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Impulsive stability of systems of second order retarded differential equations

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This paper is concerned with systems of impulsive second order delay differential equations. We prove that unstable systems can be stabilized by imposition of impulsive controls. The main tools used are Lyapunov functionals, stability theory and control by impulses. May, 2006 ICMC-USP

1. INTRODUCTION

The theory of impulsive differential equations has become an important part of differential equations. In recent years, significant progress has been made in the qualitative theory of differential equations. There are recent results dealing with the stability of solutions, for instance, see [1, 2, 3, 5, 6]. Impulses can make unstable systems stable or, otherwise stable systems can become unstable after impulse effects. The problem of stabilizing the solutions by imposing proper impulse controls has been used in many fields such as physics, pharmacokinetics, biotechnology, economics, chemical technology, population dynamics and others.

In [1], we proved that the zero solution of certain second order retarded differential equations can be made exponentially continuous with respect to initial data by impulses

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on some interval $[t_0, T)$. In the present paper, we deal with a more general equation and, by means of Lyapunov functionals, we establish sufficient conditions for the stability of solutions by imposing proper impulse controls. Our results also encompass [5].

This paper is organized as follows. In Section 2, we define exponential stabilization by impulses and exponential stabilization by periodic impulses. Section 3 is devoted to the main results. In Section 4, we give examples.

2. PRELIMINARIES

Given a continuous function $z(t) : \mathbb{R} \to \mathbb{R}$, let $z'(t)$ denote its right derivative and $z''(t) = (z'(t))'$. If $z(t)$ is piecewise continuous, then $z(s^-)$ and $z(s^+)$ denote, respectively, its left and right limits as t tends to s.

Let a constant $\tau > 0$ represent an upper bound on the time delay of our system. By $PC([-\tau, 0], \mathbb{R}^n)$ we mean the Banach space of piecewise right continuous functions taking $[-\tau, 0]$ into $\mathbb{R}^n$ with norm given by $\|\psi\| = \sup_{-r \leq s \leq 0} |\psi(s)|$, where $|\cdot|$ denotes the norm in $\mathbb{R}^n$. If $x \in PC([t_0 - \tau, \sigma], \mathbb{R}^n)$, where $t_0 \in \mathbb{R}$, $\sigma \geq t_0$, then for each $t \in [t_0, \sigma]$ we define $x_t \in PC([-\tau, 0], \mathbb{R}^n)$ by $x_t(s) = x(t+s)$ for $-\tau \leq s \leq 0$.

We consider the following equation

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) + g(t, x(t), x(t-\tau)) = 0, \quad t \geq t_0, \quad t \neq t_k, \quad k = 1, 2, \ldots \\ x_{t_0} = \varphi \\ x'(t_0) = y_0 \end{cases} \tag{1}$$

where $\{t_k\}_{k=0}^{\infty}$ is a monotone increasing unbounded sequence of real numbers, $f, g : [t_0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(t, 0, 0) = g(t, 0, 0) = 0$ and $\varphi, \varphi' : [-\tau, 0] \to \mathbb{R}$ have at most a finite number of discontinuity points all of them being of the first kind, and are right continuous at these points.

We also consider the impulses at times t_k, $k = 1, 2, \ldots$,

$$x(t_k) = I_k(x(t_k^-)) \quad \text{and} \quad x'(t_k) = J_k(x'(t_k^-)), \tag{2}$$

where I_k, $J_k : \mathbb{R} \to \mathbb{R}$ are continuous and $I_k(0) = J_k(0) = 0$, $k \in \mathbb{N}$.

Throughout this paper we assume the following hypotheses hold:

($\mathbf{H_1}$) There exists a constant $F > 0$ such that for all $t \geq t_0$ and all u, v in $\mathbb{R}$,

$$|f(t, u, v)| \leq F\,|u|.$$

($\mathbf{H_2}$) There exists a constant $G > 0$ such that for all $t \geq t_0$ and all u, v in $\mathbb{R}$,

$$|g(t, u, v)| \leq G\,|v|.$$

Now we define a solution of the impulsive problem (1)-(2).

