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Continuity of local unstable manifolds and of pullback attractors relatively to regular non-autonomous perturbations of a semilinear differential equation

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In this paper we prove a result on lower semicontinuity of *pullback attractors* for dynamical systems given by semilinear differential equations in a Banach space. The situation considered is such that the perturbed dynamical system is non-autonomous whereas the limiting dynamical system is autonomous and has an attractor given as union of unstable manifold of hyperbolic equilibrium points. Starting with a semilinear autonomous equation with a hyperbolic equilibrium solution and introducing a very small non-autonomous perturbation we prove the existence of a hyperbolic global solution for the perturbed equation near this equilibrium. Then we prove that the local unstable and stable manifolds associated to them are given as graphs (roughness of dichotomy plays a fundamental role here). Moreover, we prove the continuity of this local unstable and stable manifolds with respect to the perturbation. With that result we conclude the lower semicontinuity of *pullback attractors*. October, 2006 ICMC-USP

1. INTRODUCTION

Attractors for dynamical systems is a growing subject which has received a lot of attention in the last thirty five years. In the case of autonomous dynamical systems (nonlinear semigroups) the theory for existence and upper semicontinuity relatively to perturbations (singular or not) of attractors is quite well developed. The characterization of attractors for infinite dimensional autonomous dynamical systems remains restricted to some very specific cases (mostly gradient systems). The same can be said of lower semicontinuity of attractors for it is connected to the characterization of attractors. Perhaps the most celebrated general result known for the characterization of attractors of autonomous dynamical systems says that a gradient, asymptotically compact, nonlinear semigroup with bounded

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set of equilibria has attractor which is the the unstable manifold of the set of equilibria (if the equilibria are isolated then there is only a finite number of them and the attractor is the union of the unstable manifolds of equilibrium points). On the other hand, the corresponding development for the attractors of non-autonomous dynamical systems took place in the last fifteen years. In this period, conditions for existence of attractors for non-autonomous dynamical systems (nonlinear processes) have been established as well upper semicontinuity relatively to perturbations of attractors. The characterization of non-autonomous attractors in a Banach space is lacking and consequently results on lower semicontinuity relatively to perturbations of non-autonomous attractors are yet to be accomplished (see [16] for a finite dimensional treatment of this problem).

We note that in the proof of lower semicontinuity of attractors of an autonomous dynamical system under an autonomous perturbation (singular or not) it is only needed that the limiting semigroup system is gradient and that all of its equilibrium points are hyperbolic. That is saying that lower semicontinuity follows with the characterization of the limiting problem and no characterization is needed for the perturbed problem.

Having observed this it is natural to pursue the lower semicontinuity of attractors of non-autonomous dynamical systems for which the limiting dynamical system is an autonomous dynamical system with known (hyperbolic) structure.

Before we proceed let us define attractors for autonomous and non-autonomous dynamical systems.

Definition 1.1. Denote by $\mathcal{C}(\mathcal{Z})$ the space of the continuous (nonlinear) operators $\mathcal{T}:\mathcal{Z}\to\mathcal{Z}$. A family $\{S(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}\subset C(\mathcal{Z})$ which satisfies

$$\begin{aligned}
&1)\ S(\tau,\tau)=I,\\
&2)\ S(t,\sigma)S(\sigma,\tau)=S(t,\tau),\ \text{for each } t\geqslant\sigma\geqslant\tau,\\
&3)\ (t,\tau)\mapsto S(t,\tau)z_0 \text{ is continuous for } t\geqslant\tau,\ z_0\in\mathcal{Z}.
\end{aligned}$$

is called a nonlinear process. In the particular case when each $S(t,\tau)\in L(\mathcal{Z})$, $t\geqslant\tau$, we say that $\{S(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}$ is a linear process.

In the autonomous case; that is, $S(t,\tau)=\bar{S}(t-\tau)$ for all $t\geqslant\tau\in\mathbb{R}$, we say that $\{\bar{S}(t):t\geqslant 0\}$ is a nonlinear semigroup (we will not distinguish S and $\bar{S}$ and will write $S(t,\tau)=S(t-\tau)$).

In our context an attractor (for a nonlinear process or a nonlinear semigroup) will be what is called in the literature a *pullback attractor* (see [10, 14]).

Definition 1.2. Let $\{S(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}$ be a nonlinear process and $\{A(t):\ t\in\mathbb{R}\}$ be a family of subsets of $\mathcal{Z}$. We say that $\{A(t):\ t\in\mathbb{R}\}$ *pullback attracts* a bounded set $B\subset\mathcal{Z}$ under $\{S(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}$ if

$$\lim_{\tau\to-\infty}\operatorname{dist}(S(t,\tau)B,A(t))=0,\ \forall t\in\mathbb{R}.$$

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We say that $\{A(t): \ t\in\mathbb{R}\}$ is *invariant* under the process $\{S(t,\tau): t\geqslant\tau\in\mathbb{R}\}$ if $S(t,\tau)A(\tau)=A(t)$ for all $t\geqslant\tau\in\mathbb{R}$. We say that $\{A(t): \ t\in\mathbb{R}\}$ is a *pullback attractor* for the process $\{S(t,\tau): \ t\geqslant\tau\in\mathbb{R}\}$ if the following is satisfied

- $A(t)$ is compact for all $t\in\mathbb{R}$,
- $\{A(t): \ t\in\mathbb{R}\}$ pullback attracts each bounded set $B\subset\mathcal{Z}$ and
- $\{A(t): \ t\in\mathbb{R}\}$ is invariant.

Remark 1. 1. We note that, in the particular case for which $S(t,\tau)=S(t-\tau)$ for all $t\geqslant\tau\in\mathbb{R}$, the above definition coincides with the definition of global attractor for the nonlinear semigroup $\{S(t): t\geqslant 0\}$ given in [4, 11, 15, 18, 19]. We also note that, in the non-autonomous case, the pullback attractor does not possess any forward attraction (in general). The definition, characterization and continuity of forward attractors (different from pullback attractors) for asymptotically autonomous processes will be the subject of a forthcoming paper.

DEFINITION 1.3. Let $\eta\in[0,1]$ be a parameter and $\mathcal{Z}$ be a Banach space. For each $\eta\in[0,1]$ let $A_\eta\subset\mathcal{Z}$. We say that $\{A_\eta\}_{\eta\in[0,1]}$

- *Upper semicontinuous* at $\eta=0$ if $\lim_{\eta\to0}\operatorname{dist}(A_\eta,A_0)=0$,
- *Lower semicontinuous* at $\eta=0$ if if $\lim_{\eta\to0}\operatorname{dist}(A_0,A_\eta)=0$,
- *Continuous* at $\eta=0$ if it is upper and lower semicontinuous.

Let $\{A_\eta(t): t\in\mathbb{R}\}_{\eta\in[0,1]}$ be a family of sets in $\mathcal{Z}$ we say that this family is upper (lower) semicontinuous at $\eta=0$ if $\{A_\eta(t)\}_{\eta\in[0,1]}$ is upper (lower) semicontinuous for each $t\in\mathbb{R}$.

In this paper we prove the lower semicontinuity of pullback attractors in the situation that the perturbed dynamical system is non-autonomous whereas the limiting dynamical system is an autonomous dynamical system which has an attractor given as union of unstable manifolds of hyperbolic equilibria.

The proof of lower semicontinuity of attractors of autonomous gradient dynamical systems, for which all equilibria are hyperbolic, under autonomous perturbation relies on the continuity of the equilibria and of the local unstable manifolds under perturbation. Once we have these the continuity of the global unstable manifolds and the continuity of attractors is obtained in the following way: given a point y_0 in the limiting attractor, we follow solution through it backward in time until it enters the neighborhood of an equilibrium point where we have the continuity of local unstable manifolds; we then approximate it by a point in the unstable manifold of a hyperbolic equilibria of the perturbed problem and then follow the solution starting at this approximation forward by the same time, obtaining the approximation of y_0 by points in the perturbed attractor.

To extend this for non-autonomous perturbation the procedure is similar to the one adopted in the autonomous case (see [1, 2, 5, 8]), namely:

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• To prove that for each hyperbolic equilibrium point of the limiting dynamical system and for small enough perturbation, there is a unique global hyperbolic solution of the perturbed non-autonomous dynamical system; that is, a solution with the property that the linearization of the non-autonomous dynamical system around it has exponential dichotomy.

• To prove that these global solutions converge as the perturbation becomes smaller to the corresponding hyperbolic equilibrium point.

• To prove, in a fixed neighborhood of the equilibrium point and for each small enough perturbation, that the global hyperbolic solution has an unstable manifold.

• To prove that all the unstable manifolds of the hyperbolic solutions converge to the unstable manifold of the corresponding hyperbolic equilibria in a small enough neighborhood around the hyperbolic equilibria; that is, they behave continuously as the perturbation tends to zero.

• To prove that the continuity of the unstable manifold in a small neighborhood of the hyperbolic equilibrium is enough to guarantee the continuity of the global unstable manifold and of the pullback attractors.

This paper is devoted to follow this outline. The main step is to obtain the continuity of local unstable manifolds and that will require that we study the continuity of the time dependent projections associated to the dichotomy with respect to the parameter.

In order to be more specific about the results proved in this paper we will introduce some terminology and basic known facts.

We consider the semilinear problem on a Banach space $\mathcal{Z}$

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_0(y), \\ &y(\tau) = y_0 \end{aligned} \tag{1}$$

and a non-autonomous perturbation of it

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_\eta(t,y), \\ &y(\tau) = y_0. \end{aligned} \tag{2}$$

Assuming that, for $\eta \in [0,1]$, $f_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is continuous and uniformly (for $t \in \mathbb{R}$) Lipschitz continuous in bounded subsets of $\mathcal{Z}$ the problems (1) and (2) are locally well posed and assuming solutions exist in $[\tau,\infty)$ for any $\tau \in \mathbb{R}$ and $y_0 \in \mathcal{Z}$ we can define

$$T_\eta(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s,\tau)y_0)ds, \tag{3}$$

where $\{T_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ is a nonlinear process.

We assume $\{T_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has a *pullback attractor* $\{A_\eta(t) : t \in \mathbb{R}\}$ for each $\eta \geqslant 0$. Of course $A_0(t) = A_0$ is independent of t. Also, assume that (1) has a finite number of equilibrium points y_i^*, $1 \leqslant i \leqslant n$, all of them *hyperbolic* (see Section 2 for a definition) and

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that A_0 is the union of the *unstable manifolds* (see Section 3 for a definition) $W^u(y_i^*)$ of the equilibrium points y_i^*, $1 \leqslant i \leqslant n$.

For the perturbation we assume that

$$\lim_{\eta\to 0} \sup_{(t,z)\in\mathbb{R}\times B(0,r)} \|f_\eta(t,z) - f_0(z)\|_{\mathcal{Z}} + \|(f_\eta)_z(t,z) - f_0'(z)\|_{L(\mathcal{Z})} = 0, \tag{4}$$

for all $r > 0$.

Under (4) it is easy to see that, for each $T > 0$,

$$\sup_{\tau\in\mathbb{R}} \sup_{t\in[0,T]} \sup_{\|z\|_{\mathcal{Z}}\leqslant r} \|T_\eta(t+\tau,\tau)z - T_0(t+\tau-\tau)z\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0. \tag{5}$$

Given a hyperbolic equilibria y_i^* for (1), we prove in Section 2 that, given $\epsilon > 0$, there exists η_0 such that, for all $\eta \leqslant \eta_0$, there exists a global solution $\xi_\eta(t)$, $t \in \mathbb{R}$, of (2) uniformly (in time) ϵ-close to y_i^*. Moreover, $\xi_\eta(\cdot)$ is the unique global solution satisfying this property and it is also hyperbolic, in the sense that the linearization of (2) around $\xi_\eta(\cdot)$ possesses an exponential dichotomy with family of projections $Q(t) : \mathcal{Z} \to \mathcal{Z}$, $t \in \mathbb{R}$ (see Definition 2.1). Hence, we could say that hyperbolic global solutions are the natural generalizations in the non-autonomous framework of hyperbolic stationary solutions.

By the change of variable $z = y - \xi_\eta$, we translate all the dynamics around $\xi_\eta(\cdot)$ to the zero solution, so that we can consider

$$\dot{z} = (\mathcal{A} + B(t))z + h_\eta(t,z), \quad z(\tau) = z_0 \in \mathcal{Z} \tag{6}$$

where $\mathcal{A} = \mathfrak{B} + f_0'(y_0^*)$, $B(t) = (f_\eta)_y(\xi_\eta(t)) - f_0'(y_0^*)$ and $h_\eta : \mathbb{R}\times\mathcal{Z} \to \mathcal{Z}$ differentiable with $h_\eta(0) = 0$, $(h_\eta)_z(0) = 0 \in L(\mathcal{Z})$. Note that the limiting function $h_0 : \mathcal{Z} \to \mathcal{Z}$ is such that $h_0(0) = 0$, $h'(0) = 0 \in L(\mathcal{Z})$. Suppose that $\mathcal{A}$ is such that $\sigma(\mathcal{A})$ is disjoint of the imaginary axis, σ^+ is bounded, the projection $\mathcal{Q} \in L(\mathcal{Z})$ is defined and $\mathcal{A}^+$ and $\mathcal{A}^-$ are the restrictions of $\mathcal{A}$ to the range and kernel of $\mathcal{Q}$. The perturbation hypothesis (4) implies that

$$\lim_{\eta\to 0} \sup_{(t,z)\in\mathbb{R}\times B(0,r)} \|h_\eta(t,z) - h_0(z)\|_{\mathcal{Z}} + \|(h_\eta)_z(t,z) - h_0'(z)\|_{L(\mathcal{Z})} = 0, \tag{7}$$

for all $r > 0$. With this we obtain that the global solution found is hyperbolic.

