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Vector fields in $\mathbb{R}^2$ with maximal index*

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We consider the method of Poincaré to investigate the local index of vector fields in the plane. If m is the degree of the first non zero jet, X_m, of the vector field X at an isolated zero, we explore the geometry of the pencil generated by the coordinate functions of X_m when the absolute value of the index of X, $|\text{ind}(X)|$, is m. We also find necessary and sufficient conditions for $|\text{ind}(X)|$ to be m. October, 2006 ICMC-USP

1. INTRODUCTION

To any natural number m, there exist examples of vector fields X in $\mathbb{R}^2$, such that the absolute value of the index of X, at an isolated zero, is equal to m.

The aim of this paper is to give conditions for the index of a generic vector field, with m-jet distinct from zero, to be $\pm m$. Using Poincaré's method we describe the geometry of these vector fields and also interpret this geometry in terms of the pencil generated by the m-jet of their coordinate functions.

Our first result, Theorem 3.1, was motivated by [2], in which A. Cima, A. Gasull and J. Torregrosa construct examples of vector fields in the plane with given multiplicities and indices. Let $X = (P, Q)$ be a smooth vector field in the plane with isolated zero at the origin, $j^\infty X = X_m + X_{m+1} + \cdots$ its Taylor series at the origin, $X_i = (P_i, Q_i)$ the

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homogeneous part of degree i of X with $X_m^{-1}(0) = \{0\}$. Then m is such that $X_m \neq 0$ while $X_l = 0$ for all $l < m$ and this implies $|\text{ind}(X)| \leq m$ (see Lemma 4.2). We give necessary and sufficient conditions for the absolute value of the index of X, $|\text{ind}(X)|$, to be m.

In our second result we study vector fields X for which $X_m^{-1}\{0\} \neq \{0\}$. Under mild genericity hypothesis on the pencil generated by X_m, we obtain necessary conditions for $|\text{ind}(X)|$ to be m.

Our method is useful to classify vector fields with a fixed index and we explore this approach in the last section to obtain the classification of families of vector fields with maximal or minimal index, that is $|\text{ind}(X)| = m$.

2. BASIC RESULTS

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a continuous map such that 0 is isolated in $f^{-1}(0)$. Then the index ind(f) of f at zero is defined as follows: choose a ball B_ϵ about 0 in $\mathbb{R}^n$ so small that $f^{-1}(0) \cap B_\epsilon = \{0\}$ and let S_ϵ be its boundary $(n-1)$-sphere. Choose an orientation of each copy of $\mathbb{R}^n$. Then the index of f at zero is the degree of the mapping $(f/ \parallel f \parallel) : S_\epsilon \to S$, the unit sphere, where the spheres are oriented as $(n-1)$- spheres in $\mathbb{R}^n$. If f is differentiable, this degree can be computed as the sum of the signs of the Jacobian of f at all the f-preimages near 0 of a regular value of f near 0.

From the point of view of this paper, we can identify the germ of a C^∞ vector field in the plane with the C^∞ map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. In this context, Mather's $\mathcal{K}$-equivalence is a useful equivalence relation because it preserves the absolute value of the index of f.

We say two map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are topologically $\mathcal{K}$-equivalent (or C^0-$\mathcal{K}$-equivalent) if there exist germs of homeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ with the property $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ such that the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(id., f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (\mathbb{R}^n, 0) & \xrightarrow{(id., g)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

where id. means the identity mapping of $\mathbb{R}^n$ and π_n is the canonical projection. This equivalence relation is a topological version of Mather's $\mathcal{K}$-equivalence [6].

The following particular instance of C^0-$\mathcal{K}$-equivalence will be useful in the paper: f and g are C^0-$\mathcal{K}$-equivalent if there exist a germ of homeomorphism h as above, and an $n \times p$

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matrix $M = (m_{(i,j)})$, whose entries $m_{(i,j)}$, $1 \leq i \leq n$, $1 \leq j \leq p$ are continuous functions of x at 0, and $\det M(0) \neq 0$, such that $g(x) = M(x).f(h(x))$.

A C^∞ map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is said to be finitely C^0-$\mathcal{K}$-determined if there exists a positive number k such that for any C^∞ map germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $j^k f(0) = j^k g(0)$, f is C^0-$\mathcal{K}$-equivalent to g. Some characterizations of finite C^0-$\mathcal{K}$- determinacy are already known (see [9], p. 518). Specifically in the case $n = p$, if a C^∞ map germ X is finitely C^0-$\mathcal{K}$-determined, then $X^{-1}(0) = \{0\}$ as germs; therefore we can define the mapping degree $\mathrm{ind}(X)$ for a finitely C^0-$\mathcal{K}$-determined endomorphism germ $X : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. Nishimura (see [7]) proves the following result.