Definition 2.1. A function $x : [t_0 - \tau, t_0 + \alpha) \to \mathbb{R}$, $\alpha > 0$, is a solution of problem (1)-(2) through (t_0, φ, y_0) if

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(i) $x(t)$ and $x'(t)$ are continuous on $[t_0, t_0+\alpha) \setminus \{t_k; k \in \mathbb{N}\}$, admit lateral limits at t_k, $k \in \mathbb{N}$, and are right continuous at t_k, $k \in \mathbb{N}$;
(ii) $x(t)$ satisfies (1);
(iii) for each $k \in \mathbb{N}$, $x(t_k)$ and $x'(t_k)$ fulfill (2).

We denote by $x(t) = x(t; t_0, \varphi, y_0)$ the solution of (1)-(2) starting from (t_0, φ, y_0).

Remark 2. 1. Defining $y(t) = x'(t)$, the non-impulsive equation in (1) is transformed into the following system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -g(t, x(t), x(t-\tau)) - f(t, x(t), y(t)), \quad t \geq t_0. \end{cases} \tag{3}$$

If we consider the function $H : [t_0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$H(t, x_0, x_1, x_2) = g(t, x_0, x_1) + f(t, x_0, x_2),$$

then for all $(t, \psi) \in [t_0, \infty) \times PC([-\tau, 0], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$, we can define

$$h(t, \psi) = (\, \psi_2(0), -H(t, \psi_1(0), \psi_1(-\tau), \psi_2(0))\,).$$

By hypotheses (H_1), (H_2), $h : [t_0, \infty) \times PC([-\tau, 0], \mathbb{R}^2) \to \mathbb{R}^2$ is continuous and satisfies

$$|h(t, \psi)| \leq (1 + F + G)\|\psi\|.$$

By Theorem 3.5 in [4], there exists a (local) solution of (1)-(2) and any such solution can be continued to $[t_0 - \tau, \infty)$.

Notice that conditions $g(t, 0, 0) = f(t, 0, 0) = 0$ and $I_k(0) = J_k(0) = 0$ imply that $x \equiv 0$ is a solution of (1)-(2) with $\varphi \equiv 0$ and $y_0 = 0$.
Next we define the exponential stabilization by impulses of the solutions of (1).

Definition 2.2. The zero solution of problem (1) is said to be exponentially stabilized by impulses, if there exist $\alpha > 0$, a sequence $\{t_k\}_{k \in \mathbb{N}}$ with

$$t_0 < t_1 < t_2 < \ldots < t_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

and sequences of continuous functions, $\{I_k\}$ and $\{J_k\}$ such that for all $\varepsilon > 0$, there is $\delta > 0$ such that if a solution $x(t)$ of (1)-(2) fulfills

$$\sqrt{\|\varphi\|^2 + {y_0}^2} \leq \delta, \tag{4}$$

then

$$\sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t - t_0)], \quad t \geq t_0. \tag{5}$$

In particular, we can consider periodic impulses.

Definition 2.3. The zero solution of problem (1) is said to be exponentially stabilized by periodic impulses, if there are $\alpha > 0$, a sequence $\{t_k\}_{k\in\mathbb{N}}$ with

$$t_0 < t_1 < t_2 < \ldots < t_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

and $t_k - t_{k-1} = c > 0$, and sequences of continuous functions, $\{I_k\}$ and $\{J_k\}$ such that

$$\begin{aligned} I_1(u) = \ldots = I_k(u) = \ldots, &\quad k = 1, 2, \ldots, \forall\, u \in \mathbb{R} \\ J_1(u) = \ldots = J_k(u) = \ldots, &\quad k = 1, 2, \ldots, \forall\, u \in \mathbb{R} \end{aligned}$$

such that for all $\varepsilon > 0$, there is $\delta > 0$ such that if a solution $x(t)$ of (1)-(2) fulfills (4), then (5) holds.

