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Continuity of attractors for parabolic problems with localized large diffusion

Vera L. Carbone

Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, 13.565-905 São Carlos SP, Brazil
E-mail: carbone@dm.ufscar.br

Alexandre N. Carvalho*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: andcarva@icmsc.sc.usp.br

Karina Schiabel-Silva†

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: schiabel@icmc.usp.br

In this paper we study the continuity of asymptotics of semilinear parabolic problems of the form

$$u_t - \operatorname{div}(p(x)\nabla u) + \lambda u = f(u)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ and Dirichlet boundary conditions when the diffusion coefficient p becomes large in a subregion Ω_0 which is interior to the physical domain Ω. We prove, under suitable assumptions, that the family of attractors behave upper and lower-semicontinuously as the diffusion blows up in Ω_0. May, 2006 ICMC-USP

1. INTRODUCTION

In this paper we consider semilinear parabolic problems of the form

$$\begin{cases} u^\epsilon_t - \operatorname{div}(p_\epsilon(x)\nabla u^\epsilon) + \lambda u^\epsilon = f(u^\epsilon), & \text{in } \Omega \\ u^\epsilon = 0, & \text{in } \partial\Omega \\ u^\epsilon(0) = u^\epsilon_0, & \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $\epsilon \in (0, \epsilon_0]$ is a parameter. The parameter ϵ represents the fact that, as ϵ goes to zero, the diffusivity p_ϵ becomes very large in a region Ω_0 which is interior to Ω. This situation can be seen in models for heat conduction in composite materials where heat may diffuse much faster in one parte of the region than in others.

Such problems have been studied previously in [10], where the author studies the linear theory, and in [4], where the authors study the above semilinear problem and obtain the upper semicontinuity of attractors. In [5], the authors prove the continuity of attractors for $n = 1$ exploiting the fact that the linear operators involved have spectrum with large gaps which allow them to study the continuity properties of the attractors from the continuity properties of exponentially attracting invariant manifolds where the attractors lie. This procedure cannot be applied for $n > 1$ for the spectrum of the linear operators no longer present large gaps. Here we prove the lower-semicontinuity of attractors for the case $n > 1$.

In order to state the main results proved in this paper we introduce some terminology and for that we follow closely [10].

Let Ω_0 be a smooth sub-domain of Ω such that $\bar{\Omega}_0 \subset \Omega$. Let $m \in \mathbb{N}$ be the number of connected components of $\Omega_0 = \bigcup_{i=1}^{m} \Omega_{0,i}$ and assume that $\bar{\Omega}_{0,i} \cap \bar{\Omega}_{0,j} = \emptyset$ if $i \neq j$. Denote $\Omega_1 = \Omega \backslash \bar{\Omega}_0$, $\Gamma = \partial\Omega$, $\Gamma_{0,i} = \partial\Omega_{0,i}$, for $i = 1, \cdots, m$, and $\Gamma_0 = \bigcup_{i=1}^{m} \Gamma_{0,i}$.

The diffusivity $p_\epsilon : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is assumed to be a bounded smooth function in Ω, satisfying $0 < m_0 \leqslant p_\epsilon(x) \leqslant M_\epsilon$, for all $x \in \Omega$ and $0 < \epsilon \leqslant \epsilon_0$. In addition, assume that diffusion becomes large in Ω_0 as $\epsilon \to 0$, more precisely

$$p_\epsilon(x) \to \begin{cases} p(x), & \text{uniformly on } \Omega_1, \ (p \in C^1(\bar{\Omega}_1, (0, \infty))); \\ \infty, & \text{uniformly on compact subsets of } \ \Omega_0. \end{cases}$$

In what follows we derive heuristically the limiting problem to (1). So we will try to present arguments based on intuition and rather than provide exact proofs. The proof that it is in fact the limiting problem will be done throughout the paper. To simplify the presentation let us assume for the moment that $m = 1$.

Since large diffusivity implies fast homogenization, we expect that, for small values of ϵ, the solutions of the problem (1) become approximately constant on Ω_0. For this reason, suppose that u^ϵ converges to u as $\epsilon \to 0$, in some sense, and that u takes a time dependent spatially constant on Ω_0, which we will denote by $u_{\Omega_0}(t)$.

If we formally take the limit in problem (1), we expect that, inside Ω_1, the function u satisfies

$$\begin{cases} u_t - \text{div } (p(x)\nabla u) + \lambda u = f(u) & \text{in } \Omega_1 \\ u = 0 & \text{in } \Gamma. \end{cases}$$

Note that since limit function u is in $H^1(\Omega)$, its constant value u_{Ω_0} in Ω_0 may not be arbitrary. Integrating the equation in Ω_0 and using the inward normal in the integration

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by parts, it follows that

$$\int\limits_{\Omega_0} u_t^{\epsilon}(x,t)dx + \int\limits_{\Gamma_0} p_{\epsilon}(x)\frac{\partial u^{\epsilon}}{\partial \vec{n}}dx + \lambda \int\limits_{\Omega_0} u^{\epsilon}(x,t)dx = {}_{\Omega_0} int f(u^{\epsilon}(x,t))dx.$$

Taking the limit as $\epsilon \to 0$ and dividing both sides by $|\Omega_0|$, we have the ordinary differential equation

$$\dot{u}_{\Omega_0}(t) + \frac{1}{|\Omega_0|}\int\limits_{\Gamma_0} p(x)\frac{\partial u}{\partial \vec{n}}dx + \lambda u_{\Omega_0}(t) = f(u_{\Omega_0}(t)),$$

which is an equation relating the total heat flow from Ω_1 to Ω_0 through Γ_0 with the total heat input in Ω_0. It also relates the value of u_{Ω_0} with the values of u in Ω_1 through the integral term along Γ_0.

Therefore, when $m \geqslant 1$, the limiting problem should be

$$\begin{cases} u_t - \operatorname{div}(p(x)\nabla u) + \lambda u = f(u) & \text{in } \Omega_1 \\ u = 0 & \text{in } \Gamma \\ u_{|\Omega_{0,i}} := u_{\Omega_{0,i}} & \text{in } \Omega_{0,i},\ i = 1, \cdots, m \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int\limits_{\Gamma_{0,i}} p(x)\frac{\partial u}{\partial \vec{n}} + \lambda u_{\Omega_{0,i}} = f(u_{\Omega_{0,i}}) & i = 1, \cdots, m \\ u(0) = u_0. \end{cases} \tag{2}$$

Now that we have established the limit problem we can study the local and global well posedness of (1) and (2). To that end we introduce some more terminology to be able to write (1) and (2) as abstract equations in Banach spaces. Let $X = L^2(\Omega)$ and $A_\epsilon : D(A_\epsilon) \subset X \to X$ be the operator defined by

$$\begin{aligned} &\mathcal{D}(A_\epsilon) = \left\{u \in \mathrm{H}^1(\Omega) : -\operatorname{div}(p_\epsilon(x)\nabla u) \in X,\ u = 0 \text{ in } \Gamma\right\}, \\ &A_\epsilon u = -\operatorname{div}(p_\epsilon(x)\nabla u) + \lambda u,\ u \in D(A_\epsilon). \end{aligned}$$

In the same way, let $X_0 = L^2_{\Omega_0}(\Omega) := \{u \in X : u \text{ is constant in } \Omega_0\}$, $H^1_{\Omega_0}(\Omega) = \{u \in H^1_0(\Omega) : \nabla u = 0 \text{ in } \Omega_0\}$ and $A_0 : D(A_0) \subset X_0 \to X_0$ be the operator defined by

$$\mathcal{D}(A_0) = \left\{u \in \mathrm{H}^1_{\Omega_0}(\Omega) : -\operatorname{div}(p(x)\nabla u) \in \mathrm{L}^2(\Omega_1),\ u = 0 \text{ in } \Gamma\right\},$$

$$A_0 u = (-\operatorname{div}(p\nabla u) + \lambda u)\chi_{\Omega_1} + \left(\sum_{i=1}^{m} \frac{1}{|\Omega_{0,i}|}\int\limits_{\Gamma_{0,i}} p\frac{\partial u}{\partial \vec{n}} + \lambda u_{\Omega_{0,i}}\right)\chi_{\Omega_{0,i}},\ u \in \mathcal{D}(A_0),$$

where χ_B is the characteristic function of the set B.

It is easy to see that A_ϵ is a positive self-adjoint operator with compact resolvent, $\epsilon \in (0, \epsilon_0]$. It is shown in [10] that A_0 is a positive self-adjoint operator with compact resolvent.

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Hence, we can define the fractional power spaces X_ϵ^α associated to the operators A_ϵ, $0 \leqslant \epsilon \leqslant \epsilon_0$ and $0 \leqslant \alpha \leqslant 1$, $X_\epsilon^{\frac{1}{2}} = H_0^1(\Omega)$ with the inner product

$$\langle \phi, \psi \rangle_{X_\epsilon^{\frac{1}{2}}} = \int_\Omega p_\epsilon \nabla\phi \cdot \nabla\psi + \lambda \int_\Omega \phi\psi.$$

and $X_0^{\frac{1}{2}} = H^1_{\Omega_0}(\Omega)$ with the inner product

$$\langle \phi, \psi \rangle_{X_0^{\frac{1}{2}}} = \int_{\Omega_1} p_\epsilon \nabla\phi \cdot \nabla\psi + \lambda \int_\Omega \phi\psi.$$

Also, for the analytic semigroups generated by $-A_\epsilon$, $0 \leqslant \epsilon \leqslant 1$, we have that

$$\|e^{-A_\epsilon t}\|_{L(X_\epsilon^\alpha, X_\epsilon^\beta)} \leqslant C\, t^{\alpha-\beta} e^{-\lambda t}. \tag{3}$$

with C independent of ϵ. Now we give conditions under which the problems (1) and (2) are locally well posed. To that end we must assume some growth condition on the nonlinearity f. In fact we assume that, if $N = 2$, for every $\eta > 0$ there is a $c_\eta > 0$ such that

$$|f(u) - f(v)| \leqslant c_\eta(e^{\eta|u|^2} + e^{\eta|v|^2})|u - v|, \quad \forall u, v \in \mathbb{R} \tag{4}$$

and if $N \geqslant 3$, there is a constant $c > 0$ such that

$$|f(u) - f(v)| \leqslant c|u - v|(|u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}} + 1), \quad \forall u, v \in \mathbb{R}. \tag{5}$$

Under these assumptions the problems (1), $\epsilon \in (0, \epsilon_0]$ and (2) are locally well posed in $X_\epsilon^{\frac{1}{2}}$ and $X_0^{\frac{1}{2}}$, respectively.

We write (1) and (2) abstractly as

$$\begin{aligned} &\dot{u}^\epsilon + A_\epsilon u^\epsilon = f(u^\epsilon) \\ &u^\epsilon(0) = u_0^\epsilon \in X_\epsilon^{\frac{1}{2}}, \quad \epsilon \in [0, \epsilon_0] \end{aligned} \tag{6}$$

where f also denotes the Nemitiskiĭ map associated to f. Following [3] we obtain the local well posedness of (6).

Definition 1.1. Let Z be a Banach space, we say that a family $\{T(t) : Z \to Z : t \geqslant 0\}$ is a *continuous semigroup* if $T(t+s) = T(t)T(s)$ for all $t, s \geqslant 0$, $T(0) = I$ and $[0, \infty) \times Z \ni (t, z) \mapsto T(t)z \in Z$ is continuous. We say that A *attracts* C under the semigroup $\{T(t) : t \geqslant 0\}$ if $\sup_{c\in C} \inf_{a\in A} \|T(t)c - a\|_Z \overset{t\to\infty}{\longrightarrow} 0$. A set $\mathcal{A} \subset Z$ is said to be a *global attractor* for the semigroup $\{T(t) : t \geqslant 0\}$ if it is compact, invariant and attracts any bounded sets of Z under the semigroup (see [8]).