In Sections 3 and 4 we prove the existence of unstable and stable manifolds around the zero solution for (6). Moreover, these manifolds are locally given as graphs of Lipschitz functions; that is, if V is a suitably small neighborhood of $z = 0$ in $\mathcal{Z}$ (independent of η), with $\xi_\eta(t) \in V$, there exist $(\tau, z) \mapsto \Sigma_\eta^{*,u}(\tau, Q(\tau)z)$, $\Sigma_\eta^{*,s}(\tau, (I - Q(\tau))z) : \mathbb{R}\times V \to \mathcal{Z}$, such that the local unstable and stable manifolds $W^u_{\mathrm{loc},\eta}(\xi_\eta)$, $W^s_{\mathrm{loc},\eta}(\xi_\eta)$ (the part of the unstable and stable manifolds which are in V) for (6) are given by

$$W^u_{\mathrm{loc},\eta}(\xi_\eta)(\tau) = \{w \in V : w = Q_\eta(\tau)w + \Sigma_\eta^{*,u}(\tau, Q(\tau)w)\}.$$

and

$$W^s_{\mathrm{loc},\eta}(\xi_\eta)(\tau) = \{w \in V : w = \Sigma_\eta^{*,u}(\tau, (I - Q(\tau)w)) + (I - Q_\eta(\tau)w)\}.$$

Note that these manifolds have to be part of the associated pullback attractors for (2), so that, in particular, we are describing part of the structure of these attractors and giving lower bounds for the dimension of them.

Remark 1. 2. Observe that once we have a complete hyperbolic solution for the limiting problem, our result on the continuity of local unstable manifolds does not depend on the fact that the limiting problem is an autonomous equation. We will pursue this further to obtain continuity of attractors of semilinear autonomous problems which are unstable manifold of a finite number of global hyperbolic solutions in a future work.

Section 5 is devoted to the analysis of the relationship between these perturbed manifolds and the associated limiting one for $\eta = 0$. Actually, a result on the continuity on them can be proven as it follows that

$$\sup_{t\leqslant\tau}\sup_{z\in V}\left\{\|Q_\eta(t)z-\mathcal{Q}z\|_{\mathcal{Z}}+\|\Sigma_\eta^{*,u}(t,Q_\eta(t)z)-\Sigma_0^{*,u}(\mathcal{Q}z)\|_{\mathcal{Z}}\right\}\stackrel{\eta\to 0}{\longrightarrow}0$$

and

$$\sup_{t\geqslant\tau}\sup_{z\in V}\left\{\|Q_\eta(t)z-\mathcal{Q}z\|_{\mathcal{Z}}+\|\Sigma_\eta^{*,s}(t,(I-Q_\eta(t))z)-\Sigma_0^{*,s}((I-\mathcal{Q})z)\|_{\mathcal{Z}}\right\}\stackrel{\eta\to 0}{\longrightarrow}0.$$

As a consequence of this and of the continuity of the processes associated to our equations, we conclude in Section 7 upper and lower semicontinuity of pullback attractors with respect η at $\eta = 0$ (see [16]).

Our results, as they have been written, can be applied directly to hyperbolic partial differential equations (see Section 8), and with some minor (but followed with care throughout) changes in the proof can be also applied to parabolic differential equations.

Finally, in Section 9, we comment the results in this paper and point out some of its consequences that will be pursued in future works.

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2. EXISTENCE AND CONTINUITY OF GLOBAL HYPERBOLIC SOLUTIONS

In this section we prove that near a hyperbolic equilibrium of (1) there is a unique global hyperbolic solution of (2). To that end we first need to give meaning to a hyperbolic equilibrium for (1). We observe that since $\mathfrak{B}$ only generates a strongly continuous semigroup, it is not enough to say that the spectrum of the linearization is disjoint of the imaginary axis.

The dichotomy property that leads to the saddle point property (nonlinear dichotomy) needs to be imposed. This dichotomy property can be verified in most applications.

Suppose that y_0^* is an equilibrium solution for (1); that is, $\mathfrak{B}y_0^* + f_0(y_0^*) = 0$.

If $y(t)$ is a solution to (1) and $z = y - y_0^*$, we rewrite equation (1) as

$$\begin{aligned} \dot{z} &= \mathcal{A}z + h_0(z) \\ z(\tau) &= z_0 \end{aligned} \tag{8}$$

where $\mathcal{A} = \mathfrak{B} + f_0'(y_0^*)$ and $h_0(z) = f_0(y_0^* + z) - f_0(y_0^*) - f_0'(y_0^*)z$. Then $\mathcal{A}$ generates a strongly continuous semigroup $\{e^{\mathcal{A}t} : t \geqslant 0\}$ of bounded linear operators. Moreover, 0 is an equilibrium solution for (8) and $h_0(0) = 0$, $h_0'(0) = 0 \in L(\mathcal{Z})$.

Assume that the equilibrium solution y_0^* is hyperbolic; that is, assume that the spectrum of $\mathcal{A}$ does not intersect the imaginary axis and that the set $\sigma^+ = \{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\lambda > 0\}$ is compact. If γ is a smooth closed simple curve in $\rho(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ oriented counterclockwise and enclosing σ^+ let

$$\mathcal{Q} = \mathcal{Q}(\sigma^+) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - \mathcal{A})^{-1} d\lambda \tag{9}$$

and define $\mathcal{Z}^+ = \mathcal{Q}(\mathcal{Z})$, $\mathcal{Z}^- = (I - \mathcal{Q})(\mathcal{Z})$, and $\mathcal{A}^\pm = \mathcal{A}_{|_{\mathcal{Z}^\pm}}$. It is clear that $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$, that $\mathcal{A}^-$ generates a strongly continuous semigroup on $\mathcal{Z}^-$, and that $\mathcal{A}^+ \in L(\mathcal{Z}^+)$. Assume that, for certain $M_1 \geqslant 1$, $\beta > 0$,

$$\begin{aligned} \|e^{\mathcal{A}^+ t}\|_{L(\mathcal{Z}^+)} &\leqslant M_1 e^{\beta t}, \quad t \leqslant 0, \\ \|e^{\mathcal{A}^- t}\|_{L(\mathcal{Z}^-)} &\leqslant M_1 e^{-\beta t}, \quad t \geqslant 0. \end{aligned} \tag{10}$$

Remark 2. 1. We note that if $\mathfrak{B}$ were the generator of an analytic semigroup it would be enough to say that $\sigma(\mathfrak{B})$ is disjoint of the imaginary axis to ensure (10). In this aspect, abstract parabolic problems ($\mathfrak{B}$ generates analytic semigroup) are simpler than abstract hyperbolic problems ($\mathfrak{B}$ generates strongly continuous semigroup).

Suppose the nonlinearity f_η is a small non-autonomous perturbation of f_0, in particular assume that, for some $r > 0$,

$$\lim_{\eta \to 0} \sup_{(t,\phi) \in \mathbb{R} \times B(y_0^*, r)} \left\{ \|f_\eta(t,\phi) - f_0(\phi)\|_{\mathcal{Z}} + \|(f_\eta)_y(t,\phi) - (f_0)_y(\phi)\|_{L(\mathcal{Z})} \right\} = 0. \tag{11}$$

Then we can prove the following result

Theorem 2.1. *Let y_0^* be a hyperbolic equilibrium solution for* (1) *and assume that* (11) *holds. Then, there exists $\eta_0 > 0$ such that, for each $0 < \eta < \eta_0$ there exists*

$$\mathbb{R} \ni t \mapsto \xi_\eta^*(t) \in \mathcal{Z}$$

a complete bounded solution to (2) *such that*

$$\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} \|\xi_\eta^*(t) - y_0^*\|_{\mathcal{Z}} = 0.$$

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Proof. Denote by $y(t,\tau;y_0)$ the solution of the initial value problem (2). Then, if we define $\phi = y - y_0^*$, it satisfies

$$\dot{\phi} = \mathcal{A}\phi + g_\eta(t,\phi) \tag{12}$$

where $g_\eta(t,\phi) = f_\eta(t,\phi + y_0^*) - f(y_0^*) - f_0'(y_0^*)\phi$, so that

$$\phi(t) = e^{\mathcal{A}(t-\tau)}\phi(\tau) + \int_\tau^t e^{\mathcal{A}(t-s)} g_\eta(s,(\phi(s)))ds. \tag{13}$$

Hence, if we project by $\mathcal{Q}$ and $I - \mathcal{Q}$ and take limits as $\tau \to +\infty$ and $\tau \to -\infty$ respectively, we get

$$\mathcal{Q}\phi(t) = \int_\infty^t e^{\mathcal{A}(t-s)} \mathcal{Q} g_\eta(s,(\phi(s)))ds$$

and

$$(I - \mathcal{Q})\phi(t) = \int_{-\infty}^t e^{\mathcal{A}(t-s)}(I - \mathcal{Q}) g_\eta(s,(\phi(s)))ds.$$

Consequently, a complete solution to (13) exists in a small neighborhood of $z = 0$ if and only if

$$\mathcal{T}(\phi)(t) = \int_\infty^t e^{\mathcal{A}(t-s)} \mathcal{Q} g_\eta(s,(\phi(s)))ds + \int_{-\infty}^t e^{\mathcal{A}(t-s)}(I - \mathcal{Q}) g_\eta(s,(\phi(s)))ds$$

has a unique fixed point in the set

$$\{\phi : \mathbb{R} \to \mathcal{Z} : \sup_{t\in\mathbb{R}} \|\phi(t)\|_{\mathcal{Z}} \leqslant \epsilon\}$$

with suitably small ϵ. But note that this follows from (10) by a fixed point argument for $\mathcal{T}$ if we assume that, for $\phi, \phi_1, \phi_2 \in B(0,\epsilon)$, $\|g_\eta(t,\phi)\|_{\mathcal{Z}} \leqslant \delta$ and that $\|g_\eta(t,\phi_1) - g_\eta(t,\phi_2)\|_{\mathcal{Z}} \leqslant \delta\|\phi_1 - \phi_2\|_{\mathcal{Z}}$, with $\delta > 0$ suitably small, which follows from (11) for η small enough, since $\delta, \epsilon \to 0$ as $\eta \to 0$.

Hence, we get that $\xi_\eta^*(\cdot)$ is uniformly close to y_0^* and tends to y_0^* as $\eta \to 0$. ∎

These solutions $\xi_\eta^*(\cdot)$ plays the role of an hyperbolic equilibrium for (2). Now, proceeding as in the autonomous case we change variables $z(t) = y(t) - \xi_\eta^*(t)$ in (2) and rewrite it as

$$\begin{aligned} \dot{z} &= (\mathcal{A} + B_\eta(t))z + h_\eta(t,z) \\ z(\tau) &= z_0 \end{aligned} \tag{14}$$

where $\mathbb{R} \ni t \mapsto B_\eta(t) \in L(\mathcal{Z})$ is strongly continuous and defined as $B_\eta(t) = (f_\eta)_z(\xi_\eta^*(t)) - f_0'(y_0^*)$, and $h_\eta(t,z) = f_\eta(t,\xi_\eta^*(t) + z) - f_\eta(t,\xi_\eta^*(t)) - (f_\eta)_z(\xi_\eta^*(t))z$. Hence, 0 is a globally defined bounded solution for (14) and $h_\eta(t,0) = 0$, $(h_\eta)_y(t,0) = 0 \in L(\mathcal{Z})$.

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Consider now the linear problem associated to (14)

$$\begin{aligned} \dot{z} &= \mathcal{A}z + B_\eta(t)z \\ z(\tau) &= z_0 \in \mathcal{Z}. \end{aligned} \tag{15}$$

Note that

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|B_\eta(t)\|_{L(\mathcal{Z})} = 0. \tag{16}$$

It is well known that problem (15) has a unique mild solution $U_\eta(t,\tau)z_0$ for each $z_0 \in \mathcal{Z}$ which satisfies

$$U_\eta(t,\tau)z_0 = e^{\mathcal{A}(t-\tau)}z_0 + \int_\tau^t e^{\mathcal{A}(t-s)}B_\eta(s)U_\eta(s,\tau)z_0 ds. \tag{17}$$

The family $U_\eta(t,\tau)$ is a linear process (see Definition 1.1).
From (17) is is easy to see that, for any $T > 0$,

$$\lim_{\eta\to 0} \sup_{t\in[0,T]} \sup_{\tau\in\mathbb{R}} \|U_\eta(t+\tau,\tau) - e^{\mathcal{A}t}\|_{L(\mathcal{Z})} \to 0.$$

Next we study the existence of exponential dichotomy for (15).

