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Flows of characteristic 0^+ in impulsive semidynamical systems

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In this paper, as in [2], we continue to study the dynamics of flows defined in impulsive semidynamical systems (X, π; M, I), where X is a metric space, (X, π) is a semidynamical system, M denotes an impulsive set and I is an impulsive operator. We generalize some results of non-impulsive flows of characteristic 0^+ ($Ch0^+$) for systems with impulses. In particular, we state conditions so that the limit set of an impulsive system of $Ch0^+$ is either a periodic orbit or a single rest point. We also give conditions for a subset H in (X, π; M, I) to be globally asymptotically stable in the impulsive system, provided the flow is of $Ch0^+$. October, 2006 ICMC-USP

Key Words: Impulsive semidynamical systems; characteristic 0^+; global asymptotic stability; flows.

1. INTRODUCTION

The theory of impulsive semidynamical systems is an important and modern chapter of the theory of dynamical systems. Impulsive systems present interesting phenomena as "beating", "dying", "merging", noncontinuation of solutions, etc., which have been slowly overcome. For details of this theory, see [2], [5], [6], [7], [8], [9] and [10], for instance.

Dynamical systems of characteristic 0^+ (we write $Ch0^+$) are those dynamical systems in which all closed positively invariant sets are stable in Ura's sense. Such systems present interesting properties. In [1] and [11] the behavior of non-impulsive flows of characteristic 0^+ were studied.

In the present paper, we present important results for *impulsive* semidynamical systems of characteristic 0^+. We generalize many results of [1] for discontinuous flows. We also establish a result on asymptotic behavior of discontinuous flows of characteristic 0^+. In the next lines we describe the organization of the paper and the main results.

In Section 2, the basis of the theory of the impulsive semidynamical systems is presented. We divide this section into three subsections: 2.1, 2.2 and 2.3. In Subsection 2.1, we give

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some basic definitions and notations about impulsive semidynamical systems. In Subsection 2.2, we discuss the continuity of a function which describes the time of reaching impulsive points. In Subsection 2.3, we give some additional useful definitions.

Section 3 concerns the main results of this paper and it is divided into two subsections: 3.1 and 3.2. Subsection 3.1 deals with impulsive semidynamical systems of $Ch0^+$ on the plane. In [1], the author proved the following result for the non-impulsive case:

"Let $(\mathbb{R}^2, \pi)$ *be a flow of* $Ch0^+$*. For each* $x \in \mathbb{R}^2$*, if* $\mathrm{L}^+(x) \neq \emptyset$*, then* $\mathrm{L}^+(x)$ *is either a periodic orbit or it consists of a single rest point."*

We generalize this result for the impulsive case. See Theorem 3.1. Still in this subsection, we consider impulsive semidynamical systems $(\Omega, \widetilde{\pi})$, as presented in [10], where $\Omega \subset \mathbb{R}^2$ is an open set. We introduce the concept of dispersive discontinuous flows for such systems and then relate dispersivity and flows of $Ch0^+$ (see Lemma 3.5 and Proposition 3.2 in the sequel).

In Subsection 3.2, we discuss global asymptotic stability for impulsive systems of characteristic 0^+. Our aim is to give conditions for a subset H an impulsive semidynamical system of $Ch0^+$ to be globally asymptotically stable.

In [1], it was proved that:

"If (X, π) *is a flow of characteristic* 0^+ *on a connected locally compact space* X*,* H *is a compact positively invariant subset of* X *and* H *is a positive attractor, then* H *is globally asymptotically stable."*

We also formulate this result for the impulsive case imposing the condition that H is an attractor and is invariant in the impulsive system, that is, H is invariant by both the flow and the impulse operator.

Another result that gives conditions for H to be globally asymptotically stable in the impulsive system of $Ch0^+$ is the following: given an impulsive semidynamical system of $Ch0^+$ in the plane, if we consider S as being the set of rest points of this flow, then we can get the global asymptotic stability of a subset H of the plane, provided H is compact, invariant, H contains the impulsive set M, $S \subset \mathrm{H}$ but $S \cap \mathrm{M} = \emptyset$, there exists a neighborhood of H that contains no periodic orbits and the limit set of each trajectory in this neighborhood does not intercept M.

2. IMPULSIVE SEMIDYNAMICAL SYSTEMS

2.1. Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(\mathrm{X}, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function

$$\pi : \mathrm{X} \times \mathbb{R}_+ \longrightarrow \mathrm{X}$$

fulfills the conditions:

a) $\pi(x, 0) = x$, for all $x \in \mathrm{X}$,

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b) $\pi(\pi(x,t),s) = \pi(x,t+s)$, for all $x \in \mathrm{X}$ and $t, s \in \mathbb{R}_+$,
c) π is continuous.

We denote such system by (X, π, $\mathbb{R}_+$) or simply (X, π). When $\mathbb{R}_+$ is replaced by $\mathbb{R}$ in the definition above, the triple (X, π, $\mathbb{R}$) is a *dynamical system.* For every $x \in \mathrm{X}$, we consider the continuous function $\pi_x : \mathbb{R}_+ \longrightarrow \mathrm{X}$ given by $\pi_x(t) = \pi(x,t)$ and we call it the *trajectory* of x.

Let (X, π) be a semidynamical system. Given $x \in \mathrm{X}$, the *positive orbit* of x is given by

$$\mathrm{C}^+(x) = \{\pi(x,t) \,:\, t \in \mathbb{R}_+\}$$

which we also denote by $\pi^+(x)$. Given $x \in \mathrm{X}$ and $r \in \mathbb{R}_+$, we define

$$\mathrm{C}^+(x,r) = \{\pi(x,t) \,:\, 0 \leq t < r\}.$$

For $t \geq 0$ and $x \in \mathrm{X}$, we define

$$\mathrm{F}(x,t) = \{y : \pi(y,\, t) = x\}$$

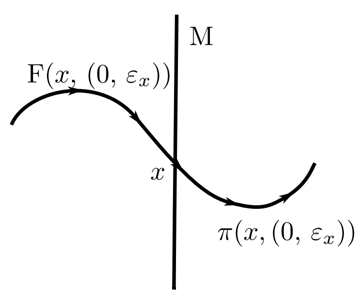
and, for $\triangle \subset [0,\, +\infty)$ and $D \subset \mathrm{X}$, we define

$$\mathrm{F}(D,\, \triangle) = \bigcup \{\mathrm{F}(x,\, t) : x \in D \text{ and } t \in \triangle\}.$$

Then a point $x \in \mathrm{X}$ is called an *initial point*, if $\mathrm{F}(x,\, t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* (X, π; M, I) consists of a semidynamical system, (X, π), a non-empty closed subset M of X and a continuous function $\mathrm{I} : \mathrm{M} \rightarrow \mathrm{X}$ such that for every $x \in \mathrm{M}$, there exists $\varepsilon_x > 0$ such that

$$\mathrm{F}(x,\, (0,\, \varepsilon_x)) \cap \mathrm{M} = \emptyset \qquad \text{and} \qquad \pi(x,\, (0,\, \varepsilon_x)) \cap \mathrm{M} = \emptyset.$$



The points of M are isolated in every trajectory of the system (X, π).

The set M is called *impulsive set*, the function I is called *impulse function* and we write $\mathrm{N} = \mathrm{I(M)}$. We also define

$$\mathrm{M}^+(x) = (\pi^+(x) \cap \mathrm{M}) \setminus \{x\}.$$

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Given an impulsive semidynamical systems (X, π; M, I) and $x \in$ X such that $\mathrm{M}^+(x) \neq \emptyset$, it is always possible to find a smallest number s_1 such that the trajectory $\pi_x(t)$ for $0 < t < s_1$ does not intercept the set M. This result is stated next and a proof of it can be found in [2], Lemma 2.1.