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To obtain the global existence and the existence of attractors we also assume that f satisfies the dissipativeness assumption

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0, \tag{7}$$

and following [2] we obtain the local and global well posedness.

For any $u_0^\epsilon \in X_\epsilon^{\frac{1}{2}}$ and $\epsilon \in [0, \epsilon_0]$, the solution $u^\epsilon(t, u_0^\epsilon)$ of (6) starting at u_0^ϵ exists for all $t \geqslant 0$. Therefore, we can define in $X_\epsilon^{\frac{1}{2}}$ the semigroup $\{T_\epsilon(t) : t \geqslant 0\}$ associated to (6) by $T_\epsilon(t)u_0^\epsilon = u^\epsilon(t, u_0^\epsilon)$, $t \geqslant 0$. To simplify we will denote the solution $u^0(t, u_0^0)$ by $u(t, u_0)$.

The existence of attractors and uniform bounds for the semigroups $\{T_\epsilon(t) : t \geqslant 0\}$ associated to (6), $\epsilon \in [0, \epsilon_0]$ are also established in [2]. In fact we have

Theorem 1.1. *The semigroup $\{T_\epsilon(t) : t \geqslant 0\}$ associated to* (1)*, $\epsilon \in [0, \epsilon_0]$, has a global attractor, $\mathcal{A}_\epsilon$, in $X_\epsilon^{\frac{1}{2}}$ if $\epsilon \in [0, \epsilon_0]$. Furthermore*

$$\sup_{\epsilon\in[0,\,\epsilon_0]} \sup_{w\in\mathcal{A}_\epsilon} \|w\|_{X_\epsilon^{\frac{1}{2}}} < \infty$$

and

$$\sup_{\epsilon\in[0,\,\epsilon_0]} \sup_{w\in\mathcal{A}_\epsilon} \|w\|_{L^\infty(\Omega)} < \infty.$$

Proof: The local existence follows from [3], the existence of global attractors as well as the uniform $L^\infty(\Omega)$ bound follows from [2]. After this, we can assume without loss of generality that f is globally Lipschitz and globally bounded. To obtain the remaining uniform estimate we can proceed as usual employing the variation of constants formula and (3). ∎

Hereafter we assume, without loss of generality, that f is bounded with bounded derivatives up to second order.

Of course $X_0^{\frac{1}{2}}$ is a closed subspace of $X_\epsilon^{\frac{1}{2}}$ and if $u \in X_0^{\frac{1}{2}}$, then $\|u\|_{X_\epsilon^{\frac{1}{2}}} \stackrel{\epsilon\to 0}{\longrightarrow} \|u\|_{X_0^{\frac{1}{2}}}$. We say that a sequence $\{u^\epsilon\}$ with $u^\epsilon \in X_\epsilon^{\frac{1}{2}}$, $X_\epsilon^{\frac{1}{2}}$**−converges** to $u \in X_0^{\frac{1}{2}}$ if $\|u^\epsilon - u\|_{X_\epsilon^{\frac{1}{2}}} \stackrel{\epsilon\to 0}{\longrightarrow} 0$ and we say that $\{u^\epsilon\}$ **weakly**$-X_\epsilon^{\frac{1}{2}}$**−converges** to $u \in X_0^{\frac{1}{2}}$ if

$$\int_\Omega p_\epsilon \nabla u^\epsilon \nabla\phi + \lambda \int_\Omega u^\epsilon \phi \to \int_{\Omega_1} p\nabla u \nabla\phi + \lambda\int_\Omega u\phi, \quad \forall \phi \in X_\epsilon^{\frac{1}{2}}.$$

We note that $\{u^\epsilon\}$ $X_\epsilon^{\frac{1}{2}}$-converges to $u \in X_0^{\frac{1}{2}}$ if, and only if,

$$\int_{\Omega_1} p_\epsilon |\nabla u^\epsilon - \nabla u|^2 + \lambda \int_\Omega (u^\epsilon - u)^2 \stackrel{\epsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad \int_{\Omega_0} p_\epsilon |\nabla u^\epsilon|^2 \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

By upper and lower semicontinuity of a family of sets we understand the following:

DEFINITION 1.2. Denoting by $d_\epsilon(\cdot,\cdot) : X_\epsilon^{\frac{1}{2}} \times X_\epsilon^{\frac{1}{2}} \to \mathbb{R}^+$ the metric in $X_\epsilon^{\frac{1}{2}}$ induced by the norm, we say that:

(a) the family $\{\mathcal{A}_\epsilon : \epsilon \in (0,\epsilon_0]\}$ is *upper semicontinuous* at $\epsilon = 0$ if $\sup_{u^\epsilon \mathcal{A}_\epsilon} d_\epsilon(u^\epsilon, \mathcal{A}_0) \stackrel{\epsilon\to 0}{\longrightarrow} 0$.

(b) the family $\{\mathcal{A}_\epsilon : \epsilon \in (0,\epsilon_0]\}$ is *lower semicontinuous* at $\epsilon = 0$ if $\sup_{u \in \mathcal{A}_0} d_\epsilon(u, \mathcal{A}_\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} 0$.

In this paper we prove upper and lower semicontinuity of the attractors $\mathcal{A}_\epsilon$ for the semigroups $\{T_\epsilon(t) : t \geqslant 0\}$ associated to (6), $0 \leqslant \epsilon \leqslant \epsilon_0$, when $\epsilon \to 0$, from the continuity properties of the linear operators involved.

This paper is organized as follows. In Section 2, we obtain the spectral convergence of the linear operators. Next, just requiring the compact convergence of the linear operators, we obtain the convergence of the resolvent operators and a type of Trotter-Kato Approximation Theorem for linear semigroups. In Section 3 we prove, using the Variation of Constants Formula and the results on the linear semigroups of Section 2, that the family of nonlinear semigroups $\{T_\epsilon(t) : t \geqslant 0\}$ associated to (1) is continuous in ϵ at $\epsilon = 0$ in compact intervals of $(0,\infty)$. From this we prove the upper semicontinuity of attractors and upper semicontinuity of the set of equilibria. In Section 4 we prove the continuity of the set of equilibria. Finally, in the Section 5, we show lower semicontinuity of attractors using the continuity of linear semigroups associated to the linearization around equilibrium points and the continuity of local unstable manifolds of the equilibrium points as $\epsilon \to 0$.

2. LINEAR THEORY

In this section we introduce the notion of compact convergence and show that A_ϵ^{-1}, $\epsilon \in [0,\epsilon_0]$, converges compactly as $\epsilon \to 0$. Then we apply these results to obtain continuity of the spectrum and convergence of the linear semigroups generated by A_ϵ. For that we follow [10] and [6].

We assume that X is a Banach space and that X_0 is a closed subspace of X.

DEFINITION 2.1. (Convergence). We say that a family of operators $\{S_\epsilon \in \mathcal{L}(X) : \epsilon \in (0,\epsilon_0]\}$ *converges* to $S \in \mathcal{L}(X_0)$ as $\epsilon \to 0$ if $X \ni S_\epsilon v^\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} Sv \in X_0$, whenever $X \ni v^\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} v \in X_0$, and we denote this convergence by $S_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} S$.

Remark 2. 1. Note that the convergence above is not just strong convergence in X since the limit vector is in a specified subspace X_0 of X. Much more general situations can be seen in [6].

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Definition 2.2. (Compact Convergence). We will say that a family of compact operators

$$\{S_\epsilon \in \mathcal{L}(X) : \epsilon \in (0, \epsilon_0]\}$$

compactly converges to $S \in \mathcal{L}(X_0)$ if $S_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} S$ and for any bounded family $\{v^\epsilon \in X : \epsilon \in (0, \epsilon_0]\}$, there is a sequence $\{\epsilon_n\}$, $\epsilon_n \stackrel{n\to\infty}{\longrightarrow} 0$, such that $\{S_{\epsilon_n} v_{\epsilon_n}\}$ converges to an element of X_0.

Lemma 2.1. *Suppose that* $\{S_\epsilon \in \mathcal{L}(X) : \epsilon \in (0, \epsilon_0]\}$ *compactly converges to* $S \in L(X_0)$ *as* $\epsilon \to 0$ *and, furthermore, that* $\mathcal{N}(I + S) = \{0\}$*. Then there are constants* $\bar\epsilon$ *and* $M > 0$ *such that*

$$\left\|(I + S_\epsilon)^{-1}\right\|_{\mathcal{L}(X)} \leqslant M, \quad \text{for all } \epsilon \in (0, \bar\epsilon]. \tag{1}$$

Proof: Because S_ϵ is compact for every $\epsilon \in [0, \epsilon_0]$, the estimate (1) is equivalent to say that

$$\|(I + S_\epsilon)u^\epsilon\|_X \geqslant \frac{1}{M}, \quad \forall \epsilon \in (0, \epsilon_0] \text{ and } \forall u^\epsilon \in X \text{ with } \|u^\epsilon\| = 1.$$

Suppose that this is not true; that is, suppose that there is a sequence $\{u_n\}$, with $u_n \in X$, $\|u_n\| = 1$ and $\epsilon_n \to 0$ such that $\|(I + S_{\epsilon_n})u_n\|_X \to 0$. Since $\{S_{\epsilon_n} u_n\}$ has a convergent subsequence, which we again denote by $\{S_{\epsilon_n} u_n\}$, to $u \in X_0$, $\|u\| = 1$, then $u_n + S_{\epsilon_n} u_n \to 0$ and $u_n \to -u$. This implies that $(I + S)u = 0$ contradicting our hypothesis. ∎

We denote by $X_\epsilon^{-\frac{1}{2}}$ the dual space of $X_\epsilon^{\frac{1}{2}}$, $0 \leqslant \epsilon \leqslant \epsilon_0$. Since $X_0^{\frac{1}{2}}$ is a closed subspace of $X_\epsilon^{\frac{1}{2}}$ given $f_\epsilon \in X_\epsilon^{-\frac{1}{2}}$ we have that f_ϵ is a bounded linear functional defined in $X_\epsilon^{\frac{1}{2}}$ and its restriction to $X_0^{\frac{1}{2}}$ (which we again denote by f_ϵ) is an element of $X_0^{-\frac{1}{2}}$. Given a sequence $\{\epsilon_n\}$ which converges to zero, a sequence $\{f_{\epsilon_n}\}$ with $f_{\epsilon_n} \in X_{\epsilon_n}^{-\frac{1}{2}}$ and $f_0 \in X_0^{-\frac{1}{2}}$ we say that $f_{\epsilon_n} \to f_0$ if $\|f_{\epsilon_n} - f_0\|_{X_0^{-\frac{1}{2}}} \stackrel{n\to\infty}{\longrightarrow} 0$ and $f_{\epsilon_n}(u_{\epsilon_n}) \to f_0(u)$ whenever u_{ϵ_n} $X_{\epsilon_n}^{\frac{1}{2}}-$converges to u.

Our next result assures that the family of solutions of the elliptic problem associated to (1) converges to a solution of the elliptic problem associated to (2) as $\epsilon \to 0$.

