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Modified C^ℓ-trivialization of real germs of functions and maps

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In this article we introduce the notion of *modified C^ℓ-triviality* for families of functions germs and maps germs of class C^ℓ. We study this kind of triviality in the quasihomogeneous and Newton non-degenerate cases. May, 2006 ICMC-USP

1. INTRODUCTION

The determinacy of families of map germs which are trivial with respect to some equivalence relation is a fundamental subject in singularity theory. In special, the topological triviality of analytic families with isolated singularity is being investigated by several authors.

However, there are few works which investigate the C^ℓ-triviality for $0 < \ell < \infty$. Kuiper in [10] showed the C^1-equivalence of functions near isolated critical points. Bromberg and Medrano gave in [2] estimates for the $C^\ell - \mathcal{R}$-triviality for families of germs of semi-quasihomogeneous functions. For families of map germs, Ruas and Saia in [13] also considered the semi-quasihomogeneous case to show estimates for the $C^\ell - G$-triviality, where G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$. More recently, the authors in [14] provided estimates on the C^ℓ-G-triviality, for $0 \leq \ell < \infty$, from the viewpoint of the Newton filtration, these estimates are given for map germs which satisfy a Newton non-degeneracy condition.

Inspired by the Whitney's example

$$W(x, y, t) = y(y - x)(y - 2x)(y - tx),\ 2 < t < \infty,$$

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Kuo in [11] introduces the notion of modified analytic trivialization of germs of real analytic functions. As this equivalence is weaker than analytic and stronger than topological triviality, Kuo initiated a new route of research for analytic germs of functions and also maps. Following Kuo's ideas, Fukui [6], Paunescu [15] and Yoshinaga [17] investigated problems of classification of germs of real analytic functions induced by the modified analytic triviality. From the viewpoint of the Newton polyhedron, Fukui and Yoshinaga in [7], gave conditions for the modified analytic trivialization of families of germs of real analytic functions which are Newton non-degenerate.

In this article we introduce the notion of *modified C^ℓ-triviality* for families of germs of real functions and mappings of class C^ℓ. We remark that this concept lies between the C^ℓ and the topological triviality. The ideas presented here are inspired in the work of Kuo [11] and Buchner-Kucharz [1], where the topological triviality of families defined by quasihomogeneous mappings of class C^2 is proved under certain conditions.

First we introduce the notion of *modified C^ℓ-triviality* for families of map germs $F_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ locally parameterized by $t = (t_1, ..., t_q)$ in $\mathbb{R}^q$. Next we consider the case of quasihomogeneous maps and functions, we prove that families of class $C^{\ell+1}$ satisfying some conditions, given in terms of the weights and degrees, have a modified C^ℓ-trivialization. In the third section we investigate the modified C^ℓ-trivialization of function germs from the viewpoint of the Newton polyhedron, we give a theorem of modified C^ℓ-trivialization for families of functions of class C^ℓ which satisfy a Newton non-degeneracy condition.

2. MODIFIED C^ℓ-TRIVIALIZATION

In this section we introduce the notion of modified C^ℓ-trivialization of maps, which is a generalization of the definition of modified analytic triviality, given by Kuo in [11]. This new definition also applies to maps of class C^ℓ while the definition given by Kuo is only for analytic germs.

We consider a family of maps $F_t : \mathbb{R}^n \to \mathbb{R}^p$ parameterized by $t = (t_1, ..., t_q) \in J$ with $F(0, t) = 0$ for all $t \in J$, where J is a compact q-dimensional cube in $\mathbb{R}^q$.

Definition 2.2.1. *F admits an* **almost modified C^ℓ-trivialization at** *J if there exists a neighborhood U of the origin in $\mathbb{R}^n$, a proper analytic map $\varphi : X \to \mathbb{R}^n$ defined in a pure n-dimensional analytic variety X, and a C^ℓ-diffeomorphism*

$$M : U_1 \times J \to U_2 \times J$$

which preserves the level t, where U_1 and U_2 are neighborhoods of $\varphi^{-1}(0)$ in $\varphi^{-1}(U)$ such that

$$F \circ (\varphi \times Id_J) \circ M$$

is constant at t, and Id_J denotes the identity map of J.

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Definition 2.2.2. *If M induces an homeomorphism $m : \varphi(U_1) \times J \to \varphi(U_2) \times J$, which preserves the level t, such that the diagram below commutes we say that the family admits a* **modified C^ℓ-trivialization along** J

$$\begin{array}{ccccc} U_1 \times J & \xrightarrow{M} & U_2 \times J & & \\ \varphi \times Id_J \downarrow & & \downarrow \varphi \times Id_J & & \\ \varphi(U_1) \times J & \xrightarrow{m} & \varphi(U_2) \times J & \xrightarrow{F} & \mathbb{R}^p. \end{array}$$

The mapping M is called the associate C^ℓ-modification to the family F.

3. THE QUASIHOMOGENEOUS CASE

Buchner and Kucharz in [1] show the topological triviality of families of maps defined by quasihomogeneous maps of class C^2 with isolated singularity at origin. The method used there is the construction of a vector fields of Kuo type.

In this section we apply this method to show that, under some conditions, a deformation of class $C^{\ell+1}$ of a quasihomogeneous function of class $C^{\ell+1}$ with isolated singularity at the origin admits a modified C^ℓ-trivialization. Next, we give a natural extension of this result for maps.

Definition 3.3.1. *We fix an n-tuple $r = (r_1, \ldots, r_n)$ in $\mathbb{Q}^n$ and say that a function germ $f(x_1, \ldots, x_n)$ is quasihomogeneous of type $(r_1, \ldots, r_n; d)$ if for all $\lambda \neq 0$,*

$$f(\lambda^{r_1} x_1, \lambda^{r_2} x_2, \ldots, \lambda^{r_n} x_n) = \lambda^d f(x_1, x_2, \ldots, x_n).$$

The numbers $r_1, \ldots, r_n$ are called weights and d is the weighted degree of f with respect to r.

For any monomial $x^m = x_1^{m_1} \ldots x_n^{m_n}$, we denote by $fil_r(x^m) = m_1 r_1 + \ldots + m_n r_n$ and for any germ g, $fil_r(g) = \min_m \{fil_r x^m\}$ for all $a_m \neq 0$ in the Taylor series at 0 of $g = \sum a_m x^m$.

A mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ is said quasihomogeneous of type $(r_1, r_2, \ldots, r_n; d_1, \ldots, d_p)$ if each coordinate f_j of f is quasihomogeneous of type $(r_1, r_2, \ldots, r_n; d_j)$ for all $j = 1, \ldots, p$.

3.1. The case of quasihomogeneous functions

We fix an n-tuple $r = (r_1, \ldots, r_n)$ and consider $F : U \times \mathbb{R}^q \to \mathbb{R}$ a family of functions defined in a neighborhood U of the origin in $\mathbb{R}^n$.