Lemma 2.1. *Let* (X, π; M, I) *be an impulsive semidynamical system. Then for every* $x \in$ X*, there is a positive number* s_1*,* $0 < s_1 \leq +\infty$*, such that* $\pi(x, t) \notin$ M*, whenever* $0 < t < s_1$*, and* $\pi(x, s_1) \in$ M *if* $\mathrm{M}^+(x) \neq \emptyset$*.*

Let (X, π; M, I) be an impulsive semidynamical system and $x \in$ X. By means of Lemma 2.1, it is possible to define a function $\phi : \mathrm{X} \to (0, +\infty]$ in the following manner: if $\mathrm{M}^+(x) = \emptyset$, then $\phi(x) = +\infty$, and if $\mathrm{M}^+(x) \neq \emptyset$, then $\phi(x)$ is the smallest number, denoted by s, such that $\pi(x, t) \notin$ M, for $t \in (0, s)$, and $\pi(x, s) \in$ M. This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Then for each $x \in$ X, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in (X, π; M, I) is a function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$) in X. The description of such trajectory follows inductively as described in [10] and reproduced in the next lines.

If $\mathrm{M}^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x, t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $\mathrm{M}^+(x) \neq \emptyset$, it follows from Lemma 2.1 that there is a smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in$ M and $\pi(x, t) \notin$ M, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = \mathrm{I}(x_1)$ and $\phi(x) = s_0$.

Since $s_0 < +\infty$, the process now continues from x_1^+ on. If $\mathrm{M}^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, $s_0 \leq t < +\infty$, and $\phi(x_1^+) = +\infty$. When $\mathrm{M}^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in$ M and $\pi(x_1^+, t - s_0) \notin$ M, for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ in $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leq t < s_0 + s_1 \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = \mathrm{I}(x_2)$ and $\phi(x_1^+) = s_1$.

Now we suppose that $\widetilde{\pi}_x$ is defined in the interval $[t_{n-1}, t_n]$ and that $\widetilde{\pi}_x(t_n) = x_n^+$, where $t_n = \sum_{i=0}^{n-1} s_i$. If $\mathrm{M}^+(x_n^+) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x_n^+, t - t_n)$, $t_n \leq t < +\infty$, and $\phi(x_n^+) = +\infty$. If $\mathrm{M}^+(x_n^+) \neq \emptyset$, then there exists $s_n \in \mathbb{R}_+$ such that $\pi(x_n^+, s_n) = x_{n+1} \in$ M and $\pi(x_n^+, t - t_n) \notin$ M, for $t_n < t < t_{n+1}$. Besides

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_n^+, t - t_n), & t_n \leq t < t_{n+1} \\ x_{n+1}^+, & t = t_{n+1}, \end{cases}$$

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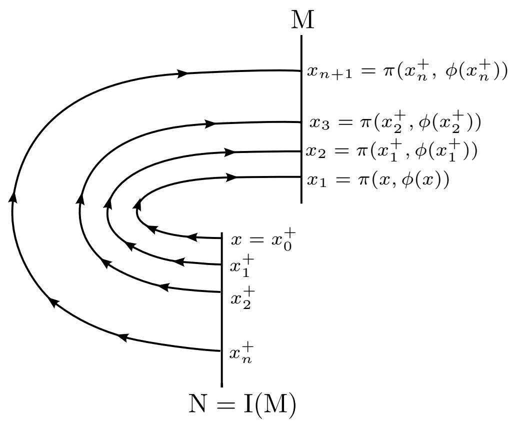
where $x_{n+1}^+ = \mathrm{I}(x_{n+1})$ and $\phi(x_n^+) = s_n$. Notice that $\widetilde{\pi}_x$ is defined in each interval $[t_n,\, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\widetilde{\pi}_x$ is defined in $[0,\, t_{n+1}]$.



Trajectory in an impulsive semidynamical system (X, π; M, I).

The process above ends after a finite number of steps, whenever $\mathrm{M}^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $\mathrm{M}^+(x_n^+) \neq \emptyset,\ n = 1, 2, 3, \ldots$, and $\widetilde{\pi}_x$ is defined in the interval $[0,\, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

It is known that given $x \in \mathrm{X}$, one of the next three properties holds:

i) $\mathrm{M}^+(x) = \emptyset$ and hence the trajectory of x has no discontinuities.

ii) For some $n \geq 1$, each x_k^+, $k = 1, 2, \ldots, n$, is defined and $\mathrm{M}^+(x_n^+) = \emptyset$. In this case, the trajectory of x has a finite number of discontinuities.

iii) For all $k \geq 1$, x_k^+ is defined and $\mathrm{M}^+(x_k^+) \neq \emptyset$. In this case, the trajectory of x has infinite discontinuities.

Let (X, π; M, I) be an impulsive semidynamical system. Given $x \in \mathrm{X}$, the *positive impulsive orbit* of x is defined by the set

$$\widetilde{\mathrm{C}}^+(x) = \{\widetilde{\pi}(x,\, t) : t \in \mathbb{R}_+\}$$

which we also denote by $\widetilde{\pi}^+(x)$ and we denote its closure in X by $\widetilde{\mathrm{K}}^+(x)$.

Analogously to the non-impulsive case, we have the next standard group properties whose proofs follow straightforwardly, but we decided to include them here to show that some special care is needed when considering the impulses and because one cannot find them easily in the literature.

Proposition 2.1. *Let* (X, π; M, I) *be an impulsive semidynamical system. If* $x \in \mathrm{X}$, *then* *i)* $\widetilde{\pi}(x, 0) = x$,

ii) $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t+s)$, *with* $t, s \in [0, T(x))$ *such that* $t + s \in [0, T(x))$.

Proof. Consider the previous description and notation for an impulsive process.

i) We have

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_n^+, t - t_n), & t_n \leq t < t_{n+1}, \\ x_{n+1}^+, & t = t_{n+1}, \end{cases}$$

where $n = 0, 1, 2, ...,$ $x_0^+ = x$ and $t_0 = 0$. It follows that

$$\widetilde{\pi}_x(0) = \widetilde{\pi}_x(t_0) = \pi(x_0^+, t_0 - t_0) = \pi(x, 0) = x.$$

ii) Given $t \in \mathbb{R}_+$, we have $t_n \leq t < t_{n+1}$ for some $n \in \mathbb{N}$. If $0 \leq s < t_{n+1} - t$, then $\phi(\pi(x_n^+, t - t_n)) = t_{n+1} - t$, since

$$\pi(\pi(x_n^+, t - t_n), t_{n+1} - t) = \pi(x_n^+, t_{n+1} - t_n) = x_{n+1} \in \mathrm{M},$$

and for $0 < \lambda < t_{n+1} - t$, we have $t_n < \lambda + t < t_{n+1}$ and

$$\pi(\pi(x_n^+, t - t_n), \lambda) = \pi(x_n^+, t + \lambda - t_n) = \widetilde{\pi}(x, \lambda + t) \notin \mathrm{M}.$$

Thus

$$\widetilde{\pi}(\pi(x_n^+, t - t_n), s) = \pi(\pi(x_n^+, t - t_n), s)$$

and therefore

$$\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(\pi(x_n^+, t - t_n), s) = \pi(\pi(x_n^+, t - t_n), s) =$$

$$= \pi(x_n^+, t + s - t_n) = \widetilde{\pi}(x, t + s).$$

But if $s \geq t_{n+1} - t$, that is, $s + t \geq t_{n+1}$, for some $k \in \{1, 2, 3, 4, ...\}$, it follows that $t_{n+k} \leq t + s < t_{n+k+1}$. Thus,

$$\widetilde{\pi}(x, t + s) = \pi(x_{n+k}^+, t + s - t_{n+k}).$$

Taking $y = \pi(x_n^+, t - t_n)$, it follows that $y_1^+ = x_{n+1}^+, ..., y_k^+ = x_{n+k}^+$, $k \in \{1, 2, 3, 4, ...\}$ and $\phi(y) = m_1 = t_{n+1} - t$. Taking $m_k = \displaystyle\sum_{j=0}^{k-1} \phi(y_j^+)$, we have

$$m_{j+1} - m_j = t_{n+j+1} - t_{n+j}, \quad j = 1, ..., k-1, \;\; k \in \{1, 2, 3, 4, ...\},$$