Theorem 2.1. *Let* $\{\epsilon_n\}$ *be a sequence in* $(0, \epsilon_0]$ *which converges to zero,* $\{f_{\epsilon_n}\}$ *with* $f_{\epsilon_n} \in X_{\epsilon_n}^{-\frac{1}{2}}$ *such that* $\|f_{\epsilon_n}\|_{X_{\epsilon_n}^{-\frac{1}{2}}} \leqslant 1$ *for all* n *and* $\{u_{\epsilon_n}\}$ *such that* $A_{\epsilon_n} u_{\epsilon_n} = f_{\epsilon_n}$*. Then, there is a subsequence of* $\{u_{\epsilon_n}\}$*, which we again denote by* $\{u_{\epsilon_n}\}$*, and* $u \in X_0^{\frac{1}{2}}$ *such that* $\{u_{\epsilon_n}\}$ *converges to* u $X_\epsilon^{\frac{1}{2}}$*-weakly and strongly in* X*. Furthermore, if* $\{f_{\epsilon_n}\} \subset X_\epsilon^{-\frac{1}{2}}$ *converges* $X_\epsilon^{-\frac{1}{2}}-$*weakly to* $f \in X_0^{-\frac{1}{2}}$*, then* $\{u_{\epsilon_n}\}$ *converges to* u $X_\epsilon^{\frac{1}{2}}-$*weakly and strongly in* X*, where* u *is the solution of*

$$A_0 u = f.$$

Proof: First note that if $\|f_{\epsilon_n}\|_{X_{\epsilon_n}^{-\frac{1}{2}}} \leqslant 1$ there is a constant M such that $\|f_{\epsilon_n}\|_{X_0^{-\frac{1}{2}}} \leqslant M$. Hence, there is a $f_0 \in X_0^{-\frac{1}{2}}$ such that $f_{\epsilon_n}(\phi_0) \stackrel{n\to\infty}{\longrightarrow} f_0(\phi_0)$ for all $\phi_0 \in X_0^{\frac{1}{2}}$. It is easy to see that $f_{\epsilon_n}(\phi_{\epsilon_n}) \to f_0(\phi_0)$ whenever $\{\phi_{\epsilon_n}\}$ $X_{\epsilon_n}^{\frac{1}{2}}$−converges to ϕ_0.

Note that

$$\|u_{\epsilon_n}\|^2_{X_{\epsilon_n}^{\frac{1}{2}}} = \int_\Omega p_{\epsilon_n}(x)|\nabla u_{\epsilon_n}|^2\, dx + \lambda\int_\Omega |u_{\epsilon_n}|^2\, dx \leqslant \|f_{\epsilon_n}\|_{X_{\epsilon_n}^{-\frac{1}{2}}} \|u_{\epsilon_n}\|_{X_{\epsilon_n}^{\frac{1}{2}}},$$

and therefore $\|u_{\epsilon_n}\|_{X_{\epsilon_n}^{\frac{1}{2}}} \leqslant 1$.

Since $0 < m_0 \leqslant p_\epsilon(x)$, for all $x \in \Omega$ and for all $0 < \epsilon \leqslant \epsilon_0$, we have that

$$m_0\int_\Omega |\nabla u_{\epsilon_n}|^2\, dx + \lambda\int_\Omega |u_{\epsilon_n}|^2\, dx \leqslant \|u_{\epsilon_n}\|^2_{X_{\epsilon_n}^{\frac{1}{2}}} \leqslant 1$$

that is, $\left\{\|u_{\epsilon_n}\|_{H^1(\Omega)}\right\}$ is bounded. Hence, there are a subsequence $\left\{u_{\epsilon_{n_k}}\right\}$ and $u \in H^1(\Omega)$ such that $u_{\epsilon_{n_k}} \to u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. It is easy to see that $u_{\epsilon_{n_k}} \stackrel{k\to\infty}{\longrightarrow} u$ weakly in $H^1(K)$ for any open set K with $K \subset\subset \Omega$. Then, it follows that $\int_K |\nabla u_{\epsilon_n}|^2\, dx \leqslant \liminf_{\epsilon\to 0} \int_K |\nabla u_{\epsilon_n}|^2\, dx$. On the other hand, for some $M > 0$,

$$\inf_{x\in K} \{p_{\epsilon_n}(x)\} \int_K |\nabla u_{\epsilon_n}|^2\, dx \leqslant \int_K p_{\epsilon_n}(x)|\nabla u_{\epsilon_n}|^2\, dx \leqslant M,$$

and, since $p_{\epsilon_n} \to \infty$ uniformly on compact subsets of Ω_0, we have that $\lim_{\epsilon\to 0}\int_K |\nabla u_{\epsilon_n}|^2\, dx = 0$. So, $\int_K |\nabla u|^2 = 0$, for all $K \subset\subset \Omega_0$. Consequently, u is constant in K for all $K \subset\subset \Omega_0$. From this and from the fact that $\Omega_0 = \bigcup_{\substack{i=1 \\ K_i \subset\subset \Omega_0}}^{\infty} K_i$, we obtain that $u \in X_0^{\frac{1}{2}}$.

Assume that $X_{\epsilon_n}^{\frac{1}{2}} \ni \phi_{\epsilon_n}$ $X_{\epsilon_n}^{\frac{1}{2}}$-converges to $\phi_0 \in X_0^{\frac{1}{2}}$. It follows that ϕ_{ϵ_n} converges to ϕ_0 in X and since $u_{\epsilon_n} \to u$ in X, it follows that $\int_\Omega u_{\epsilon_n}\phi_{\epsilon_n}\, dx \stackrel{n\to\infty}{\longrightarrow} \int_\Omega u\phi\, dx$ (in particular if $\phi_{\epsilon_n} = \phi_0$ for all $n \in \mathbb{N}$). Hence,

$$\int_{\Omega_1} p_{\epsilon_n}(x)\nabla u_{\epsilon_n}\nabla\phi_{\epsilon_n}\, dx + \int_\Omega \lambda u_{\epsilon_n}\phi_{\epsilon_n}\, dx \stackrel{n\to\infty}{\longrightarrow} \int_{\Omega_1} p(x)\nabla u\nabla\phi_0\, dx + \int_\Omega \lambda u\phi_0\, dx.$$

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In this manner, if $\phi \in X_0^{\frac{1}{2}}$,

$$\int_\Omega p_{\epsilon_n}(x)\nabla u_{\epsilon_n}\nabla\phi\,dx + \int_\Omega \lambda u_{\epsilon_n}\phi\,dx = \int_{\Omega_1} p_{\epsilon_n}(x)\nabla u_{\epsilon_n}\nabla\phi\,dx + \int_\Omega \lambda u_{\epsilon_n}\phi\,dx$$
$$\to \int_{\Omega_1} p(x)\nabla u\nabla\phi\,dx + \int_\Omega \lambda u\phi\,dx,$$

and, from $A_{\epsilon_n}u_{\epsilon_n} = f_{\epsilon_n}$ and from the convergence $\langle f_{\epsilon_n}, \phi\rangle \to \langle f, \phi\rangle$, it follows that $\langle A_0 u, \phi\rangle = \langle f, \phi\rangle$, for every $\phi \in X_0^{\frac{1}{2}}$, that is,

$$\langle A_0 u, \phi\rangle = \int_{\Omega_1} p(x)\nabla u\nabla\phi\,dx + \int_\Omega \lambda u\phi\,dx = \langle f, \phi\rangle.$$

For $f \in X_0$ we have that

$$0 = \int_{\Omega_1} p(x)\nabla u\nabla\phi + \int_{\Omega_1}\lambda u\phi\,dx - \int_{\Omega_1} f\phi + \int_{\Omega_0}\lambda u\phi - \int_{\Omega_0} f\phi, \quad \forall\phi \in X_0^{\frac{1}{2}}. \tag{2}$$

And if $\phi \in C_c^1(\Omega_1) \subset X_0^{\frac{1}{2}}$, then $pu_{x_j} \in H^1(\Omega_1)$, $1 \leqslant j \leqslant N$, and

$$\begin{aligned} 0 &= \int_{\Omega_1} p(x)\nabla u\nabla\phi + \int_{\Omega_1}\lambda u\phi\,dx - \int_{\Omega_1} f\phi \\ &= -\int_{\Omega_1} \operatorname{div}(p\nabla u)\phi\,dx + \int_{\Omega_1}\lambda u\phi - \int_{\Omega_1} f\phi. \end{aligned}$$

This implies that

$$-\operatorname{div}(p\nabla u) + \lambda u = f \quad \text{almost everywhere in } \Omega_1. \tag{3}$$

Using (3) and integrating by parts in (2) we have that

$$0 = \int_{\partial\Omega_0}\phi\frac{\partial u}{\partial n} + \int_{\Omega_0}\lambda u\phi - \int_{\Omega_0} f\phi = \phi_{\Omega_0}\left(\int_{\partial\Omega_0}\frac{\partial u}{\partial n} + |\Omega_0|\lambda u_{\Omega_0} - |\Omega_0| f_{\Omega_0}\right)$$

and

$$\frac{1}{|\Omega_0|}\int_{\partial\Omega_0}\frac{\partial u}{\partial n} + \lambda u_{\Omega_0} = f_{\Omega_0}.$$

The proof is complete. ∎

From compact convergence of resolvents we can obtain spectral convergence and the convergence of the projections associated to determined part of the spectrum, which allows us to show that unstable manifolds have the same dimension.

THEOREM 2.2. *The family* $\left\{A_\epsilon^{-1} : X \to X, 0 < \epsilon \leqslant \epsilon_0\right\}$ *compactly converges to* $A_0^{-1} : X_0 \to X_0$.

Proof: Let $\{h_\epsilon\} \subset X$ such that $h_\epsilon \to h \in X_0$ in X. Then $X_\epsilon^{-\frac{1}{2}} \ni h_\epsilon \to h \in X_0^{-\frac{1}{2}}$ and $\{\|h_\epsilon\|_{X_\epsilon^{-\frac{1}{2}}}\}$ is bounded. Since $A_\epsilon : X_\epsilon^{\frac{1}{2}} \to X_\epsilon^{-\frac{1}{2}}$ is an isomorphism, we have $A_\epsilon^{-1} h_\epsilon := u^\epsilon \in X_\epsilon^{\frac{1}{2}}$. Furthermore, from Theorem 2.1, there exists $u \in X_0^{\frac{1}{2}}$ such that $\{u^\epsilon\}$ converges to u $X_\epsilon^{\frac{1}{2}}$-weakly and strongly in X where $A_0 u = h$. Therefore,

$$A_\epsilon^{-1} h_\epsilon \overset{\epsilon \to 0}{\longrightarrow} A_0^{-1} h \text{ in } X,$$

and we obtain the convergence ${A_\epsilon}^{-1} \overset{\epsilon \to 0}{\longrightarrow} A_0^{-1}$.

Next we show that the family $\left\{{A_\epsilon}^{-1} : X \to X, 0 < \epsilon \leqslant \epsilon_0\right\}$ is collectively compact. For each $0 < \epsilon \leqslant \epsilon_0$ fixed, the operator ${A_\epsilon}^{-1} : X \to X$ is compact, since ${A_\epsilon}^{-1} : X_\epsilon^{-\frac{1}{2}} \to X_\epsilon^{\frac{1}{2}}$ is an isomorphism and $X_\epsilon^{\frac{1}{2}} \hookrightarrow X$ compactly. Let $\{g_\epsilon\}$ be a bounded sequence in X. It follows that $\{\|g_\epsilon\|_{X_\epsilon^{-\frac{1}{2}}}\}$ is bounded and, if v^ϵ is such that $A_\epsilon v^\epsilon = g_\epsilon$, we have

$$\int_\Omega p_\epsilon(x)|\nabla v^\epsilon|^2\, dx + \int_\Omega \lambda |v^\epsilon|^2\, dx = \langle g_\epsilon, v^\epsilon\rangle \leqslant \|g_\epsilon\|_{X_\epsilon^{-\frac{1}{2}}} \|v^\epsilon\|_{X_\epsilon^{\frac{1}{2}}},$$

and $\left\{\|v^\epsilon\|_{X_\epsilon^{\frac{1}{2}}}\right\}$ is bounded and $\{v^\epsilon\}$ has a convergent subsequence in X to $u \in X_0^{\frac{1}{2}}$, concluding the proof. ∎

2.1. Spectral convergence

The spectral behavior of the linear operators plays a fundamental role in the study of the nonlinear dynamics of semilinear parabolic problems. Next we prove compact convergence of resolvents and as a consequence we prove that eigenvalues and eigenfunctions of A_ϵ converge as $\epsilon \to 0$ to the eigenvalues and eigenfunctions of the linear operators A_0. This is proved in [10] but here is a direct consequence of the compact convergence.