Definition 2.1. We say that (15) has exponential dichotomy with exponent ω and constant M if there exists a family of projections $\{Q_\eta(t): \ t\in\mathbb{R}\} \subset L(\mathcal{Z})$ such that

i) $Q_\eta(t)U_\eta(t,s) = U_\eta(t,s)Q_\eta(s)$, for all $t \geqslant s$;
ii) The restriction $U_\eta(t,s)_{|_{R(Q_\eta(s))}}$, $t \geqslant s$ is an isomorphism from $R(Q_\eta(s))$ into $R(Q_\eta(t))$ and its inverse is denoted by $U_\eta(s,t) : R(Q_\eta(t)) \to R(Q_\eta(s))$.
iii)

$$\begin{aligned} \|U_\eta(t,s)(I - Q_\eta(s))\| &\leqslant Me^{-\omega(t-s)} \quad t \geqslant s \\ \|U_\eta(t,s)Q_\eta(s)\| &\leqslant Me^{\omega(t-s)}, \quad t \leqslant s. \end{aligned} \tag{18}$$

The existence of an exponential dichotomy for (15) follows from Theorem 7.6.11. in [12], i.e., it holds that given $\omega < \beta$ and $M > M_1$, there exists $\delta_0 > 0$ such that, if $\|B_\eta(t)\|_{L(\mathcal{Z})} = \delta < \delta_0$ then (15) has exponential dichotomy.

A global solution of (14) $\xi^*_\eta(\cdot)$ satisfying it is defined for all $t \in \mathbb{R}$ and that the linearization around it has exponential dichotomy will be called a hyperbolic solution.

3. EXISTENCE OF UNSTABLE MANIFOLDS AS A GRAPH

Now we are ready to study the unstable an stable manifolds of a hyperbolic solution $\xi^*_\eta(\cdot)$.

Definition 3.1. The unstable manifold of a hyperbolic solution ξ^*_η to (2) is the set

$$W^u(\xi^*_\eta) = \{\,(\tau,\zeta)\in\mathbb{R}\times\mathcal{Z}:\ \text{there is a backward solution } z(t,\tau,\zeta) \text{ of } (2)\\ \text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to-\infty}\|z(t,\tau,\zeta)-\xi^*_\eta(t)\|=0\}.$$

The stable manifold of a hyperbolic solution $\xi^*_\eta(\cdot)$ to (2) is the set

$$W^s(\xi^*_\eta) = \{\,(\tau,\zeta)\in\mathbb{R}\times\mathcal{Z}:\ \text{there is a forward solution } z(t,\tau,\zeta) \text{ of } (2)\\ \text{satisfying } z(\tau,\tau,\zeta)=\zeta \text{ and such that } \lim_{t\to+\infty}\|z(t,\tau,\zeta)-\xi^*_\eta(t)\|=0\}.$$

We will show that the unstable and stable manifolds of ξ^*_η are given by maps

$$\mathbb{R}\times\mathcal{Z}\ni(t,z)\mapsto\Sigma^u(t,Q_\eta(t)z)\in(I-Q_\eta(t))\mathcal{Z}$$

$$\mathbb{R}\times\mathcal{Z}\ni(t,z)\mapsto\Sigma^s(t,(I-Q_\eta(t))z)\in Q_\eta(t)\mathcal{Z}.$$

The points in the unstable manifold will be those of the form

$$(t,Q_\eta(t)z+\Sigma^u(t,Q_\eta(t)z))\in\mathbb{R}\times\mathcal{Z} \text{ with } (t,z)\in\mathbb{R}\times\mathcal{Z} \text{ and } z \text{ small},$$

and the points on the stable manifold those of the form

$$(t,(I-Q_\eta(t))z+\Sigma^s(t,(I-Q_\eta(t))z))\in\mathbb{R}\times\mathcal{Z} \text{ with } (t,z)\in\mathbb{R}\times\mathcal{Z} \text{ and } z \text{ small}.$$

After that we will show, under suitable assumptions that, as η tends to zero, the unstable and stable manifolds above approach the unstable manifold and stable manifolds of the autonomous problem (1).

Note that the way we got (14) allows us to concentrate on the existence of invariant manifolds of complete hyperbolic trajectories around the zero stationary solution. Thus, if z is a solution of (14) we write $z^+(t)=Q_\eta(t)z$ and $z^-(t)=z-z^+(t)$. Then we have

$$\begin{aligned}\dot z^+ &= \mathcal{A}^+(t)z^+ + H(t,z^+,z^-),\\ \dot z^- &= \mathcal{A}^-(t)z^- + G(t,z^+,z^-),\end{aligned}\tag{19}$$

where

$$\mathcal{A}^+(t)=(\mathcal{A}+B_\eta(t))Q_\eta(t),$$

$$\mathcal{A}^-(t)=(\mathcal{A}+B_\eta(t)))(I-Q_\eta(t)),$$

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$$H(t,z^+,z^-) = Q_\eta(t)h_\eta(t,z^+ + z^-)$$

and

$$G(t,z^+,z^-) = (I - Q_\eta(t))h_\eta(t,z^+ + z^-)$$

(as there is no possible ambiguity, to simplify the notation we do not write the dependence on η of H and G).

Since at $(t,0,0)$ the functions H and G are zero with zero derivatives (with respect to z^+ and z^-), from the continuous differentiability of H and G, uniform with respect to t, we obtain that given $\rho > 0$ there exists $\delta > 0$ such that if $\|z\|_{\mathcal{Z}} = \|(z^+ + z^-)\|_{\mathcal{Z}} < \delta$, then

$$\begin{aligned}
&\|H(t,z^+,z^-)\|_{\mathcal{Z}} \leqslant \rho,\\
&\|G(t,z^+,z^-)\|_{\mathcal{Z}} \leqslant \rho,\\
&\|H(t,z^+,z^-) - H(t,\tilde{z}^+,\tilde{z}^-)\|_{\mathcal{Z}} \leqslant \rho(\|z^+ - \tilde{z}^+\|_{\mathcal{Z}} + \|z^- - \tilde{z}^-\|_{\mathcal{Z}}),\\
&\|G(t,z^+,z^-) - G(t,\tilde{z}^+,\tilde{z}^-)\|_{\mathcal{Z}} \leqslant \rho(\|z^+ - \tilde{z}^+\|_{\mathcal{Z}} + \|z^- - \tilde{z}^-\|_{\mathcal{Z}}).
\end{aligned} \tag{20}$$

Remark 3. 1. It is possible to extend H and G outside a ball of radius δ in such a way that the conditions (20) hold for all $z^+ \in \mathcal{Z}^+$, $z^- \in \mathcal{Z}^-$. Indeed, given a W-valued function g on the ball of radius δ contained in $V \times Z$, where V, Z, W are Banach spaces, define $g_\delta : V \times Z \to W$

$$g_\delta(z^+,z^-) = \begin{cases} g(z^+,z^-), & \|z^+ + z^-\|_{\mathcal{Z}} \leqslant \delta \\ g\left(\frac{\delta\, z^+}{\|z^+ + z^-\|_{\mathcal{Z}}}, \frac{\delta\, z^-}{\|z^+ + z^-\|_{\mathcal{Z}}}\right), & \|z^+ + z^-\|_{\mathcal{Z}} > \delta. \end{cases}$$

The extension g_δ becomes globally Lipschitz and simultaneously its Lipschitz constant is the Lipshitz constant for g restricted to the ball of radius δ.

The above consideration suggests to obtain first the result concerning existence of the unstable and stable manifolds under the assumption that (20) holds for all $z = (z^+, z^-) \in \mathcal{Z}$ with some suitably small $\rho > 0$. Also assuming that (20) holds for all $z = (z^+, z^-) \in \mathcal{Z}$, we will prove continuity of the unstable and stable manifolds. Finally, we will conclude the existence and continuity of local unstable and stable manifolds for the case when h_η only satisfies (20) for $\|z\|_{\mathcal{Z}} = \|z^+ + z^-\|_{\mathcal{Z}} < \delta$ with $\delta > 0$ suitably small.

Thus, assume additionally that h_η is such that H and G satisfy (20) for all $z^+(t) \in Q_\eta(t)\mathcal{Z}$, $z^-(t) \in (I - Q_\eta(t))\mathcal{Z}$ with certain $\rho > 0$, which will be defined below via (22). Let $W^u(t,0,0)$ be the unstable manifold of equilibrium solution $(0,0)$ to (19). We will show that there exists a bounded and Lipschitz continuous function $\Sigma_\eta^{*,u}(t,\cdot) : Q_\eta(t)\mathcal{Z} \to (I - Q_\eta(t))\mathcal{Z}$ such that

$$W^u_\eta(t,0,0) = \{(t,z^+,z^-) : z^- = \Sigma_\eta^{*,u}(t,z^+),\ z^+ \in Q_\eta(t)\mathcal{Z}\}.$$

Remark 3. 2. Observe that we are looking for a function $\Sigma_\eta^{*,u}(t)$ such that, if $\tau \in \mathbb{R}$ and $(\zeta, \Sigma_\eta^{*,u}(\tau,\zeta)) \in \mathcal{Z}$, then the solution of (19) such that $z^+(\tau) = \zeta$, $z^-(\tau) = \Sigma_\eta^{*,u}(\zeta)$ is such that $z(t)$ is in the graph of $\Sigma_\eta^{*,u}(t,\cdot)$ for all positive and all negative time t. This means that $z^-(t) = \Sigma_\eta^{*,u}(t, z^+(t))$ for all t and thus (19) becomes

$$\begin{aligned}
\dot{z}^+ &= \mathcal{A}^+(t)z^+ + H(t,z^+,\Sigma_\eta^{*,u}(t,z^+)),\\
\dot{z}^- &= \mathcal{A}^-(t)z^- + G(t,z^+,\Sigma_\eta^{*,u}(t,z^+)).
\end{aligned} \tag{21}$$

Also, the solution $(z^+(t), z^-(t))$ should tend to zero as $t \to -\infty$ (in particular, it should stay bounded as $t \to -\infty$). Since

$$z^-(t) = U_\eta(t,t_0)(I - Q_\eta(t_0))z^-(t_0) + \int_{t_0}^t U_\eta(t,s)(I - Q_\eta(s))G(s, z^+(s), \Sigma_\eta^{*,u}(s, z^+(s)))ds,$$

letting $t_0 \to -\infty$ we have that

$$z^-(t) = \Sigma_\eta^{*,u}(t, z^+(t)) = \int_{-\infty}^t U_\eta(t,s)(I - Q_\eta(s))G(s, z^+(s), \Sigma_\eta^{*,u}(s, z^+(s)))ds$$

and, in particular,

$$\begin{aligned}\Sigma_\eta^{*,u}(\tau,\zeta) &= \Sigma_\eta^{*,u}(\tau, z^+(\tau)) = z^-(\tau)\\ &= \int_{-\infty}^\tau U_\eta(\tau,s)(I - Q_\eta(s))G(s, z^+(s), \Sigma_\eta^{*,u}(s, z^+(s)))ds.\end{aligned}$$

In order to show existence of the function $\Sigma_\eta^{*,u}(\tau,\cdot)$ we will use the Banach contraction principle. For this we fix $D > 0$, $L > 0$, $0 < \vartheta < 1$ and choose $\rho > 0$ such that

$$\begin{array}{ll}\frac{\rho M}{\omega} \leqslant D, & \frac{\rho M}{\omega}(1+L) \leqslant \vartheta < 1,\\ \dfrac{\rho M^2(1+L)}{\omega - \rho M(1+L)} \leqslant L, & \rho M + \frac{\rho^2 M^2(1+L)(1+M)}{2\omega - \rho M(1+L)} < \omega.\end{array} \tag{22}$$

DEFINITION 3.2. Given $\eta > 0$, denote by $\mathcal{LB}(D,L)$ the complete metric space of all bounded and globally Lipschitz continuous functions $\mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ defined as $(\tau, z) \mapsto \Sigma(\tau, Q_\eta(\tau)z) \in (I - Q_\eta(\tau))\mathcal{Z}$ satisfying

$$\begin{array}{l}\sup\{\|\Sigma(\tau, Q_\eta(\tau)z)\|_{\mathcal{Z}},\ (\tau,z) \in \mathbb{R}\times\mathcal{Z}\} \leqslant D,\\ \|\Sigma(\tau, Q_\eta(\tau)z) - \Sigma(\tau, Q_\eta(\tau)\tilde{z})\|_{\mathcal{Z}} \leqslant L\|Q_\eta(\tau)z - Q_\eta(\tau)\tilde{z}\|_{\mathcal{Z}}, \forall(\tau, z, \tilde{z}) \in \mathbb{R}\times\mathcal{Z}\times\mathcal{Z},\end{array} \tag{23}$$

where the distance of $\Sigma, \tilde{\Sigma} \in \mathcal{LB}(D,L)$ is defined as

$$|\!|\!|\Sigma(\cdot,\cdot) - \tilde{\Sigma}(\cdot,\cdot)|\!|\!| := \sup\{\|\Sigma(\tau, Q_\eta(\tau)z) - \tilde{\Sigma}(\tau, Q_\eta(\tau)z)\|_{\mathcal{Z}},\ (\tau,z) \in \mathbb{R}\times\mathcal{Z}\}.$$

THEOREM 3.1. *Suppose that the above conditions are satisfied. Then, there exist a* $\Sigma_\eta^{*,u}(\tau,\cdot) \in \mathcal{LB}(D,L)$, *such that the unstable manifold* $W_\eta^u(\tau,0,0)$ *to* (19) *is given by*

$$W_\eta^u(0,0) = \{(\tau, w) \in \mathbb{R}\times\mathcal{Z} : w = (Q_\eta(\tau)w, \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w))\}. \tag{24}$$

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Proof: For $\tau \in \mathbb{R}$ and arbitrary $\zeta \in Q_\eta(\tau)\mathcal{Z}$, $\Sigma \in \mathcal{LB}(D,L)$ denote by $z^+(t) = \psi(t,\tau,\zeta,\Sigma)$ the solution of

$$\frac{dz^+}{dt} = \mathcal{A}^+(t)z^+ + H(t,z^+,\Sigma(t,z^+)),\ t<\tau, \quad z^+(\tau) = \zeta \in Q_\eta(\tau)\mathcal{Z}. \tag{25}$$

Next we define, for $\Sigma \in \mathcal{LB}(D,L)$,

$$\Phi(\Sigma)(\tau,\zeta) = \int_{-\infty}^{\tau} U_\eta(\tau,s)(I-Q_\eta(s))G(s,z^+(s),\Sigma(s,z^+(s)))ds,\ (\tau,\zeta) \in \mathbb{R}\times Q_\eta(\tau)\mathcal{Z}. \tag{26}$$

We will show that, for $\rho > 0$ satisfying (22), the map Φ takes $\mathcal{LB}(D,L)$ into itself, is a strict contraction, and hence possesses a unique fixed point in $\mathcal{LB}(D,L)$.