Theorem 2.1. *Let X, $Y\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be two finitely C^0-$\mathcal{K}$-determined map germs and $n \neq 4$. Then the following statements are equivalent:*
(1) *X and Y are topologically $\mathcal{K}$-equivalent,*
(2) $|\mathrm{ind}(X)| = |\mathrm{ind}(Y)|$.

We now recall the Poincaré definition of the index of a vector field in the plane (see [1]). Let C be a simple closed curve in $\mathbb{R}^2$, let X be a vector field defined on a simply connected open region of $\mathbb{R}^2$ which contains the curve C and let r be some straight line in the (x, y)-plane. Suppose that there exist only finitely many points M_k (for $k = 1, 2, ..., n$) on C at which the vector $X(M_k)$ is parallel to r. Let M be a point describing the curve in the counterclockwise sense, and let p (respectively q) be the number of points of M_k at which the vector $X(M)$ passes through the direction of r in the counterclockwise (respectively clockwise) sense. Points M_k at which the vector field $X(M)$ assumes the direction of r while moving, say, in the clockwise sense and then begins to move in the opposite direction (or vice versa) are not counted. Then the index of C, $i(C)$, is defined by $i(C) = (p - q)/2$. If we have a zero N of X, we define the index $\mathrm{ind}_N(X)$ of X at N by $\mathrm{ind}_N(X) = i(C)$, where C is a simple closed curve on which there are no zeros of X and which is such that it surrounds only the point N.

We also use the following result of Gutierrez and Ruas, which shows that the index of a Newton non-degenerate vector field, in the sense of [3], is determined by its principal part (see [3]).

Theorem 2.2. *Let $X = (P, Q)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the germ of a smooth vector field in the plane. Let $X_\Gamma = (P|_{\Gamma(P)}, Q|_{\Gamma(Q)})$ be the principal part of the vector field X. If X is Newton non-degenerate, then, there exists a neighborhood U of $(0, 0)$ such that, $\forall (x, y) \in U \backslash \{(0, 0)\}$ and $\forall s \in [0, 1]$,*

$$0 < |sX(x, y) + (1 - s)X_\Gamma(x, y)|.$$

In particular, $(0,0)$ *is an isolated singularity of both* X *and* X_Γ *and*

$$\operatorname{ind}(X) = \operatorname{ind}(X_\Gamma).$$

3. MAIN RESULTS

Let $X = (P, Q)$ be the germ of a C^∞ vector field at the origin such that X is finitely C^0-$\mathcal{K}$-determined, $j^\infty X$ its Taylor series at the origin, $j^\infty X = X_m + X_{m+1} + \cdots$ where $X_i = (P_i, Q_i)$ is the homogeneous part of degree i of the Taylor expansion of X. We want to find conditions on X such that $|\operatorname{ind}(X)| = m$, based on the geometry of the pencil generated by the homogeneous polynomials P_m and Q_m.

Let $H^m(2,1)$ be the space of the real homogeneous polynomials $a_0x^m + a_1x^{m-1}y + \cdots + a_my^m$. We denote by Ω_m the open set of $H^m(2,1)$, consisting of polynomials with m distinct (real or complex) roots. Then $\Omega_m = \Omega_{\{m,0\}} \cup \Omega_{\{m,1\}} \cup \cdots \cup \Omega_{\{m,m\}}$ where $\Omega_{\{m,i\}}$ is the set of the elements in Ω_m with i distinct real roots. We denote the m-jet of X by X_m, which we think as a pencil $\mathcal{L} = \{\alpha P_m + \beta Q_m : \alpha, \beta \in \mathbb{R}\}$ in $\mathcal{P}^m$, the m-real projective space, or alternatively its complexification, $\mathcal{L}_\mathbb{C} = \{\alpha P_m + \beta Q_m : \alpha, \beta \in \mathbb{C}\}$ in the complex projective space $\mathcal{P}^m_\mathbb{C}$. The non-degenerate pencil $\mathcal{L}_\mathbb{C}$ is thought as a line in $\mathcal{P}^m_\mathbb{C}$. The discriminant variety of forms with repeated roots is a hypersurface $\Delta_\mathbb{C}$ with singular locus that we denote by $C_\mathbb{C}$ (we denote the corresponding sets over the real by Δ and C). The obvious invariant of the pencil is the way in which the line $\mathcal{L}_\mathbb{C}$ meets $\Delta_\mathbb{C}$ and $C_\mathbb{C}$.