Lemma 2.2. *For any $\mu \in \rho(-A_0)$, there exists ϵ_μ such that $\mu \in \rho(-A_\epsilon)$, for all $\epsilon \in (0, \epsilon_\mu]$, and*

$$\sup_{\epsilon \in (0,\epsilon_\mu]} \left\|(\mu + A_\epsilon)^{-1}\right\|_{\mathcal{L}(X)} < \infty.$$

Furthermore, $(\mu + A_\epsilon)^{-1}$ compactly converges to $(\mu + A_0)^{-1}$ as $\epsilon \to 0$. From this we obtain that $\sup_{\epsilon\in(0,\epsilon_\mu]} \|A_\epsilon(\mu + A_\epsilon)^{-1}\|_{\mathcal{L}(X)} < \infty$ and consequently

$$\sup_{\epsilon \in (0,\epsilon_\mu]} \left\|(\mu + A_\epsilon)^{-1}\right\|_{\mathcal{L}(X, X_\epsilon^{\frac{1}{2}})} < \infty.$$

Proof: Let $\mu \in \rho(-A_0)$. Then the equality $(\mu + A_0) = A_0(I + \mu {A_0}^{-1})$ implies the existence of $(I + \mu {A_0}^{-1})^{-1}$. From Lemma 2.1, there is $\epsilon_\mu \in (0, \epsilon_0]$ such that $(I + \mu {A_\epsilon}^{-1})^{-1}$ is well-defined, for all $\epsilon \in (0, \epsilon_\mu]$, and it is uniformly bounded. Thus, $(\mu + A_\epsilon)^{-1} =$

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$(I+\mu {A_\epsilon}^{-1})^{-1}{A_\epsilon}^{-1}$ exists. Since A_ϵ^{-1} compactly converges to A_0^{-1} as $\epsilon \to 0$, we have that $\left\{\left\|A_\epsilon^{-1}\right\| : \epsilon \in (0,\epsilon_\mu]\right\}$ is bounded. Hence, the limitation of $\left\{(I+\mu A_\epsilon^{-1})^{-1} : 0 < \epsilon \leqslant \epsilon_\mu\right\}$ allows us to conclude that $(\mu + A_\epsilon)^{-1}$ compactly converges to $(\mu + A_0)^{-1}$ as $\epsilon \to 0$. ∎

COROLLARY 2.1. *Given a compact subset K of $\rho(A_0)$, there exists ϵ_K such that $K \subset \rho(A_\epsilon)$ for $0 \leqslant \epsilon \leqslant \epsilon_K$ and*

$$\sup_{\alpha\in[0,1]} \sup_{\epsilon\in(0,\epsilon_K]} \sup_{\mu\in K} \|(\mu + A_\epsilon)^{-1}\|_{\mathcal{L}(X,X_\epsilon^\alpha)} < \infty. \tag{4}$$

Furthermore, if J is a compact subset of X_0, we have

$$\sup_{\substack{\mu\in K\\ f\in J}} \left\|(\mu + A_\epsilon)^{-1}f - (\mu + A_0)^{-1}f\right\|_{X_\epsilon^{\frac{1}{2}}} \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{5}$$

Proof: We first prove that

$$\sup_{\epsilon\in(0,\epsilon_K]} \sup_{\mu\in K} \|(\mu + A_\epsilon)^{-1}\|_{\mathcal{L}(X)} < \infty.$$

Suppose that there are sequences $\epsilon_n \to 0$ and $\mu_n \in K$ such that $N(\mu_n + A_{\epsilon_n}) \neq \{0\}$. We may assume, taking subsequences if necessary, that $\mu_n \to \mu \in K$. Since $\mu_n A_{\epsilon_n}^{-1}$ converges compactly to μA_0^{-1} we have a contradiction with Lemma 2.1. Thus, there is $\epsilon_K > 0$ such that $K \subset \rho(A_\epsilon)$ for all $0 < \epsilon \leqslant \epsilon_K$. To prove the uniform bound we proceed in a similar manner taking sequences $\epsilon_n \to 0$ and $\mu_n \in K$ with $\mu_n \to \mu \in K$ such that

$$\|(\mu_n + A_{\epsilon_n})^{-1}\| \to \infty.$$

Again we obtain a contradiction from the compact convergence of $\mu_n A_{\epsilon_n}^{-1}$ to μA_0^{-1}. To conclude the proof of (4) we note that

$$A_\epsilon(\mu + A_\epsilon)^{-1} = I - \mu(\mu + A_\epsilon)^{-1} \tag{6}$$

and apply the Momentum Inequality. The proof of (5) is obtained in the following manner. Let $u^\epsilon = (\mu + A_\epsilon)^{-1}f$ and $u = (\mu + A_0)^{-1}f$, then

$$\int_\Omega p_\epsilon |\nabla u^\epsilon|^2 + (\lambda + \mu)\int_\Omega {u^\epsilon}^2 = \int_\Omega f u^\epsilon$$
$$\int_{\Omega_1} p_\epsilon \nabla u^\epsilon \cdot \nabla u + (\lambda + \mu)\int_\Omega u^\epsilon u = \int_\Omega f u$$
$$\int_{\Omega_1} p|\nabla u|^2 + (\lambda + \mu)\int_\Omega (u)^2 = \int_\Omega f u.$$

Hence

$$\begin{aligned}\int_\Omega f(u^\epsilon-u) &= \int_\Omega p_\epsilon|\nabla u^\epsilon|^2-\int_{\Omega_1} p|\nabla u|^2+(\lambda+\mu)\int_\Omega u^{\epsilon 2}-(\lambda+\mu)\int_\Omega (u)^2\\ &= \int_\Omega p_\epsilon|\nabla u^\epsilon-\nabla u|^2+(\lambda+\mu)\int_\Omega (u^\epsilon-u)^2-\int_{\Omega_1}(p_\epsilon-p)|\nabla u|^2\\ &+2\int_{\Omega_1} p_\epsilon\nabla u^\epsilon\cdot\nabla u+2(\lambda+\mu)\int_\Omega u^\epsilon u-2\int_{\Omega_1} p|\nabla u|^2-2(\lambda+\mu)\int_\Omega (u)^2\\ &= \int_\Omega p_\epsilon|\nabla u^\epsilon-\nabla u|^2+(\lambda+\mu)\int_\Omega (u^\epsilon-u)^2-\int_{\Omega_1}(p_\epsilon-p)|\nabla u|^2\end{aligned}$$

Now the result follows from the fact that $\sup_{f\in J}\|u^\epsilon-u\|_X\to 0$, $p_\epsilon\to p$ uniformly in Ω_1 and $\sup_{f\in J}\int_{\Omega_1}|\nabla u|^2<\infty$. ∎

LEMMA 2.3. *Assume that* $\{u_*^\epsilon\}$ $X_\epsilon^{\frac{1}{2}}$*–converges to* $u_*\in X_0^{\frac{1}{2}}$ *and that* $0\notin\sigma(A_0-f'(u_*))$. *Then, for any* $0\leqslant\theta<1$, *there is* $\bar\epsilon\in(0,\epsilon_0]$ *such that*

$$\left\{(A_\epsilon)^\theta(A_\epsilon-f'(u_*^\epsilon))^{-1}:0\leqslant\epsilon\leqslant\epsilon_0\right\}$$

is collectively compact, uniformly bounded and $(A_\epsilon)^\theta(A_\epsilon-f'(u_*^\epsilon))^{-1}\overset{\epsilon\to 0}{\longrightarrow}(A_0)^\theta(A_0-f'(u_*))^{-1}$.

Proof: Note that $A_\epsilon(A_\epsilon-f'(u_*^\epsilon))^{-1}=(I-f'(u_*^\epsilon)A_\epsilon^{-1})^{-1}$ and that $N(I-f'(u_*)A_0^{-1})=\{0\}$. If there is a sequence z^ϵ with $\|z^\epsilon\|_X=1$ such that $(I-f'(u_*^\epsilon)A_\epsilon^{-1})z^\epsilon=0$, then from Corollary (2.1) there is $w\in X_0^{\frac{1}{2}}$ such that $\{A_\epsilon^{-1}z^\epsilon\}$ $X_\epsilon^{\frac{1}{2}}$–converges to w. Consequently $f'(u_*^\epsilon)A_\epsilon^{-1}z^\epsilon\to f'(u_*)w$ in X and $z^\epsilon\to z=f'(u_*)w$. This implies that $f'(u_*^\epsilon)A_\epsilon^{-1}z^\epsilon\to f'(u_*)A_0^{-1}z$ in X, $\|z\|_X=1$ and

$$(I-f'(u_*)A_0^{-1})z=0,$$

contradicting the assumption $N(I-f'(u_*)A_0^{-1})=\{0\}$. Hence $(I-f'(u_*^\epsilon)A_\epsilon^{-1})$ has bounded inverse for suitably small ϵ. To prove that $(I-f'(u_*^\epsilon)A_\epsilon^{-1})^{-1}$ is bounded uniformly we proceed in a similar manner assuming that there is a sequence z^ϵ with $\|z^\epsilon\|_X=1$ and such that $(I-f'(u_*^\epsilon)A_\epsilon^{-1})z^\epsilon\to 0$. Using compact convergence and proceeding as above we obtain a contradiction with $N(I-f'(u_*)A_0^{-1})=\{0\}$.

It is easy to see that $(A_\epsilon-f'(u_*^\epsilon))^{-1}=A_\epsilon^{-1}(I-f'(u_*^\epsilon)A_\epsilon^{-1})^{-1}$ is bounded uniformly in ϵ and the proof of the lemma follows from Moment's inequality. ∎

With this, given a closed, simple and rectifiable curve γ in $\rho(A_0)$ around $\mu_0\in\sigma(A_0)$, oriented counterclockwise we have that there is $\epsilon_\gamma>0$ such that γ is in $\rho(A_\epsilon)$ for $0\leqslant\epsilon\leqslant\epsilon_\gamma$. Define

$$Q_\epsilon(\mu_0)=\frac{1}{2\pi i}\int_\gamma(\lambda+A_\epsilon)^{-1}d\lambda$$

and $W(\mu, -A_\epsilon) = Q_\epsilon(\mu_0)X$, $0 \leqslant \epsilon \leqslant \epsilon_\gamma$.

THEOREM 2.3. *The following statements hold:*

(i)For any $\mu_0 \in \sigma(-A_0)$, there are sequences $\epsilon_n \to 0$ and $\{\mu_n\}, \mu_n \in \sigma(-A_{\epsilon_n}), n \in \mathbb{N}$, such that $\mu_n \to \mu_0$ as $n \to \infty$.