First note that, by (18), one has

$$\|\Phi(\Sigma)(\tau,\cdot)\|_{\mathcal{Z}} \leqslant \int_{-\infty}^{\tau} \rho M e^{-\omega(\tau-s)}ds = \frac{\rho M}{\omega}, \tag{27}$$

and from (22) we have $\sup\{\|\Phi(\Sigma)(\tau,Q_\eta(\tau)z)\|_{\mathcal{Z}},\ (\tau,z) \in \mathbb{R}\times\mathcal{Z}\} \leqslant D$.

Next, suppose that Σ and $\tilde{\Sigma}$ are functions satisfying (23), ζ, $\tilde{\zeta} \in Q_\eta(\tau)\mathcal{Z}$ and denote by $z^+(t) = \psi(t,\tau,\zeta,\Sigma)$, $\tilde{z}^+(t) = \psi(t,\tau,\tilde{\zeta},\tilde{\Sigma})$. Then

$$\begin{aligned} z^+(t) - \tilde{z}^+(t) &= U_\eta(t,\tau)Q_\eta(\tau)(\zeta-\tilde{\zeta}) \\ &\quad + \int_\tau^t U_\eta(t,s)Q_\eta(s)[H(s,z^+(s),\Sigma(s,z^+(s))) - H(s,\tilde{z}^+(s),\tilde{\Sigma}(s,\tilde{z}^+(s)))]ds, \end{aligned}$$

and with (18), (20) we obtain

$$\begin{aligned} \|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Z}} &\leqslant M e^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}} \\ &\quad + M\int_t^\tau e^{\omega(t-s)}\|H(s,z^+(s),\Sigma(s,z^+(s))) - H(s,\tilde{z}^+(s),\tilde{\Sigma}(s,\tilde{z}^+(s)))\|_{\mathcal{Z}}ds \\ &\leqslant M e^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}} + \rho M |\!|\!|\Sigma-\tilde{\Sigma}|\!|\!| \int_t^\tau e^{\omega(t-s)}ds \\ &\quad + \rho M(1+L)\int_t^\tau e^{\omega(t-s)}\|z^+(s)-\tilde{z}^+(s)\|_{\mathcal{Z}}ds. \end{aligned}$$

Let $\phi(t) = e^{-\omega(t-\tau)}\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Z}}$. Then,

$$\phi(t) \leqslant M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}} + \rho M\int_t^\tau e^{\omega(\tau-s)}ds |\!|\!|\Sigma-\tilde{\Sigma}|\!|\!| + \rho M(1+L)\int_t^\tau \phi(s)ds.$$

By Gronwall's inequality

$$\begin{aligned} \|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Z}} &\leqslant [M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}} e^{\omega(t-\tau)} + \rho M\int_t^\tau e^{\omega(t-s)}ds |\!|\!|\Sigma-\tilde{\Sigma}|\!|\!|]e^{-\rho M(1+L)(t-\tau)} \\ &\leqslant [M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}} + \rho M\omega^{-1}|\!|\!|\Sigma-\tilde{\Sigma}|\!|\!|]e^{-\rho M(1+L)(t-\tau)}. \end{aligned} \tag{28}$$

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Thus,

$$\begin{aligned}\|\Phi(\Sigma)(\tau,\zeta)-\Phi(\tilde{\Sigma})(\tau,\tilde{\zeta})\|_{\mathcal{Z}} \\ &\leqslant M\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\|G(s,z^{+}(s),\Sigma(s,z^{+}(s)))-G(s,\tilde{z}^{+}(s),\tilde{\Sigma}(s,\tilde{z}^{+}(s)))\|_{\mathcal{Z}}ds \\ &\leqslant \rho M\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\left[(1+L)\|z^{+}(s)-\tilde{z}^{+}(s)\|_{\mathcal{Z}}+\|\!|\Sigma-\tilde{\Sigma}|\!\|\right]ds.\end{aligned}$$

Using the estimates for $\|z^{+}-\tilde{z}^{+}\|_{\mathcal{Z}}$ we obtain

$$\begin{aligned}\|\Phi(\Sigma)(\tau,\zeta)-\Phi(\tilde{\Sigma})(\tau,\tilde{\zeta})\|_{\mathcal{Z}} \\ &\leqslant \frac{\rho M}{\omega}\left[1+\frac{\rho M(1+L)}{\omega-\rho M(1+L)}\right]\|\!|\Sigma-\tilde{\Sigma}|\!\|+\frac{\rho M^{2}(1+L)}{\omega-\rho M(1+L)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}.\end{aligned} \tag{29}$$

Let

$$I_{\Sigma}=\frac{\rho M}{\omega}\left[1+\frac{\rho M(1+L)}{\omega-\rho M(1+L)}\right] \quad \text{and} \quad I_{\zeta}=\frac{\rho M^{2}(1+L)}{\omega-\rho M(1+L)}.$$

Since $I_{\Sigma}\leqslant \frac{\rho M}{\omega}(1+L)$, it follows from (22), (29) that $I_{\Sigma}\leqslant\vartheta$, $I_{\zeta}\leqslant L$ and

$$\|\Phi(\Sigma)(\tau,\zeta)-\Phi(\tau,\tilde{\Sigma})(\tilde{\zeta})\|_{\mathcal{Z}}\leqslant L\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\vartheta\|\!|\Sigma-\tilde{\Sigma}|\!\|. \tag{30}$$

The inequality (30) with $\Sigma=\tilde{\Sigma}$ and (27) imply that Φ takes $\mathcal{LB}(D,L)$ into $\mathcal{LB}(D,L)$. Due to (22), estimate (30) with $\zeta=\tilde{\zeta}$ shows that Φ is a contraction map. Therefore, there exists a unique fixed point $\Sigma_{\eta}^{*,u}=\Phi(\Sigma_{\eta}^{*,u})$ in $\mathcal{LB}(D,L)$.

In what follows we will prove that, if $(z^{+}(t),z^{-}(t))$, $t\in\mathbb{R}$, is a global solution of (19) bounded as $t\to-\infty$, then there are constants $M\geqslant 1$ and $\gamma>0$ such that

$$\|z^{-}(t)-\Sigma_{\eta}^{*,u}(t,z^{+}(t))\|_{\mathcal{Z}}\leqslant Me^{-\gamma(t-t_0)}\|z^{-}(t_0)-\Sigma_{\eta}^{*,u}(t,z^{+}(t_0))\|_{\mathcal{Z}},\quad t_0\leqslant t. \tag{31}$$

Making $t_0\to-\infty$ in (31) we will obtain that $z^{-}(t)=\Sigma_{\eta}^{*,u}(t,z^{+}(t))$ for each $t\in\mathbb{R}$. That also ensures that $\Sigma_{\eta}^{*,u}(t,0)=0$, since $(0,0)$ is a stationary solution to (19).

Let $\zeta(t)=z^{-}(t)-\Sigma_{\eta}^{*,u}(t,z^{+}(t))$ and $y^{+}(s,t)$, $s\leqslant t$, be the solution of

$$\dot{y}^{+}=\mathcal{A}^{+}(s)y^{+}+H(s,y^{+},\Sigma_{\eta}^{*,u}(s,y^{+})),\ s<t,\quad y^{+}(t,t)=z^{+}(t).$$

Hence,

$$\begin{aligned}\|y^{+}(s,t)-z^{+}(s)\|_{\mathcal{Z}}&=\left\|\int_{t}^{s}U_{\eta}(s,\theta)[H(\theta,y^{+}(\theta,t),\Sigma_{\eta}^{*,u}(\theta,y^{+}(\theta,t)))-H(\theta,z^{+}(\theta),z^{-}(\theta))]d\theta\right\|_{\mathcal{Z}} \\ &\leqslant \rho M\int_{s}^{t}e^{\omega(s-\theta)}\left[(1+L)\|y^{+}(\theta,t)-z^{+}(\theta)\|_{\mathcal{Z}}+\|\zeta(\theta)\|_{\mathcal{Z}}\right]d\theta.\end{aligned}$$

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If $\psi(s) = e^{-\omega s}\|y^+(s,t) - z^+(s)\|_{\mathcal{Z}}$, then

$$\psi(s) \leqslant \rho M(1+L)\int_s^t \psi(\theta)d\theta + \rho M\int_s^t e^{-\omega\theta}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta, \quad s \leqslant t.$$

Using Gronwall's Lemma we have

$$\|y^+(s,t) - z^+(s)\|_{\mathcal{Z}} \leqslant \rho M\int_s^t e^{-(\omega-\rho M(1+L))(\theta-s)}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta, \quad s \leqslant t. \tag{32}$$

If $s \leqslant t_0 \leqslant t$, then

$$\begin{aligned}
&\|y^+(s,t) - y^+(s,t_0)\|_{\mathcal{Z}} = \big\|U_\eta(s,t_0)Q_\eta(t_0)[y^+(t_0,t) - z^+(t_0)]\big\|_{\mathcal{Z}} \\
&+ \left\|\int_{t_0}^s U_\eta(s,\theta)Q_\eta(\theta)[H(\theta,y^+(\theta,t),\Sigma_\eta^{*,u}(\theta,y^+(\theta,t))) - H(\theta,y^+(\theta,t_0),\Sigma_\eta^{*,u}(\theta,y^+(\theta,t_0)))]d\theta\right\|_{\mathcal{Z}} \\
&\leqslant \rho M^2 e^{\omega(s-t_0)}\int_{t_0}^t e^{-(\omega-\rho M(1+L))(\theta-t_0)}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta \\
&+ \rho M\int_s^{t_0} e^{\omega(s-\theta)}(1+L)\|y^+(\theta,t) - y^+(\theta,t_0)\|_{\mathcal{Z}}d\theta
\end{aligned}$$

and using Gronwall's Lemma we have

$$\|y^+(s,t) - y^+(s,t_0)\|_{\mathcal{Z}} \leqslant \rho M^2\int_{t_0}^t e^{-(\omega-\rho M(1+L))(\theta-s)}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta. \tag{33}$$

We use this to estimate $\zeta(t)$. Note that

$$\begin{aligned}
&\zeta(t) - U_\eta(t,t_0)(I - Q_\eta(t_0))\zeta(t_0) \\
&= z^-(t) - \Sigma_\eta^{*,u}(t,z^+(t)) - U_\eta(t,t_0)(I - Q_\eta(t_0))[z^-(t_0) - \Sigma_\eta^{*,u}(t_0,z^+(t_0))] \\
&= \int_{t_0}^t U_\eta(t,s)(I - Q_\eta(s))G(s,z^+(s),z^-(s))ds \\
&- \Sigma_\eta^{*,u}(t,z^+(t)) + U_\eta(t,t_0)(I - Q_\eta(t_0))\Sigma_\eta^{*,u}(t_0,z^+(t_0)) \\
&= \int_{t_0}^t U_\eta(t,s)(I - Q_\eta(s))[G(s,z^+(s),z^-(s)) - G(s,y^+(s,t),\Sigma_\eta^{*,u}(s,y^+(s,t)))]ds - \\
&\int_{-\infty}^{t_0} U_\eta(t,s)(I-Q_\eta(s))[G(s,y^+(s,t),\Sigma_\eta^{*,u}(s,y^+(s,t))) - G(s,y^+(s,t_0),\Sigma_\eta^{*,u}(s,y^+(s,t_0)))]ds.
\end{aligned}$$