THEOREM 3.3.2. *Suppose that we can write F as $F = f + g$ with $f, g : U \times \mathbb{R}^q \to \mathbb{R}$ satisfying the following conditions:*

(1) *f and g are of class $C^{\ell+1}$; g and $\dfrac{\partial g}{\partial t_s}$ are of class C^k on the variables $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$, where $k > \max\left\{\dfrac{d}{r_1}, \ldots, \dfrac{d}{r_n}\right\} + \ell$, and $1 \leq s \leq q$;*

(2) *for each $t \in \mathbb{R}^q$, $f_t = f(.,t)$ is the restriction to U of a quasihomogeneous function of type $(r; d)$, with isolated singularity at origin;*

(3) *$fil(j^{k-1}(g_t)) > d$, $\forall t \in \mathbb{R}^q$, where $j^k(.)$ denotes the Taylor's polynomial of degree k at origin.*

Then F admits a modified C^ℓ-trivialization along J.

To prove this result we construct a Kuo's type vector field which has an appropriate class of differentiability.

First we consider the variety $X = S^{n-1} \times (-\epsilon, \epsilon)$ and define the weighted homogenous action $\varphi : S^{n-1} \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ in terms of the of the weights $r_1, \ldots, r_n$:

$$\varphi(x, \lambda) = (\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n)$$

with $\epsilon > 0$ chosen such that φ has its values in U.

To compensate this action, we define the map

$$H : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^q \to \mathbb{R},$$

in terms of the degree d:

$$H(x, \lambda, t) = \lambda^{-d} g(\varphi(x, \lambda), t)$$

for $\lambda \neq 0$ and $H(x, 0, t) = 0$.

We call $\pi : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^q \to S^{n-1} \times \mathbb{R}^q$ the standard projection $\pi(x, \lambda, t) = (x, t)$.

The next lemma is essential to prove that the vector field which we will construct has the appropriate class of differentiability.

LEMMA 3.3.3. *The conditions (1) and (3) above imply that $H, \dfrac{\partial H}{\partial x_i}, \dfrac{\partial H}{\partial t_s}$ are of class C^ℓ, and are zero at $\lambda = 0$.*

Proof: First we remark that from the item *1.*, $fil(j^k(g_t)) > d$, since for any $i_1 + ... + i_n \geq k$, we have $fil(x_1^{i_1} \cdots x_n^{i_n}) > d$. Therefore,

$$\begin{aligned} j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1} \cdots \partial x_{i_q}}\right) &= \frac{\partial^q}{\partial x_{i_1} \cdots \partial x_{i_q}} j^k(g_t) \\ fil\left(j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1} \cdots \partial x_{i_q}}\right)\right) &> d - \sum_{j=1}^{q} r_{i_j} \end{aligned} \qquad (1)$$

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Applying the infinitesimal Taylor's formula, we obtain

$$\frac{\partial^q}{\partial x_{i_1}\cdots\partial x_{i_q}}(g_t\circ\varphi(x,\lambda)) = j^{k-q}\left(\frac{\partial^q}{\partial x_{i_1}\cdots\partial x_{i_q}}(g_t\circ\varphi(x,\lambda))\right) + R_{k-q}(\varphi(x,\lambda))$$

with $\lim\limits_{\lambda\to 0}\frac{R_{k-q}(\varphi(x,\lambda))}{\|\varphi(x,\lambda)\|^{k-q}} = 0$. Hence,

$$\frac{\partial^q H}{\partial x_{i_1}\cdots\partial x_{i_q}} = \frac{1}{\lambda^{(d-\sum\limits_{j=1}^{q} r_{i_j})}}\left(j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}}\right)(\varphi(x,\lambda))\right) + \frac{R_{k-q}(\varphi(x,\lambda))}{\lambda^d}$$

for $q = 0,1,...,\ell+1$.

From the equation (0) we have

$$\lim_{\lambda\to 0}\frac{1}{\lambda^{(d-\sum\limits_{j=1}^{q} r_{i_j})}}\left(j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}}\right)(\varphi(x,\lambda))\right) = 0,$$

then it is enough to show that $\lim\limits_{\lambda\to 0}\dfrac{R_{k-q}(\varphi(x,\lambda))}{\lambda^d} = 0$. Since

$$\frac{R_{k-q}(\varphi(x,\lambda))}{\lambda^d} = \frac{R_{k-q}(\varphi(x,\lambda))}{\|\varphi(x,\lambda)\|^{k-q}}\frac{\|\varphi(x,\lambda)\|^{k-q}}{\lambda^d}$$

the result follows from the equality $\lim\limits_{\lambda\to 0}\dfrac{R_{k-q}(\varphi(x,\lambda))}{\|\varphi(x,\lambda)\|^{k-q}} = 0$ and

$$\left(\frac{\|\varphi(x,\lambda)\|^{k-q}}{\lambda^d}\right)^2 = \left[\frac{\lambda^{2r_1}x_1^2+...+\lambda^{2r_n}x_n^2}{\lambda^{\frac{2d}{k-q}}}\right]^{k-q} \le \lambda^{2(r'(k-q)-d)}$$

with $r' = \min\{r_1,...,r_n\}$ and

$$k-q \ge k-\ell-1 > \frac{d}{r'}-1 \Rightarrow k-q \ge \frac{d}{r'} \Rightarrow r'(k-q)-d \ge 0.$$

The proof is complete. ▌

Proof of the Theorem 3.3.2: Without loss of generality, we may assume that J is the cube $[-L,L]^q$, for some positive real number L.

We consider the map $\widetilde{F}: S^{n-1}\times(-\epsilon,\epsilon)\times\mathbb{R}^q\to\mathbb{R}^n$ defined as $\widetilde{F}(x,\lambda,t) = f\circ\pi(x,\lambda,t)+H(x,\lambda,t)$.

Since f_t has isolated singularity at 0,

$$\nabla_x\widetilde{F}: S^{n-1}\times(-\epsilon,\epsilon)\times\mathbb{R}^q\to\mathbb{R}^n$$

is surjective, where ϵ is chosen as small as needed and $\mathbb{R}^q$ is replaced by $(-L-\delta, L+\delta)^q$.

Therefore we can write $\dfrac{\partial \widetilde{F}}{\partial t_q}$ as a linear combination of the coordinate functions of $\nabla_x \widetilde{F}$ whose coefficients are functions of class C^ℓ in $S^{n-1} \times (-\epsilon, \epsilon) \times (-L-\delta, L+\delta)^q$. Then we obtain

$$\frac{\partial \widetilde{F}}{\partial t_q} = \langle \nabla_x \widetilde{F}, u \rangle \tag{2}$$

where $u(x, \lambda, t) = (u_1(x, \lambda, t), ..., u_n(x, \lambda, t))$ is a map of class C^ℓ in $S^{n-1} \times (-\epsilon, \epsilon) \times (-L-\delta, L+\delta)^q$.

Since $\nabla_x \widetilde{F}$ is of class C^ℓ, the implicit function theorem says that u is of class C^ℓ.