(ii)If for some sequence $\epsilon_n \to 0$, onde has $\mu_n \to \mu_0$ as $n \to \infty$, with $\mu_n \in \sigma(-A_{\epsilon_n}), n \in \mathbb{N}$, then $\mu_0 \in \sigma(-A_0)$.

(iii)There is $\epsilon_0 > 0$ such that $\dim W(\mu, -A_\epsilon) = \dim W(\mu, -A_0)$, for all $0 < \epsilon \leqslant \epsilon_0$.

(iv)For any $u \in W(\mu_0, -A_0)$, there is a sequence $\{u^\epsilon\}$, $u^\epsilon \in W(\mu_0, -A_\epsilon)$, such that $u^\epsilon \xrightarrow{\epsilon \to 0} u$.

(v)For any sequence $\epsilon_n \to 0$, the sequence $\{u_n\}, u_n \in W(\mu, -A_{\epsilon_n}), n \in \mathbb{N}$, with $\|u_n\|_X = 1$, has a convergent subsequence and any limit point of this sequence belongs to $W(\mu_0, -A_0)$.

Proof: (i) Let $\mu_0 \in \sigma(-A_0)$ and consider $U = U(\mu_0, \delta) = \{\mu \in \mathbb{C} : |\mu - \mu_0| \leqslant \delta\}$, where δ is chosen such that $U \cap \sigma(-A_0) = \{\mu_0\}$. Our aim is to show that there exists $\bar{\epsilon} \in (0, \epsilon_0]$ such that $U \cap \sigma(-A_\epsilon) \neq \varnothing$, for all $\epsilon \in (0, \bar{\epsilon}]$.

is bounded for $\epsilon\mu \in \partial U$, since contradiction, that there are

Suppose that $U \subset \rho(-A_\epsilon), \epsilon \in (0, \bar{\epsilon}]$. The function $\mu \mapsto (I + \mu A_\epsilon^{-1})^{-1}$ is holomorphic, for all $\mu \in \partial U$. So, it follows from Maximum Modulus Theorem that

$$\left\|(I + \mu_0 A_\epsilon^{-1})^{-1}\right\| \leqslant \max_{|\mu - \mu_0| = \delta} \left\|(I + \mu A_\epsilon^{-1})^{-1}\right\| \leqslant \sup_{\substack{|\mu - \mu_0| = \delta \\ \epsilon \in (0, \epsilon_0]}} \left\|(I + \mu A_\epsilon^{-1})^{-1}\right\| = C < \infty,$$

for some constant $C > 0$. Therefore, for $u \in X$ and $\epsilon_n \to 0$, we obtain

$$\|u\| = \left\|(I + \mu_0 A_{\epsilon_n}^{-1})^{-1}(I + \mu_0 A_{\epsilon_n}^{-1})u\right\| \leqslant C \left\|(I + \mu_0 A_{\epsilon_n}^{-1})u\right\|,$$

and then $\left\|(I + \mu_0 A_0^{-1})u\right\| \geqslant \dfrac{1}{C} \|u\|$ and $\mu_0 \in \rho(-A_0)$, a contradiction. Hence, $U(\mu_0, \delta)$ contains some point of $\sigma(-A_\epsilon)$, for ϵ appropriately small.

(ii) Consider the sequences $\epsilon_n \to 0$ and $\{\mu_n\}, \mu_n \in \sigma(-A_{\epsilon_n})$, such that $\mu_n \to \mu_0$ as $n \to \infty$. Let $\{u_n\} \subset X$, with $\|u_n\| = 1$ and $(I + \mu_n A_{\epsilon_n}^{-1})u_n = 0$. Then

$$\left\|(I + \mu_0 A_{\epsilon_n}^{-1})u_n\right\| \leqslant \left\|(\mu - \mu_n)A_{\epsilon_n}^{-1}u_n\right\| \to 0$$

as $n \to \infty$. Since $\left\{A_{\epsilon_n}^{-1} : n \in \mathbb{N}\right\}$ is collectively compact and $\|u_n\| = 1$, taking subsequences if necessary, we obtain that $\mu_0(A_{\epsilon_n})^{-1}u_n \to v$ and $u_n \to -v$, $\|v\| = 1$. Therefore, $(I + \mu_0 A_0^{-1})v = 0$, $\|v\| = 1$, that is, $\mu \in \sigma(-A_0)$.

(iii) Let us see that there is $\bar{\epsilon}$ such that rank $(Q_\epsilon(\mu_0)) \geqslant$ rank $(Q_0(\mu_0))$, $\epsilon \in (0, \bar{\epsilon}]$. Let $\{v_1, \cdots, v_k\}$ be a basis for $W(\mu_0, -A_0) = Q_0(\mu_0)X$; then it is easy to see that, for suitably small ϵ,

$$\{Q_\epsilon(\mu_0)v_1, \cdots Q_\epsilon(\mu_0)v_k\}$$

is a linearly dependent set in $W(\mu_0, A_\epsilon) = Q_\epsilon(\mu_0)X$.

Now we assume that for some sequence $\epsilon_n \to 0$ we have rank $(Q_{\epsilon_n}(\mu_0)) >$ rank $(Q_0(\mu_0))$. Then, for each $n \in \mathbb{N}$, there is $u_n \in W(\mu_0, -A_{\epsilon_n})$, $\|u_n\| = 1$, such that dist $(u_n, W(\mu_0, A_0))$ $= \|u_n\| = 1$. From compact convergence, we can assume that $Q_{\epsilon_n}(\mu_0)u_n = u_n \to Q_0(\mu_0)u = u$, and then,

$$1 \leqslant \|u_n - Q_0(\mu_0)u_n\| = \|Q_{\epsilon_n}(\mu_0)u_n - Q_0(\mu_0)u_n\| \to 0,$$

a contradiction.

(iv) Let $u \in W(\mu_0, -A_0)$ and $u^\epsilon = Q_\epsilon(\mu_0)u$. Since $Q_\epsilon(\mu_0)u \to Q_0(\mu_0)u$ and $u = Q_0(\mu_0)u$, we have $u^\epsilon \to u$.

(v) Suppose that $\epsilon_n \to 0$ and consider $\{u_n\} \subset X$, $u_n \in W(\mu_0, -A_{\epsilon_n})$, $n \in \mathbb{N}$, $\|u_n\| = 1$. Since $Q_\epsilon(\mu_0)$ compactly converges to $Q_0(\mu_0)$ as $\epsilon \to 0$, $u_n = Q_{\epsilon_n}(\mu_0)u_n$ has a convergent subsequence $\{u_{n_k}\}$. Let us say that $u_{n_k} \to u$, then

$$u \leftarrow u_{n_k} = Q_{\epsilon_{n_k}}(\mu_0)u_{n_k} \to Q_0(\mu_0)u$$

and $u \in W(\mu_0, -A_0)$. ∎

2.2. Convergence of Linear Semigroups

The properties of the operators $A_\epsilon, \epsilon \in [0, \epsilon_0]$, imply the existence of a constant M_ω, independent of ϵ, such that

$$\left\| t^{\frac{1}{2}} e^{-A_\epsilon t} \right\|_{\mathcal{L}(X, X_\epsilon^{\frac{1}{2}})} \leqslant M_\omega e^{-\omega t} \tag{7}$$

for all $t > 0$ and for all $\epsilon \in [0, \epsilon_0]$. We use these estimates to show that linear semigroups associated to these operators behave continuously at $\epsilon \to 0$.

Theorem 2.4. *Let K be a compact subset in X_0 and $0 < \theta < \frac{1}{2}$. Then there is a function $\nu : (0, \epsilon_0] \to \mathbb{R}^+$, $\nu(\epsilon) \overset{\epsilon \to 0}{\longrightarrow} 0$, such that*

$$\sup_{u \in K} \left\| e^{-A_\epsilon t}u - e^{-A_0 t}u \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \nu(\epsilon)\, e^{-\omega t} t^{-\theta - \frac{1}{2}}, \quad \textit{for all}\, t > 0. \tag{8}$$

Proof: Let $0 < \theta < \frac{1}{2}$ be fixed and $\delta \in (0,1)$. We deal with the cases $0 < t < \delta$ and $t \geqslant \delta$ separately.

Thus, since u lies in a compact subset of X_0 and $0 < t < \delta$, we have that

$$\left\| e^{-A_\epsilon t}u - e^{-A_0 t}u \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant 2M_\omega\, t^{-\theta - \frac{1}{2}} e^{-\omega t} \delta^\theta.$$

Recall that

$$e^{-A_\epsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\mu t} (\mu + A_\epsilon)^{-1} d\mu, \quad \epsilon \in [0, \epsilon_0],$$

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where Γ is the boundary of $\Sigma_{-\omega,\phi} = \{\mu \in \mathbb{C} : |\arg(\mu+\omega)| \leqslant \phi\}$ with $\frac{\pi}{2} > \phi > \pi$ is oriented in such a way that the imaginary part of μ increases as μ runs in Γ and $\omega < \lambda$. In this case, we obtain

$$e^{(-A_\epsilon+\omega)t} = \frac{1}{2\pi i}\int_\Gamma e^{(\mu+\omega)t}(\mu+\omega+A_\epsilon-\omega)^{-1}d\mu, \quad \epsilon \in [0,\epsilon_0].$$

Changing variables $\mu+\omega \mapsto \mu$ and denoting $B_\epsilon := A_\epsilon - \omega I, \epsilon \in [0,\epsilon_0]$, we can write

$$e^{-B_\epsilon t} = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\mu t}(\mu+B_\epsilon)^{-1}d\mu, \quad \epsilon \in [0,\epsilon_0],$$

where Γ_0 is the boundary of $\Sigma_{0,\phi}$. Our aim is to estimate, to $t \geqslant \delta$, the difference

$$2\pi t^{\frac{1}{2}} \left\| e^{-B_\epsilon t}u - e^{-B_0 t}u \right\|_{X_\epsilon^{\frac{1}{2}}} = \left\| \int_{\Gamma_0} t^{\frac{1}{2}} e^{\mu t}[(\mu+B_\epsilon)^{-1}u - (\mu+B_0)^{-1}u]d\mu \right\|_{X_\epsilon^{\frac{1}{2}}}. \tag{9}$$

Let us see that the integral on the right hand in (9) is convergent. For $\mu \in \Gamma_0$, we previously obtained the sectorial estimative $\left\|(\mu+B_\epsilon)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{M}{1+|\mu|}$, for all $\epsilon \in [0,\epsilon_0]$. Hence,

$$\left\|B_\epsilon(\mu+B_\epsilon)^{-1}u\right\|_X \leqslant \|u\|_X + |\mu| \left\|(\mu+B_\epsilon)^{-1}u\right\|_X \leqslant M_1 \|u\|_X .$$

From Moment's Inequality, we obtain

$$\left\|(\mu+B_\epsilon)^{-1}u\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{M_2}{1+|\mu|^{\frac{1}{2}}} \|u\|_X ,$$

and then

$$\left\|(\mu+B_\epsilon)^{-1}u - (\mu+B_0)^{-1}u\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{2M_2}{1+|\mu|^{\frac{1}{2}}} \|u\|_X. \tag{10}$$

Therefore, we have

$$\left\| \int_{\Gamma_0} t^{\frac{1}{2}} e^{\mu t}[(\mu+B_\epsilon)^{-1}u - (\mu+B_0)^{-1}u]d\mu \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant t^{\frac{1}{2}} \int_{\Gamma_0} |e^{\mu t}| \frac{2M_2}{1+|\mu|^{\frac{1}{2}}} \|u\|_X \, d\mu = M_3 \|u\|_X .$$