Thus, using (32) and (33), we obtain

$$\begin{aligned}
&\|\zeta(t)-U_\eta(t,t_0)(I-Q_\eta(t_0))\zeta(t_0)\|_{\mathcal{Z}}\\
&\quad\leqslant\ \rho M\int_{t_0}^{t} e^{-\omega(t-s)}\left[\|z^+(s)-y^+(s,t)\|_{\mathcal{Z}}+\|z^-(s))-\Sigma_\eta^{*,u}(s,y^+(s,t))\|_{\mathcal{Z}}\right]ds\\
&\quad+\ \rho M(1+L)\int_{-\infty}^{t_0} e^{-\omega(t-s)}\|y^+(s,t)-y^+(s,t_0)\|_{\mathcal{Z}}ds\\
&\quad\leqslant\ \rho M\int_{t_0}^{t} e^{-\omega(t-s)}\|\zeta(s)\|_{\mathcal{Z}}ds\\
&\quad+\ \rho^2M^2(1+L)\int_{t_0}^{t} e^{-\omega(t-s)}\int_{s}^{t} e^{-(\omega-\rho M(1+L))(\theta-s)}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta ds\\
&\quad+\ \rho^2M^3(1+L)\int_{-\infty}^{t_0} e^{-\omega(t-s)}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))(\theta-s)}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta ds,
\end{aligned}$$

so that

$$\begin{aligned}
&\|\zeta(t)-U_\eta(t,t_0)(I-Q_\eta(t_0))\zeta(t_0)\|_{\mathcal{Z}}\leqslant\rho M\int_{t_0}^{t} e^{-\omega(t-s)}\|\zeta(s)\|_{\mathcal{Z}}ds\\
&\quad+\ \rho^2M^2(1+L)e^{-\omega t}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))\theta}\|\zeta(\theta)\|_{\mathcal{Z}}\int_{t_0}^{\theta} e^{(2\omega-\rho M(1+L))s}ds\,d\theta\\
&\quad+\ \rho^2M^3(1+L)e^{-\omega t}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))\theta}\|\zeta(\theta)\|_{\mathcal{Z}}\int_{-\infty}^{t_0} e^{(2\omega-\rho M(1+L))s}ds\,d\theta\\
&\quad\leqslant\ \left[\rho M+\frac{\rho^2M^2(1+L)}{2\omega-\rho M(1+L)}\right]\int_{t_0}^{t} e^{-\omega(t-s)}\|\zeta(s)\|_{\mathcal{Z}}ds\\
&\quad+\ \frac{\rho^2M^3(1+L)}{2\omega-\rho M(1+L)}e^{-\omega(t-t_0)}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))(\theta-t_0)}\|\zeta(\theta)\|_{\mathcal{Z}}d\theta
\end{aligned}$$

and therefore

$$\begin{aligned}
e^{\omega(t-t_0)}\|\zeta(t)\|_{\mathcal{Z}}\ &\leqslant\ M\|\zeta(t_0)\|_{\mathcal{Z}}+\left[\rho M+\frac{\rho^2M^2(1+L)}{2\omega-\rho M(1+L)}\right]\int_{t_0}^{t} e^{\omega(s-t_0)}\|\zeta(s)\|_{\mathcal{Z}}ds\\
&+\ \frac{\rho^2M^3(1+L)}{2\omega-\rho M(1+L)}\int_{t_0}^{t} e^{-(2\omega-\rho M(1+L))(s-t_0)}e^{\omega(s-t_0)}\|\zeta(s)\|_{\mathcal{Z}}ds\\
&\leqslant\ M\|\zeta(t_0)\|_{\mathcal{Z}}+\left[\rho M+\frac{\rho^2M^2(1+L)(1+M)}{2\omega-\rho M(1+L)}\right]\int_{t_0}^{t} e^{\omega(s-t_0)}\|\zeta(s)\|_{\mathcal{Z}}ds.
\end{aligned}$$

Due to Gronwall's inequality we have that

$$\|\zeta(t)\|_{\mathcal{Z}}\leqslant M\|\zeta(t_0)\|_{\mathcal{Z}}\,e^{-\gamma(t-t_0)},\tag{34}$$

where

$$\gamma = \omega - \left[\rho M + \frac{\rho^2 M^2 (1+L)(1+M)}{2\omega - \rho M(1+L)}\right].$$

This proves (31) and consequently

$$W_\eta^u(0,0) \subset \{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = (Q_\eta(\tau)w, \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w))\}.$$

We now prove that $\{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = (Q_\eta(\tau)w, \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w))\} \subset W_\eta^u(0,0)$. Consider $z_0^+ \in \mathcal{Q}_\eta(\tau)\mathcal{Z}$ and the solution $z^{+*}(t)$ of the initial value problem

$$\dot{z}^+ = \mathcal{A}^+(t)z^+ + H(t, z^+, \Sigma_\eta^{*,u}(t, z^+)), \quad z^+(\tau) = z_0^+.$$

This defines a curve $(z^{+*}(t), \Sigma_\eta^{*,u}(t, z^{+*}(t)))$, $t \in \mathbb{R}$. Recalling (26) one can check that

$$\Sigma_\eta^{*,u}(t, z^{+*}(t)) = \int_{-\infty}^t U_\eta(t,s)(I - Q_\eta(s))G(s, z^{+*}(s), \Sigma_\eta^{*,u}(s, z^{+*}(s))ds, \quad t \in \mathbb{R}.$$

Thus $\Sigma_\eta^{*,u}(t, z^{+*}(t))$ solves

$$\dot{z}^- = \mathcal{A}^-(t)z^- + G(t, z^{+*}(t), \Sigma_\eta^{*,u}(t, z^{+*}(t))), \; t \in \mathbb{R},$$

and we conclude that $(z^{+*}(t), \Sigma_\eta^{*,u}(t, z^{+*}(t)))$, $t \in \mathbb{R}$, is a solution of (19), passing through $(z_0^+, \Sigma_\eta^{*,u}(\tau, z_0^+))$ at time τ, with $\Sigma_\eta^{*,u}(t, z^{+*}(t)) \to 0$ as $t \to -\infty$. Since $\Sigma_\eta^{*,u}(t,0) = 0$, the reasoning that lead to (28) can be used now to ensure that

$$\|z^+(t)\|_\mathcal{Z} \leqslant M\|z_0^+\|_\mathcal{Z} e^{(\omega - \rho M(1+L))(t-\tau)}.$$

As a consequence $z^{+*}(t) \to 0$ as $t \to -\infty$ and the proof is complete. □

4. EXISTENCE OF STABLE MANIFOLDS AS A GRAPH

We assume again that h_η is such that H and G satisfy (20) for all $z^+(t) \in Q_\eta(t)\mathcal{Z}$, $z^-(t) \in (I - Q_\eta(t))\mathcal{Z}$ with certain $\rho > 0$, which is defined below via (22). Let $W^s(t,0,0)$ be the stable manifold of equilibrium solution $(0,0)$ to (19). In a similar way as in the previous Section, and for this reason simplifying all the computations, we show that there is a bounded and Lipschitz continuous function $\Sigma^{*,s}(t,\cdot) : (I - Q_\eta(t))\mathcal{Z} \to Q_\eta(t)\mathcal{Z}$ such that

$$W_\eta^s(t,0,0) = \{(t, z^+, z^-) : z^+ = \Sigma^{*,s}(t, z^-), \; z^- \in (I - Q_\eta(t))\mathcal{Z}\}.$$

Remark 4. 1. In this case we look for a function $\Sigma^{*,s}(t,\cdot)$ with the property that, if $\tau \in \mathbb{R}$ and $(\Sigma^{*,s}(\tau,\zeta),\zeta) \in \mathcal{Z}$, then the solution of (19) such that $z^+(\tau) = \Sigma^{*,s}(\tau,\zeta)$, $z^-(\tau) = \zeta$, is such that $z(t)$ is in the graph of $\Sigma^{*,s}(t,\cdot)$ for all positive and all negative time t. This means that $z^+(t) = \Sigma^{*,s}(t, z^-(t))$ for all t. Also, the solution $(z^+(t), z^-(t))$ should tend to zero as $t \to +\infty$ (in particular, it should stay bounded as $t \to +\infty$). Since

$$z^+(t) = U_\eta(t,t_0)Q_\eta(t_0)z^+(t_0) + \int_{t_0}^{t} U_\eta(t,s)Q_\eta(s)H(s, \Sigma^{*,s}(z^-(s)), z^-(s))ds,$$

letting $t_0 \to +\infty$ we have that

$$z^+(t) = \Sigma^{*,s}(t, z^-(t)) = -\int_{t}^{+\infty} U_\eta(t,s)Q_\eta(s)H(s, \Sigma^{*,s}(s, z^-(s)), z^-(s))ds.$$

Fix $D > 0$, $L > 0$, $0 < \vartheta < 1$ and choose $\rho > 0$ such that (22) be satisfied

Definition 4.1. Given $\eta > 0$, denote by $\mathcal{LB}^s(D, L)$ a complete metric space of all bounded and globally Lipschitz continuous functions $\mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ defined as $(\tau, z) \mapsto \Sigma(\tau, (I - Q_\eta(\tau))z) \in Q_\eta(\tau)\mathcal{Z}$ satisfying, for all $(\tau, z, \tilde{z}) \in \mathbb{R} \times \mathcal{Z} \times \mathcal{Z}$,

$$\begin{aligned} &\sup\{\|\Sigma(\tau, (I - Q_\eta(\tau))z)\|_{\mathcal{Z}},\ (\tau, z) \in \mathbb{R} \times \mathcal{Z}\} \leqslant D, \\ &\|\Sigma(\tau, (I - Q_\eta(\tau))z) - \Sigma(\tau, (I - Q_\eta(\tau))\tilde{z})\|_{\mathcal{Z}} \leqslant L\|(I - Q_\eta(\tau))(z - \tilde{z})\|_{\mathcal{Z}}, \end{aligned} \tag{35}$$

where the distance between Σ and $\tilde{\Sigma}$ in $\mathcal{LB}^s(D, L)$ is defined as

$$|\!|\!|\Sigma(\cdot,\cdot) - \tilde{\Sigma}(\cdot,\cdot)|\!|\!| := \sup\{\|\Sigma(\tau, (I - Q_\eta(\tau))z) - \tilde{\Sigma}(\tau, (I - Q_\eta(\tau)z))\|_{\mathcal{Z}},\ (\tau, z) \in \mathbb{R} \times \mathcal{Z}\}.$$

Theorem 4.1. *Suppose that the above conditions are satisfied. Then, there exist a* $\Sigma^{*,s}(\tau, \cdot) \in \mathcal{LB}^s(D, L)$*, such that the stable manifold* $W^s(\tau, 0, 0)$ *to* (19) *is given by*

$$W^s(0, 0) = \{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = (\Sigma^{*,s}(\tau, (I - Q_\eta(\tau))w), (I - Q_\eta(\tau))w)\}. \tag{36}$$

Proof: For $\tau \in \mathbb{R}$ and arbitrary $\zeta \in (I - Q_\eta(\tau))\mathcal{Z}$, $\Sigma \in \mathcal{L}B^s(D, L)$ denote by $z^-(t) = \psi(t, \tau, \zeta, \Sigma)$ the solution of

$$\frac{dz^-}{dt} = \mathcal{A}^-(t)z^- + G(t, \Sigma(t, z^-), z^-),\ t > \tau, \quad z^-(\tau) = \zeta \in (I - Q_\eta(\tau))\mathcal{Z}. \tag{37}$$

Next we define, for $\Sigma \in \mathcal{L}B^s(D, L)$,

$$\Phi(\Sigma)(\tau, \zeta) = -\int_{\tau}^{+\infty} U_\eta(\tau, s)Q_\eta(s)H(s, \Sigma(s, z^-(s)), z^-(s))ds,\ (\tau, \zeta) \in \mathbb{R} \times Q_\eta(\tau)\mathcal{Z}. \tag{38}$$

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Following the lines of Theorem 3.1, for $\rho > 0$ satisfying (22), it can be proved that the map Φ takes $\mathcal{LB}^s(D, L)$ into itself, is a strict contraction, and hence possesses a unique fixed point in $\mathcal{LB}^s(D, L)$.

First note that, by (18), one has

$$\|\Phi(\Sigma)(\tau, \cdot)\|_{\mathcal{Z}} \leqslant \int_\tau^{+\infty} \rho M e^{\omega(\tau-s)} ds = \frac{\rho M}{\omega}, \tag{39}$$

and from (22) we have $\sup\{\|\Phi(\Sigma)(\tau, (I - Q_\eta(\tau))z)\|_{\mathcal{Z}},\ (\tau, z) \in \mathbb{R} \times \mathcal{Z}\} \leqslant D$.