Multiplying the equation (2) by λ^d we obtain

$$\frac{\partial F}{\partial t_p}(\varphi(x,\lambda), t) = \langle \nabla_{\varphi(x,\lambda)} F(\varphi(x,\lambda), t), w(x, \lambda, t) \rangle \tag{3}$$

with $(x, \lambda, t) \in S^{n-1} \times ((-\epsilon, 0) \cup (0, \epsilon)) \times (-L-\delta, L+\delta)^q$ and

$$w(x, \lambda, t) = \varphi(u(x, \lambda, t)) = (\lambda^{r_1} u_1(x, \lambda, t), ..., \lambda^{r_n} u_n(x, \lambda, t))$$

is of class C^ℓ in $S^{n-1} \times (-\epsilon, \epsilon) \times (-L-\delta, L+\delta)^q$. Then we rewrite the equation (3) as

$$\frac{\partial F}{\partial t_q}(y, t) = \langle \nabla_y F(y, t), w(\varphi^{-1}(y), t) \rangle \tag{4}$$

where $y \in \varphi(S^{n-1} \times (-\epsilon, \epsilon)) \backslash \{0\}$.

Let W be the C^ℓ-vector field defined in $\left\{\varphi(S^{n-1} \times (-\epsilon, \epsilon)) \backslash \{0\}\right\} \times \mathbb{R}^q$ as:

$$W(y, t) = \frac{\partial}{\partial t_q} - w(\varphi^{-1}(y), t).$$

We remark that W is orthogonal to ∇F, since

$$\langle W, \nabla F \rangle = \frac{\partial F}{\partial t_q} - \langle \nabla_y F, w(\varphi^{-1}(y), t) \rangle = 0.$$

As $\lim\limits_{y \to 0} W(y, t) = \dfrac{\partial}{\partial t_q} - \lim\limits_{\lambda \to 0} w(x, \lambda, t) = \dfrac{\partial}{\partial t_q}$, we conclude that W has a continuous extension to $(0, t)$ with $W(0, t) = \frac{\partial}{\partial t_q}$.

Now we consider the following initial problem value

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$$\begin{cases} \dfrac{\partial \tau^q}{\partial t_q}(y,t) & = \; W(\tau^q(y,t)) \\ \\ \tau^q(y,t_1,...,t_{q-1},0) & = \; (y,t_1,...,t_{q-1},0). \end{cases} \tag{5}$$

Let us suppose that there exist ϵ_q, δ_q and τ^q such that $0 < \epsilon_q \leq \epsilon$, $0 < \delta_q \leq \delta$ and τ^q is a map from

$$\left\{\varphi(S^{n-1} \times (-\epsilon_q, \epsilon_q))\backslash\{0\}\right\} \times (-L-\delta_q, L+\delta_q)^q \to \varphi(S^{n-1} \times (-\epsilon, \epsilon))\backslash\{0\}$$

satisfying (5).

Since W is constant along the parameter t_q we obtain $\tau^q(y,t) = (\overline{\tau}^q(y,t),t)$. As W is orthogonal to ∇F, we have that

$$F(\overline{\tau}^q(y,t),t) = F(y,t_1,...,t_{q-1},0).$$

In fact

$$\frac{\partial}{\partial t_q}(F(\tau^q(y,t))) = \langle \nabla F(\tau^q(y,t)), \frac{\partial \tau^q}{\partial t_q}(y,t)\rangle = \langle \nabla F, W\rangle = 0.$$

Therefore, applying the theorem of existence and uniqueness of ordinary differential equations, it follows that there exists only one solution

$$\overline{\tau}^q_t : \varphi(S^{n-1} \times (-\epsilon_q, \epsilon_q))\backslash\{0\} \to \varphi(S^{n-1} \times (-\epsilon, \epsilon))\backslash\{0\}$$

of class C^ℓ.

In order to continue the construction of $\overline{\tau}^q(y,t)$, we show that if we define $\overline{\tau}^q_t(0) = 0$, i.e., $\tau^q_t(0,t) = (0,t)$ then

$$\overline{\tau}^q_t : \varphi(S^{n-1} \times (-\epsilon_q, \epsilon_q)) \to \varphi(S^{n-1} \times (-\epsilon, \epsilon))$$

is an homeomorphism which is induced by a C^ℓ-diffeomorphism of $S^{n-1} \times (-\epsilon_q, \epsilon_q) \to S^{n-1} \times (-\epsilon, \epsilon)$.

Therefore, let us consider the lifting of $w_t \circ \varphi^{-1}$ to $S^{n-1} \times [(-\epsilon, 0) \cup (0, \epsilon)]$, or in other words, the vector field

$$V_t : S^{n-1} \times [(-\epsilon, 0) \cup (0, \epsilon)] \to T(S^{n-1} \times [(-\epsilon, 0) \cup (0, \epsilon)])$$

such that $d_{(x,\lambda)}\varphi.V_t = w_t(x, \lambda)$.

Lemma 3.3.4. *V_t has an extension of class C^ℓ to $S^{n-1} \times (-\epsilon, \epsilon)$ which is tangent to $S^{n-1} \times \{0\}$ at all points of $S^{n-1} \times \{0\}$.*

Proof: Let us identify the tangent bundle of S^{n-1} with a sub bundle of the trivial bundle $S^{n-1} \times \mathbb{R}^n$.

Let $\gamma(s) = (x(s), r(s))$ with $\gamma'(0) = (x'(0), r'(0)) = v = (v_1, ..., v_n, v_{n+1})$ and $\gamma(0) = (x, \lambda)$. Then

$$\begin{aligned}
\frac{\partial}{\partial s}(\varphi \circ \gamma)_{|s=0} &= \frac{\partial}{\partial s}(r(s)^{r_1} x_1(s), ..., r(s)^{r_n} x_n(s))_{|s=0} = \\
&= (r_1 x_1 \lambda^{r_1 - 1} v_{n+1} + \lambda^{r_1} v_1, ..., r_n x_n \lambda^{r_n - 1} v_{n+1} + \lambda^{r_n} v_n) = \\
&= \begin{bmatrix} \lambda^{r_1} & 0 & \cdots & 0 & r_1 x_1 \lambda^{r_1 - 1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda^{r_n} & r_n x_n \lambda^{r_n - 1} \\ x_1 & x_2 & \cdots & x_n & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ v_{(n+1)} \end{bmatrix}.
\end{aligned}$$

Therefore the equation of V_t is given as

$$\begin{bmatrix} \lambda^{r_1} & 0 & ... & 0 & r_1 x_1 \lambda^{r_1 - 1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & ... & \lambda^{r_n} & r_n x_n \lambda^{r_n - 1} \\ x_1 & x_2 & ... & x_n & 0 \end{bmatrix} \begin{bmatrix} V_{1t} \\ \vdots \\ V_{nt} \\ V_{(n+1)t} \end{bmatrix} = \begin{bmatrix} \lambda^{r_1} u_1(x, \lambda, t) \\ \vdots \\ \lambda^{r_n} u_n(x, \lambda, t) \\ 0 \end{bmatrix}. \qquad (6)$$

Solving the equation (6) by the Crammer's rule we obtain

$$V_{it} = \frac{\Delta_i}{\Delta}, i = 1, ..., n, \text{ e } V_{(n+1)t} = \frac{\Delta_{n+1}}{\Delta}.$$

Hence Δ is the product of $\lambda^{(\sum_i r_i) - 1}$ with a nonvanishing term and $\lambda^{(\sum_i r_i) - 1}$ is a factor of each Δ_i, for $i = 1, ..., n$.