For $t \geqslant \delta$, we proceed as follows: since the integral on the right hand side of (9) is uniformly convergent in ϵ, given $\eta > 0$, we can write $\Gamma_0 = \Gamma_1^\eta \cup \Gamma_2^\eta$, with Γ_1^η bounded and

Γ_2^η satisfying

$$\sup_{\epsilon\in(0,\epsilon_0]}\sup_{u\in K}\left\| t^{\frac{1}{2}}\int_{\Gamma_2^\eta} e^{\mu t}\left[(\mu+B_\epsilon)^{-1}u-(\mu+B_0)^{-1}u\right]d\mu\right\|_{X_\epsilon^{\frac{1}{2}}}<\frac{\eta}{2}.$$

For Γ_1^η we note that, from Corollary 2.1,

$$s_\eta(\epsilon):=\sup_{\mu\in\Gamma_1^\eta}\sup_{u\in K}\left\|(\mu+B_\epsilon)^{-1}u-(\mu+B_0)^{-1}u\right\|_{X_\epsilon^{\frac{1}{2}}}\overset{\epsilon\to 0}{\longrightarrow}0.$$

It follows from the fact that $t\geqslant\delta$ that there is $\bar\epsilon>0$ such that, for $\epsilon\in(0,\bar\epsilon]$

$$\left\| t^{\theta+\frac{1}{2}}\int_{\Gamma_1^\eta} e^{\mu t}\left[(\mu+B_\epsilon)^{-1}u-(\mu+B_0)^{-1}u\right]d\mu\right\|_{X_\epsilon^{\frac{1}{2}}}\leqslant s_\eta(\epsilon)\,t^{\theta+\frac{1}{2}}\int_{\Gamma_1^\eta}e^{\mathrm{Re}\mu\, t}\,d|\mu|\leqslant\frac{\eta}{2}.$$

From this we have that, for $\epsilon<\bar\epsilon$,

$$\sup_{u\in K}\left\|e^{-B_\epsilon t}u-e^{-B_0 t}u\right\|_{X_\epsilon^{\frac{1}{2}}}\leqslant\frac{\eta}{2\pi}t^{-\theta-\frac{1}{2}},\quad\text{for } t\geqslant\delta.$$

Therefore, there is a function $\nu:(0,\epsilon_0]\to\mathbb{R}^+$, with $\nu(\epsilon)\overset{\epsilon\to 0}{\longrightarrow}0$, satisfying

$$\sup_{u\in K}\left\|e^{-A_\epsilon t}u-e^{-A_0 t}u\right\|_{X_\epsilon^{\frac{1}{2}}}\leqslant\nu(\epsilon)e^{-\omega t}t^{-\frac{1}{2}-\theta},\ \text{ for all } t>0.$$

The proof is complete. ∎

3. UPPER SEMICONTINUITY OF ATTRACTORS

In this section we obtain the continuity of nonlinear semigroups by using the continuity of linear semigroups and Variation of Constants Formula. We also prove that the attractors are upper semicontinuous at $\epsilon\to 0$.

THEOREM 3.1. *Let $u\in X_0$, $\{u^\epsilon\}$ a sequence in X with $u^\epsilon\to u$ in X and $\tau>0$. Then there is a function $\bar\nu:(0,\epsilon_0]\to\mathbb{R}^+$, with $\bar\nu(\epsilon)\overset{\epsilon\to 0}{\longrightarrow}0$ satisfying*

$$\|T_\epsilon(t)u^\epsilon-T_0(t)u\|_{X_\epsilon^{\frac{1}{2}}}\leqslant\bar\nu(\epsilon)t^{-\frac{1}{2}}e^{-\omega t},\quad t\in(0,\tau).$$

Proof: Nonlinear semigroups $T_\epsilon(t)u^\epsilon$ are given by Variation of Constants Formula,

$$T_\epsilon(t)u^\epsilon=e^{-A_\epsilon t}u^\epsilon+\int_0^t e^{-A_\epsilon(t-s)}f(T_\epsilon(s)u^\epsilon)\,ds,\quad\epsilon\in[0,\epsilon_0].$$

Then,

$$\|T_\epsilon(t)u^\epsilon - T_0(t)u\|_{X_\epsilon^{\frac12}} \leqslant \left\|e^{-A_\epsilon t}u^\epsilon - e^{-A_0 t}u\right\|_{X_\epsilon^{\frac12}} + \int_0^t \left\|e^{-A_\epsilon(t-s)}f(T_\epsilon(s)u^\epsilon) - e^{-A_0(t-s)}f(T_0(s)u)\right\|_{X_\epsilon^{\frac12}} ds.$$

From (8) we have that

$$\begin{aligned}\left\|e^{-A_\epsilon t}u^\epsilon - e^{-A_0 t}u\right\|_{X_\epsilon^{\frac12}} &\leqslant \left\|e^{-A_\epsilon t}u^\epsilon - e^{-A_\epsilon t}u\right\|_{X_\epsilon^{\frac12}} + \left\|e^{-A_\epsilon t}u - e^{-A_0 t}u\right\|_{X_\epsilon^{\frac12}}\\ &\leqslant M_\omega\, e^{-\omega t}t^{-\frac12}\ \|u^\epsilon - u\|_X + \nu(\epsilon)\ e^{-\omega t}t^{-\theta-\frac12}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\int_0^t \left\|e^{-A_\epsilon(t-s)}f(T_\epsilon(s)u^\epsilon) - e^{-A_0(t-s)}f(T_0(s)u)\right\|_{X_\epsilon^{\frac12}} ds \leqslant\\ &\leqslant \int_0^t \left\|e^{-A_\epsilon(t-s)}f(T_\epsilon(s)u^\epsilon) - e^{-A_\epsilon(t-s)}f(T_0(s)u)\right\|_{X_\epsilon^{\frac12}} ds\\ &+ \int_0^t \left\|e^{-A_\epsilon(t-s)}f(T_0(s)u) - e^{-A_0(t-s)}f(T_0(s)u)\right\|_{X_\epsilon^{\frac12}} ds.\end{aligned}$$

Since $\{f(T_0(s)u) \in X_0 : s \in [0,t]\}$ is a compact subset in X_0 we have that

$$\begin{aligned}\int_0^t \left\|[e^{-A_\epsilon(t-s)} - e^{-A_0(t-s)}]f(T_0(s)u)\right\|_{X_\epsilon^{\frac12}} ds &\leqslant \int_0^t \nu(\epsilon)\ (t-s)^{-\frac12-\theta}e^{-\omega(t-s)}ds\\ &\leqslant \nu(\epsilon)\omega^{\theta-\frac12}\ \Gamma\left(\frac12-\theta\right).\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_0^t \left\|e^{-A_\epsilon(t-s)}[f(T_\epsilon(s)u^\epsilon) - f(T_0(s)u)]\right\|_{X_\epsilon^{\frac12}} ds\\ &\leqslant \int_0^t M_\omega\ (t-s)^{-\frac12}e^{-\omega(t-s)}L\,\|T_\epsilon(s)u^\epsilon - T_0(s)u)\|_{X_\epsilon^{\frac12}} ds.\end{aligned}$$

Then,

$$\begin{aligned}e^{\omega t}\|T_\epsilon(t)u^\epsilon - T_0(t)u\|_{X_\epsilon^{\frac12}} &\leqslant M_\omega t^{-\theta-\frac12}\ \|u^\epsilon - u\|_X + \nu(\epsilon)\ e^{-\omega t}t^{-\theta-\frac12}\\ &+ M_\omega\, L\int_0^t (t-s)^{-\frac12}e^{\omega s}\|T_\epsilon(s)u^\epsilon - T_0(s)u\|_{X_\epsilon^{\frac12}} ds.\end{aligned}$$

Note that, in the right hand side of the last inequality, the first two terms are bounded by $\tilde M\tilde\nu(\epsilon)t^{-\theta-\frac12}$, $\tilde M = \tilde M(\tau)$, as $t \in (0,\tau)$ and $\tilde\nu(\epsilon) \stackrel{\epsilon\to0}{\longrightarrow} 0$. If $\varphi(t) = e^{\omega t}\|T_\epsilon(t)u^\epsilon - T_0(t)u\|_{X_\epsilon^{\frac12}}$, we have

$$\varphi(t) \leqslant \tilde{M}\tilde{\nu}(\epsilon)\, t^{-\theta-\frac{1}{2}} + M_\omega\, L \int_0^t (t-s)^{-\frac{1}{2}}\, \varphi(s)\, ds \quad \text{em } (0,\tau),\ \tau<\infty.$$

From Gronwall's Inequality it follows that

$$\varphi(t) \leqslant C\tilde{M}\tilde{\nu}(\epsilon) t^{-\theta-\frac{1}{2}},$$

with $C = C(M_\omega, L, \tau)$ constant.

Therefore, there is a function $\bar{\nu} : (0, \epsilon_0] \to \mathbb{R}^+$, with $\bar{\nu}(\epsilon) \overset{\epsilon\to 0}{\longrightarrow} 0$ satisfying

$$\|T_\epsilon(t)u^\epsilon - T_0(t)u\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \bar{\nu}(\epsilon) t^{-\theta-\frac{1}{2}} e^{-\omega t}, \quad t \in (0,\tau).$$

This concludes the proof. ∎

THEOREM 3.2. *The family of attractors* $\{\mathcal{A}_\epsilon : \epsilon \in (0,\epsilon_0]\}$ *is upper semicontinuous at* $\epsilon = 0$.

Proof: Let us consider a sequence $\{u_n\}$, $u_n \in \mathcal{A}_{\epsilon_n}$, $n \in \mathbb{N}$, and $\epsilon_n \to 0$. Since

$$\sup\{\|u^\epsilon\|_{X_\epsilon^{\frac{1}{2}}} : u^\epsilon \in \mathcal{A}_\epsilon,\ \epsilon \in [0,\epsilon_0]\} < \infty,$$

$\underset{\epsilon\in[0,\epsilon_0]}{\cup} \mathcal{A}_\epsilon$ is relatively compact in X, there is $u \in X$ such that $u_n \to u$ strongly in X. Furthermore, we have

$$\int_\Omega p_{\epsilon_n}(x)|\nabla u_n^2|\, dx \leqslant M, \text{ for some constant } M > 0.$$

Thus, it follows that $u \in X_0^{\frac{1}{2}}$. It remains to show that $u \in \mathcal{A}_0$. For this, it is enough to prove that there is a bounded complete orbit through u. For each $n \in \mathbb{N}$, we have a bounded complete orbit $\varphi_n(\cdot, u_n) : \mathbb{R} \to X_\epsilon^{\frac{1}{2}}$ through u_n. For $t \geqslant 0$, it follows from the continuity of nonlinear semigroups that

$$\varphi_n(t,u_n) = T_{\epsilon_n}(t,u_n) \to T_0(t,u).$$

For $t < 0$, we set up the orbit through u in the following way: for $t \in (-k, -k+1]$, $k \in \mathbb{Z}^+$, we consider the sequence $\{\varphi_n(-k,u_n)\}_{n\in\mathbb{N}}$ in $\underset{\epsilon\in(0,\epsilon_0]}{\cup} \mathcal{A}_\epsilon$. Following the argument used previously for the sequence $\{u_n\}$, we obtain that $\varphi_n(-k,u_n) \to \varphi_0(-k,u)$. Hence,

$$\varphi_n(t,u_n) = \varphi_n(t+k, \varphi_n(-k,u_n)) = T_{\epsilon_n}(t+k, \varphi_n(-k,u_n)) \to T_0(t+k, \varphi_0(-k,u)) := \tilde{\varphi}_0(t,u).$$

Finally, defining

$$\varphi_0(t,u) = \begin{cases} T_0(t,u), & \text{for } t \geqslant 0; \\ \tilde{\varphi}_0(t,u), & \text{for } t < 0, \end{cases}$$

it follows that $\varphi_0(t,u)$ is a bounded complete orbit through u. Therefore, $u \in \mathcal{A}_0$. ∎

4. CONTINUITY OF THE SET OF EQUILIBRIA

In order to obtain the lower semicontinuity of attractors we will need to obtain the continuity of the set of equilibria and then study the continuity of the linearization around each equilibrium. In this section we prove that the family $\{\mathcal{E}_\epsilon : \epsilon \in [0,\epsilon_0]\}$ is continuous at $\epsilon = 0$. We start with the proof of upper semicontinuity.