Under generic conditions, the pencil intercepts $\Omega_{\{m,m\}}$. In fact, if $\mathcal{L} \cap \Omega_m = \emptyset$ then $\mathcal{L} \subset \Delta$ and the pencil is singular. We will prove in Lemma 4.3 that if $\mathcal{L} \cap \Omega_{\{m,j\}} \neq \emptyset$, then $|\operatorname{ind}(X)| \leq j$. Since we are interested in vector fields X such that $|\operatorname{ind}(X)| = m$ then, the generic situation $\mathcal{L} \cap \Omega_m \neq \emptyset$ implies that

$$\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset \qquad (*)$$

In this work, we give necessary and sufficient conditions for $|\operatorname{ind}(X)|$ to be m, assuming that condition $(*)$ holds. If P_m and Q_m are in $\Omega_{\{m,m\}}$, we can write $P_m = \prod_{i=1}^m L_i$, $Q_m = \prod_{j=1}^m K_j$, where L_i and K_j, $i, j = 1, \ldots m$ are the linear branches of P_m and Q_m respectively. Let $p_1 < p_2 < \ldots < p_m$ and $q_1 < q_2 < \ldots < q_m$ denote the slopes of the branches $L_i = 0$ and $K_j = 0$, respectively (we assign slope equals to ∞ to the line $x = 0$.) Under these conditions, if $p_1 < q_1 < p_2 < q_2 < \ldots < p_m < q_m$ (or $q_1 < p_1 < q_2 < p_2 < \ldots < q_m < p_m$) we shall say that P_m and Q_m have alternating branches. The main results are:

THEOREM 3.1. *Let us assume that* $X_m^{-1}(0) = \{0\}$. *Then the following conditions are equivalent*

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(i) $|\text{ind}(X)| = m$

(ii) $\mathcal{L} \subset \Omega_{\{m,m\}}$

(iii) P_m, $Q_m \in \Omega_{\{m,m\}}$, *and the branches of* P_m *and* Q_m *alternate.*

We observe that $(iii) \Rightarrow (i)$ in Theorem 3.1 was previously given in [4].

If $X_m^{-1}(0) \neq \{0\}$ then P_m, Q_m have a common factor $c(x,y)$, that is $X_m = c.(p,q)$, degree$(c(x,y)) = d$ with $1 \leq d \leq m$. If $d = m$, $\mathcal{L}$ is degenerate, that is, $\mathcal{L}$ reduces to a point. Otherwise, $1 \leq d \leq m-1$, and the hypothesis $|\text{ind}(X)| = m$ imposes geometric conditions on the pencil $\mathcal{L}$. In this case we prove that:

THEOREM 3.2. *Let* X *be finitely* C^0-$\mathcal{K}$*-determined,* $X_m = c.(p,q)$ *with* $p^{-1}(0) \cap q^{-1}(0) = \{0\}$*,* degree$(c(x,y)) = d$*. If* $\mathcal{L} \cap \Omega_m \neq \emptyset$ *and* $|\text{ind}(X)| = m$ *then* $\mathcal{L}$ *meets* Δ *at exactly* d *points and* $\mathcal{L} \cap \mathcal{C} = \emptyset$.

The condition $|\text{ind}(X)| = m$ imposes restrictions on the maximum degree d of $c(x,y)$ when $d \geq 3$ (see Proposition 5.4).

We give in Propositions 5.2 and 5.3 a complete characterization of vector fields with $|\text{ind}(X)| = m$, when $d = 1$ and $d = 2$.

Remark 3. 1. The hypothesis $\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset$ is not a necessary condition for $|\text{ind}(X)| = m$. For instance, let $X = ((x-y^4)(x-y^8), (x-y^3)(x-y^6))$. The pencil associated to X_2 is degenerate, but the index of X is equal to -2 (see Lemma 4.1).

4. PROOF OF THE MAIN RESULTS

Lemmas 4.1, 4.2 and 4.3 are consequences of Poincaré's method.

LEMMA 4.1. *Let* $X = (P,Q)$ *be a finitely* C^0-$\mathcal{K}$*-determined germ at the origin,* $j^\infty X = X_m + X_{m+1} + \cdots$ *with* $X_m \neq 0$*. The absolute value of the index of* X *is maximal,* $|\text{ind}(X)| = m$*, if and only if the following holds:*

(i) the zero sets of $P(x,y)$ *and* $Q(x,y)$ *decomposes into* $2m$ *semi-algebraic branches, and the complement of the set* $P^{-1}(0) \cup Q^{-1}(0)$ *in a sufficiently small open ball* $B_\epsilon(0)$ *centered at* 0*, is a union of* $4m$ *sectors,*

(ii) the boundary of each sector in $B_\epsilon(0)$ *is the union of one half-branch of* P *and one half-branch of* Q*,*

(iii) the sign of $P.Q$ *alternates on each pair of consecutive sectors.*

Proof. We apply Poincaré's method, taking a small circle C centered at 0, and the y-axes as the line r. Then, M_k are the points $(x,y) \in C$ at which the vector field $X = (P,Q)$ is ver-

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tical, that is, $P(x,y)=0$. If the conditions (i), (ii) and (iii) hold, then $p=2m$, (or $q=2m$), and there are no points M_k at which the vector field X assumes the vertical direction while moving in the counterclockwise sense and then begins to move in the opposite direction (or vice versa). Then $|\text{ind}(X)|=m$. On the other hand, if any of the conditions (i), (ii) or (iii) fails, then $|\text{ind}(X)|\neq m$. ▮

Lemma 4.2. *Let $X:\mathbb{R}^2,0\to\mathbb{R}^2,0$ be finitely C^0-$\mathcal{K}$-determined at the origin, then*
(i) if $j^mX\neq(0,0)$ and $j^lX=(0,0)$ for all $l<m$ then $|\text{ind}(X)|\leq m$;
(ii) if $|\text{ind}(X)|=m$ then $j^lX=0$ for all $l<m$.