Since each Δ_i is a function which depends of $u_1, ..., u_n$, this proves that V_i has an extension of class C^ℓ.

To finish, since $\lambda^{\sum_i r_i}$, is a factor of Δ_{n+1}, this justifies the claim that the extension is tangent to $S^{n-1} \times \{0\}$. ▮

We consider now the initial problem value

$$\begin{cases} \dfrac{\partial \eta^q}{\partial t_q}(x, \lambda, t) &= -V_t(\eta^q(x, \lambda, t)) \\ \eta(x, \lambda, t_1, ..., t_{q-1}, 0) &= (x, \lambda) \end{cases} \qquad (7)$$

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This problem has a unique solution of class C^ℓ in $S^{n-1} \times (-\epsilon_q, \epsilon_q) \times (-L - \delta_q, L + \delta_q)^q$ where $0 < \epsilon_q \leq \epsilon$ e $0 < \delta_q \leq \delta$.

η_t^q is an embedding of class C^ℓ of $S^{n-1} \times (-\epsilon_q, \epsilon_q) \to S^{n-1} \times (-\epsilon, \epsilon)$ such that

$$\eta_t^q(S^{n-1} \times \{0\}) = S^{n-1} \times \{0\}, \quad \eta_t^q(S^{n-1} \times (-\epsilon_q, 0)) \subset S^{n-1} \times (-\epsilon, 0)$$

and

$$\eta_t^q(S^{n-1} \times (0, \epsilon_q)) \subset S^{n-1} \times (0, \epsilon)$$

We define ρ_t^q as $\rho_t^q = \varphi \circ \eta_t^q \circ \varphi^{-1}$ in $\varphi(S^{n-1} \times (0, \epsilon_q))$. Hence ρ_t^q satisfies (5). Moreover $\rho^q = (\rho_t^q, t)$ has a continuous extension to the t axis, since $\lim_{(y,t)\to(0,t_0)} \rho_t^q = 0$. A similar result is also valid for ${\rho^q}^{-1}$ and the theorem follows by induction on the number of parameters. ∎

Example 1: Let $F = f + g$ where

$$f(x, y) = x^6 - y^{12} + \frac{x^{10}}{x^4 + y^8} \quad \text{and} \quad g(x, y, t_1, t_2) = \frac{t_1^5 t_2^9}{t_1^2 + t_2^4} \cdot \frac{x^{17} y^7}{x^4 + y^8}.$$

Here f is quasihomogeneous of type $(2, 1; 20)$ and from the results of [13], we obtain that f and g are of class C^5, g and $\dfrac{\partial g}{\partial t_i}$ are of class C^{17} on the variables x, y, for $i = 1, 2$.

Since $fil(j^{16}(g_{(t_1,t_2)}(x, y))) = \infty$ and $17 > \max\{\frac{12}{2}, \frac{12}{1}\} + 4$, we apply the Theorem 3.3.2, to obtain that

$$F(x, y, t_1, t_2) = x^6 - y^{12} + \frac{x^{10}}{x^4 + y^8} + \frac{t_1^5 t_2^9}{t_1^2 + t_2^4} \cdot \frac{x^{17} y^7}{x^4 + y^8}$$

admits a modified C^4-trivialization.

Example 2: Let $f(x, y) = x^2 - y^5$ which is quasihomogeneous of type $(5, 2; 10)$ and $F(x, y, t) = f + g = x^2 - y^5 + (t^7 \sin(\frac{1}{t}))(y^{17} \sin(\frac{1}{y}))$ be a deformation of f.

Since $g(x, y, t) = (t^7 \sin(\frac{1}{t}))(y^{17} \sin(\frac{1}{y}))$ is of class C^3 the deformation $F(x, y, t)$ is also of class C^3.

Since $fil(j^8(g_t)) = \infty$, g_t and $\dfrac{\partial g}{\partial t}$ are of class C^8 in the variables (x, y) and also $8 > \max\left\{\dfrac{10}{5}, \dfrac{10}{2}\right\} + 2$, we can apply the Theorem 3.3.2, to obtain that F admits a modified C^2-trivialization, hence the topological triviality of F.

We remark that the estimates shown in [2] also show the topological triviality for this example, but does not guarantee the C^1-triviality of F.

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Example 3: *The C^ℓ-modification of the Briançon-Speder example*

Let $f : \mathbb{R}^3 \to \mathbb{R}$ defined as $f(x, y, z) = x^{15} + xy^7 + z^5$ which quasihomogeneous of type $(1, 2, 3; 15)$ and the family $F : \mathbb{R}^3 \times \mathbb{R}, 0 \to \mathbb{R}, 0$ with $F(x, y, z, t) = f(x, y, z) + tx^a y^b z^c$.

Ruas and Saia in [13] showed that for small values of t, deformations of type $f_t(x, y, z) = f(x, y, z) + tx^a y^b z^c$ are topologically trivial if $fil(x^a y^b z^c) = a + 2b + 3c \geq 15$, and are C^ℓ-$\mathcal{R}$-trivial $(\ell \geq 1)$ if $fil(x^a y^b z^c) \geq 15 + 3\ell$.

Applying the Theorem 3.3.2 we see that for $15 < fil(x^a y^b z^c) < 15 + 3\ell$, the family f_t is modified C^1-trivial.

3.2. The case of quasihomogeneous maps

In this section we show that the Theorem (3.3.2) has a general setup and is also valid in the case of families of quasihomogeneous maps.

We fix a set of weights $r = (r_1, \ldots, r_n)$, degrees $d = (d_1, \ldots, d_p)$ and consider $F : U \times \mathbb{R}^q \to \mathbb{R}^p$ be a family of maps defined in a neighborhood U of 0 in $\mathbb{R}^n$.

Theorem 3.3.5. *Suppose that we can write F in the form $F = f + g$ with $f, g : U \times \mathbb{R}^p \to \mathbb{R}^p$ satisfying the following conditions:*

(1) *f and g are of class $C^{\ell+1}$ and $g, \dfrac{\partial g}{\partial t_s}$ are of class C^k in the variables $x_1, \ldots, x_n$ and $k > \max\left\{\frac{d_j}{r_1}, \ldots, \frac{d_j}{r_n}\right\} + l$, for all $j = 1, ..., p$;*
(2) *for each t in $\mathbb{R}^q$, $f_t = f(., t)$ is the restriction to U of a quasihomogeneous mapping of type $(r; d)$ with isolated singularity at 0;*
(3) *$fil(j^{k-1}(g_{j,t})) > d_j$, $\forall t \in \mathbb{R}^q$.*

Then F admits a modified C^ℓ-trivialization along J.

Proof: The proof of this result is already equal to the proof of the Theorem 3.3.2.

Let $\varphi : S^{n-1} \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ the mapping defined as $\varphi(x, \lambda) = (\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n)$.