Next, suppose that Σ and $\tilde{\Sigma}$ are functions satisfying (23), $\zeta, \tilde{\zeta} \in (I - Q_\eta(\tau))\mathcal{Z}$ and denote $z^-(t) = \psi(t, \tau, \zeta, \Sigma)$, $\tilde{z}^-(t) = \psi(t, \tau, \tilde{\zeta}, \tilde{\Sigma})$. Then

$$\begin{aligned} z^-(t) - \tilde{z}^-(t) &= U_\eta(t, \tau)(I - Q_\eta(\tau))(\zeta - \tilde{\zeta}) \\ &+ \int_\tau^t U_\eta(t, s)(I - Q_\eta(s))[G(s, \Sigma(s, z^-(s)), z^-(s)) - G(s, \tilde{\Sigma}(s, \tilde{z}^-(s)), \tilde{z}^-(s))]ds, \end{aligned}$$

for which we can prove that

$$\begin{aligned} \|z^-(t) - \tilde{z}^-(t)\|_{\mathcal{Z}} &\leqslant [M\|\zeta - \tilde{\zeta}\|_{\mathcal{Z}} e^{-\omega\,(t-\tau)} \\ &\quad + \rho M \int_\tau^t e^{-\omega(t-s)} ds |\!|\!| \Sigma - \tilde{\Sigma} |\!|\!|] e^{-\rho M(1+L)(t-\tau)} \\ &\leqslant [M\|\zeta - \tilde{\zeta}\|_{\mathcal{Z}} + \rho M \omega^{-1} |\!|\!| \Sigma - \tilde{\Sigma} |\!|\!|] e^{\rho M(1+L)(t-\tau)}. \end{aligned} \tag{40}$$

Thus, we can arrive to

$$\begin{aligned} &\|\Phi(\Sigma)(\tau, \zeta) - \Phi(\tilde{\Sigma})(\tau, \tilde{\zeta})\|_{\mathcal{Z}} \\ &\qquad \leqslant \frac{\rho M}{\omega}\left[1 + \frac{\rho M(1+L)}{\omega - \rho M(1+L)}\right] |\!|\!| \Sigma - \tilde{\Sigma} |\!|\!| + \frac{\rho M^2(1+L)}{\omega - \rho M(1+L)} \|\zeta - \tilde{\zeta}\|_{\mathcal{Z}}. \end{aligned} \tag{41}$$

Let

$$I_\Sigma = \frac{\rho M}{\omega}\left[1 + \frac{\rho M(1+L)}{\omega - \rho M(1+L)}\right] \quad \text{and} \quad I_\zeta = \frac{\rho M^2(1+L)}{\omega - \rho M(1+L)}.$$

Since $I_\Sigma \leqslant \frac{\rho M}{\omega}(1 + L)$, it follows from (22), (41) that $I_\Sigma \leqslant \vartheta$, $I_\zeta \leqslant L$ and

$$\|\Phi(\Sigma)(\tau, \zeta) - \Phi(\tau, \tilde{\Sigma})(\tilde{\zeta})\|_{\mathcal{Z}} \leqslant L\|\zeta - \tilde{\zeta}\|_{\mathcal{Z}} + \vartheta |\!|\!| \Sigma - \tilde{\Sigma} |\!|\!|. \tag{42}$$

The inequality (42) with $\Sigma = \tilde{\Sigma}$ and (39) imply that Φ takes $\mathcal{LB}^s(D, L)$ into $\mathcal{LB}^s(D, L)$. Due to (22), estimate (42) with $\zeta = \tilde{\zeta}$ shows that Φ is a contraction map. Therefore, there exists a unique fixed point $\Sigma^{*,s} = \Phi(\Sigma^{*,s})$ in $\mathcal{LB}^s(D, L)$.

Now, if $(z^-(t), z^+(t))$, $t \in \mathbb{R}$, is a global solution of (19) bounded as $t \to +\infty$, then it can be proved that there are constants $M \geqslant 1$ and $\gamma > 0$ such that

$$\|z^+(t) - \Sigma^{*,s}(t, z^-(t))\|_{\mathcal{Z}} \leqslant M e^{\gamma(t-t_0)} \|z^+(t_0) - \Sigma^{*,s}(t, z^-(t_0))\|_{\mathcal{Z}}, \quad t_0 \geqslant t. \tag{43}$$

Making $t_0 \to +\infty$ in (31) we obtain that $z^+(t) = \Sigma^{*,s}(t, z^-(t))$ for each $t \in \mathbb{R}$. That also ensures that $\Sigma^{*,s}(t,0) = 0$, since $(0,0)$ is a stationary solution to (19).

We now prove that $\{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = (\Sigma_\eta^{*,u}(\tau, (I - Q_\eta(\tau))w), (I - Q_\eta(\tau))w)\} \subset W_\eta^u(0,0)$. Consider $z_0^- \in (I - Q_\eta(\tau))\mathcal{Z}$ and the solution $z^{-*}(t)$ of the initial value problem

$$\dot{z}^- = \mathcal{A}^-(t)z^- + G(t, \Sigma^{*,s}(t, z^-), z^-), \quad z^-(\tau) = z_0^-.$$

This defines a curve $(\Sigma^{*,s}(t, z^{-*}(t)), z^{-*}(t))$, $t \geqslant \tau$. Recalling (38) one can check that $\Sigma^{*,s}(t, z^{-*}(t)) = \int_{+\infty}^t U_\eta(t,s)Q_\eta(s)H(s, \Sigma^{*,s}(s, z^{-*}(s)), z^{-*}(s))ds$, $t \geqslant \tau$. Therefore $\Sigma^{*,s}(t, z^{-*}(t))$ solves

$$\dot{z}^+ = \mathcal{A}^+(t)z^+ + H(t, \Sigma^{*,s}(t, z^{-*}(t)), z^{-*}(t)), \ t \geqslant \tau,$$

and we conclude that $(\Sigma^{*,s}(t, z^{-*}(t)), z^{-*}(t))$, $t \geqslant \tau$, is a solution of (19), passing through $(\Sigma^{*,s}(\tau, z_0^-), z_0^-)$ at time τ, with $\Sigma^{*,s}(t, z^{-*}(t)) \to 0$ as $t \to \infty$. Since $\Sigma^{*,s}(t,0) = 0$, the reasoning that lead to (40) can be used now to ensure that

$$\|z^-(t)\|_{\mathcal{Z}} \leqslant M\|z_0^-\|_{\mathcal{Z}} e^{-(\omega - \rho M(1+L))(t-\tau)}.$$

As a consequence $z^{-*}(t) \to 0$ as $t \to +\infty$. □

5. CONTINUITY OF UNSTABLE AND STABLE MANIFOLDS

In this section we will prove only the continuity of unstable manifolds with respect to parameter η, since the continuity of the stable manifolds follows in a completely similar manner.

First of all, we will characterize $Q_\eta(t)$ as a function of $B_\eta(t)$ and $\mathcal{Q}$ in such a way that we can obtain the continuity result as follows

LEMMA 5.1. *Let $Q_\eta, \mathcal{Q}$, the projections associated to the linear dichotomies introduced in Section 2. Then we have that, for $\tau \in \mathbb{R}$ given*

$$\lim_{\eta \to 0} \sup_{s \leqslant \tau} \|Q_\eta(s) - \mathcal{Q}\|_{L(\mathcal{Z})} \to 0 \tag{44}$$

and

$$\lim_{\eta \to 0} \sup_{s \geqslant \tau} \|Q_\eta(s) - \mathcal{Q}\|_{L(\mathcal{Z})} \to 0. \tag{45}$$

Proof: For $\zeta \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto z_\eta(t) \in \mathcal{Z}$ be the unique solution of

$$\begin{aligned} \dot{z} &= (\mathcal{A} + B_\eta(t))z \\ z(\tau) &= Q_\eta(\tau)\zeta. \end{aligned}$$

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This solution must stay bounded as $t \to -\infty$ and satisfy the variation of constants formula

$$z(t) = e^{\mathcal{A}(t-t_0)} z(t_0) + \int_{t_0}^{t} e^{\mathcal{A}(t-s)} B_\eta(s) z(s) ds.$$

Applying $(I - \mathcal{Q})$ to the above equation and letting $t_0 \to -\infty$ we have

$$(I - \mathcal{Q}) Q_\eta(\tau)\zeta = \int_{-\infty}^{\tau} e^{\mathcal{A}(\tau-s)} (I - \mathcal{Q}) B_\eta(s) z(s) ds, \text{ for all } \zeta \in \mathcal{Z}.$$

¿From this it follows that

$$\lim_{\eta \to 0} \sup_{s \leqslant \tau} \|(I - \mathcal{Q}) Q_\eta(s)\|_{L(\mathcal{Z})} \to 0. \tag{46}$$

Proceeding in a similar manner we consider $\xi \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto x(t) \in \mathcal{Z}$ be the unique solution of

$$\begin{aligned} &\dot{x} = \mathcal{A}x = (\mathcal{A} + B_\eta(t))x - B_\eta(t)x \\ &x(\tau) = \mathcal{Q}\xi. \end{aligned}$$

This solution must stay bounded as $t \to -\infty$ and satisfy the variation of constants formula

$$x(t) = U_\eta(t, t_0) x(t_0) - \int_{t_0}^{t} U_\eta(t, s) B_\eta(s) x(s) ds.$$

Applying $(I - Q_\eta(\tau))$ to the above equation and letting $t_0 \to -\infty$ we have

$$(I - Q_\eta(\tau)) \mathcal{Q}\xi = - \int_{-\infty}^{\tau} U_\eta(\tau, s)(I - Q_\eta(s)) B_\eta(s) z(s) ds, \text{ for all } \xi \in \mathcal{Z}.$$

¿From this it follows that

$$\lim_{\eta \to 0} \sup_{s \leqslant \tau} \|(I - Q_\eta(s))\mathcal{Q}\|_{L(\mathcal{Z})} \to 0. \tag{47}$$

Now we are ready to prove (44). In fact, it is enough to observe that

$$Q_\eta(s) - \mathcal{Q} = (I - \mathcal{Q}) Q_\eta(s) - \mathcal{Q}(I - Q_\eta(s)). \tag{48}$$

and the result follows from (46) and (47).

To prove (45) we proceed in a similar way. For $\zeta \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto z_\eta(t) \in \mathcal{Z}$ be the unique solution of

$$\begin{aligned} &\dot{z} = (\mathcal{A} + B_\eta(t))z \\ &z(\tau) = (I - Q_\eta(\tau))\zeta. \end{aligned}$$

This solution must stay bounded as $t \to \infty$ and satisfy the variation of constants formula

$$z(t) = e^{\mathcal{A}(t-t_0)} z(t_0) + \int_{t_0}^{t} e^{\mathcal{A}(t-s)} B_\eta(s) z(s) ds.$$

Applying $\mathcal{Q}$ to the above equation and letting $t_0 \to +\infty$ we have

$$\mathcal{Q}(I - Q_\eta(\tau))\zeta = \int_{\infty}^{\tau} e^{\mathcal{A}(\tau-s)} \mathcal{Q} B_\eta(s) z(s) ds, \text{ for all } \zeta \in \mathcal{Z}.$$

¿From this it follows that

$$\lim_{\eta\to 0} \sup_{s\geq\tau} \|\mathcal{Q}(I - Q_\eta(s))\|_{L(\mathcal{Z})} \to 0. \tag{49}$$

Proceeding in a similar manner we consider $\xi \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto x(t) \in \mathcal{Z}$ be the unique solution of

$$\begin{aligned} &\dot{x} = \mathcal{A}x = (\mathcal{A} + B_\eta(t))x - B_\eta(t)x \\ &x(\tau) = (I - \mathcal{Q})\xi. \end{aligned}$$

This solution must stay bounded as $t \to \infty$ and satisfy the variation of constants formula

$$x(\tau) = U_\eta(\tau, t_0)x(t_0) - \int_{t_0}^{\tau} U_\eta(\tau, s) B_\eta(s) x(s) ds.$$

Applying $Q_\eta(\tau)$ to the above equation and letting $t_0 \to \infty$ we have

$$Q_\eta(\tau)(I - \mathcal{Q})\xi = -\int_{\infty}^{\tau} U_\eta(\tau, s) Q_\eta(s) B_\eta(s) x(s) ds, \text{ for all } \xi \in \mathcal{Z}$$

¿From this it follows that

$$\lim_{\eta\to 0} \sup_{s\geq\tau} \|Q_\eta(s)(I - \mathcal{Q})\|_{L(\mathcal{Z})} \to 0, \tag{50}$$

and (45) follows (48), (49) and (50).□

Remember that we can decompose a solution $z_\eta(t)$ of (14) as $z_\eta^+(t) = Q_\eta(t)(z(t))$ and $z_\eta^-(t) = (I - Q_\eta(t))(z(t))$. Then

$$\begin{aligned} \dot{z}_\eta^+ &= \mathcal{A}^+(t) z_\eta^+ + H_\eta(t, z_\eta^+, z_\eta^-), \\ \dot{z}_\eta^- &= \mathcal{A}^-(t) z_\eta^- + G_\eta(t, z_\eta^+, z_\eta^-), \end{aligned} \tag{51}$$

where $H_\eta(t, z_\eta^+, z_\eta^-) = Q_\eta(t) h_\eta(t, z_\eta^+ + z_\eta^-)$ and $G(t, z_\eta^+, z_\eta^-) = (I - Q_\eta(t)) h_\eta(t, z_\eta^+ + z_\eta^-)$.

Let $D > 0$, $L > 0$, $0 < \theta < 1$, $\rho > 0$ satisfy (22) and assume (20) for all $z^+ \in \mathcal{Z}^+$ and $z^- \in \mathcal{Z}^-$.