3. MAIN RESULTS

Consider the following equation

$$x''(t) + a\,x(t-\tau) + b\,x(t) = 0, \quad t \geq t_0 \tag{6}$$

We will prove that (6) is unstable for suitable coefficients $a, b \in \mathbb{R}$ by means of its characteristic equation

$$\lambda^2 + a\,\mathrm{e}^{-\tau\lambda} + b = 0. \tag{7}$$

The fact that (6) is unstable follows from the existence of a characteristic root with positive real part. Taking a, b such that $1 + a\,\mathrm{e}^{-\tau} + b = 0$ one gets that $\lambda = 1$ is a characteristic root of (7). Therefore (6) is unstable. Notice that $f(t, u, v) = bu$ and $g(t, u, v) = av$ satisfy (H_1) and (H_2).

In what follows, we prove that the zero solution of the more general problem (1) can be exponentially stabilized by impulses.

Theorem 3.1. *Suppose (H_1), (H_2) hold and*

$$G\tau < \exp[-\beta\tau], \tag{8}$$

where $\beta = \max\{1, F + G\}$. Then the zero solution of problem (1) can be exponentially stabilized by impulses.

Proof: Suppose (8) holds. Then there exist $\alpha > 0$ and $\ell \geq \tau$ such that

$$G\tau \leq \exp[-\alpha(\ell + \tau)]\exp[-\beta\ell]. \tag{9}$$

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Let α and ℓ be as in (9). Thus we can choose a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_0 < t_1 < \ldots < t_k < \ldots$ and $\lim\limits_{k\to\infty} t_k = \infty$, with $\tau \le t_k - t_{k-1} \le \ell$. Let

$$I_k(u) = d_k u, \qquad J_k(v) = d_k v, \qquad k = 1, 2, \ldots, \tag{10}$$

where

$$d_k = p_k - G\tau$$

and

$$p_k = \exp[-\alpha(t_{k+1} - t_k + \tau)] \exp[-\beta(t_{k+1} - t_k)].$$

Then d_k is a non-negative real number, since $p_k \ge G\tau$ from (9).

For every $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{\sqrt{2}(1 + G\tau)} \exp[-\alpha(t_1 - t_0)] \exp[-\beta(t_1 - t_0)]. \tag{11}$$

We will show that, for each solution $x(t) = x(t; t_0, \varphi, y_0)$ of (1)-(2) such that

$$\sqrt{\|\varphi\|^2 + {y_0}^2} \le \delta,$$

we have

$$\sqrt{x^2(t) + x'^2(t)} \le \varepsilon \exp[-\alpha(t - t_0)], \qquad t \ge t_0.$$

Consider the Lyapunov functional

$$V(\psi) = |\psi_1(0)| + |\psi_2(0)| + G\int_{-\tau}^{0} |\psi_1(s)| ds,$$

for $\psi \in PC([-\tau, 0], \mathbb{R}^2)$, $\psi(t) = (\psi_1(t), \psi_2(t))$.

Given any solution $z_t = (x_t, y_t)$ of (3) taking $\psi = z_t$ and recalling that $y = x'$, we find

$$V(t) = V(z_t) = |x(t)| + |x'(t)| + G\int_{-\tau}^{0} |x_t(s)| ds.$$

We can easily see that

$$\sqrt{x^2(t) + x'^2(t)} \le |x(t)| + |x'(t)| \le \sqrt{2\left[x^2(t) + x'^2(t)\right]}\,.$$

Also $V(t)$ satisfies

(i) $V(t) \ge |x(t)| + |x'(t)|$.

(ii) $V(t) \le (1+G\tau)\,(\,\|x_t\| + |x'(t)|\,)$, since

$$V(t) \le |x(t)| + |x'(t)| + \|x_t\| G \int_{-\tau}^{0} ds \le (1+G\tau)(\,\|x_t\| + |x'(t)|).$$