We define $H_j : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^q \to \mathbb{R}$ as $H_j(x, \lambda, t) = \dfrac{g(\varphi(x, \lambda), t)}{\lambda^{d_j}}$, and $H : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^q \to \mathbb{R}^p$ by $H = (H_1, ..., H_p)$.

First we remark here that $H_j, \dfrac{\partial H_j}{\partial x_i}$ and $\dfrac{\partial H_j}{\partial t_s}$ are of class C^ℓ, for all $j = 1, ..., p$ and are zero if $\lambda = 0$, as a consequence of the Lemma 3.3.3.

Let $\widetilde{F} = f \circ \pi + H$, since each f_t has isolated singularity at 0 the matrix

$$d_x \widetilde{F} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \circ \pi + \frac{\partial H_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \circ \pi + \frac{\partial H_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1} \circ \pi + \frac{\partial H_p}{\partial x_1} & \cdots & \frac{\partial f_p}{\partial x_n} \circ \pi + \frac{\partial H_p}{\partial x_n} \end{pmatrix}$$

has rank p for all $t \in J$. Hence we can write $\frac{\partial \widetilde{F}}{\partial t_q}$ as a linear combination of the columns of $d_x\widetilde{F}$ with coefficient functions of class C^ℓ.

Therefore,

$$\frac{\partial \widetilde{F}}{\partial t_q} = d_x\widetilde{F}.u, \tag{8}$$

where $u(x, \lambda, t) = (u_1(x, \lambda, t), ..., u_n(x, \lambda, t))$ is a mapping of class C^ℓ in $S^{n-1} \times (-\epsilon, \epsilon) \times (-L - \delta, L + \delta)^q$.

Now we compare the equation (8) with the equation (2) in the proof of the Theorem 3.3.2, to obtain the proof in an analogous way to the proof of the Theorem 3.3.2. ▌

4. THE NEWTON NON-DEGENERATE CASE

Fukui and Yoshinaga showed in [7] that families of Newton non-degenerate functions with constant Newton polyhedron for any parameter, admit an analytic modified trivialization at the parameter space.

In this section we show, in an analog way, that this result also is valid in the category of families of class C^ℓ, or in other words, under the same hypothesis these families admit a modified C^ℓ-trivialization.

4.1. Newton non-degeneracy

We recover here the concepts of Newton polyhedron and Newton non-degeneracy of analytic functions in order to give such definitions for functions of class C^ℓ.

For an analytic germ $g(x) = \sum a_k x^k$, let us define the support of g by supp $g = \{k \in \mathbb{Z}^n \,:\, a_k \neq 0\}$.

Definition 4.4.1. *The Newton polyhedron of an analytic germ g, denoted by $\Gamma_+(g)$, is the convex hull in $\mathbb{R}^n_+$ of the set*

$$\bigcup \left\{k + v \,:\, k \in supp\ g,\ v \in \mathbb{R}^n_+\right\}.$$

$\Gamma(g)$ denotes the union of all compact faces of $\Gamma_+(g)$.

Definition 4.4.2. *The principal part of a germ g is defined as the polynomial $inf(x) = \sum\limits_{k\in\Gamma(g)} a_k x^k$.*

If γ is a compact face of $\Gamma_+(g)$ and $g = \sum_k a_k x^k$, we set $g_\gamma = \sum\limits_{k\in\gamma} a_k x^k$.

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Definition 4.4.3. *We say that g is Newton non-degenerate if for any compact face* $\gamma \in \Gamma(g)$

$$\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 \frac{\partial g_\gamma}{\partial x_1} = \cdots = x_n \frac{\partial g_\gamma}{\partial x_n} = 0\} \subset S,$$

where $S = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 \cdots x_n = 0\}$.

The analogous definitions for functions of class C^ℓ, are done in terms of the ℓ-jet of the function.

Definition 4.4.4. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function of class* C^ℓ. *We define the Newton polyhedron of* f, *denoted by* $\Gamma_+(f)$, *as the Newton polyhedron of its* ℓ*-jet* $\Gamma_+(j^\ell(f))$. *We denote* $\Gamma(f)$ *the union of the compact faces of* $\Gamma_+(f)$.

Definition 4.4.5. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *of class* C^ℓ *is Newton non-degenerate if its* ℓ*-jet* $j^\ell(f)$ *is Newton non-degenerate.*

4.2. The toroidal embedding

The main results of this section, Theorems 4.4.12 and 4.4.13, are done in terms of the toric variety X and of the toroidal embedding π associate to $\Gamma_+ = \Gamma_+(f_t)$, concepts that we recover here.

The procedure of constructing the toroidal embedding associated to a given Newton polyhedron, due to Varchenko [16, 183–184], is a local modification of Khovanskii's method of assigning a compact complex nonsingular toroidal manifold to an integer-valued compact convex polyhedron in $\mathbb{R}^n$.

We summarize here the construction of the toroidal embedding.

Let us fix a Newton polyhedron $\Gamma_+(g)$, and $(a_1, \dots, a_n)$ a system of coordinates in the dual space $\mathbb{R}^{n*}$.

Definition 4.4.6. *For each* $a = (a_1, \dots, a_n) \in \mathbb{R}^{n*}_+$ *call:*

(1) $\ell(a) = min\{\langle a, k\rangle \ : \ k \in \Gamma_+(g)\}$, *where* $\langle a, k\rangle = \sum_{i=1}^n a_i k_i$;

(2) $\Delta(a) = \{k \in \Gamma_+(I) \ : \ \langle a, k\rangle = \ell(a)\}$;

(3) *Two vectors* $a, a' \in \mathbb{R}^{n*}_+$ *are equivalent if* $\Delta(a) = \Delta(a')$.

A vector a is called a primitive integer vector if a is the vector with minimum length in $C(a) \cap (\mathbb{Z}^n_+ - \{0\})$, where C($a$) is the half ray emanating from 0 passing through a.

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From the equivalence defined in the item (3) of this definition, we see that any equivalence class is naturally identified with a convex cone with its vertex at zero, that is specified by finitely many linear equations and strictly linear inequalities with rational coefficients.

The closures of equivalence classes specify a partition Σ_0 of the positive cone $\mathbb{R}_+^{n*}$ into closed convex cones that have the properties:

1. If σ_1 is a face of a cone $\sigma \in \Sigma_0$, then $\sigma_1 \in \Sigma_0$.

2. For any cones σ_1 and σ_2 in Σ_0, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Following the algorithm described in the proof of Theorem 11 of [9, p.32], we construct on the basis of Σ_0, a partition Σ of the cone $\mathbb{R}_+^{n*}$ into finitely many closed convex with their vertices at zero such that:

1. Any cone belonging to Σ lies in one of the cones in Σ_0 and is specified by finitely many linear equalities and linear inequalities with rational coefficients.
2. If σ_1 is a face of a cone σ in Σ, then $\sigma_1 \in \Sigma$.
3. For any cones σ_1 and σ_2 in Σ, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.
4. Any cone σ in Σ is simplicial and unimodular, *i.e.*, if the dimension of σ is q, there exist a set of primitive integer vectors $a^1(\sigma), \ldots, a^q(\sigma)$ which are linearly independent over $\mathbb{R}$ and $n-q$ primitive integer vectors $a^{q+1}(\sigma), \ldots, a^n(\sigma)$ such that $\mathbb{Z}a^1(\sigma) \oplus \ldots \oplus \mathbb{Z}a^n(\sigma) = \mathbb{Z}^n$.