Proof. (i) If $j^mX\neq(0,0)$ and $j^lX=(0,0)$ for all $l<m$ then each component function of X has at most m semi-algebraic branches. Therefore taking the axis y or x as the line r, it follows from Poincaré's method that $|\text{ind}(X)|\leq m$.

(ii) If $j^lX\neq(0,0)$ for some $l<m$, it follows from (i) that $|\text{ind}(X)|\leq l<m$. ▮

Lemma 4.3. *Let $\mathcal{L}$ be the pencil $\alpha P_m+\beta Q_m$, with α, $\beta\in\mathbb{R}$. If there exist j such that $\mathcal{L}\cap\Omega_{\{m,j\}}\neq\emptyset$, $0\leq j\leq m$, then $|\text{ind}(X)|\leq j$.*

Proof. If $\mathcal{L}\cap\Omega_{\{m,j\}}\neq\emptyset$ then making, if necessary, a linear change of coordinates in the target, we can assume that X is C^0-$\mathcal{K}$-equivalent to (P,Q), $j^mP=P_m$, with $P_m\in\Omega_{\{m,j\}}$. The assumption that $P_m\in\Omega_{\{m,j\}}$ implies that the function-germ $P:\mathbb{R}^2,0\to\mathbb{R},0$ is C^0-determined by its m-jet, that is, there exists a germ of homeomorphism $h:\mathbb{R}^2,0\to\mathbb{R}^2,0$ such that $P_m=P\circ h$, (see [5], [8] or [9]). Then, it follows that $X\circ h=(P_m,Q_m+R)$, where $P_m\in\Omega_{\{m,j\}}$ and $R(x,y)$ is a smooth function germ with $j^mR(0,0)=0$. When the vector field $X=(P,Q)$ is finitely C^0-$\mathcal{K}$-determined, we can assume that R is a polynomial function-germ.

To complete the proof, we apply Poincaré's method, taking as in Lemma 4.1, the y axis as the line r, to conclude that $|\text{ind}(X)|\leq j$. ▮

Let $Y(x,y)=(\pm\, x\prod_{i=1}^{m-1}(y-p_ix),\prod_{i=1}^{m}(y-q_ix))$ where $p_1<p_2<\cdots<p_{m-1}$, $q_1<q_2<\cdots<q_m$, $q_i\neq p_j\ \forall\ i,j$.

The proof of Theorem 3.1 follows from the following results.

Lemma 4.4. *The absolute value of the index of Y is m if and only if $q_1<p_1<q_2<p_2<\cdots<p_{m-1}<q_m$.*

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Proof. The result follows from Lemma 4.1. ∎

LEMMA 4.5. *Let us suppose $X_m^{-1}(0) = \{0\}$. Then $|\mathrm{ind}(X)| = m$ implies that X is C^0-$\mathcal{K}$-equivalent to Y.*

Proof. If $X_m^{-1}(0) = \{0\}$ then $\mathrm{ind}(X) = \mathrm{ind}(X_m)$ (see [2] or [3]) and by Nishimura's result X is C^0-$\mathcal{K}$-equivalent to X_m. If $|\mathrm{ind}(X)| = m$ then X_m is equivalent to Y by Lemma 4.1. ∎

PROPOSITION 4.1. *Let us suppose $X_m^{-1}(0) = \{0\}$. If $\mathcal{L} \subset \Omega_{\{m,m\}}$, then the roots of P_m and Q_m alternate, for every pair P_m, Q_m of generators of the pencil.*

Proof. If $\mathcal{L} \subset \Omega_{\{m,m\}}$, with a convenient choice of affine coordinates, we can write the pencil in the form $(y - \lambda_1(t)x) \cdots (y - \lambda_m(t)x)$, with $t \in \mathbb{R}$ and $\lambda_i(t)$ continuous real functions for any i. Let $\mathrm{Im}(\lambda_i) \subset \mathbb{R}$ be the image set of the function λ_i.

Since $X_m^{-1}(0) = \{0\}$, it follows that

(i) $\forall\ i = 1, \cdots, m$, $\lambda_i(t)$ is injective, consequently it is monotonic;

(ii) $\mathrm{Im}(\lambda_i) \cap \mathrm{Im}(\lambda_j) = \emptyset$, $\forall\ i \neq j$.

Let $m = 2$, and $\mathcal{L} : (y - \lambda_1(t)x)(y - \lambda_2(t)x)$, $\lambda_i(0) = p_i$, $\lambda_i(1) = q_i$, $i = 1, 2$. If $p_1 < p_2$ it also follows that $q_1 < q_2$ (otherwise, by the Intermediate Value Theorem, there would be a point $t_0 \in (0, 1)$ such that $\lambda_1(t_0) = \lambda_2(t_0)$, and this contradicts (ii)).