THEOREM 5.1. *Suppose that the above conditions are satisfied and* (22) *holds, so that there exists a function* $\Sigma_\eta^{*,u} : \mathcal{Z}^+ \to \mathcal{Z}^-$, *such that the unstable manifold* $W_\eta^u(0,0)$ *of the equilibrium solution* $(0,0)$ *to* (51) *is given by*

$$W_\eta^u(0,0) = \{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = (Q_\eta(\tau)w, \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w))\};$$

also, for any $\zeta \in \mathcal{Z}^+$ *and* $\eta \in [0, \eta_0]$,

$$\Sigma_\eta^{*,u}(\tau,\zeta) = \int_{-\infty}^{\tau} U_\eta(t,s)(I - Q_\eta(s))G_\eta(s, z^+(s), \Sigma_\eta^{*,u}(s, z^+(s)))ds.$$

If in addition

$$\left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2(1+L)}{\omega(2\omega - \rho M(1+L)}\right] \leqslant \frac{1}{2}$$

then, for any $r > 0$,

$$\sup_{t\leqslant\tau} \sup_{\substack{z\in\mathcal{Z} \\ \|z\|_{\mathcal{Z}}\leqslant r}} \left\{\|Q_\eta(t)(z) - \mathcal{Q}(z)\|_{\mathcal{Z}} + \|\Sigma_\eta^{*,u}(t, Q_\eta(t)(z)) - \Sigma_0^{*,u}(\mathcal{Q}(z))\|_{\mathcal{Z}}\right\} \overset{\eta\to 0}{\longrightarrow} 0.$$

Proof: Note that we only have to prove that

$$\sup_{t\leqslant\tau} \sup_{\substack{z\in\mathcal{Z}^+ \\ \|z\|_{\mathcal{Z}}\leqslant r}} \|\Sigma_\eta^{*,u}(t, Q_\eta(t)(z)) - \Sigma_0^{*,u}(\mathcal{Q}(z))\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0.$$

If $z \in \mathcal{QZ}$ with $\|z\|_{\mathcal{Z}} \leqslant r$, then

$$\begin{aligned}
&\Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)z) - \Sigma_0^{*,u}(\mathcal{Q}z) \\
&= \int_{-\infty}^{\tau} [U_\eta(t,s)(I - Q_\eta(s)) - e^{\mathcal{A}(\tau-s)}(I - \mathcal{Q})]h_\eta(s, z_\eta^+, \Sigma_\eta^{*,u}(s, z_\eta^+))ds \\
&- \int_{-\infty}^{\tau} e^{\mathcal{A}(\tau-s)}(I - \mathcal{Q})[h(z_0^+, \Sigma_0^{*,u}(z_0^+)) - h_\eta(s, z_0^+, \Sigma_0^{*,u}(z_0^+))]ds \qquad (52)\\
&- \int_{-\infty}^{\tau} e^{\mathcal{A}(\tau-s)}(I - \mathcal{Q})[h_\eta(s, z_0^+, \Sigma_0^{*,u}(z_0^+)) - h_\eta(s, z_\eta^+, \Sigma_\eta^{*,u}(s, z_\eta^+))]ds \\
&=: I_1(\eta) + I_2(\eta) + I_3(\eta).
\end{aligned}$$

Recalling $\sup_{s\leqslant\tau} \|Q_\eta(s) - \mathcal{Q}\|_{L(\mathcal{Z})}$ converges to zero and

$$U_\eta(t,\tau)(I - Q_\eta(\tau)) - e^{\mathcal{A}(t-\tau)}(I - Q_\eta(\tau)) = \int_\tau^t e^{\mathcal{A}(t-s)}B_\eta(s)U_\eta(s,\tau)(I - Q_\eta(\tau))ds$$

we have from (16) and from (18) that $I_1(\eta) \to 0$ as $\eta \to 0$. Also, $I_2(\eta) \to 0$ as $\eta \to 0$ from (11). Next let us estimate $I_3(\eta)$. Recall that

$$z_\eta^+(t) = U_\eta(t,\tau)Q_\eta(\tau)z + \int_\tau^t U_\eta(t,s)Q_\eta(s)h_\eta(s, z_\eta^+(s), \Sigma_\eta^{*,u}(s, z_\eta^+(s)))ds,$$

from which similarly as in case of (28) one can get

$$\|z_\eta^+(t)\|_{\mathcal{Z}} \leqslant M e^{(\omega-\rho M(1+L))(t-\tau)}\|Q_\eta(\tau)z\|_{\mathcal{Z}}. \tag{53}$$

When $\eta = 0$ we may also assume, without loss of generality, that $M = 1$. Since

$$\begin{aligned}
&\|I_3(\eta)\|_{\mathcal{Z}} \leqslant \rho M\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\left[\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}+\|\Sigma_\eta^{*,u}(s,z_\eta^+(s))-\Sigma_0^{*,u}(z_0^+(s))\|_{\mathcal{Z}}\right]ds\\
&\leqslant \rho M\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\left[(1+L)\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}+\|\Sigma_\eta^{*,u}(s,Q_\eta(s)z_0^+(s))-\Sigma_0^{*,u}(z_0^+(s))\|_{\mathcal{Z}}\right]ds\\
&\leqslant \rho M(1+L)\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}ds+\frac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u}-\Sigma_0^{*,u}|\!|\!|_r,
\end{aligned}$$

where

$$|\!|\!|\Sigma_\eta^{*,u}-\Sigma_0^{*,u}|\!|\!|_r=\sup_{s\leqslant\tau}\sup_{\substack{z_0^+\in\mathcal{Z}_0^+\\ \|z_0^+\|_{\mathcal{Z}}\leqslant r}}\|\Sigma_\eta^{*,u}(s,Q_\eta(s)z_0^+)-\Sigma_0^{*,u}(z_0^+)\|_{\mathcal{Z}}. \tag{54}$$

Replacing this in (52) we have

$$\begin{aligned}
\left\|\Sigma_\eta^{*,u}(\tau,Q_\eta(\tau)z)-\Sigma_0^{*,u}(\mathcal{Q}z)\right\|_{\mathcal{Z}}&\leqslant o(1)+\tfrac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u}-\Sigma_0^{*,u}|\!|\!|_r\\
&\quad+\rho M(1+L)\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}ds.
\end{aligned} \tag{55}$$

We next have

$$\begin{aligned}
&\|z_\eta^+(t)-z_0^+(t)\|_{\mathcal{Z}}\leqslant\left\|U_\eta(t,\tau)Q_\eta(\tau)z-e^{\mathcal{A}(t-\tau)}\mathcal{Q}z\right\|_{\mathcal{Z}}\\
&+\Big\|\int_\tau^t[U_\eta(t,s)Q_\eta(s)h_\eta(s,z_\eta^+(s),\Sigma_\eta^{*,u}(s,z_\eta^+(s)))ds-e^{\mathcal{A}(t-s)}\mathcal{Q}h(z_0^+(s),\Sigma_0^{*,u}(z_0^+(s)))]ds\Big\|_{\mathcal{Z}}\\
&\leqslant\left\|U_\eta(t,\tau)Q_\eta(\tau)z-e^{\mathcal{A}(t-\tau)}\mathcal{Q}z\right\|_{\mathcal{Z}}\\
&+\Big\|\int_\tau^t[U_\eta(t,s)Q_\eta(s)-e^{\mathcal{A}(t-s)}\mathcal{Q}]h(z_0^+(s),\Sigma_0^{*,u}(z_0^+(s)))ds\Big\|_{\mathcal{Z}}\\
&+\Big\|\int_\tau^t U_\eta(t,s)Q_\eta(s)\left[h_\eta(s,z_0^+(s),\Sigma_0^{*,u}(z_0^+(s)))-h(z_0^+(s),\Sigma_0^{*,u}(z_0^+(s)))\right]ds\Big\|_{\mathcal{Z}}\\
&+\Big\|\int_\tau^t U_\eta(t,s)Q_\eta(s)\left[h_\eta(s,z_\eta^+(s),\Sigma_\eta^{*,u}(s,z_\eta^+(s)))-h_\eta(s,z_0^+(s),\Sigma_0^{*,u}(z_0^+(s)))\right]ds\Big\|_{\mathcal{Z}}\\
&\leqslant o(1)+\rho M\int_t^\tau e^{\omega(t-s)}[(1+L)\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}\\
&+\|\Sigma_\eta^{*,u}(s,Q_\eta(s)z_0^+(s))-\Sigma_0^{*,u}(z_0^+(s))\|_{\mathcal{Z}}]ds\\
&\leqslant o(1)+\frac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u}-\Sigma_0^{*,u}|\!|\!|_r+\rho M(1+L)\int_t^\tau e^{\omega(t-s)}\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}ds
\end{aligned}$$

and, from Gronwall's inequality,

$$\|z^{+}(t)-z_{0}^{+}(t)\|_{\mathcal{Z}} \leqslant \big(o(1)+\frac{\rho M}{\omega}\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r}\big)e^{(\omega-\rho M(1+L))(t-\tau)}. \tag{56}$$

Applying (56) to (55) we obtain that

$$\begin{aligned}\|\Sigma_{\eta}^{*,u}(\tau,Q_{\eta}(\tau)z)-\Sigma_{0}^{*,u}(\mathcal{Q}z)\|_{\mathcal{Z}} &\leqslant o(1)+\frac{\rho M}{\omega}\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r}\\ &+\rho M(1+L)\int_{-\infty}^{\tau}e^{-(2\omega-\rho M(1+L))(\tau-s)}\left[o(1)+\frac{\rho M}{\omega}\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r}\right]ds\\ &\leqslant o(1)+\left[\frac{\rho M}{\omega}+\frac{\rho^{2}M^{2}(1+L)}{\omega(2\omega-\rho M(1+L))}\right]\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r}\\ &=: o(1)+\tilde{\theta}\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r},\end{aligned} \tag{57}$$

where $\tilde{\theta}\in(0,1)$ as a consequence of (22).

It follows from (57) that

$$\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r}\leqslant o(1)+\tilde{\theta}\|\Sigma_{\eta}^{*,u}-\Sigma_{0}^{*,u}\|_{r}, \tag{58}$$

which completes the proof.□

6. EXISTENCE AND CONTINUITY OF LOCAL STABLE AND UNSTABLE MANIFOLDS

In this section we use the results in Sections 3 and 5 to obtain the existence and continuity of local unstable manifolds when h, h_η only satisfy (20) for $\|z\|_{\mathcal{Z}}=\|z^{+}+z^{-}\|_{\mathcal{Z}}<\delta$ with $\delta>0$ suitably small. We only consider the unstable manifold case since the stable manifold case is completely similar.

Theorem 6.1. *Let $\eta\in[0,1]$, $h_\eta:\mathbb{R}\times\mathcal{Z}\to\mathcal{Z}$ differentiable. Consider initial value problem*

$$\dot{z}=\mathcal{A}z+h_{\eta}(t,z),\quad z(\tau)=z_{0}\in\mathcal{Z}. \tag{59}$$

Assume that $h_0:\mathcal{Z}\to\mathcal{Z}$ is such that where $h_0(0)=0$, $h'(0)=0\in L(\mathcal{Z})$, that $\mathcal{A}$ is such that $\sigma(\mathcal{A})$ is disjoint of the imaginary axis, σ^{+} is bounded, the projection (9) *is defined and the restrictions $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$ of $\mathcal{A}$ to the range and kernel of $\mathcal{Q}$ satisfy* (10). *Suppose that* (7) *holds for some $r>0$. Under these assumptions the following holds:*

1. *For each suitably small η, there exists a globally defined solution of* (59) *$\xi_{\eta}^{*}:\mathbb{R}\to\mathcal{Z}$ with $\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|\xi_{\eta}^{*}(t)\|_{L(\mathcal{Z})}=0$ and such that*

$$\dot{z}=\mathcal{A}+(h_{\eta})_{z}(t,\xi(t))z \tag{60}$$

has exponential dichotomy; that is, there is a a family of projections $\{Q_{\eta}(t):\ t\in\mathbb{R}\}$ such that the conditions in Definition 2.1 are satisfied and $U_{\eta}(t,\tau)$ is the solution operator associated to (60).

2. *For any* $T > 0$

$$\lim_{\eta\to 0}\sup_{t\in[0,T]}\sup_{\tau\in\mathbb{R}}\|U_\eta(t+\tau,\tau)-e^{\mathcal{A}(t-\tau)}\|_{L(\mathcal{Z})}\to 0.$$

3. *The family of projections* $\{Q_\eta(t) : t\in\mathbb{R}\}$ *satisfies*

$$\lim_{\eta\to 0}\sup_{t\in\mathbb{R}}\|Q_\eta(t)-\mathcal{Q}\|_{L(\mathcal{Z})}=0.$$

4. *There exists* $\eta_0>0$, *neighborhood* V *of* $z=0$ *in* $\mathcal{Z}$ *(independent of* η*) with* $\xi^*_\eta(t)\in V$, $\forall t\in\mathbb{R}$, $\eta\in[0,\eta_0]$ *and, for each* $0\leqslant\eta\leqslant\eta_0$, *a function* $(\tau,z)\mapsto\varSigma^{*,u}_\eta(\tau,Q(\tau)z):\mathbb{R}\times V\to\mathcal{Z}$, *such that the local unstable manifold* $W^u_{\mathrm{loc},\eta}(\xi^*_\eta)=W^u_\eta(\xi^*_\eta)\cap V$ *to* (59) *is given by*

$$W^u_{\mathrm{loc},\eta}(\xi^*_\eta)=\{(\tau,w)\in V : w=Q_\eta(\tau)w+\varSigma^{*,u}_\eta(\tau,Q(\tau)w)\}.$$

5. *Finally, the unstable manifolds behave continuously at* $\eta=0$ *in the sense that*

$$\sup_{t\leqslant\tau}\sup_{z\in V}\left\{\|Q_\eta(t)z-\mathcal{Q}z\|_{\mathcal{Z}}+\|\varSigma^{*,u}_\eta(t,Q_\eta(t)z)-\varSigma^{*,u}_0(\mathcal{Q}z)\|_{\mathcal{Z}}\right\}\overset{\eta\to 0}{\longrightarrow}0.$$

