{r,\{B_j\}}(\Omega) \times H^{1^-}_{r,\{B_j\}}(\Omega)), \ r \ge p. \tag{39}$$

Therefore, we have that $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \in C^1((0,\tau], C^{2+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega}))$, $\mu \in (0,1)$. Using Schauder type estimate it is not difficult to see that $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \in C((0,\tau], C^{4+\mu}(\bar{\Omega}) \times C^{2+\mu}(\bar{\Omega}))$, $\mu \in (0,\mu_0)$. This proves that $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix}$ is a classical solution.

The cases i) and ii) are of much simpler nature and checking that Proposition 2.2, Theorem 3.1 and Corollary 3.4 apply is omitted.

Remark 4. 1. If one wishes to consider the classical parabolic Cahn-Hilliard problem the restrictions on the nonlinear term will be exactly the same as the ones obtained in Corollary 3.4 for (38). That is a direct consequence of the results in [2].

REFERENCES

1. H. Amann, *Linear and Quasilinear Parabolic Problems*, Birkhäuser, Basel, 1995.
2. J. M. Arietta, A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Trans. Amer. Math. Soc.* **352** (2000), 285–310.
3. J. Arrieta, A. Rodriguez-Bernal, J. Cholewa, T. Dlotko, Linear parabolic equations in locally uniform spaces, *Math. Models Methods Appl. Sci.* **14** (2004), 253–294.
4. A. N. Carvalho, J. W. Cholewa, Local well posedness for strongly damped wave equations with critical nonlinearities, *Bull. Austral. Math. Soc.* **66** (2002), 443–463.
5. A. N. Carvalho, J. W. Cholewa, Attractors for strongly damped wave equations with critical nonlinearities, *Pacific J. Math.* **207** (2002), 287–310.
6. A. N. Carvalho, J. W. Cholewa, Continuation and asymptotics of solutions to semilinear parabolic equations with critical nonlinearities, *J. Math. Anal. Appl.* **310** (2005), 557–578.
7. A. N. Carvalho, J. W. Cholewa, Strongly damped wave equations in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, submitted.
8. S. Chen, R. Triggiani, Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems: The case $\alpha = \frac{1}{2}$, in: *Lecture Notes in Mathematics* **1354**, Springer, 1988, 234–256.
9. S. Chen, R. Triggiani, Proof of extension of two conjectures on structural damping for elastic systems: The case $\frac{1}{2} \leq \alpha \leq 1$, *Pacific J. Math.* **136** (1989), 15–55.
10. S. Chen, R. Triggiani, Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications, *J. Differential Equations* **88** (1990), 279–293.
11. J. W. Cholewa, T. Dlotko, Remarks on the powers of elliptic operators, *Rev. Mat. Complut.* **13** (2000) 1–12.
12. J. W. Cholewa, T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, 2000.
13. I. Chueshov, I. Lasiecka, Attractors for second-order evolution equations with a nonlinear damping, *J. Dynam. Differential Equations* **16** (2004), 469–512.
14. R. Denk, G. Dore, M. Hieber, J. Prüss, A. Venni, New thoughts on old results of R. T. Seeley, *Math. Ann.* **328** (2004), 545–583.
15. A. Friedman, *Partial Differential Equations,* Holt, Rinehart and Winston, New York, 1969.
16. J.-M. Ghidaglia, A. Marzocchi, Longtime behaviour of strongly damped wave equations, global attractors and their dimension. *SIAM J. Math. Anal.* **22** (1991), 879–895.

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17. D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Lecture Notes in Mathematics **840**, Springer, Berlin, 1981.

18. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta. Math. **104** (1960), 93–140.

19. T. Kato, Note on fractional powers of linear operators, *Proc. Japan Acad.*, **36** (1960), 94–96.

20. T. Kato, Fractional powers of dissipative operators, *J. Math. Soc. Japan*, **13** (1961), 246–574.

21. H. Komatsu, Fractional powers of operators, *Pacific J. Math.* **19** (1966), 285–346.

22. De-Sheng Li, Chen-Kui Zhong, Global attractor for the Cahn-Hilliard system with fast growing nonlinearity, *J. Differential Equations* **149** (1998), 191–210.

23. P. Massat, Limiting behavior for strongly damped nonlinear wave equations, *J. Differential Equations* **48** (1983), 334–349.

24. A. Rodriguez-Bernal, On the generation of analytic semigroups by a class of damped wave equations, preprint.

25. K. Taira, *Analytic Semigroups and Semilinear Initial Boundary Value Problems*, Cambridge University Press, Cambridge, 1995.

26. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* Springer, New York, 1988.

27. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Veb Deutscher, Berlin 1978.

28. S. Zheng, A. Milani, Global attractors for singular perturbations of the Cahn-Hilliard equations, *J. Differential Equations* **209** (2005), 101–139.

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