Since $t_{n+k} \leq t + s < t_{n+k+1}$, it follows that $m_k \leq s < m_{k+1}$, because $t_{n+k} - t \leq s < t_{n+k+1} - t$ and, besides,

$$t_{n+k} - t = \sum_{j=2}^{k} (t_{n+j} - t_{n+j-1}) + t_{n+1} - t = \sum_{j=2}^{k} (m_j - m_{j-1}) + m_1 = m_k$$

and

$$t_{n+k+1} - t = \sum_{j=2}^{k+1} (t_{n+j} - t_{n+j-1}) + t_{n+1} - t = \sum_{j=2}^{k+1} (m_j - m_{j-1}) + m_1 = m_{k+1}.$$

Hence $\widetilde{\pi}(y,\, s) = \pi(y_k^+,\, s - m_k)$. But since

$$s - m_k = s - (m_k - m_1) - m_1 = s - (t_{n+k} - t_{n+1}) - m_1 =$$

$$= s - t_{n+k} + t_{n+1} - (t_{n+1} - t) = t + s - t_{n+k},$$

we have

$$\pi(y_k^+,\, s - m_k) = \pi(x_{n+k}^+,\, t + s - t_{n+k}).$$

Recall that $y = \pi(x_n^+,\, t - t_n)$. Hence,

$$\widetilde{\pi}(\widetilde{\pi}(x,\, t),\, s) = \widetilde{\pi}(\pi(x_n^+,\, t - t_n),\, s) = \widetilde{\pi}(y,\, s) = \pi(y_k^+,\, s - m_k)$$

$$= \pi(x_{n+k}^+,\, t + s - t_{n+k}) = \widetilde{\pi}(x,\, t + s)$$

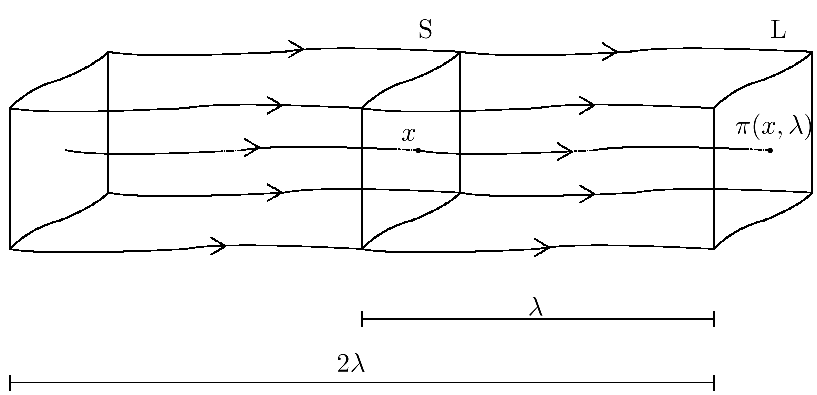
and the proof is complete. □

2.2. Semicontinuity and continuity of ϕ

The results of this section are borrowed from [5]. They are applied intrinsically in the proofs of the main theorems of the next section.

Let $(X,\, \pi)$ be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a $\lambda-$*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

a) $F(L,\, \lambda) = S$;
b) $F(L,\, [0,\, 2\lambda])$ is a neighborhood of x;
c) $F(L,\, \mu) \cap F(L,\, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.



The set F(L, $[0, 2\lambda]$) is called a *tube* or a λ-*tube* and the set L is called a *bar*.

If (X, π) is a semidynamical system and S is a $\lambda-$section through x, $x \in$ X, then S is a $\mu-$section through x, for $\mu \leq \lambda$. A proof of this fact can be found in [3], Lemma 1.9 (see also [2], Lemma 2.2).

Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube. Any tube F(L, $[0, 2\lambda]$) given by the section S through $x \in$ X such that

$$\mathrm{S} \subset \mathrm{M} \cap \mathrm{F}(\mathrm{L},\ [0,\ 2\lambda])$$

is called *TC-tube* on x. We say that a point $x \in$ M fulfills the *Tube Condition* and we write TC, if there exists a TC-tube F(L, $[0, 2\lambda]$) through x. In particular, if

$$\mathrm{S} = \mathrm{M} \cap \mathrm{F}(\mathrm{L},\ [0,\ 2\lambda]),$$

we have a *STC-tube* on x and we say that a point $x \in$ M fulfills the *Strong Tube Condition* (we write STC), if there exists a STC-tube F(L, $[0, 2\lambda]$) through x.

The following theorem concerns the continuity of the function ϕ which is accomplished outside M for M satisfying the condition TC. See [5], Theorem 3.8.

Theorem 2.1. *Consider an impulsive semidynamical system* (X, π; M, I). *Assume that no initial point belongs to the impulsive set* M *and that each element of* M *satisfies the condition* TC. *Then ϕ is continuous at x if and only if $x \notin$* M.

2.3. Additional definitions

Let us consider the metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in$ X and ratio $\delta > 0$. Let $B(A, \delta) = \{x \in \mathrm{X} : \rho_A(x) < \delta\}$ and $B[A, \delta] = \{x \in \mathrm{X} : \rho_A(x) \leq \delta\}$, where $\rho_A(x) = \inf\{\rho(x, y) : y \in A\}$. Throughout this paper, we use ∂A and $\overline{A}$ to represent the boundary and closure of A in X respectively.

In what follows, (X, π; M, I) is an impulsive semidynamical system and $x \in$ X.

We define the *limit set* of x in (X, π; M, I) by

$$\widetilde{\mathrm{L}}^+(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x,\ t_n) \to y,\ \text{ for some }\ t_n \to +\infty\}.$$

The *prolongational limit set* of x in (X, π; M, I) is given by

$$\widetilde{\mathrm{J}}^+(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x_n,\ t_n) \to y,\ \text{ for some }\ x_n \to x\ \text{ and }\ t_n \to +\infty\};$$

and the *prolongation set* of x in (X, π; M, I) is defined by

$$\widetilde{\mathrm{D}}^+(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x_n,\ t_n) \to y,\ \text{ for some }\ x_n \to x\ \text{ and }\ t_n \in [0,\ +\infty)\}.$$

For the case of semidynamical systems without impulses, we denote by $\mathrm{L}^+(x)$, $\mathrm{J}^+(x)$ and $\mathrm{D}^+(x)$ respectively the limit set, the prolongational limit set and the prolongation set of a point $x \in$ X.

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For a set $K \subset \mathrm{X}$ we consider $\widetilde{\mathrm{D}}^{+}(K) = \bigcup\{\widetilde{\mathrm{D}}^{+}(x) : x \in K\}$.

We say that $\mathrm{C} \subset \mathrm{X}$ is *minimal* in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, whenever $\mathrm{C} = \widetilde{\mathrm{K}}^{+}(x)$ for each $x \in \mathrm{C} \setminus \mathrm{M}$. This definition is due to S. K. Kaul (see [8]).

A point $x \in \mathrm{X}$ is called *stationary* or *rest point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for all $t \geq 0$, it is a *periodic point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for some $t > 0$ and x is not stationary, and it is a *regular point* if it is neither a rest point nor a periodic point.