Let σ be an n-dimensional cone in Σ and $a^1(\sigma), a^2(\sigma), \ldots, a^n(\sigma)$ the corresponding set of primitive integer vectors of σ that has been ordered once and for all. We associate to each such σ a copy of $\mathbb{R}^n$ denoted by $\mathbb{R}^n(\sigma)$. Let us denote by $\pi_\sigma\colon \mathbb{R}^n(\sigma) \to \mathbb{R}^n$ the mapping given by the formulae

$$x_i = y_1^{a_i^1(\sigma)} \cdot \ldots \cdot y_n^{a_i^n(\sigma)}$$

where $x_1, x_2, \ldots, x_n$ are coordinates in $\mathbb{R}^n$, $y_1, y_2, \ldots, y_n$ are coordinates in $\mathbb{R}^n(\sigma)$ and $a_1^j(\sigma), \ldots, a_n^j(\sigma)$ denote the coordinates of the vector $a^j(\sigma)$.

We shall glue any two copies $\mathbb{R}^n(\sigma)$ and $\mathbb{R}^n(\tau)$ via the following equivalence relation. Let $y_\sigma \in \mathbb{R}^n(\sigma)$ and $y_\tau \in \mathbb{R}^n(\tau)$, then $y_\sigma \sim y_\tau$ if and only if $\pi_\sigma(y_\sigma) = \pi_\tau(y_\tau)$.

We denote this set by $X = X(\Gamma_+(g)) = \bigcup \mathbb{R}^n(\sigma)/\sim$, where $\bigcup \mathbb{R}^n(\sigma)$ denotes the disjoint union of $\mathbb{R}^n(\sigma)$.

It follows from the properties 1-4 of the partition Σ and Theorems 6,7 and 8 of [9], pp.24–26 that X is a nonsingular n-dimensional algebraic variety and that $\pi\colon X \to \mathbb{R}^n$ defined by $\pi(y) = \pi_\sigma(y_\sigma)$ is a proper analytic mapping onto $\mathbb{R}^n$ (y_σ denotes a representative in $\mathbb{R}^n(\sigma)$ of the equivalence class y in X).

Definition 4.4.7. *We call a non compact face γ a coordinate face if γ contains a non empty open subset of some coordinate space.*

For a fixed n-dimensional cone σ in Σ with respective primitive integers $a^1, a^2, \ldots, a^n$ and for any subset J of $(1, \ldots, n)$, let:

$$E_{\sigma,J} = \{y_\sigma \in \mathbb{R}^n(\sigma) \,:\, y_{\sigma,j} = 0, \text{ if } j \in J\}, \text{ and}$$

$$E^*_{\sigma,J} = \{y_\sigma \in E_{\sigma,J} \,:\, y_{\sigma,j} \neq 0, \text{ if } j \notin J\}.$$

Proposition 4.4.8 (Kaneko, [8]). *$\pi_\sigma(E_{\sigma,I}) = \{0\}$ if, and only if, $\gamma = \bigcap_{i\in I} \gamma(a^i(\sigma))$ is a compact face of $\Gamma_+(f)$.*

Proposition 4.4.9 (Fukui, [6]). *If $\gamma = \bigcap_{i\in I} \gamma(a^i(\sigma))$ is a coordinate face, then the map $\pi_{\sigma|E^*_{\sigma,I}}$ is injective.*

Proposition 4.4.10 (Varchenko, [16]). *Let $f(x_1, ..., x_n)$ be a real analytic function. Let $\sigma \in \Sigma$, $\dim \sigma = n$. Then*

$$f \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} g(y_\sigma)$$

where $y_\sigma = (y_{\sigma,1}, ..., y_{\sigma,n})$ are the coordinates of $\mathbb{R}^n(\sigma)$ and $g(0, ..., 0) \neq 0$.

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}$ be an analytic function. We define $g_i(y_\sigma)$ by

$$(x_i \frac{\partial f}{\partial x_1}) \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} g_i(y_\sigma).$$

Proposition 4.4.11 (Fukui e Yoshinaga, [7]). *Let us suppose that f is Newton non degenerate, then $\sum_{i=1}^{n} g_i(y_\sigma)^2 > 0$ for all $y_\sigma \in \pi_\sigma^{-1}(0)$.*

4.3. Main results

Now we show the main results of this section, they are analogous to the Theorems **A** and **B** of [7], but are done for families of functions of class C^ℓ.

Theorem 4.4.12. *Let $F : \mathbb{R}^n \times I \to \mathbb{R}$ be a family of Newton non-degenerate functions $f_t(x) = F(x,t)$ of class C^ℓ, where $I \subset \mathbb{R}^q$ is the compact cube $I = \prod_{i=1}^q [a_i, b_i]$. Suppose that the Newton polyhedra $\Gamma_+(f_t)$ are constant for all t and $\ell \geq \max\left\{\frac{\ell(a^j(\sigma))}{a_i^j(\sigma)}\right\}$, for all $\sigma \in \Sigma$. Then the family $F(x,t)$ admits an almost modified $C^{\ell-1}$-trivialization along I.*

THEOREM 4.4.13. *Let $F(x,t)$ be a family satisfying the conditions of the Theorem 4.4.12. Suppose that, for each compact face γ of $\Gamma_+(f_t)$, $f_{t\gamma}(x)$ is constant in the t-variables. Then the family $F(x,t)$ admits a modified $C^{\ell-1}$-trivialization along I.*

LEMMA 4.4.14. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Newton non-degenerate function of class C^ℓ, with $\ell \geq \sup\left\{\dfrac{\ell(a^j(\sigma))}{a_i^r(\sigma)}\right\}$, $\forall\sigma\in\Sigma$. Then*

$$\sum_{i=1}^{n}(h_i(y_\sigma))^2>0,\ \forall\ y_\sigma\in\pi_\sigma^{-1}(0),$$

where $h_i(y_\sigma)$ is defined as

$$h_i(y_\sigma)=\frac{\left(x_i\frac{\partial f}{\partial x_i}\right)\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}.$$

Proof: The following is a consequence of the theorem of Taylor with infinitesimal remainder and from the uniqueness of the Taylor series.