(iii) $V'(t) \le \beta V(t)$, for all $t \in (t_0, t_1)$, where $V'(t)$ denotes the right upper derivate of $V(t)$ along the solution of (1)-(2). Indeed, we have

$$\begin{aligned}
V'(t) &= x'(t)\mathrm{sgn}[x(t)] + x''(t)\mathrm{sgn}[x'(t)] + G|x(t)| - G|x(t-\tau)| \\
&= x'(t)\mathrm{sgn}[x(t)] + [-g(t,x(t),x(t-\tau)) - f(t,x(t),x'(t))]\,\mathrm{sgn}[x'(t)] \\
&\quad + G|x(t)| - G|x(t-\tau)| \\
&\le |x'(t)| + |g(t,x(t),x(t-\tau))| + |f(t,x(t),x'(t))| + G|x(t)| - G|x(t-\tau)| \\
&\le |x'(t)| + G|x(t-\tau)| + F|x(t)| + G|x(t)| - G|x(t-\tau)| \\
&= |x'(t)| + (F+G)|x(t)| \\
&\le \beta\,(|x(t)| + |x'(t)|) \le \beta V(t).
\end{aligned}$$

Solving $V'(t) \le \beta V(t)$, we obtain

$$V(t) \le V(t_0)\exp\left[\beta(t-t_0)\right], \text{ for all } t \in (t_0,t_1). \tag{12}$$

Therefore

$$\begin{aligned}
|x(t)| + |x'(t)| &\le V(t) \le V(t_0)\exp\left[\beta(t-t_0)\right] \le V(t_0)\exp\left[\beta(t_1-t_0)\right] \\
&\le (1+G\tau)(\,\|x_{t_0}\| + |x'(t_0)|\,)\exp\left[\beta(t_1-t_0)\right] \\
&\le (1+G\tau)\sqrt{2}\delta\exp\left[\beta(t_1-t_0)\right] \\
&= \varepsilon\exp\left[-\alpha(t_1-t_0)\right] \\
&\le \varepsilon\exp\left[-\alpha(t-t_0)\right]
\end{aligned}$$

and hence

$$\sqrt{x^2(t) + {x'}^2(t)} \le |x(t)| + |x'(t)| \le \varepsilon\exp\left[-\alpha(t-t_0)\right], \qquad t \in (t_0,t_1). \tag{13}$$

But from the right continuity of $x(t)$ and $x'(t)$, (13) also holds on $[t_0,t_1)$, that is,

$$\sqrt{x^2(t) + {x'}^2(t)} \le \varepsilon\exp\left[-\alpha(t-t_0)\right], \qquad t \in [t_0,t_1). \tag{14}$$

Now, we repeat the procedure above for $t \in (t_1,t_2)$. Properties (i) and (iii) still hold for $t \in (t_1,t_2)$. Similarly to (12), by the definition of p_1 and knowing that $\tau \le t_1 - t_0$, we

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obtain

$$
\begin{aligned}
V(t) &\le V(t_1^+)\exp[\beta(t_2-t_1)] \\
&= \left(|x(t_1^+)|+|x'(t_1^+)|+G\int_{-\tau}^{0}|x(t_1+s)|ds\right)\exp[\beta(t_2-t_1)] \\
&= \left\{|x(t_1)|+|x'(t_1)|+G\int_{t_1-\tau}^{t_1}|x(s)|ds\right)\exp[\beta(t_2-t_1)] \\
&= \left\{|I_1(x(t_1{}^-)|+|J_1(x'(t_1{}^-))|+G\int_{t_1-\tau}^{t_1}|x(s)|ds\right)\exp[\beta(t_2-t_1)] \\
&= \left\{d_1[|x(t_1{}^-)|+|x'(t_1{}^-)|]+G\int_{t_1-\tau}^{t_1}|x(s)|ds\right)\exp[\beta(t_2-t_1)] \\
&\le d_1\sup_{t_1-\tau\le t\le t_1}[|x(t)|+|x'(t)|]\exp[\beta(t_2-t_1)] \\
&\quad + \sup_{t_1-\tau\le t\le t_1}|x(t)|G\tau\exp[\beta(t_2-t_1)] \\
&\le (d_1+G\tau)\sup_{t_1-\tau\le t\le t_1}[|x(t)|+|x'(t)|]\exp[\beta(t_2-t_1)]. \\
&\le (d_1+G\tau)\varepsilon\exp[-\alpha(t_1-t_0-\tau)]\exp[\beta(t_2-t_1)] \\
&= p_1\varepsilon\exp[-\alpha(t_1-t_0-\tau)]\exp[\beta(t_2-t_1)] \\
&= \varepsilon\exp[-\alpha(t_2-t_0)]\le\varepsilon\exp[-\alpha(t-t_0)].
\end{aligned}
$$