Then, assuming that the function λ_1 is increasing, it is sufficient to analyze the following two possibilities for the roots of the generators of the pencil: $p_1 < p_2 < q_1 < q_2$, or $p_1 < q_1 < p_2 < q_2$. If the first condition holds, we again get a contradiction, hence the result follows for $m = 2$.

For any m, if the roots of P_m and Q_m do not alternate, we can find i and j, such that the roots of λ_i and λ_j do not alternate, and we get a contradiction. ∎

The following result follows as a consequence of the Theorems 3.1 and 2.1.

COROLLARY 4.1. *The vector fields X such that $j^\infty X = X_m + X_{m+1} + \cdots$, X_m has an isolated zero and $\mathcal{L} \subset \Omega_{\{m,m\}}$ form a unique C^0-$\mathcal{K}$-orbit.*

We now assume that X is finitely C^0-$\mathcal{K}$-determined, $X_m^{-1}(0) \neq \{0\}$ and the pencil $\mathcal{L}$, $\alpha P_m + \beta Q_m$, does not reduce to a point. Then P_m, Q_m have a common factor $c(x, y)$, that is $X_m = c.(p, q)$ with $p^{-1}(0) \cap q^{-1}(0) = \{0\}$. In others words, the pencil $\alpha P_m + \beta Q_m$ is of the form $c(x, y)(\alpha p(x, y) + \beta q(x, y))$. The condition $(*)$ implies that $c(x, y)$ is non-degenerate, in the sense that it is a homogeneous polynomial of degree d, with $1 \leq d \leq m-1$, having d distinct real roots and degree$(p) =$ degree$(q) = m - d$.

The proof of the Theorem 3.2 follows from Lemmas 4.6, 4.7 and 4.8 below.

LEMMA 4.6. *If the pencil $\mathcal{L}$ is of the form $\alpha P_m + \beta Q_m = c.(\alpha p + \beta q)$, $c(x, y)$ has at least one linear factor and* degree$(c) = d$, $1 \leq d \leq m-1$, *then $\mathcal{L} \cap \Delta \neq \emptyset$.*

Proof. We can change coordinates to obtain $c(x, y) = x\bar{c}(x, y)$, $p(x, y) = a_0 x^{m-d} + \cdots + a_{m-d} y^{m-d}$ and $q(x, y) = b_0 x^{m-d} + \cdots + b_{m-d} y^{m-d}$. If $a_{m-d} = 0$, then x is a factor of multiplicity 2 of $P_m = cp$, hence the result follows. If $a_{m-d} \neq 0$, by a change of coordinates we can eliminate the term $c(x, y) b_{m-d} y^{m-d}$ of the second coordinate function, and this implies that x is a factor of multiplicity 2 of Q_m. ∎

Notice that the Lemma 4.6 applies to any linear factor of $c(x, y)$, then we can state the following:

LEMMA 4.7. *If the pencil is of the form $c(x, y)(\alpha p(x, y) + \beta q(x, y))$, where $c \in \Omega_{\{d,d\}}$, $1 \leq d \leq m-1$, then $\mathcal{L}$ intercepts Δ in at least d points.*

Let $X_m^{-1}(0) \neq \{0\}$, $|\text{ind}(X)| = m$ and the pencil intercepts $\Omega_{\{m,m\}}$ at P_m. Then X is C^0-$\mathcal{K}$-equivalent to $(P_m, Q_m + R)$, $j^m R(0, 0) = 0$ and $\mathcal{L} \cap \Delta \neq \emptyset$. That is X_m is C^0-$\mathcal{K}$-equivalent to $c(x, y)(p(x, y), q(x, y))$, degree$(c(x, y)) = d$, where one of the roots of q may coincide with some root of $c(x, y)$ by the proof of Lemma 4.6. With the following Lemma we conclude that the pencil intercepts Δ on exactly d points and $\forall\ \alpha, \beta$, $\alpha P_m + \beta Q_m \notin \mathcal{C}$.

LEMMA 4.8. *With the above conditions $\forall\ \alpha, \beta$, $\alpha p(x, y) + \beta q(x, y) \in \Omega_{\{m-d,m-d\}}$.*

Proof. Since $|\text{ind}(X)| = m$, the result is a consequence of Poincaré's method. ∎

5. CLASSIFICATION

In this section we classify finitely C^0-$\mathcal{K}$-determined vector fields with $|\text{ind}(X)| = m$ under the hypothesis that $\mathcal{L} \cap \Omega_m \neq \emptyset$ and $X_m^{-1}(0) \neq \{0\}$. This means that $\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset$ and $\alpha P_m + \beta Q_m = c(x, y)(\alpha p(x, y) + \beta q(x, y))$, with degree$(c(x, y)) = d$, $p^{-1}(0) \cap q^{-1}(0) = \{0\}$, $1 \leq d \leq m$. If $d = 1$, by changes of coordinates, we can take $c(x, y) = x$, otherwise if $d \geq 2$, $c(x, y) = xy \prod_{i=1}^{d-2} (y - r_i x)$ with $r_i \in \mathbb{R}$.