We write

$$\frac{\partial f}{\partial x_i}=j^{\ell-1}\left(\frac{\partial f}{\partial x_i}\right)+R_{\ell-1}\text{ with }\lim_{v\to 0}\frac{R_{\ell-1}(v)}{\|v\|^{\ell-1}}=0.$$

Then

$$\frac{\partial f}{\partial x_i}=\frac{\partial}{\partial x_i}(j^\ell(f))+R_{\ell-1}\Rightarrow x_i\frac{\partial f}{\partial x_i}=x_i\frac{\partial}{\partial x_i}(j^\ell(f))+x_iR_{\ell-1}.$$

We remark that $\lim\limits_{v\to 0}\frac{v_iR_{\ell-1}(v)}{\|v\|^\ell}=\lim\limits_{v\to 0}\frac{v_i}{\|v\|}\frac{R_{\ell-1}(v)}{\|v\|^{\ell-1}}=0$, since $\frac{v_i}{\|v\|}$ is limited in a neighborhood of the origin.

Now we can write

$$\frac{\left(x_i\frac{\partial f}{\partial x_i}\right)\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}=g_i(y_\sigma)+\frac{(x_iR_{\ell-1})\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}},$$

where

$$g_i(y_\sigma)=\frac{\left(x_i\frac{\partial}{\partial x_i}(j^\ell(f))\right)\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}.$$

Then we obtain

$$\lim_{y_\sigma\to\pi_\sigma^{-1}(0)}\frac{(x_iR_{\ell-1})\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}=0.$$

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Since

$$\frac{(x_iR_{\ell-1})\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}=\frac{(x_iR_{\ell-1})\circ\pi_\sigma(y_\sigma)}{\|\pi_\sigma(y_\sigma)\|^\ell}\frac{\|\pi_\sigma(y_\sigma)\|^\ell}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}$$

and

$$\frac{\|\pi_\sigma(y_\sigma)\|^\ell}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}} = \left(\frac{\sum y_{\sigma,1}^{2a_i^1(\sigma)}\cdots y_{\sigma,n}^{2a_i^n(\sigma)}}{y_{\sigma,1}^{\frac{2\ell(a^1(\sigma))}{\ell}}\cdots y_{\sigma,n}^{\frac{2\ell(a^n(\sigma))}{\ell}}}\right)^{\frac{\ell}{2}} =$$

$$= \sum y_{\sigma,1}^{\frac{2a_i^1(\sigma)\ell-2\ell(a^1(\sigma))}{\ell}}\cdots y_{\sigma,n}^{\frac{2a_i^n(\sigma)\ell-2\ell(a^n(\sigma))}{\ell}}$$

and from the hypothesis, it follows that $\frac{2a_i^r(\sigma)\ell-2\ell(a^r(\sigma))}{\ell}\geq 0$.

Therefore for all $y_\sigma\in\pi_\sigma^{-1}(0)$ we have $h_i(y_\sigma)=g_i(y_\sigma)$, and the result follows applying the Proposition 4.4.11. ▌

LEMMA 4.4.15. *Let $S=\{(x_1,...,x_n)\in\mathbb{R}^n\ /\ x_1\cdots x_n=0\}$ and f be as in the Lema (4.4.14), then*

$$\{x\in\mathbb{R}^n\ /\ x_i\frac{\partial f}{\partial x_i}=\cdots=x_n\frac{\partial f}{\partial x_n}=0\}\subset S$$

as a germ of set in the origin.

Proof: For each σ, $\pi_\sigma:\mathbb{R}^n-\pi_\sigma^{-1}(S)\to\mathbb{R}-S$ is an analytic isomorphism, see [16] Lemma 2.13, then, this Lemma is a direct consequence of the Lemma 4.4.14. ▌

4.4. The Kuo type vector field

Next we shall construct the Kuo type vector field using a convenient Riemmanian singular metric in $\mathbb{R}^n$. We denote $\Gamma_+(f_t)=\Gamma_+$ for all $t\in I$.

We shall construct a vector field V of class $C^{\ell-1}$ defined in $\pi^{-1}(U)\times I$, where U is a neighborhood of the origin in $\mathbb{R}^n$, satisfying the following:

(V_1) $V(p)$ is tangent to the level set of $F\circ(\pi\times id_I)$ at any regular point p of any level set of $F\circ(\pi\times id_I)$ in $\pi^{-1}(U)\times I$, where U is a sufficiently small neighborhood of the origin in $\mathbb{R}^n$;

(V_2) the t component of V is equal to $\frac{\partial}{\partial t}$,

The first step is to introduce a singular Riemmanian metric in $\mathbb{R}^n$, which is in fact a metric on $\mathbb{R}^n-S$.

DEFINITION 4.4.16. *Consider the Riemmanian metric defined by*

$$\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right\rangle:=\delta_{ij}\frac{\prod_{k=1}^n x_k^2}{x_ix_j}$$

where $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *if* $i \neq j$.

From this definition we have the following equalities:

$$\nabla_x F = \frac{\sum\limits_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i}}{\prod\limits_{k=1}^{n} x_k^2},$$

$$\|\nabla_x F\|^2 = \frac{\sum\limits_{i=1}^{n} \left(x_i \frac{\partial F}{\partial x_i} \right)^2}{\prod\limits_{k=1}^{n} x_k^2},$$

where $\nabla_x F$ denotes the gradient vector of $f_t(x) = F(x,t)$ with respect to the metric defined above, see [7] for more details.

We define now the Kuo type vector field

Definition 4.4.17. (cf. [11]) *Let* $W(x,t)$ *the vector field defined by*

$$W(x,t) = \frac{-\frac{\partial F}{\partial t}}{\|\nabla_x F\|^2} \nabla_x F + \frac{\partial}{\partial t}.$$

This vector field $W(x,t)$ is tangent at all regular points of each level set of $F(x,t)$. From the formulae obtained above, it follows that:

$$W(x,t) = \frac{-\frac{\partial F}{\partial t} \sum\limits_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i}}{\sum\limits_{k=1}^{n} \left(x_k \frac{\partial F}{\partial x_k} \right)^2} + \frac{\partial}{\partial t},$$

which is well defined in $U - S$, by the Lemma (4.4.15), where U is a neighborhood of the origin in $\mathbb{R}^n$.

Our next step is the construction of modification, which is done in terms of the toroidal variety.

Let X the toroidal variety and π the toroidal embedding associate to $\Gamma_+ = \Gamma_+(f_t)$. To construct the modification we consider the mapping $\varphi := (\pi \times id_I|_{(X - \pi^{-1}(S)) \times I})^{-1}$.

From the definition of π_σ we see that $x_i = y_{\sigma,1}^{a_i^1(\sigma)} \cdots y_{\sigma,n}^{a_i^n(\sigma)}$, and the matrix of the differential of π_σ is given as $d\pi_\sigma(y_\sigma) = (A_{ij})$, where

$$A_{ij} = a_i^j(\sigma).y_{\sigma,1}^{a_i^1(\sigma)} \cdots y_{\sigma,j-1}^{a_i^{j-1}(\sigma)}.y_{\sigma,j}^{a_i^j(\sigma)-1}.y_{\sigma,j+1}^{a_i^{j+1}(\sigma)} \cdots y_{\sigma,n}^{a_i^n(\sigma)},\ i,j = 1, ..., n.$$

Therefore,

$$d\pi_\sigma\left(y_{\sigma,j}\frac{\partial}{\partial y_{\sigma,j}}\right)=\sum_{i=1}^{n}a_i^j(\sigma)x_i\frac{\partial}{\partial x_i}$$

for all $1\le j\le n$, hence

$$d\varphi\left(x_i\frac{\partial}{\partial x_i}\right)=\sum_{j=1}^{n}b_j^i(\sigma)y_{\sigma,j}\frac{\partial}{\partial y_{\sigma,j}}$$

for $1\le i\le n$ where the matrix $(b_j^i(\sigma))$ is the inverse of $(a_i^j(\sigma))$.