Thus for $t\in(t_1,t_2)$

$$\sqrt{x^2(t)+x'^2(t)}\le|x(t)|+|x'(t)|\le V(t)\le\varepsilon\exp[-\alpha(t-t_0)].$$

Hence

$$\sqrt{x^2(t)+x'^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)],\qquad t\in(t_1,t_2). \tag{15}$$

In fact we have from the right continuity of $x(t)$ and $x'(t)$ that (15) holds for $t\in[t_1,t_2)$, that is,

$$\sqrt{x^2(t)+x'^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)],\qquad t\in[t_1,t_2). \tag{16}$$

Therefore by (14) and (16)

$$\sqrt{x^2(t)+x'^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)],\qquad t\in[t_0,t_2). \tag{17}$$

In this way, it follows that for $k\in\mathbb{N}$,

$$\sqrt{x^2(t)+x'^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)],\qquad t\in[t_0,t_k),\quad k=1,2,\dots.$$

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Hence

$$\sqrt{x^2(t)+x'^2(t)} \leq \varepsilon \exp\left[-\alpha(t-t_0)\right], \qquad t \geq t_0$$

and the proof is complete. □

COROLLARY 3.1. *Suppose that the hypotheses in Theorem 3.1 hold. Then the zero solution of problem* (1) *can be exponentially stabilized by periodical impulses.*

Proof: Choose a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_0 < t_1 < \ldots < t_k < \ldots$ and $\lim\limits_{k\to\infty} t_k = \infty$, with $t_k - t_{k-1} = \ell$. Let

$$I_k(u) = d\,u, \;\; J_k(v) = d\,v, \;\; k = 1,2,\ldots, \tag{18}$$

where

$$d = p - G\tau \quad \text{and} \quad p = \exp[-\alpha(\ell+\tau)]\exp\left[-\beta\ell\right].$$

The proof follows as in Theorem 3.1. □

The procedure in Theorem 3.1 can be specialized to prove the exponential stabilization for the second order differential equations

$$\begin{cases} x''(t) + \displaystyle\sum_{i=1}^{N} a_i(t)\,x(t-\tau_i) + f(t,x(t),x'(t)) = 0, \quad t \geq t_0, \quad t \neq t_k, \quad k = 1,2,\ldots \\ x_{t_0}(s) = \varphi(s), \quad -\tau_N \leq s \leq 0, \\ x'(t_0) = y_0, \end{cases} \tag{19}$$

and

$$\begin{cases} x''(t) + \displaystyle\sum_{i=1}^{N} \int_{t-\tau_i}^{t} b_i(t-u)x(u)du + f(t,x(t),x'(t)) = 0, \quad t \geq t_0, \quad t \neq t_k, \quad k = 1,2,\ldots \\ x_{t_0}(s) = \varphi(s), \quad -\tau_N \leq s \leq 0, \\ x'(t_0) = y_0, \end{cases} \tag{20}$$

under the following hypotheses: $0 < \tau_1 < \tau_2 < \ldots < \tau_N$, $a_i : [t_0,\infty) \to \mathbb{R}$, $i = 1,\ldots,N$, are piecewise continuous functions, $b_i : [0,\tau_i] \to \mathbb{R}$, $i = 1,\ldots,N$, are Lebesgue integrable, f is continuous, $f(t,0,0) = 0$, for all $t \geq t_0$ and (H_1) holds. Conditions (21) and (23) below are also assumed.

We also consider the impulses at times t_k, $k = 0,1,2,\ldots$, given by (2).