As in the previous section (see proof of Lemma 4.3), the hypothesis $\mathcal{L} \cap \Omega_{\{m,m\}} \neq \emptyset$ implies that the vector field $X = (P, Q)$ is $\mathcal{K}$-equivalent to $(P_m, Q_m + R)$, where $P_m \in$

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$\Omega_{\{m,m\}}$ and $j^m R(0,0) = 0$. In what follows, we use the symbol $\simeq$ to indicate the C^0-$\mathcal{K}$-equivalence.

We have the following as pre-normal forms for vector fields X which satisfy these conditions.

(Notice that if $m = 1$, $X_m^{-1}(0) \neq \{0\}$ implies that $m = d = 1$.)

LEMMA 5.1. *Assume* $|\text{ind}(X)| = m$. *Let* $m \geq 1$ *and* $d \leq m$*:*

(i) If $d = 1$*, then* $X \simeq (xp(x,y), x^2q(x,y) + \epsilon y^k)$*, with* $\epsilon = \pm 1$, $\text{degree}(p) = m - 1$, $\text{degree}(q) = m - 2$, $k > m$*, where* p *and* q *have only simple real roots, all distinct from* x*.*

(ii) If $d \geqslant 2$*, then* $X \simeq (xyp(x,y) \prod_{i=1}^{d-2}(y - r_i x), x^2y(f(x,y) + q(x,y) \prod_{i=1}^{d-2}(y - r_i x)) + \epsilon_1 x^l + \epsilon_2 y^k)$*, where* $\epsilon_i = \pm 1$, $l > m$, $k > m$, $\text{degree}(p) = m - d$, $\text{degree}(q) = m - d - 1$ *(when* $m = d$*, we take* $q = 0$*). The non zero polynomials* p *and* q *have only simple real roots that are distinct from* x*,* y *and* $(y - r_i x)$*,* $i = 1, \cdots, d - 2$*.*

When $d = 2$*, we take* $f = 0$*, and interpret* $\prod_{i=1}^{0} = 1$*. In the case that* $d \geqslant 3$*, if* $f \neq 0$*,* $\text{order}(x^2yf(x,y)) = n > m$*,* $n \leq l$ *and* $n \leq k$*, and when* $d = m$ *we always have* $f \neq 0$*.*

Proof. When $m \geq 2$, $d \leq m - 1$, by the proof of the Lemma 4.6, we can take $X_m = (xp(x,y), x^2\bar{q}(x,y))$, $X \simeq X_m + (0, R)$. Let $\Gamma_+(X)$ be the Newton polyhedra of X (see [3]). In the cases $d = 1$ and $d = 2$, we can see that the principal part X_Γ of X, with respect to the Newton filtration coincides with the given pre-normal forms. Furthermore, X is Newton non-degenerate in the sense of [3]. Hence it follows from the Theorem 2.2 of [3] that $\text{ind}(X) = \text{ind}(X_\Gamma)$. In case $d \geq 3$, X is always Newton degenerate. With simple coordinate changes we can reduce $R(x,y)$ to the following form: $R(x,y) = \epsilon_1 x^l + \epsilon_2 y^k + x^2yf(x,y)$, $k, l > m$. There are two cases to be considered:

1. $\text{order}(x^2yf) > min\{l,k\}$,
2. $\text{order}(x^2yf) \leq min\{l,k\}$.

Using Lemma 4.1 and Nishimura's result, we obtain that conditions 1 and 2 lead respectively, to the pre-normal forms with $f = 0$ or $f \neq 0$ of (ii).

In case $d = m$, we can take $X_m = (xp(x,y), xp(x,y))$, then $X \simeq (xp(x,y), R(x,y))$, $j^m R(0,0) = 0$. If $d = 1$ $(d = 2)$ by a change of coordinates we can eliminate the term $x\psi(x,y)$ (resp. $xy\psi(x,y)$) of $R(x,y)$. If $d \geq 3$ the proof is similar to the proof of the case $d \geq 3$, $d \leq m - 1$, but we observe that since $(P_m, Q_m + R)$ is C^0-$\mathcal{K}$- equivalent to (P_m, R), the condition $d \geq 3$ does not imply that X is always Newton degenerate, this can occur only when $l = k$ and $n \leq l$. The pre-normal form (ii) with $f = 0$ does not occur for $d = m$, since in this case, we can use the Lemma 4.1 to show that the absolute value of the index is strictly less than m. ▮

Remark 5. 1.