Let $A \subset \mathrm{X}$. If $\widetilde{\pi}^{+}(A) \subset A$, we say that A is $\widetilde{\pi}-$*invariant.* If for every $\varepsilon > 0$ and every $x \in A$, there is $\delta > 0$ such that

$$\widetilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \varepsilon),$$

then A is called $\widetilde{\pi}-$*stable.* The set A is $\widetilde{\pi}-$*orbitally stable* if for every neighborhood U of A, there is a $\widetilde{\pi}-$invariant neighborhood V of A, $V \subset U$. Finally, the set A is $\mathrm{I}-$*invariant* in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, whenever $\mathrm{I}(x) \in A$ for all $x \in \mathrm{M} \cap A$, and A is $\mathrm{I}-$*stable* if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mathrm{I}(\mathrm{M} \cap B(A, \delta)) \subset B(A, \varepsilon).$$

Let $\mathrm{H} \subset \mathrm{X}$. We define the sets

$$\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{H}) = \{x \in \mathrm{X} : \text{ for every neighborhood } U \text{ of } \mathrm{H}, \text{ there is a sequence}$$

$$\{t_n\} \subset \mathbb{R}_{+}, t_n \to +\infty \text{ such that } \widetilde{\pi}(x, t_n) \in U\},$$

and

$$\widetilde{\mathrm{P}}^{+}(\mathrm{H}) = \{x \in \mathrm{X} : \text{ for every neighborhood } U \text{ of } \mathrm{H}, \text{ there is } \tau \in \mathbb{R}_{+}$$

$$\text{such that } \widetilde{\pi}(x, [\tau, +\infty)) \subset U\}.$$

The set $\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{H})$ is called *region of weak attraction* of H with respect to $\widetilde{\pi}$ and the set $\widetilde{\mathrm{P}}^{+}(\mathrm{H})$ is called *region of attraction* of H with respect to $\widetilde{\pi}$. If $x \in \widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{H})$ or $x \in \widetilde{\mathrm{P}}^{+}(\mathrm{H})$, then we say that x is $\widetilde{\pi}-$*weakly attracted* or $\widetilde{\pi}-$*attracted* to H respectively. A set $\mathrm{H} \subset \mathrm{X}$ is called a *weak* $\widetilde{\pi}-$*attractor*, if $\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{H})$ is a neighborhood of H, and it is called a $\widetilde{\pi}-$*attractor*, if $\widetilde{\mathrm{P}}^{+}(\mathrm{H})$ is a neighborhood of H. A set $\mathrm{H} \subset \mathrm{X}$ is called *asymptotically* $\widetilde{\pi}-$*stable*, if it is both a weak $\widetilde{\pi}-$attractor and $\widetilde{\pi}-$orbitally stable.

3. MAIN RESULTS

We divide this section into two parts. The first subsection concerns impulsive semidynamical systems of $Ch0^{+}$ on the plane. In the second subsection, we consider the global asymptotic stability for impulsive semidynamical systems of $Ch0^{+}$.

Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. We assume that each element of M satisfies the condition TC and that no initial point belongs to the impulsive set M. Under these conditions, ϕ is always continuous on $\mathrm{X} \setminus \mathrm{M}$ (Theorem 2.1).

3.1. Impulsive Flows of Characteristic 0^+ on the plane

Dynamical systems of characteristic 0^+ are systems in which all closed positively invariant sets are stable in Ura's sense. In [9], flows of characteristic 0^+ are defined for the impulsive case and such definiton is mentioned next.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We say that a point $x \in X$ is of characteristic 0^+ or simply that $x \in X$ is of $Ch0^+$ if $\widetilde{D}^+(x) = \widetilde{K}^+(x)$. We say that the impulsive system $(X, \pi; M, I)$ is of $Ch0^+$, if $\widetilde{D}^+(x) = \widetilde{K}^+(x)$ for all $x \in X$.

In the sequel, we consider the space $X = \mathbb{R}^2$ and we study the behavior of an impulsive system $(\mathbb{R}^2, \pi; M, I)$ of $Ch0^+$. The next lemma says that in an impulsive system of $Ch0^+$, if a limit set contains a rest point, then this limit set is the rest point itself.

Lemma 3.1. *Let* $(\mathbb{R}^2, \pi; M, I)$ *be an impulsive semidynamical system of* $Ch0^+$ *and* $x \in \mathbb{R}^2$*. If* $\widetilde{L}^+(x)$ *contains a rest point* s_0*, then* $\widetilde{L}^+(x) = \{s_0\}$*.*

Proof. Suppose $\widetilde{L}^+(x)$ contains a rest point s_0. Let $y \in \widetilde{L}^+(x)$. We assert that if $y_1, y_2 \in \widetilde{L}^+(x)$, then $y_1 \in \widetilde{D}^+(y_2)$ and $y_2 \in \widetilde{D}^+(y_1)$. Indeed. Since $y_1, y_2 \in \widetilde{L}^+(x)$, there exist sequences of positive real numbers $\{t_n\}_{n\geq 1}$, $\{\tau_m\}_{m\geq 1}$, with $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ and $\tau_m \stackrel{m\to+\infty}{\longrightarrow} +\infty$ such that

$$\widetilde{\pi}(x, t_n) \stackrel{n\to+\infty}{\longrightarrow} y_1 \quad \text{and} \quad \widetilde{\pi}(x, \tau_m) \stackrel{m\to+\infty}{\longrightarrow} y_2.$$

Since t_n and τ_m tend to $+\infty$, we can choose a sequence ℓ_k such that $\{t_{\ell_k}\}_{k\geq 1}$ is a subsequence of $\{t_n\}_{n\geq 1}$ and $\{\tau_{\ell_k}\}_{k\geq 1}$ is a subsequence of $\{\tau_m\}_{m\geq 1}$ with $t_{\ell_k} - \tau_{\ell_k} > 0$. In this manner

$$\widetilde{\pi}(\widetilde{\pi}(x, \tau_{\ell_k}), t_{\ell_k} - \tau_{\ell_k}) = \widetilde{\pi}(x, t_{\ell_k}) \stackrel{k\to+\infty}{\longrightarrow} y_1.$$

Consequently $y_1 \in \widetilde{D}^+(y_2)$. Similarly, one can prove that $y_2 \in \widetilde{D}^+(y_1)$.

By the assertion above, $y \in \widetilde{D}^+(s_0) = \widetilde{K}^+(s_0) = \{s_0\}$. Hence $y = s_0$ and $\widetilde{L}^+(x) = \{s_0\}$. The proof is complete. □

Let us present a version of Lemma 2.3 in [10] for the impulsive system $(X, \pi; M, I)$. The proof follows analogously.

Lemma 3.2. *Given an impulsive semidynamical system* $(X, \pi; M, I)$*, where* X *is a metric space. Suppose* $w \in X \setminus M$ *and* $\{w_n\}_{n\geq 1}$ *is a sequence convergent to the point* w*. Then for any* $t \in [0, T(w))$*, there exists a sequence of real numbers* $\{\varepsilon_n\}_{n\geq 1}$*,* $\varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$*, such that* $t + \varepsilon_n < T(w_n)$ *and* $\widetilde{\pi}(w_n, t + \varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$*.*

Proof. Since $w \in X \setminus M$ and M is closed, there exists $\eta > 0$ such that $B(w, \eta) \cap M = \emptyset$. Taking subsequence, if necessary, we can assume that $w_n \in B(w, \eta)$ for all n. Since ϕ is continuous on $X \setminus M$, because we are supposing in this paper that each element of M satisfies the condition TC and that no initial point belongs to the impulsive set M, then the proof follows similarly as in [10], Lemma 2.3. □

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Lemma 3.3 below concerns the invariancy and closeness of the limit set. The proof that it is closed can be found in [7], page 122, and the proof that is $\widetilde{\pi}-$invariant is analogous to the proof of [10], Lemma 2.6 using Lemma 3.2 above.

Lemma 3.3. *Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$*, where* X *is a metric space. If for* $x \in \mathrm{X}$*,* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$*, then* $\widetilde{\mathrm{L}}^+(x)$ *is closed and* $\widetilde{\pi}-$*invariant.*

The next result characterizes the orbits of the limit set of any given point in an impulsive semidynamical system of $Ch0^+$.

Proposition 3.1. *Given an impulsive semidynamical system* $(\mathbb{R}^2, \pi; \mathrm{M}, \mathrm{I})$ *of* $Ch0^+$ *and* $x \in \mathbb{R}^2$*, if* $\widetilde{\mathrm{L}}^+(x)$ *is nonempty and* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$*, then* $\widetilde{\mathrm{K}}^+(y) = \widetilde{\mathrm{L}}^+(x)$ *for all* $y \in \widetilde{\mathrm{L}}^+(x)$*.*

Proof. If $\widetilde{\mathrm{L}}^+(x)$ is a singleton, the result follows. Let us suppose that $\widetilde{\mathrm{L}}^+(x)$ is not a singleton. Given $y \in \widetilde{\mathrm{L}}^+(x)$, we have $\widetilde{\mathrm{K}}^+(y) \subseteq \widetilde{\mathrm{L}}^+(x)$, once $\widetilde{\mathrm{L}}^+(x)$ is $\widetilde{\pi}-$invariant and closed by Lemma 3.3.