Proof: According to Remark 3.1 and Theorem 3.1, we only need to ensure that, given $\delta>0$ there is $0<\delta'\leqslant\delta$ such that any solution $(z^+(t),\varSigma^*_\eta(t,z^+(t))$ on the unstable manifold which satisfies $\|z^+(t_0)\|_{\mathcal{Z}}+\|\varSigma^*_\eta(t_0,z^+(t_0))\|_{\mathcal{Z}}<\delta'$ satisfies $\|z^+(t)\|_{\mathcal{Z}}+\|\varSigma^*_\eta(t,z^+(t))\|_{\mathcal{Z}}<\delta$, for all $t\leqslant t_0$. However, since $z^+(t)$ is the solution of

$$\dot z^+=\mathcal{A}^+(t)z^++Q(t)h_\eta(t,z^+,\varSigma^{*,u}_\eta(z^+(t))),\quad t\leqslant t_0,$$

the variation of constants formula implies that

$$\|z^+(t)\|_{\mathcal{Z}}\leqslant Me^{(\omega-\rho M(1+L))(t-t_0)}\|z^+(t_0)\|_{\mathcal{Z}}. \tag{61}$$

Thus we have

$$\|\varSigma^{*,u}_\eta(t,z^+(t))\|_{\mathcal{Z}}\leqslant MLe^{(\omega-\rho M(1+L))(t-t_0)}\|z^+(t_0)\|_{\mathcal{Z}}$$

and the proof now follows easily. □

7. ON THE CONTINUITY AND STRUCTURE OF PULLBACK ATTRACTORS

The upper semicontinuity of pullback attractors for small non-autonomous and stochastic perturbations of autonomous dynamical systems has been proved in [6, 7]. On the other hand, in [16] it is proved the following theorem on the continuity (upper and lower semicontinuity) of pullback attractors

THEOREM 7.1. *Consider the family* $\{T_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, *of nonlinear processes such that* $T_0(t,\tau) = T_0(t-\tau)$, *for all* $t \geqslant \tau \in \mathbb{R}$, *and assume that* (5) *is satisfied. Suppose that, for all* $\eta \in [0,1]$, *the processes* $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ *have pullback attractors* $\{A_\eta(t) : t \in \mathbb{R}\}$ *and that* $\{T_0(t) : t \geq 0\}$ *is a gradient dynamical system* (*in the sense of [11]*) *for which all the stationary points* $\{y_i^* : 1 \leq i \leq n\}$ *are hyperbolic. Assume that the unstable manifolds of* y_j^* *behave continuously in the following sense: there exists a* $\delta > 0$ *such that for any* $\epsilon > 0$ *there exists an* η_0 *such that for all* $0 < \eta < \eta_0$ *there exists a global hyperbolic solution* $\xi_{j,\eta}^*(\cdot)$ *of* T_η *with*

$$\sup_j \sup_{t\in\mathbb{R}} |\xi_{j,\eta}^*(t) - y_j^*| < \epsilon$$

and within a δ *neighbourhood of* y_j^*

$$\sup_j dist_{\mathrm{H}}(W^u(\xi_{j,\eta}^*), W^u(y_j^*(\cdot))(t)) < \epsilon \qquad \textit{for all} \qquad t \in \mathbb{R}.$$

Then

$$\sup_{t\in\mathbb{R}} dist_{\mathrm{H}}(A_\eta(t), A_0) \to 0 \qquad \textit{as} \qquad \eta \to 0, \tag{62}$$

where $dist_H$ *denotes the Hausdorff distance between bounded sets* $B, C \subset \mathcal{Z}$, *defined as* $dist_H(B,C) = \max\{dist(B,C), dist(C,B)\}$.

This theorem is then applied in [16] to finite dimensional gradient systems. Note that, in our case, we easily get the convergence of nonlinear operators in (5), related to (2) and (1), from (11). Hence, from Theorem 6.1, it is now a straightforward application of the above result to our non-autonomous perturbation of differential equations in a Banach space. Actually, if we follow the proof in [16], we note that what it is proved there is the lower semicontinuity of each of the perturbed unstable manifold to the associated limited one; that is, for all $i = 1, ..., n$

$$\sup_{t\in\mathbb{R}} \mathrm{dist}_{\mathrm{H}}(W_\eta^u(\xi_{i,\eta}^*)(t), W^u(y_i^*)) \to 0 \qquad \text{as} \qquad \eta \to 0,$$

and that is proved as follows: given any $y \in W^u(y_i^*)$, there exists a sequence of points $y_{\eta_k} \in W_{\eta_k}^u(\xi_{i,\eta}^*)(t)$, such that $\lim_{k\to\infty} y_{\eta_k} = y$. On the other hand, observe that the above continuity in the Hausdorff metric implies that

$$\sup_{t\in\mathbb{R}} \mathrm{dist}_{\mathrm{H}}(A_\eta(t), A_\eta(\tau)) \to 0 \qquad \text{as} \qquad \eta \to 0, \qquad \text{for all } \tau \in \mathbb{R}. \tag{63}$$

Note that (62) and (63) implies that all the attractors $A_0, A_\eta(t)$ are very similar for η small enough.

On the other hand, bounded global solutions and consequently, unstable manifolds for the perturbed systems have to be part of the pullback attractor (Lemma 2.1 in [16]); that is, for all $t \in \mathbb{R}$,

$$\bigcup_{i=0}^{n} W_\eta^u(\xi_\eta^i(\cdot)(t) \subset A_\eta(t),$$

so that, we are describing part of the structure on these pullback attractors, which, in particular, would give us lower bounds for the dimension of these sets.

8. APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

Let Ω be a bounded smooth domain in $\mathbb{R}^3$. For $\eta \in [0,1]$, assume that $g_\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in the first variable and twice differentiable in the second variable such that it satisfies

$$\begin{aligned}
&\frac{\partial g_\eta}{\partial u} : \mathbb{R}^2 \to \mathbb{R} \text{ is continuous,}\\
&|\frac{\partial^2 g_\eta}{\partial u^2}(t,u)| \leqslant c\,(1+|u|), \quad \forall u \in \mathbb{R} \text{ and for some } c > 0,\\
&\sup_{t\in\mathbb{R}}\left(|g_\eta(t,u) - g_0(u)| + |\frac{\partial g_\eta}{\partial u}(t,u) - g_0'(u)|\right) \overset{\eta\to 0}{\longrightarrow} 0, \quad \forall u \in \mathbb{R},\\
&\sup_{t\in\mathbb{R}}\ \sup_{u\in\mathbb{R}} |g_\eta(t,u) - g_0(u)| < \infty,
\end{aligned} \tag{64}$$

and

$$\limsup_{|u|\to\infty} \frac{g_0(u)}{u} < \delta_0 < \lambda_0 \tag{65}$$

where $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is $-\Delta$ with homogeneous Dirichlet boundary condition and λ_0 is the first eigenvalue of A.

Consider the non-autonomous damped hyperbolic equation

$$u_{tt} + \beta u_t - \Delta u = g_\eta(t,u) \qquad \text{in} \quad \Omega \tag{66}$$

with the boundary condition

$$u = 0 \quad \text{in} \quad \partial\Omega. \tag{67}$$

The initial data for (66), (67) will be taken in the space $\mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$, where the norm in $H_0^1(\Omega)$ is defined by $\|\varphi\|_{H_0^1(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)}$, $\varphi \in H_0^1(\Omega)$.

It is known (see [3]) that (66), (67) defines a nonlinear process $\{T_\eta(t,\tau), t \geqslant \tau \in \mathbb{R}\}$ on $\mathcal{Z}$ where $T(t,\tau)(\varphi,\psi) = (u(t), u_t(t))$ with $(u(t), u_t(t))$ being the solution of (66), (67) such that $u(\tau) = \varphi$ and $u_t(\tau) = \psi$.

We consider (66), (67) as an abstract evolutionary equation in $\mathcal{Z}$:

$$\begin{aligned}
&\dot{z} = \mathcal{C}z + f_\eta(t,z),\\
&z(\tau) = z_0 \in \mathcal{Z}
\end{aligned} \tag{68}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{Z},$$

$$C = \begin{pmatrix} 0 & I \\ -A & -\beta \end{pmatrix}$$

$$f_\eta(z) = \begin{pmatrix} 0 \\ g_\eta^e(z_1) \end{pmatrix},$$

where $g_\eta^e(t, \cdot) : H_0^1(\Omega) \to L^2(\Omega)$ is given by $g_\eta^e(t, z_1)(x) = g_\eta(t, z_1(x))$ for $x \in \Omega$, and $A : H^2(\Omega) \cap H_0^1(\Omega) = D(A) \subset L^2(\Omega) \to L^2(\Omega)$.

Under the assumption (64) we have that g_η^e is continuous in the first variable and continuously differentiable in the second variable (see [3]). Hence, the same holds for f_η. It is not difficult to see that (4) is satisfied.

Using the energy $V : \mathcal{Z} \to \mathbb{R}$ defined by

$$V(z) = \frac{1}{2} \int_\Omega |\nabla z_1|^2 + \delta \int_\Omega z_1 z_2 + \frac{1}{2} \int_\Omega z_2^2 + \int_\Omega G_0(z_1)$$

for suitably chosen $\delta > 0$, where

$$G_0(z_1) = \int_0^{z_1} g_0(s) ds$$

it follows in a similar way as in [3] that (68) has a pullback attractor in $\mathcal{Z}$.

We note that the equilibrium points of (66) with $\eta = 0$ are of the form $z_0^* = (u_0^*, 0)$ where u_0^* is a solution of

$$-\Delta u = g_0(u). \tag{69}$$

Furthermore, if u_0^* is a solution of (69) such that $0 \notin \sigma(-\Delta + g_0'(u_0^*)I)$ (which is true generically) then, $(u_0^*, 0)$ is a hyperbolic equilibrium point of (66) with $\eta = 0$.

As a consequence of the results in the previous sections the following result holds

THEOREM 8.1. *Assume that* (64) *and* (65) *hold and that* $0 \notin \sigma(-\Delta + g_0'(u_0^*)I)$ *whenever* u_0^* *is a solution of* (69). *Then, the nonlinear processes associated to* (68) *have pullback attractors* $\{A_\eta(t) : t \in \mathbb{R}\}$, $\eta \in [0, 1]$, *and this family of pullback attractors is upper and lower semicontinuous at* $\eta = 0$.

9. FINAL REMARKS AND CONCLUSIONS

All the results in this paper have been written for semilinear equations with the unbounded operator $\mathfrak{B}$ generating a strongly continuous semigroup which lead us to consider an application to an hyperbolic partial differential equation. As a first remark we note that everything proved here can be carried out for the case when $\mathfrak{B}$ generates analytic semigroup. In this case many of the computations would have been a little more involved to deal with singularities that appear in the integrals due to the fact that the kind of bounds for linear operators are now of the type

$$\|U_\eta(t,s)u_s\|_{\mathcal{Z}^\sigma} \leqslant Me^{\delta(t-s)}(t-s)^{-\sigma}\|u_s\|_{\mathcal{Z}}, \tag{70}$$

for $0 < \sigma < 1$ and positive constants δ and M. The roughness of dichotomy theorem (Theorem 7.6.11. in [12]) will also apply. Some aspects are simpler when $\mathfrak{B}$ is sectorial as it is the definition of hyperbolic equilibrium point.

Another aspect that deserves to be mentioned is the following: Our proof of existence and continuity of unstable manifolds for the perturbed dynamical system does not use in a essential way the fact that the limiting dynamical system is autonomous. Instead, what we use is that the limiting problem has a hyperbolic orbit (in the case described a hyperbolic equilibrium). One could start from a limiting non-autonomous semilinear equation and study the continuity of the union of its unstable manifolds under small perturbations. Since our general expectation is that the attractor for a dynamical system consists of unstable manifolds of hyperbolic orbits the results presented here are quite general. In particular, we are now able to study the continuity of attractors for autonomous dynamical systems under autonomous perturbation (singular or not) for situations which are more general than the case when the attractor of the limiting dynamical system is the union of unstable manifolds of hyperbolic equilibria.

Of course, pullback attractors do not enjoy (in general) any forward attraction property. That is saying that the non-autonomous dynamical systems will posses additional structures that need to be studied. In a forthcoming article, using the results developed here, we give a characterization of pullback attractors for non-autonomous dynamical systems which are perturbation of autonomous problems when the attractor for the later is the union of unstable manifolds of hyperbolic equilibria. Using this we study asymptotically autonomous problems (when t tends to $-\infty$ and t tends to $+\infty$) and prove that if the autonomous limits are gradient and their attractors are the union of the unstable manifolds of hyperbolic equilibria (possibly different in $-\infty$ and $+\infty$) then the pullback attractors can be characterized as the union of the unstable manifolds of the backwards limit whereas the forward attractor (in the sense that it is invariant and attracts bounded sets forward in time) is the union of the unstable manifold hyperbolic orbits (which converges to the equilibria). In this case we also prove that any solution converges forward to a (forward) hyperbolic solution (in particular to equilibrium points of the forward limit); that is, any point in the phase space is in the stable manifold of some (forward) hyperbolic solution.

The sort of perturbations considered here are of very simple nature. In a forthcoming article we will consider the case when the perturbations are of more singular nature. In particular we will consider situations for which the unbounded operator $\mathcal{B}$ also depends upon the parameter.

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