1. Given a vector field of types (i), (ii) (with $f \neq 0$), for all $d \leq m-1$, we can choose l, k, ϵ_1, ϵ_2 and f such that $|\text{ind}(X)| = m$. This is not true for vector fields of type (ii) with $f(x,y) = 0$ when $d \geq 3$. In fact, in this case there are values of d for which $|\text{ind}(X)| < m$, $\forall\ \epsilon_1, \epsilon_2, l, m$ (see Proposition 5.4).
2. If $m = d$, for any m we can choose l, k, ϵ_1, ϵ_2 and f such that $|\text{ind}(X)| = m$.

In what follows we obtain necessary and sufficient conditions for $|\text{ind}(X)| = m$ in the cases d equal to 1 or 2. Our method applies to the cases $d \geq 3$, but the statement of the general results become too technical. In Lemma 5.2 and Proposition 5.4 we give some partial information for this case.

5.1. Case $d = 1$

In this case, if $|\text{ind}(X)| = m$, the pre-normal forms of X are given by (i) of the Lemma 5.1 and we have the following proposition that extends a result of A. Cima, A. Gasull and J. Torregrosa, [2], and give a method to construct vector fields with index equal to $\pm m$.

Proposition 5.1. (1) *If* $X \simeq (xp(x,y), x^2q(x,y)+\epsilon y^k)$ *with* $k > m \geq 2$ *then* $|\text{ind}(X)| = m$ *if and only if the following holds:*

(i) $p(x,y)$ *has* $m-1$ *distinct real branches,* $p(x,y) = \prod_{i=1}^{m-1}(y-p_ix)$, $p_1 < p_2 < \cdots < p_{m-1}$,

(ii) $q(x,y)$ *has* $m-2$ *distinct real branches,* $q(x,y) = \prod_{i=1}^{m-2}(y-q_ix)$, $q_1 < q_2 < \cdots < q_{m-2}$,

(iii) $m \equiv k$ *mod* 2, $\epsilon < 0$ *and* $p_1 < q_1 < p_2 < \cdots < p_{m-2} < q_{m-2} < p_{m-1}$.

(2) *When* $m = d = 1$, $X \simeq (x, \epsilon y^k)$, $k > 1$ *and* $|\text{ind}(X)| = 1$ *if and only if* $k \equiv 1$ *mod* 2.

Let B_ϵ be a small open ball centered at 0. We denote by S_1 (respectively S_2) the sector in B_ϵ bounded by the two consecutive half branches of p, containing one of the half branches $\bar{c}_1$ (resp. $\bar{c}_2$) of $c(x,y) = 0$, see Figure 1. Denote by S^i_{cq} the closed subsector of S_i bounded by $\bar{c}_i$ and the unique half branch of q inside S_i (we can change coordinates, as in Lemma 4.6, in such way that the branches q and c coincide). In the pre-normal forms in Lemma 5.1 q and c are taken to be equal to x and in this case S^i_{cq} is the half branch $\bar{c}_i$, $i = 1, 2$. The notation $F|_{\bar{c}_i}$ means F restricted to $\bar{c}_i$ and $S^i \backslash_{cq}$ indicates the complement of S^i_{cq} in S_i.

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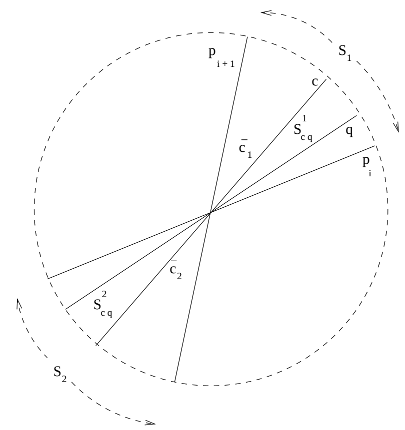

FIG. 1. Sectors and subsectors

The Proposition 5.2 follows from Proposition 5.1 and Lemma 4.1.

PROPOSITION 5.2. (1) *If* $X \simeq (P_m, Q)$, $m \geq 2$, $P_m = cp$, $Q_m = cq$, *c linear polynomial,* $Q|_c = R(x,y)$ *with* $j^m R(0,0) = 0$, *then* $|\text{ind}(X)| = m$ *if and only if* $\alpha p(x,y) + \beta q(x,y) \in \Omega_{\{m-1,m-1\}}$, $\forall\ \alpha, \beta$, *and the sign of* Q_m *in* $S^i \backslash_{cq}$ *is the opposite of the sign of* $R|_{\bar{c}_i}$, *for each* $i = 1, 2$.

(2) *In case m=1,* $X \simeq (c, R)$ *with c a linear polynomial, then* $|\text{ind}(X)| = m$ *if and only if the signs of* $R|_{\bar{c}_1}$ *and* $R|_{\bar{c}_2}$ *are opposite.*

5.2. Cases $d = 2$ and $d \geq 3$

In this case, if $|\text{ind}(X)| = m$, the pre-normal forms of X are given by (ii) of the Lemma 5.1. The results are similar to that in 5.1.