Now we will prove the other inclusion. Let $z \in \widetilde{\mathrm{L}}^+(x)$. Then there exists a sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$\widetilde{\pi}(x,\, t_n) \stackrel{n\to+\infty}{\longrightarrow} z \quad \text{as} \quad t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty.$$

Since $y \in \widetilde{\mathrm{L}}^+(x)$, there exists a sequence $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$\widetilde{\pi}(x,\, \tau_n) \stackrel{n\to+\infty}{\longrightarrow} y \quad \text{as} \quad \tau_n \stackrel{n\to+\infty}{\longrightarrow} +\infty.$$

We can assume that $t_n - \tau_n > 0$ and thus we have

$$\widetilde{\pi}(\widetilde{\pi}(x,\, \tau_n),\, t_n - \tau_n) = \widetilde{\pi}(x,\, t_n) \stackrel{n\to+\infty}{\longrightarrow} z.$$

Therefore $z \in \widetilde{\mathrm{D}}^+(y) = \widetilde{\mathrm{K}}^+(y)$, since $(\mathbb{R}^2, \pi; \mathrm{M}, \mathrm{I})$ is of $Ch0^+$. Hence the result follows. □

Corollary 3.1. *Given an impulsive semidynamical system* $(\mathbb{R}^2, \pi; \mathrm{M}, \mathrm{I})$ *of* $Ch0^+$ *and* $x \in \mathbb{R}^2$*, if* $\widetilde{\mathrm{L}}^+(x)$ *is nonempty and* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$*, then* $\widetilde{\mathrm{L}}^+(x)$ *is minimal.*

Remark 3. 1. Proposition 3.1 and Corollary 3.1 remain true for any impulsive semidynamical system $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ of $Ch0^+$, where X is a metric space.

The next lemma is a result for non-impulsive systems which will be important to next Theorem.

Lemma 3.4. *Let* $(\mathbb{R}^2, \pi)$ *be a non-impulsive semidynamical system. If* $\mathrm{L}^+(y)$ *is minimal and it does not contain rest points for each* $y \in \mathbb{R}^2$*, then* $\mathrm{L}^+(y)$ *is a periodic orbit.*

Proof. Since $L^+(y)$ is minimal (i.e., closed, positively invariant and contains no nonempty proper subset with these properties), given $z \in L^+(y)$, we have $L^+(z) = L^+(y) = K^+(z)$. Thus, $z \in L^+(z)$. By [4], Theorem 5.8 and by the hypothesis, z is a periodic point. Then, by Lemma 5.1 from [4], $L^+(y) = C^+(z)$ is a periodic orbit. □

Theorem 3.1 below says that the limit set of a flow of $Ch0^+$ can be classified as being a rest point or a periodic orbit.

THEOREM 3.1. *Let* $(\mathbb{R}^2, \pi; M, I)$ *be an impulsive semidynamical system of* $Ch0^+$. *Suppose* $x \in \mathbb{R}^2$, $\widetilde{L}^+(x) \neq \emptyset$, $\widetilde{L}^+(x) \cap M = \emptyset$ *and* M *is compact or* $\widetilde{L}^+(x)$ *is compact. Then* $\widetilde{L}^+(x)$ *is either a periodic orbit or it consists of a single rest point.*

Proof. By Lemma 3.1, if $\widetilde{L}^+(x)$ contains a rest point s_0, then $\widetilde{L}^+(x) = \{s_0\}$. Suppose that $\widetilde{L}^+(x)$ consists of regular points only. Since $\mathbb{R}^2$ is a normal space, $\widetilde{L}^+(x) \cap M = \emptyset$ and M is compact or $\widetilde{L}^+(x)$ is compact, it follows that there exists $\eta > 0$ such that $B(\widetilde{L}^+(x), \eta) \cap M = \emptyset$. Thus there exists $\beta > 0$ such that

$$\widetilde{\pi}(x, t) = \pi(\widetilde{\pi}(x, \beta), t - \beta) \subset B(\widetilde{L}^+(x), \eta), \quad \text{for all} \quad t > \beta.$$

Let $y = \widetilde{\pi}(x, \beta)$. Then the orbit $\widetilde{C}^+(y)$ equals $C^+(y)$ and $L^+(y) = \widetilde{L}^+(x)$. Note that $\widetilde{L}^+(x)$ is minimal by Corollary 3.1. Since $C^+(y)$ is the trajectory without impulses and $L^+(y)$ is minimal and has no rest points, it follows that $L^+(y)$ is a periodic orbit by Lemma 3.4. Hence $\widetilde{L}^+(x)$ is a periodic orbit and the theorem is proved. □

Let us now present the concept of dispersive impulsive semidynamical systems. Then, using Theorem 3.1 we will be able to prove a result, namely Proposition 3.2, which states conditions for an impulsive semidynamical system of $Ch0^+$ to be dispersive.

We shall consider $\Omega \subset \mathbb{R}^2$ as an open set. We will assume that the boundary of Ω in $\mathbb{R}^2$ is the impulsive set M, that is, $\partial\Omega = M$, and the impulse function I is such that $I(M) \subset \Omega$. In this case, we will denote such impulsive semidynamical system by $(\Omega, \widetilde{\pi})$ as in [10].

Remark 3. 2. When we are treating impulsive semidynamical systems as $(\Omega, \widetilde{\pi})$, with $\Omega \subset \mathbb{R}^2$ open, we use the notation and terminology from [10]. Hence, the corresponding non-impulsive semidynamical system is in fact $(\mathbb{R}^2, \pi)$ and not (Ω, π). Indeed. Consider, for instance, $X = \mathbb{R}^2$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$ and the system given by the equation $\frac{dx}{dt} = 0$ with $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$ and $I(x_1, x_2) = (0, x_2)$ for all $(x_1, x_2) \in M$. The solution of this impulsive system is in the phase space Ω with initial condition $(t_0, x(t_0))$ in Ω. But the solution of the corresponding non-impulsive system is in the phase space $\mathbb{R}^2$.

An impulsive semidynamical system $(\Omega, \widetilde{\pi})$ is called dispersive if for each $x \in \Omega$, $\widetilde{J}^+(x) = \emptyset$. We will prove that any impulsive semidynamical system which is dispersive is also of $Ch0^+$.

Lemma 3.5. *Consider an impulsive semidynamical system* $(\Omega, \widetilde{\pi})$. *If* $(\Omega, \widetilde{\pi})$ *is dispersive, then it is of* $Ch0^+$.

Proof. Suppose $\widetilde{\mathrm{J}}^+(x) = \emptyset$ for all $x \in \Omega$. Since $\widetilde{\mathrm{D}}^+(x) = \widetilde{\mathrm{K}}^+(x) \cup \widetilde{\mathrm{J}}^+(x)$ [see [10], Lemma 2.11], we have $\widetilde{\mathrm{D}}^+(x) = \widetilde{\mathrm{K}}^+(x)$, $x \in \Omega$. Hence $(\Omega, \widetilde{\pi})$ is of $Ch0^+$. □

Lemma 3.6. *If* $(\Omega, \widetilde{\pi})$ *is a dispersive impulsive semidynamical system, then it admits neither rest points nor periodic orbits in* Ω.

Proof. Let $x \in \Omega$. If x is a rest point or $\widetilde{\mathrm{C}}^+(x)$ is periodic, then $\widetilde{\mathrm{L}}^+(x) = \widetilde{\mathrm{K}}^+(x)$ and since $\widetilde{\mathrm{L}}^+(x) \subset \widetilde{\mathrm{J}}^+(x) = \emptyset$, we have a contradiction. Then there are no rest points and no periodic orbits in Ω and the result follows. □

The converse of Lemma 3.5 holds for impulsive semidynamical system without rest points. See the next proposition.