Definition 4.1. The *equilibrium solutions* of (6), $\epsilon \in [0,\epsilon_0]$, are those which are independent of time; that is, the solutions of the elliptic problems

$$A_\epsilon u - f(u) = 0. \tag{1}$$

We denote by $\mathcal{E}_\epsilon$ the set of solutions to (1), $\epsilon \in [0,\epsilon_0]$.

The upper semicontinuity of the family $\{\mathcal{E}_\epsilon : \epsilon \in [0,\epsilon_0]\}$ is an easy consequence of the compact convergence. In fact

Proposition 4.1. *The family $\{\mathcal{E}_\epsilon : \epsilon \in [0,\epsilon_0]\}$ is upper semicontinuous at $\epsilon = 0$.*

Proof: Note that $\mathcal{E}_\epsilon \subset \mathcal{A}_\epsilon$ and therefore $\sup\{\|u^\epsilon\|_{X_\epsilon^{\frac{1}{2}}} : u^\epsilon \in \mathcal{E}_\epsilon,\ \epsilon \in [0,\epsilon_0]\} < \infty$ and that $f : X_\epsilon^{\frac{1}{2}} \to X$ is bounded. If $u^\epsilon \in \mathcal{E}_\epsilon$, we have that $u^\epsilon = A_\epsilon^{-1} f(u^\epsilon)$ and the result follows from compact convergence of A_ϵ^{-1} to A_0^{-1}. ∎

The proof of lower semicontinuity requires additional assumptions. We need to assume that the equilibrium points of (2) are stable under perturbation. This stability under perturbation will be given by the hyperbolicity.

Definition 4.2. We say that an equilibrium u_* of (2) is *hyperbolic* if the spectrum $\sigma(A_0 - f'(u_*))$ of $A_0 - f'(u_*)$ is disjoint from the imaginary axis.

Proposition 4.2. *If all equilibrium points of* (2) *are isolated then, there is only a finite number of them. Any hyperbolic equilibrium point u_* of* (2) *is isolated.*

Proof: Since $\mathcal{E}_0$ is compact we only need to prove that hyperbolic equilibria are isolated. We note that $u \in \mathcal{E}_0$ is a solution of (1) with $\epsilon = 0$ if and only if u is a fixed point of

$$\Phi(u) = -(A_0 - f'(u_*))^{-1}(f(u) - f'(u_*)u).$$

It's not difficult to see that there is $\delta > 0$ such that Φ is a contraction map from $\bar{B}_\delta(u_*) = \{u \in X_0^{\frac{1}{2}} : \|u - u_*\|_{X_0^{\frac{1}{2}}} \leqslant \delta\}$ into itself. Thus we obtain that u_* is the only element in $\mathcal{E}_0$ in $\bar{B}_\delta(u_*)$. ∎

Proposition 4.3. *Suppose that u_* is an equilibrium solution for* (2) *and that $0 \notin \sigma(A_0 - f'(u_*))$. Then there are $\bar{\epsilon} > 0$ and $\delta > 0$ such that the problem* (1) *has exactly one* *equilibrium solution,* u_*^ϵ*, in* $\{w^\epsilon \in X_\epsilon^{\frac{1}{2}} : \|w^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \delta\}$*, for* $\epsilon \in (0, \bar{\epsilon}]$*. Furthermore,* $\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \to 0$ *as* $\epsilon \to 0$*.*

Proof: First of all, we note that from Lemma 2.3, from the hyperbolicity of u_* and from Corollary 2.1 it follows that $\sigma(A_\epsilon - f'(u_*))$ is disjoint from imaginary axis. From Lemma 2.3 with $\theta = \frac{1}{2}$, we obtain a constant $M > 0$ such that

$$\left\|(A_\epsilon - f'(u_*))^{-1}\right\|_{\mathcal{L}(X, X_\epsilon^{\frac{1}{2}})} \leqslant M, \quad \text{for } \epsilon \in (0, \epsilon_0].$$

Now we note that if u^ϵ is a solution of (1), then

$$0 = (A_\epsilon - f'(u_*))[u^\epsilon + (A_\epsilon - f'(u_*))^{-1}(f'(u_*)u^\epsilon - f(u^\epsilon))].$$

Since $(A_\epsilon - f'(u_*))$ is invertible, to show that u^ϵ is a solution of (1) we may equivalently to show that u^ϵ is a fixed point of the application

$$\Phi_\epsilon(u^\epsilon) = -(A_\epsilon - f'(u_*))^{-1}(f'(u_*)u^\epsilon - f(u^\epsilon)).$$

Observe that

$$\Phi_\epsilon(u_*) = (A_\epsilon - f'(u_*))^{-1}(A_0 - f'(u_*))u_* \stackrel{\epsilon \to 0}{\longrightarrow} u_*. \tag{2}$$

Next we prove that there are $\delta > 0$ and $\bar{\epsilon} \in (0, \epsilon_0]$ such that Φ_ϵ is a contraction map from $\bar{B}_\delta^\epsilon(u_*) = \{u^\epsilon \in X_\epsilon^{\frac{1}{2}} : \|u^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \delta\}$ into itself, uniformly in $(0, \bar{\epsilon}]$. First, let us see that Φ_ϵ is a contraction. To do it, let $u^\epsilon, v^\epsilon \in \bar{B}_\delta^\epsilon(u_*)$ and note that

$$\|\Phi_\epsilon(u^\epsilon) - \Phi_\epsilon(v^\epsilon)\|_{X_\epsilon^{\frac{1}{2}}} \leqslant M \, \|[f'(su^\epsilon + (1-s)v^\epsilon) - f'(u_*)](u^\epsilon - v^\epsilon)\|_X \, .$$

Note that $\|f'(z) - f'(u_*)\|_X \leqslant C\|z - u_*\|_X$ and $\|f'(z) - f'(u_*)\|_{L^\infty(\Omega)} \leqslant C$, for some constant C. Hence, we can choose $\delta > 0$ such that

$$M \sup_{z \in \bar{B}_\delta^\epsilon(u_*)} \|f'(z) - f'(u_*)\|_{L^n(\Omega)} \leqslant \frac{1}{2}$$

and

$$\|\Phi_\epsilon(u^\epsilon) - \Phi_\epsilon(v^\epsilon)\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{1}{2} \|u^\epsilon - v^\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} \leqslant \frac{1}{2} \|u^\epsilon - v^\epsilon\|_{X_\epsilon^{\frac{1}{2}}} \, .$$

To show that $\Phi_\epsilon(\bar{B}_\delta^\epsilon(u_*)) \subset \bar{B}_\delta^\epsilon(u_*)$, observe that if $u^\epsilon \in \bar{B}_\delta^\epsilon(u_*)$ then

$$\|\Phi_\epsilon(u^\epsilon) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{1}{2} \|u^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} + \|\Phi_\epsilon(u_*) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \, .$$

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From (2) it follows that there is $\bar{\epsilon}$ such that $\|\Phi_\epsilon(u_*) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \dfrac{\delta}{2}$, and, for any $u^\epsilon \in \bar{B}_\delta^\epsilon(u_*)$, we have $\|\Phi_\epsilon(u^\epsilon) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \delta$.

Thus, $\Phi_\epsilon : \bar{B}_\delta^\epsilon(u_*) \to \bar{B}_\delta^\epsilon(u_*)$ is a contraction, for all $\epsilon \in (0, \bar{\epsilon}]$ and then there is only one fixed point of Φ_ϵ in $\bar{B}_\delta^\epsilon(u_*)$, that we will call u_*^ϵ.

Finally, let us see that $u_*^\epsilon \to u_*$ in $X_\epsilon^{\frac{1}{2}}$ as $\epsilon \to 0$. Indeed,

$$\begin{aligned}\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} &= \|\Phi_\epsilon(u_*^\epsilon) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \le \|\Phi_\epsilon(u_*^\epsilon) - \Phi_\epsilon(u_*)\|_{X_\epsilon^{\frac{1}{2}}} + \|\Phi_\epsilon(u_*) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \\ &\leqslant \frac{1}{2}\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} + \|\Phi_\epsilon(u_*) - u_*\|_{X_\epsilon^{\frac{1}{2}}},\end{aligned}$$

and then $\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant 2\ \|\Phi_\epsilon(u_*) - u_*\|_{X_\epsilon^{\frac{1}{2}}} \to 0$ as $\epsilon \to 0$. ∎

5. LOWER SEMICONTINUITY OF ATTRACTORS

Next we show that the local unstable manifolds of u_k, for $k = 1, \dots, m$ fixed, are continuous in $X_\epsilon^{\frac{1}{2}}$ as $\epsilon \to 0$. This fact and the continuity of the set of equilibria enable us to prove the lower semicontinuity of the attractors at $\epsilon = 0$. For this we will use the convergence results of the previous sections and the convergence of the linearized semigroups proved next.

5.1. Continuity of the linearized semigroups

If $\bar{A}_\epsilon = A_\epsilon - f'(u_*^\epsilon)$, $0 < \epsilon \leqslant \epsilon_0$, it follows from Lemma 2.3 that $\bar{A}_\epsilon^{-1}$ converges compactly to $\bar{A}_0^{-1}$. Hence it follows that Theorem 2.3 holds with A_ϵ replaced by $\bar{A}_\epsilon$. Again, if Q_ϵ^+ is the projection determined by the part of the spectrum of $\bar{A}_\epsilon$ to the right of the imaginary axis and $\tilde{A}_\epsilon$ is the restriction of $\bar{A}_\epsilon$ to the kernel of Q_ϵ^+ then, from the compact convergence of resolvents we obtain a result analogous to Theorem 2.4 for $e^{\tilde{A}_\epsilon t}$.

Hereafter we assume that all equilibrium points of (6) with $\epsilon = 0$ are hyperbolic.

Proposition 5.1. *Let $\{u_*^\epsilon : \epsilon \in (0, \epsilon_0]\}$ be a sequence of solutions of* (1) *such that $\{u_*^\epsilon\}$ $X_\epsilon^{\frac{1}{2}}$–converges to u_*. If $\bar{A}_\epsilon = A_\epsilon - f'(u_*^\epsilon)$, then there exists $\bar{\epsilon} > 0$ such that $\sigma(-\bar{A}_\epsilon)$ doesn't intercept the imaginary axis, for $0 \leqslant \epsilon \leqslant \bar{\epsilon}$, and $\left\|(A_\epsilon - f'(u_*^\epsilon))^{-1}\right\| \leqslant C$, with C independent of ϵ. Moreover, if Q_ϵ^+ denotes the projection defined by spectral set $\sigma_\epsilon^+ = \left\{\mu \in \sigma(-\bar{A}_\epsilon) : \mathrm{Re}\mu > 0\right\}$, then Q_ϵ^+ converges compactly to Q_0^+ as $\epsilon \to 0$ and the family of sets σ_ϵ^+ is upper and lower semicontinuous at $\epsilon = 0$. Furthermore, there are $\beta > 0$ and $M \geqslant 1$ such that, for $\epsilon \in [0, \epsilon_0]$, we have*

$$\begin{aligned}&\left\|e^{-\bar{A}_\epsilon t} Q_\epsilon^+\right\|_{\mathcal{L}(X)} \leqslant M e^{\beta t}, \quad t \leqslant 0 \\ &\left\|e^{-\bar{A}_\epsilon t}(I - Q_\epsilon^+)\right\|_{\mathcal{L}(X, X_\epsilon^{\frac{1}{2}})} \leqslant M\, t^{-\frac{1}{2}} e^{-\beta t}, \quad t > 0. \qquad (1)\end{aligned}$$

5.2. Continuity of local unstable manifolds

In this section we consider the continuity of local unstable manifolds. Assume that $\{u_*^\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ is a sequence of solutions of (1) with $\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \to 0$ as $\epsilon \to 0$. It follows from the Proposition 5.1 that $\sigma(-\bar{A}_\epsilon)$ behave continuously as $\epsilon \to 0$.