In [1], we restricted ourselves to an interval $[t_0-\tau_N, T]$, for a certain T. Now we get the existence of solutions for (19)-(2) and (20)-(2) on $[t_0-\tau_N,\infty)$ by taking

$$H(t,x_0,x_1,\ldots,x_N,x_{N+1}) = -\sum_{i=1}^{N} a_i(t)\,x_i - f(t,x_0,x_{N+1})$$

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for problem (19) and

$$H(t,x_0,x_1,\ldots,x_N,x_{N+1}) = \sum_{i=1}^{N} \int_t^{t+\tau_i} b_i(t-s+\tau_i)x_i\,du - f(t,x_0,x_{N+1})$$

for (20) in Remark 2.1. Then the process of exponential stabilization of the zero solution of (19) and (20) follows the steps of the proof of Theorem 3.1 with the Lyapunov functionals

$$V(t) = |x(t)| + |x'(t)| + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |a_i(s+\tau_i)||x(s)|ds$$

and

$$V(t) = |x(t)| + |x'(t)| + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} \left[\int_u^t |b_i(u-s+\tau_i)||x(s)|ds\right] du.$$

In what follows we formalize such results.

THEOREM 3.2. *Suppose there exists a positive constant A such that*

$$|\, a_i(t)| \le A, \quad i = 1,\ldots,N \tag{21}$$

and (H_1) holds. If

$$A\tau < \exp[-\beta\tau], \tag{22}$$

where $\tau = \sum_{i=1}^{N}\tau_i$ and $\beta = \max\{1, AN+F\}$, then the zero solution of problem (19) can be exponentially stabilized by impulses.

THEOREM 3.3. *Suppose there is a positive constant B such that for each $i = 1,\ldots,N$,*

$$\int_0^{\tau_i} |\, b_i(s)|\, ds \le B \tag{23}$$

and (H_1) holds. If

$$B\tau < \exp[-\beta\tau], \tag{24}$$

where $\tau = \sum_{i=1}^{N}\tau_i$ and $\beta = \max\{1, BN+F\}$, then the zero solution of problem (20) can be exponentially stabilized by impulses.

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Remark 3. 1. Also following the steps of the proof of Corollary 3.1, one can prove that the zero solution of problems (19) and (20) can be exponentially stabilized by periodic impulses.

4. EXAMPLES

EXAMPLE 4.1. Let us consider equation (6) again. Taking $F = |b|$, $G = |a|$, $\beta = \max\{1, F+G\}$ and $\ell = \tau$, suppose

$$G\tau \exp[\beta\tau] < 1.$$

In this way, one can choose $\alpha \geq 0$ in order to obtain

$$G\tau < \exp[-2\alpha\tau] \exp[-\beta\tau].$$

Therefore

$$G\tau < \exp[-\beta\tau].$$

We can choose the impulses at the instants t_k, with $t_k - t_{k-1} = \ell = \tau$, given by

$$I_k(x(t_k^-)) = d\,x(t_k^-), \quad J_k(x'(t_k^-)) = d\,x'(t_k^-), \quad k = 1, 2, \ldots,$$

where $d = \exp[-(2\alpha+\beta)\tau] - G\tau$. Then the hypotheses in Corollary 3.1 are satisfied and, therefore, problem (6) can be exponentially stabilized by periodic impulses.

EXAMPLE 4.2. Consider the equation

$$x''(t) - 0.00324\,x(t-2) - 0.00512\,x(t-1) = 0, \quad t \geq 0 \tag{25}$$

whose characteristic equation is

$$\lambda^2 - 0.00324\,\mathrm{e}^{-2\lambda} - 0.00512\,\mathrm{e}^{-\lambda} = 0. \tag{26}$$

By using the software Maple, one can find a characteristic root with positive real part. Hence the non-impulsive equation (25) is unstable.

Consider $A = 0.00512$, $\ell = \tau = \tau_1 + \tau_2 = 3$ and $\alpha = \dfrac{1}{9}$. Then $\beta = \max\{1, AN\} = 1$ and we can also verify that

$$A\tau < \exp[-\alpha(\ell+\tau)] \exp[-\beta\ell] < \exp[-\beta\ell] = \exp[-\ell].$$

We can choose the impulses at the instants t_k, with $t_k - t_{k-1} = l$, given by

$$I_k(x(t_k^-)) = d\,x(t_k^-), \quad J_k(x'(t_k^-)) = d\,x'(t_k^-), \quad k = 1, 2, \ldots,$$

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where $d = \exp(-3.66667) - 0.01536$, then the hypotheses in Theorem 3.2 are satisfied and hence problem (25) can be exponentially stabilized by periodic impulses.

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