Let $c(x,y) = c_1(x,y)c_2(x,y)$, with c_i $i = 1,2$ linear polynomials. If $d = 2$, $m > 2$, $X \simeq (P_m, Q) = (cp, cq + R)$, if $m = d = 2$, $X \simeq (c(x,y), R(x,y))$, with $j^m R(0,0) = 0$. Denote respectively by $\bar{c}_{1i}$ and $\bar{c}_{2i}$ the half branches given by $c_1(x,y) = 0$ and $c_2(x,y) = 0$. They define four sectors C_{1i} and C_{2i}, $i = 1,2$, such that $C_j = C_{j1} \cup C_{j2}$ has the boundary given by c_1 and c_2. Let $m - 2 = m_1 + m_2$ with m_1 and m_2 being the number of branches of $p(x,y)$, respectively, in C_1 and C_2. As in the case $d = 1$, we denote S_{ij} the sector containing the half branch $\bar{c}_{ij}$, and $S^{ij}\backslash_{c_i q}$ the complement of $S^{ij}_{c_i q}$ in S_{ij}, $i, j \in \{1,2\}$. We have the following result:

PROPOSITION 5.3. (1) *If* $d = 2$ *and* $m > 2$, *the necessary and sufficient conditions for* $|\text{ind}(X)|$ *to be m are*

(i) $\alpha p(x,y) + \beta q(x,y) \in \Omega_{\{m-2,m-2\}}$, $\forall\ \alpha, \beta$;

(ii) the sign of $R|_{\bar{c}_{1i}}$ *is the opposite of the sign of* Q_m *in the sector* $S^{1i}\backslash_{c_{1q}}$*, for each* $i = 1, 2$*;*

(iii) If m_i *is even (odd) then the sign of* $R|_{\bar{c}_{2j}}$ *is the opposite of (same as) the sign of* $R|_{\bar{c}_{1k}}$*, where* $\bar{c}_{2j}$ *and* $\bar{c}_{1k}$*,* $j, k \in \{1, 2\}$*, are boundary of* C_{il}*, for each* $l = 1, 2$*,* $i = 1, 2$*.*

(2) *If* $m = d = 2$ *then* $|\text{ind}(X)| = 2$ *if and only if the signs of* $R|_{\bar{c}_{1i}}$ *and* $R|_{\bar{c}_{2j}}$*, for each* $i, j \in \{1, 2\}$*, are opposite.*

The next result holds for any $d \geq 2$.

LEMMA 5.2. *Let* X *be a vector field as in Lemma 5.1, with* $d \geq 2$*. If* $|\text{ind}(X)| = m$ *then* l*,* k *and* m *are congruent modulo* 2*.*

Proof. The result is a consequence of Poincaré's method. We choose the line r as the axis y, and examine the pre-normal form of the vector field to conclude that the condition l, k and m congruent modulo 2 is necessary for $|\text{ind}(X)|$ to be m. The sense of the vectors at the axes x and y plays an essential role in the argument. ▌

We can always describe necessary and sufficient conditions to construct X with $j^\infty X = X_m + \cdots$ and $|\text{ind}(X)| = m$ if the condition $(*)$ holds. When degree$(c) = d$ with $3 \leq d \leq m$, the conditions are very technical, but the following result is useful to get a better version of the Theorem 3.2, when the vector field X is C^0-$\mathcal{K}$- equivalent to the pre-normal form (ii) of Lemma 5.1, with $d \geq 3$, $d \neq m$ and $f = 0$.

PROPOSITION 5.4. *Let* X *be the pre-normal form (ii), with* $f(x, y) = 0$*, of the Lemma 5.1,* $3 \leq d \leq m - 1$*. Then* $|\text{ind}(X)| < m$ *when one of the following conditions hold:*

(i) l*,* k*,* m *even,* $\epsilon_1\epsilon_2 < 0$*, and* $d > \frac{m+2}{2}$*; or*

(ii) l*,* k*,* m *even,* $\epsilon_1\epsilon_2 > 0$*, and* $d > \frac{m}{2}$*; or*

(iii) l*,* k*,* m *odd, and* $d > \frac{m+1}{2}$*;*

Otherwise, taking into account the conditions in Lemma 5.2, to all values of d *not satisfying the above condition, we can construct vector fields* X *with* $|\text{ind}(X)| = m$*.*

For the other pre-normal forms of the Lemma 5.1, given d with $1 \leq d \leq m - 1$, we can always construct vector fields X such that $|\text{ind}(X)| = m$ where $\sharp(\mathcal{L} \cap \Delta) = d$. When $d = m \geq 3$, $\mathcal{L}$ reduces to a point (degenerated pencil), the pre-normal form of X is (ii) where $f(x, y) \neq 0$ and with appropriated choice of $f(x, y)$ we can always get $|\text{ind}(X)| = m$. Again, our method applies in this case but the conditions are very technical.

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