Proposition 3.2. *Let* $(\Omega, \widetilde{\pi})$ *be an impulsive semidynamical system of* $Ch0^+$ *with* M *compact and the underlying planar semidynamical system associated,* (X, π), *does not contain rest points. If for each* $x \in \Omega$ *we have* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$, *then* $(\Omega, \widetilde{\pi})$ *is dispersive.*

Proof. Firstly, we will prove that $\widetilde{\mathrm{L}}^+(x) = \emptyset$ for $x \in \Omega$. Suppose $\widetilde{\mathrm{L}}^+(x) \neq \emptyset$ for some $x \in \Omega$. Then by Theorem 3.1, $\widetilde{\mathrm{L}}^+(x)$ must be a periodic orbit, since it consists of regular points only. Thus $\widetilde{\mathrm{L}}^+(x)$ is compact.

Since $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$, there exists $\eta > 0$ such that $B(\widetilde{\mathrm{L}}^+(x), \eta) \cap \mathrm{M} = \emptyset$. Thus there exists $\lambda > 0$ such that

$$\widetilde{\pi}(x, t) = \pi(\widetilde{\pi}(x, \lambda), t - \lambda) \subset B(\widetilde{\mathrm{L}}^+(x), \eta) \quad \text{for} \quad t > \lambda.$$

Taking $y = \widetilde{\pi}(x, \lambda)$, we have

$$\widetilde{\mathrm{C}}^+(y) = \mathrm{C}^+(y) \subset B(\widetilde{\mathrm{L}}^+(x), \eta),$$

where $B[\widetilde{\mathrm{L}}^+(x), \eta]$ is compact. Hence $\mathrm{L}^+(y) = \widetilde{\mathrm{L}}^+(x)$ is connected and periodic. Now we are in the non-impulsive case and since the bounded component in (X, π) of a periodic orbit contains a rest point, we have a contradiction (see [4], Lemma 5.11). Thus $\widetilde{\mathrm{L}}^+(x) = \emptyset$ for all $x \in \Omega$. Now, it is enough to prove that $\widetilde{\mathrm{J}}^+(x) = \emptyset$. Indeed, since $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$, there exists an integer $\ell > 0$ such that $\mathrm{M}^+(x_\ell^+) = \emptyset$, that is

$$\widetilde{\pi}(x_\ell^+, t) = \pi(x_\ell^+, t) \quad \text{for all} \quad t \geq 0.$$

Since $\widetilde{\pi}(x, t_\ell) = x_\ell^+$, $t_\ell = \sum_{j=0}^{\ell-1} \phi(x_j^+)$, we have $\mathrm{L}^+(x_\ell^+) = \widetilde{\mathrm{L}}^+(x)$ and as the proved above $\widetilde{\mathrm{L}}^+(x) = \emptyset$. We have $\widetilde{\mathrm{J}}^+(x_\ell^+) \subset \widetilde{\mathrm{D}}^+(x_\ell^+) = \widetilde{\mathrm{K}}^+(x_\ell^+)$ (as the system is of $Ch0^+$). Note also that

$$\widetilde{\mathrm{K}}^+(x_\ell^+) = \mathrm{K}^+(x_\ell^+) = \mathrm{C}^+(x_\ell^+) \cup \mathrm{L}^+(x_\ell^+).$$

But, $\mathrm{L}^+(x_\ell^+) = \emptyset$, thus

$$\widetilde{\mathrm{J}}^+(x_\ell^+) \subset \mathrm{C}^+(x_\ell^+).$$

Moreover, for all $z \in \widetilde{\mathrm{C}}^+(x_\ell^+) = \mathrm{C}^+(x_\ell^+)$, there exists an $t_z > 0$ such that $\pi(x_\ell^+, t_z) = \widetilde{\pi}(x_\ell^+, t_z) = z$. Now taking $w \in \widetilde{\mathrm{J}}^+(x_\ell^+)$, then there exist sequences $\{w_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ such that $w_n \stackrel{n\to+\infty}{\longrightarrow} x_\ell^+$, $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ and $\widetilde{\pi}(w_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} w$. Since $x_\ell^+ \notin \mathrm{M}$, by [10] Lemma 2.3, there exists a sequence $\{\varepsilon_n\}_{n\geq 1} \subset \mathbb{R}$, $\varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$, such that

$$\widetilde{\pi}(w_n, \varepsilon_n + t_z) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(x_\ell^+, t_z) = z.$$

Then

$$\widetilde{\pi}(\widetilde{\pi}(w_n, t_z + \varepsilon_n), t_n - t_z - \varepsilon_n) = \widetilde{\pi}(w_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} w,$$

with $t_n - t_z - \varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$. Hence, $w \in \widetilde{\mathrm{J}}^+(z)$. Thus,

$$\widetilde{\mathrm{J}}^+(x_\ell^+) \subset \widetilde{\mathrm{J}}^+(z) \subset \mathrm{C}^+(z) \quad \text{for all} \quad z \in \mathrm{C}^+(x_\ell^+).$$

Taking the intersection for all $z \in \mathrm{C}^+(x_\ell^+)$ we get $\widetilde{\mathrm{J}}^+(x_\ell^+) = \emptyset$.

Now, it is enough to show that $\widetilde{\mathrm{J}}^+(x) \subset \widetilde{\mathrm{J}}^+(x_\ell^+)$. In fact, let $y \in \widetilde{\mathrm{J}}^+(x)$ then there are sequences $\{x_n\}$, $\{t_n\}$ with $x_n \stackrel{n\to+\infty}{\longrightarrow} x$ and $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\widetilde{\pi}(x_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} y.$$

But, since $x \notin \mathrm{M}$, it follows from [10], Lemma 2.3 that there exists a sequence of real numbers $\{\varepsilon_n\}_{n\geq 1}$, $\varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$, such that

$$\widetilde{\pi}(x_n, t_\ell + \varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(x, t_\ell) = x_\ell^+.$$

Then,

$$\widetilde{\pi}(\widetilde{\pi}(x_n, t_\ell + \varepsilon_n), t_n - t_\ell - \varepsilon_n) = \widetilde{\pi}(x_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} y$$

with $\widetilde{\pi}(x_n, t_\ell + \varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} x_\ell^+$ and $(t_n - t_\ell - \varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} +\infty$. Thus, $y \in \widetilde{\mathrm{J}}^+(x_\ell^+)$.

Hence $\widetilde{\mathrm{J}}^+(x) = \emptyset$ and $(\Omega, \widetilde{\pi})$ is dispersive. □

3.2. Global asymptotic stability for Impulsive Semidynamical Systems of Characteristic 0^+

In what follows $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is an impulsive semidynamical system on a metric space X. In [2] the asymptotic stability of subsets of impulsive semidynamical systems was investigated. In this subsection we present some results on global asymptotic stability for impulsive systems of $Ch0^+$. This means that a subset of an impulsive semidynamical system is not only a weak $\widetilde{\pi}-$attractor and $\widetilde{\pi}-$orbitally stable, but also its region of weak attraction is the whole space. Formally, we say that a set $\mathrm{H} \subset \mathrm{X}$ is *globally asymptotically* $\widetilde{\pi}-$*stable*, if it is both asymptotically $\widetilde{\pi}-$stable and $\widetilde{\mathrm{P}}^+(\mathrm{H}) = \mathrm{X}$.

In [2], we proved (Proposition 3.7) that if $\mathrm{H} \subset \mathrm{X}$ is a $\widetilde{\pi}$-attractor, then $\widetilde{\mathrm{P}}^+(\mathrm{H}) = \widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$. Therefore we can replace $\widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$ by $\widetilde{\mathrm{P}}^+(\mathrm{H})$. This means that when $\mathrm{H} \subset \mathrm{X}$ is a $\widetilde{\pi}-$attractor, then instead of $\widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$ one can consider $\widetilde{\mathrm{P}}^+(\mathrm{H})$.

Before presenting the main results of this subsection which states conditions for a subset of an impulsive semidynamical system of $Ch0^+$ to be globally asymptotically $\widetilde{\pi}-$stable, we will give a lemma and a proposition concerning flows in general.

The next lemma presents another property of the set $\widetilde{\mathrm{P}}^+(\mathrm{H})$ which will be useful in the proofs of some of the followings results.

Lemma 3.7. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. Assume that* H *is a compact* $\widetilde{\pi}-$*attractor subset of* X *and* $\mathrm{M} \subset \mathrm{H}$. *Then* $\widetilde{\mathrm{P}}^+(\mathrm{H})$ *is open in* X.