As a consequence we have

$$d\varphi(W)|\mathbb{R}^n(\sigma)=\frac{-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}b_j^i(\sigma)G_tG_iy_{\sigma,j}\frac{\partial}{\partial y_{\sigma,j}}}{\sum\limits_{k=1}^{n}G_k^2}+\frac{\partial}{\partial t}, \tag{9}$$

where

$$G_i(y_\sigma):=\frac{\left(x_i\frac{\partial F}{\partial x_i}\right)\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}\text{ e }G_t(y_\sigma):=\frac{\frac{\partial F}{\partial t}\circ\pi_\sigma(y_\sigma)}{y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}}$$

In the next Lemma we show that the vector field $V=d\varphi(W)$ is well defined in a neighborhood of $\pi^{-1}(0)\times I$

Lemma 4.4.18. *The vector field $V=d\varphi(W)$ has an extension of class $C^{\ell-1}$ to a neighborhood of $\pi^{-1}(0)\times I$ in $X\times I$ and satisfies the properties (V_1) and (V_2).*

Proof: We know that the vector field W is of class $C^{\ell-1}$ in $(\mathbb{R}^n-S)\times I$. Hence V is of class $C^{\ell-1}$ in $(X-\pi^{-1}(S))\times I$. Moreover by the Lemma (4.4.14) and applying the formula (9), V has an extension of class $C^{\ell-1}$ to a neighborhood of $\pi^{-1}(0)\times I$ em $X\times I$.

By construction we see that the vector field V satisfies the properties (V_1) and (V_2). ▌

4.5. Proof of the Theorem 4.4.12

By the Lemma (4.4.18) we see that there exist vector fields of class $C^{\ell-1}$, $V_1,...,V_q$ in $\pi^{-1}(U)\times I$, where U is a neighborhood of the origin in $\mathbb{R}^n$, satisfying the conditions (V_1) and (V_2).

We denote by $\tau_j(t_j,y,c)$ the integral curves of V_j with $\tau_j(0,y,c)=(0,c)$ and consider

$$M(y,t):=\tau_q(t_q-a_q,\tau_{q-1}(...,\tau_1(t_1-a_1,y,a_1)...))$$

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where $t = (t_1, ..., t_q) \in I$. Therefore M is a diffeomorphism of class $C^{\ell-1}$ between two neighborhoods of $\pi^{-1}(0) \times I$ and $F \circ (\pi \times Id_I) \circ M$ does not depend of the parameter t and this proves that Theorem 4.4.12. ▮

4.6. Proof of the Theorem 4.4.13

To prove the Theorem 4.4.13 it is enough to show that the $C^{\ell-1}$-diffeomorphism M induces an homeomorphism between two neighborhoods of $\{0\} \times I$. The next two Lemmas will be used to show this.

LEMMA 4.4.19. *Let* $\pi_\sigma^{-1}(0) = \bigcup_{J \in N} E_{\sigma,J}$*, then each vector field* $V_{k|\mathbb{R}^n(\sigma)}$, $1 \leq k \leq q$ *is tangent to each* $E_{\sigma,J} \times I$ *for any subset* J *of* $\{1, \ldots, n\}$.

Proof: Since the coefficient of $\partial/\partial y_{\sigma,j}$ in the formula 9 is zero for each $j \in J$, the vector field $V_{k|\mathbb{R}^n(\sigma)}$ is tangent to $E_{\sigma,J} \times I$. ▮

Now, for each $p \in \mathbb{R}^n$ we denote

$$E_{\sigma,J}(p) := \pi_\sigma^{-1}(p) \cap E_{\sigma,J},$$

$$E^*_{\sigma,J}(p) := \pi_\sigma^{-1}(p) \cap E^*_{\sigma,J}.$$

Let $N(p)$ the set formed by the minimal elements, in the inclusion sense, of

$$\{J \subset \{1, 2, ..., n\} \ / \ E^*_{\sigma,J}(p) \neq \emptyset\}.$$

Therefore we have

$$\pi_\sigma^{-1}(p) = \bigcup_{J \in N(p)} E_{\sigma,J}(p).$$

Now we suppose that $p \in S - \{0\}$, if $\pi_\sigma^{-1}(p)$ is not empty and is not a unitary set, then $\pi_\sigma^{-1}(p)$ has not discrete points and $E_{\sigma,J}(p)$ is an algebraic set of $\mathbb{R}^n(\sigma)$.

We remark that each component of $\pi_\sigma(y_\sigma)$ is a monomial in the variables $y_{\sigma,1}, ..., y_{\sigma,n}$.

LEMMA 4.4.20. *Suppose that* $\pi_\sigma^{-1}(p)$ *is not empty and is not a unitary set, then if the hypothesis of the Theorem 4.4.13 hold, each vector field* $V_{k|\mathbb{R}^n(\sigma)}$, $1 \leq k \leq q$, *is tangent to each* $E_{\sigma,J}(p) \times I$. *Moreover* $V_{k|\mathbb{R}^n(\sigma)} = \partial/\partial t_k$ *in each point of* $E_{\sigma,J}(p) \times I$.

Proof: First we remark that the set $E^*_{\sigma,J}(p)$ is not empty and not unitary, then by the Propositions 4.4.8 and 4.4.9, $\gamma = \bigcap_{j \in J} \gamma(a^j(\sigma))$ is not a compact face for $p \neq 0$ and is not a coordinate face.

From the hypothesis of the Theorem 4.4.13, each $F_\gamma(x,t)$ is independent of the parameter t. Hence $\dfrac{\partial F_\gamma}{\partial t_i} = 0$ and $G_{t_i}(y_\sigma) = 0$ for all $y_\sigma \in E_{\sigma,J}$. Then the equation (9) implies that $V_k(y_\sigma) = \dfrac{\partial}{\partial t_k}$ for all $y_\sigma \in E_{\sigma,J}$. ▮

Now, from the Lemmas 4.4.19 and 4.4.20 we see that M induces an homeomorphism between two neighborhoods of $\{0\}\times I$ and this completes the proof of the Theorem 4.4.13. ∎

Example 4: Let $f(x,y) = y^2 + 2x^2y + g(x,y)$ with $g(x,y) = \sum c_{a,b}x^ay^b$ and $a + 2b > 4$. We consider here $F_t(x,y) = f(x,y) + t\dfrac{x^5+y^5}{x^2+y^2}$ which is of class C^2. Doing the following exchange of coordinates $X = x,\ Y = y + x^2$, the family $F(X,Y,t)$ satisfies the conditions of Theorem 4.4.13. Therefore we obtain that F_t admits a modified C^1-trivialization.

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