Proof. By the compactness of H and by the fact that $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is a neighborhood of H, there exists an open set $U \subset \mathrm{X}$ such that $\mathrm{H} \subset U \subset \widetilde{\mathrm{P}}^+(\mathrm{H})$.

Suppose $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is not open in X. Then there exists $x \in \widetilde{\mathrm{P}}^+(\mathrm{H}) \setminus U$ such that for all $\delta > 0$, $B(x, \delta)$ does not belong to $\widetilde{\mathrm{P}}^+(\mathrm{H})$, that is, for each $\delta > 0$, there exists $y_\delta \in B(x, \delta)$ such that $y_\delta \notin \widetilde{\mathrm{P}}^+(\mathrm{H})$. Note that since $\mathrm{M} \subset \mathrm{H}$, then there exists $\tau > 0$ such that $\widetilde{\pi}(x, \tau) = \pi(x, \tau) \in U$. Since U is open, there exists an open subset V of X such that $\pi(x, \tau) \in V \subset U \setminus \mathrm{H}$.

By the continuity of π, it follows that for a sufficiently small $\mu > 0$, $\pi(y_\mu, T_{y_\mu}) \in V$ for all $y_\mu \in B(x, \mu)$ and some $T_{y_\mu} > 0$ and $\pi(y_\mu, T_{y_\mu}) = \widetilde{\pi}(y_\mu, T_{y_\mu})$. But $V \subset \widetilde{\mathrm{P}}^+(\mathrm{H})$. Thus $\widetilde{\pi}(y_\mu, T_{y_\mu}) \in \widetilde{\mathrm{P}}^+(\mathrm{H})$ for all $y_\mu \in B(x, \mu)$. Then for every neighborhood W of H, there is a $\tau_W^{y_\mu} > 0$ such that $\widetilde{\pi}(y_\mu, [T_{y_\mu} + \tau_W^{y_\mu}, +\infty)) = \widetilde{\pi}(\widetilde{\pi}(y_\mu, T_{y_\mu}), [\tau_W^{y_\mu}, +\infty)) \subset W$. Hence $y_\mu \in \widetilde{\mathrm{P}}^+(\mathrm{H})$ for all $y_\mu \in B(x, \mu)$ and this contradicts the hypotheses. Thus $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is open in X. □

Proposition 3.3. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system on a normal connected space* X. *Suppose* H *is a compact globally asymptotically* $\widetilde{\pi}-$*stable subset of* X, $\mathrm{M} \subset \mathrm{H}$ *and every component* A *of* H *has the property* $\mathrm{I}(A) \subset A$. *Then* H *is connected.*

Proof. Suppose H is not connected. Then there exist two nonempty disjoint closed sets H_1 and H_2 such that $\mathrm{H} = \mathrm{H}_1 \cup \mathrm{H}_2$. Since X is normal, there exist two disjoint open neighborhoods U_1 and U_2 of H_1 and H_2 respectively. On the other hand, since H is $\widetilde{\pi}-$orbitally stable and $U = U_1 \cup U_2$ is a neighborhood of H, there is an open neigborhood V of H such that $\widetilde{\pi}^+(V) \subset V \subset U$. Therefore, if we let $V_1 = V \cap U_1$ and $V_2 = V \cap U_2$, then for each $x \in V_i$, we have $\widetilde{\pi}^+(x) \subset V_i$, $i = 1, 2$, for if $x \in V_i$, $i = 1, 2$, we can assume, without loss of generality, that $\phi(x) < +\infty$. In this manner, there exists $t_1^i > 0$ such that

$\pi(x,\, t_1^i) \in \mathrm{M} \subset \mathrm{H}_i$, $i = 1, 2$. Since every component A of H satisfies $\mathrm{I}(A) \subset A$, then

$$\mathrm{I}(\pi(x,\, t_1^i)) \in \mathrm{H}_i,$$

and therefore $\widetilde{\pi}(x,\, t) \in V_i$ for all $t \geq 0$. Hence if $x \in V_i$ then $\widetilde{\mathrm{C}}^+(x) \subset V_i \subset U_i$, $i = 1, 2$.

Since H is globally asymptotically $\widetilde{\pi}-$stable, we have $\widetilde{\mathrm{P}}^+(\mathrm{H}) = \mathrm{X}$. Thus for all $x \in \mathrm{X}$ and for every neighborhood W of H, there is $\tau_W^x > 0$ such that $\widetilde{\pi}(x,\, [\tau_W^x,\, +\infty)) \subset W$. Since V is a neighborhood of H, given $x \in \mathrm{X}$ there exists $\gamma > 0$ such that either $\widetilde{\pi}(x,\, \gamma) \in V_1$ or $\widetilde{\pi}(x,\, \gamma) \in V_2$. If $\widetilde{\pi}(x,\, \gamma) \in V_1$, then $x \in \widetilde{\mathrm{P}}^+(\mathrm{H}_1)$. But if $\widetilde{\pi}(x,\, \gamma) \in V_2$, then $x \in \widetilde{\mathrm{P}}^+(\mathrm{H}_2)$. Here, we have $\mathrm{X} = \widetilde{\mathrm{P}}^+(\mathrm{H}) = \widetilde{\mathrm{P}}^+(\mathrm{H}_1) \cup \widetilde{\mathrm{P}}^+(\mathrm{H}_2)$. Notice that $\mathrm{H}_1 \subset V_1 \subset \widetilde{\mathrm{P}}^+(\mathrm{H}_1)$ and $\mathrm{H}_2 \subset V_2 \subset \widetilde{\mathrm{P}}^+(\mathrm{H}_2)$. Thus it follows that H_1 and H_2 are $\widetilde{\pi}-$attractors. Then by Lemma 3.7, it follows that $\widetilde{\mathrm{P}}^+(\mathrm{H}_1)$ and $\widetilde{\mathrm{P}}^+(\mathrm{H}_2)$ are open sets. But this is a contradiction, since by hypothesis X is connected and

$$\mathrm{X} = \widetilde{\mathrm{P}}^+(\mathrm{H}) = \widetilde{\mathrm{P}}^+(\mathrm{H}_1) \cup \widetilde{\mathrm{P}}^+(\mathrm{H}_2),$$

where $\widetilde{\mathrm{P}}^+(\mathrm{H}_1)$ and $\widetilde{\mathrm{P}}^+(\mathrm{H}_2)$ are non-empty disjoint open sets. The proof is complete. □

In [2], we proved (Theorem 3.2) the following result: Suppose X is locally connected. Let H be a non-empty compact subset of X and suppose every component A of H has the property $\mathrm{I}(A) \subset A$ and $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is open. Then H is asymptotically $\widetilde{\pi}-$stable if and only if H has a finite number of components each of them being asymptotically $\widetilde{\pi}-$stable. This result can be formulated in the following manner for the case of global asymptotic stability.

Theorem 3.2. *Suppose* X *is locally connected. Let* H *be a nonempty compact subset of* X *and suppose every component A of* H *has the property* $\mathrm{I}(A) \subset A$ *and* $\widetilde{\mathrm{P}}^+(\mathrm{H})$ *is open. Then* H *is globally asymptotically $\widetilde{\pi}-$stable if and only if* H *has a finite number of components,* $\mathrm{H}_1, ..., \mathrm{H}_k$ *each of them being asymptotically $\widetilde{\pi}-$stable and* $\widetilde{\mathrm{P}}^+(\mathrm{H}_1) \cup ... \cup \widetilde{\mathrm{P}}^+(\mathrm{H}_k) = \mathrm{X}$.

The next result is Ura's Theorem for impulsive systems and it can be found in [6].

Theorem 3.3. *Let* $(\mathrm{X},\, \pi; \mathrm{M},\, \mathrm{I})$ *be an impulsive semidynamical system on a locally compact space* X *and let A be a compact subset of* X*. Then A is $\widetilde{\pi}-$stable if and only if* $\widetilde{\mathrm{D}}^+(A) = A$.

Now, we present a result that gives conditions for $\mathrm{H} \subset \mathrm{X}$ to be globally asymptotically $\widetilde{\pi}-$stable, provided $(\mathrm{X},\, \pi; \mathrm{M},\, \mathrm{I})$ is of $Ch0^+$.

Theorem 3.4. *Let* $(\mathrm{X},\, \pi; \mathrm{M},\, \mathrm{I})$ *be an impulsive semidynamical system of $Ch0^+$ on a connected locally compact space* X*. If* H *is a compact $\widetilde{\pi}-$invariant subset of* X*,* $\mathrm{M} \subset \mathrm{H}$ *and* H *is a $\widetilde{\pi}-$attractor, then* H *is globally asymptotically $\widetilde{\pi}-$stable.*

Proof. Clearly $\mathrm{H} \subset \widetilde{\mathrm{D}}^+(\mathrm{H})$. We will show that $\widetilde{\mathrm{D}}^+(\mathrm{H}) \subset \mathrm{H}$. Let $x \in \widetilde{\mathrm{D}}^+(\mathrm{H})$. Then there exists $y \in \mathrm{H}$ such that $x \in \widetilde{\mathrm{D}}^+(y) = \widetilde{\mathrm{K}}^+(y)$, since $(\mathrm{X},\, \pi; \mathrm{M},\, \mathrm{I})$ is of $Ch0^+$. Thus

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since $y \in \mathrm{H}$ and H is closed and $\widetilde{\pi}-$invariant, it follows that $x \in \mathrm{H}$. Therefore $\widetilde{\mathrm{D}}^+(\mathrm{H}) = \mathrm{H}$ and by Ura's Theorem (Theorem 3.3) H is $\widetilde{\pi}-$stable and hence $\widetilde{\pi}$-orbitally stable [see [6], Theorem 4.1]. Since H is a $\widetilde{\pi}-$attractor, then H is asymptotically $\widetilde{\pi}-$stable.

Its remains to prove that $\widetilde{\mathrm{P}}^+(\mathrm{H}) = \mathrm{X}$. But it is enough to show that $\partial\widetilde{\mathrm{P}}^+(\mathrm{H}) = \emptyset$. Suppose $\partial\widetilde{\mathrm{P}}^+(\mathrm{H}) \neq \emptyset$. Then given $x \in \partial\widetilde{\mathrm{P}}^+(\mathrm{H})$, there exists a sequence $\{x_n\}_{n\geq 1} \subset \widetilde{\mathrm{P}}^+(\mathrm{H})$ such that $x_n \overset{n\to+\infty}{\longrightarrow} x$. Besides $\widetilde{\mathrm{L}}^+(x_n) \subset \mathrm{H}$, because $x_n \in \widetilde{\mathrm{P}}^+(\mathrm{H})$ for all n. Since $x_n \overset{n\to+\infty}{\longrightarrow} x$ and $\widetilde{\mathrm{L}}^+(x_n) \subset \mathrm{H}$, $n = 1, 2, ...$, it follows that $\widetilde{\mathrm{J}}^+(x) \cap \mathrm{H} \neq \emptyset$. In this manner $\widetilde{\mathrm{D}}^+(x) \cap \mathrm{H} \neq \emptyset$. But $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is of $Ch0^+$. Thus $\widetilde{\mathrm{D}}^+(x) = \widetilde{\mathrm{K}}^+(x)$ and therefore $x \in \widetilde{\mathrm{P}}^+(\mathrm{H})$. But this is a contradiction, since $\partial\widetilde{\mathrm{P}}^+(\mathrm{H}) \cap \widetilde{\mathrm{P}}^+(\mathrm{H}) = \emptyset$ whenever $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is open in X. □

In the same manner we prove the following proposition.

Proposition 3.4. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system of* $Ch0^+$ *on a connected space* X. *If* $\mathrm{H} \subset \mathrm{X}$ *is compact,* $\widetilde{\pi}-$*invariant, asymptotically* $\widetilde{\pi}-$*stable and* $\mathrm{M} \subset \mathrm{H}$*, then* H *is globally asymptotically* $\widetilde{\pi}-$*stable.*

The next corollary follows from Theorem 3.4 and Proposition 3.3.

Corollary 3.2. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system of* $Ch0^+$ *on a normal connected locally compact space* X. *Suppose* H *is a compact,* $\widetilde{\pi}-$*invariant, asymptotically* $\widetilde{\pi}-$*stable subset of* X, $\mathrm{M} \subset \mathrm{H}$ *and every component* A *of* H *has the property* $\mathrm{I}(A) \subset A$. *Then* H *is connected.*

Another result that gives conditions for $\mathrm{H} \subset \mathrm{X}$ to be globally asymptotically $\widetilde{\pi}-$stable is the following.

Theorem 3.5. *Let* $(\mathbb{R}^2, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system of* $Ch0^+$ *and let* S *be the set of rest points of this flow. Suppose* $\mathrm{H} \subset \mathbb{R}^2$ *is compact,* $\widetilde{\pi}-$*invariant and* $\mathrm{M} \subset \mathrm{H}$. *Suppose also there exists a neighborhood* $B(\mathrm{H}, \eta)$ *of* H *that contains no periodic orbits and for all* $y \in B(\mathrm{H}, \eta)$, $\widetilde{\mathrm{L}}^+(y) \cap \mathrm{M} = \emptyset$ *and* $B[\mathrm{H}, \eta] \cap S = \{s_0\}$, $s_0 \in \mathrm{H}$. *Then* H *is globally asymptotically* $\widetilde{\pi} -$ *stable and, consequently,* $S = \{s_0\}$.

Proof. As in the beginning of the proof of Theorem 3.4, it follows clearly that $\widetilde{\mathrm{D}}^+(\mathrm{H}) = \mathrm{H}$ and by Ura's Theorem (Theorem 3.3), H is $\widetilde{\pi}-$stable and hence $\widetilde{\pi}$-orbitally stable [see [6], Theorem 4.1]. Then, since $B(\mathrm{H}, \eta)$ is a neighborhood of H, there exists an open neighborhood V of H such that $\widetilde{\pi}^+(V) \subset V \subset B(\mathrm{H}, \eta)$. Notice that for all $x \in V$, $\widetilde{\mathrm{L}}^+(x) \neq \emptyset$. Moreover, $\widetilde{\mathrm{L}}^+(x)$ is compact. Since there are no periodic orbits in $B(\mathrm{H}, \eta)$ and for all $y \in B(\mathrm{H}, \eta)$ we have $\widetilde{\mathrm{L}}^+(y) \cap \mathrm{M} = \emptyset$, it follows that if $x \in V$ then $\widetilde{\mathrm{L}}^+(x)$ consists of a single rest point, by Theorem 3.1. But $B[\mathrm{H}, \eta] \cap S = \{s_0\}$. Therefore $\widetilde{\mathrm{L}}^+(x) = \{s_0\}$, $x \in V$. Hence H is a $\widetilde{\pi}-$attractor and thus H is asymptotically $\widetilde{\pi}-$stable. By Theorem 3.4, it follows that H is globally asymptotically $\widetilde{\pi}-$stable. □

If in Theorem 3.5 we replace the assumption that $B[\mathrm{H}, \eta] \cap S = \{s_0\}$ by $S \subset \mathrm{H} \setminus \mathrm{M}$, then the result follows as well see the next corollary.

Corollary 3.3. *Let* $(\mathbb{R}^2, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system of* $Ch0^+$ *and let* S *be the set of rest points of this flow. Suppose* $\mathrm{H} \subset \mathbb{R}^2$ *is compact,* $\widetilde{\pi}-$*invariant,* $\mathrm{M} \subset \mathrm{H}$ *and* $S \subset \mathrm{H} \backslash \mathrm{M}$*. If there exists a neighborhood* $B(\mathrm{H}, \eta)$ *of* H *that contains no periodic orbits such that for all* $y \in B(\mathrm{H}, \eta)$ *we have* $\widetilde{\mathrm{L}}^+(y) \cap \mathrm{M} = \emptyset$*, then* H *is globally asymptotically* $\widetilde{\pi} -$ *stable.*

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