To deal in a neighborhood of the equilibrium point u_*^ϵ, we rewrite the problem (1) as

$$\dot{w}^\epsilon + \bar{A}_\epsilon w^\epsilon = f(w^\epsilon + u_*^\epsilon) - f(u_*^\epsilon) - f'(u_*^\epsilon)w^\epsilon, \tag{2}$$

where $w^\epsilon = u^\epsilon - u_*^\epsilon$ and $\bar{A}_\epsilon = A_\epsilon - f'(u_*^\epsilon)$.

To obtain the projections Q_ϵ^+ we proceed on the following way: we denote by $\{\mu_i^\epsilon\}_{i=1}^\infty$ the eigenvalues of $\bar{A}_\epsilon$, $\epsilon \in [0,\epsilon_0]$ (ordered and counting multiplicity), and by $\{\phi_i^\epsilon\}_{i=1}^\infty$ a corresponding orthonormal system of eigenfunctions. We order the eigenvalues such that $\mu_1^\epsilon \geqslant \ldots \geqslant \mu_r^\epsilon > \omega > 0 > -\omega > \mu_{r+1}^\epsilon \geqslant \mu_{r+2}^\epsilon \geqslant \ldots$, $\epsilon \in [0,\bar{\epsilon}]$, for some $\beta > 0$ and $\bar{\epsilon} \in (0,\epsilon_0]$. We consider $W_\epsilon = [\phi_1^\epsilon, \ldots, \phi_r^\epsilon]$ and $W_\epsilon^\perp = \{\psi \in X_\epsilon^{\frac{1}{2}} : \int_\Omega \psi\phi = 0, \text{ for all } \phi \in W_\epsilon\}$. Let $Q_\epsilon^+ : X_\epsilon^{\frac{1}{2}} \to X_\epsilon^{\frac{1}{2}}$ be the orthogonal projections on W_ϵ, $\epsilon \in [0,\epsilon_0]$,

$$Q_\epsilon^+\psi = \sum_{i=1}^r \left(\int_\Omega \psi\phi_i^\epsilon\right)\phi_i^\epsilon.$$

If $\psi \in W_\epsilon$, then $\psi = \sum_{i=1}^r \left(\int_\Omega \psi\phi_i^\epsilon\right)\phi_i^\epsilon$ and

$$\|\psi\|_{X_\epsilon^{\frac{1}{2}}} = \left(\sum_{i=1}^r (1+\mu_i^\epsilon)\left(\int_\Omega \psi\phi_i^\epsilon\right)^2\right)^{\frac{1}{2}}.$$

Since the eigenvalues $\mu_i^\epsilon \to \mu_i^0$ as $\epsilon \to 0$ for $1 \leqslant i < \infty$, we have that W_ϵ is isomorphic to W_0, for $\epsilon \in (0,\bar{\epsilon}]$. Moreover, $W_\epsilon, \epsilon \in [0,\bar{\epsilon}]$, is isomorphic to $\mathbb{R}^r$ through the isomorphism I_ϵ given by

$$W_\epsilon \ni \psi \mapsto \left(\int_\Omega \psi\phi_1^\epsilon, \ldots, \int_\Omega \psi\phi_r^\epsilon\right) \in \mathbb{R}^r.$$

Note that I_ϵ is a bounded operator with bounded inverse I_ϵ^{-1} and that the norms of I_ϵ and I_ϵ^{-1} are uniformly bounded, for $\epsilon \in [0,\bar{\epsilon}]$.

This fact allows us to decompose $X_\epsilon^{\frac{1}{2}}$ in two spaces, $\mathbb{R}^r$ and $(I - Q_\epsilon^+)X_\epsilon^{\frac{1}{2}}$. This space decomposition induces a decomposition of the solution w^ϵ to (2), in the following way: $w^\epsilon = v^\epsilon + z^\epsilon$, with $v^\epsilon = \sum_{i=1}^r v_i\phi_i^\epsilon$, $v_i = \int_\Omega w\phi_i^\epsilon$ and $z^\epsilon = (I - Q_\epsilon^+)w^\epsilon$.

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Hence, since $\dot{v^\epsilon} = Q_\epsilon^+ \dot{w^\epsilon}$, $\dot{z^\epsilon} = (I - Q_\epsilon^+)\dot{w^\epsilon}$ and $Q_\epsilon^+, (I - Q_\epsilon^+)$ commute with $\bar{A}_\epsilon$, we have that

$$\dot{v}_i + \mu_i^\epsilon v_i = \int_\Omega [(f(v^\epsilon + z^\epsilon + u_*^\epsilon) - f(u_*^\epsilon) - f'(u_*^\epsilon)(v^\epsilon + z^\epsilon))]\phi_i^\epsilon$$

and

$$\begin{aligned} \dot{z}^\epsilon + \bar{A}_\epsilon z^\epsilon &= (I - Q_\epsilon^+)(f(v^\epsilon + z^\epsilon + u_*^\epsilon) - f(u_*^\epsilon) - f'(u_*^\epsilon)(v^\epsilon + z^\epsilon)) \\ &= f(w^\epsilon + u_*^\epsilon) - f(u_*^\epsilon) - f'(u_*^\epsilon)(w^\epsilon) \\ &- \sum_{i=1}^r \left(\int_\Omega [f(w^\epsilon + u_*^\epsilon) - f(u_*^\epsilon) - f'(u_*^\epsilon)(w^\epsilon)]\phi_i^\epsilon \right) \phi_i^\epsilon. \end{aligned} \tag{3}$$

We denote $v = (v_1, \ldots, v_r)^t$ and $H_\epsilon(v, z) = (H_1(v, z), \ldots, H_r(v, z))^t$, where

$$H_j(v, z) = \int_\Omega \left[f\left(\sum_{i=1}^r v_i \phi_i^\epsilon + z^\epsilon + u_*^\epsilon \right) - f(u_*^\epsilon) - f'(u_*^\epsilon) \left(\sum_{i=1}^r v_i \phi_i^\epsilon + z^\epsilon \right) \right] \phi_j^\epsilon$$

and also

$$G_\epsilon(v, z) = (I - Q_\epsilon^+)(f(v^\epsilon + z^\epsilon + u_*^\epsilon) - f(u_*^\epsilon) - f'(u_*^\epsilon)(v^\epsilon + z^\epsilon)),$$

it results that $H_\epsilon(0, 0) = 0 = G_\epsilon(0, 0)$. Furthermore, H_ϵ and G_ϵ are continuously differentiable and satisfy $\dfrac{\partial H_\epsilon}{\partial v^\epsilon}(0, 0) = 0 = \dfrac{\partial G_\epsilon}{\partial v^\epsilon}(0, 0)$. Since the projections Q_ϵ^+, $\epsilon \in (0, \epsilon_0]$, are uniformly bounded in Ω and the function f not depends of ϵ it follows that, given $\rho > 0$, there are $\bar{\epsilon} > 0$ and $\delta > 0$ such that if $\|v\|_{\mathbb{R}^r} + \|z\|_{X_\epsilon^{\frac{1}{2}}} < \delta$ and $\epsilon \leqslant \bar{\epsilon}$, we have

$$\begin{aligned} &\|H_\epsilon(v, z)\|_{\mathbb{R}^r} \leqslant \rho, \\ &\|G_\epsilon(v, z)\|_X \leqslant \rho, \\ &\|H_\epsilon(v, z) - H_\epsilon(\tilde{v}, \tilde{z})\|_{\mathbb{R}^r} \leqslant \rho \left(\|v - \tilde{v}\|_{\mathbb{R}^r} + \|z - \tilde{z}\|_{X_\epsilon^{\frac{1}{2}}} \right), \\ &\|G_\epsilon(v, z) - G_\epsilon(\tilde{v}, \tilde{z})\|_X \leqslant \rho \left(\|v - \tilde{v}\|_{\mathbb{R}^r} + \|z - \tilde{z}\|_{X_\epsilon^{\frac{1}{2}}} \right). \end{aligned} \tag{4}$$

The fact that we can choose ρ and δ uniformly for $\epsilon \in (0, \bar{\epsilon}]$ satisfying the inequalities above is the key point to obtain that the local unstable manifolds are defined in a small neighborhood of the equilibrium point u_*^ϵ, uniformly for $\epsilon \leqslant \bar{\epsilon}$.

We can extend H_ϵ and G_ϵ outside $B_\delta(u_*^\epsilon)$ in such a way that the bounds in (4) hold for any $v^\epsilon \in W$ and $z^\epsilon \in W_\epsilon^\perp$.

Denote by $\bar{A}_\epsilon^+ := \mathrm{diag}(\mu_1^\epsilon, \ldots, \mu_r^\epsilon)$ and $\bar{A}_\epsilon^- := \bar{A}_\epsilon|_{(I-Q_\epsilon^+)X_\epsilon^{\frac{1}{2}}}$. Then we can rewrite the equation (2) as

$$\begin{aligned} \dot{v}^\epsilon + \bar{A}_\epsilon^+ v^\epsilon &= H_\epsilon(v^\epsilon, z^\epsilon) \\ \dot{z}^\epsilon + \bar{A}_\epsilon^- z^\epsilon &= G_\epsilon(v^\epsilon, z^\epsilon), \end{aligned} \tag{5}$$

where H_ϵ and G_ϵ satisfy (4), for any $v^\epsilon \in \mathbb{R}^r$ and $z^\epsilon \in (I - Q_\epsilon^+)X_\epsilon^{\frac{1}{2}}$.

Moreover, for some positive M, β, independent of ϵ, $\epsilon \leqslant \bar{\epsilon}$, we have

$$\begin{aligned} \left\| e^{-\bar{A}_\epsilon^- t} z \right\|_{X_\epsilon^{\frac{1}{2}}} &\leqslant M e^{-\beta t} \left\| z \right\|_{X_\epsilon^{\frac{1}{2}}}, \quad t \geqslant 0, \\ \left\| e^{-\bar{A}_\epsilon^- t} z \right\|_{X_\epsilon^{\frac{1}{2}}} &\leqslant M t^{-\frac{1}{2}} e^{-\beta t} \left\| z \right\|_X, \quad t > 0, \\ \left\| e^{-\bar{A}_\epsilon^+ t} v \right\|_{\mathbb{R}^r} &\leqslant M e^{\beta t} \left\| v \right\|_{\mathbb{R}^r}, \quad t \leqslant 0. \end{aligned}$$

Next we show that for a suitable small $\rho > 0$ the an unstable manifold for u_*^ϵ is given by

$$\mathrm{W}_\epsilon^u = \left\{ (v, z) : z = \mathcal{S}_\epsilon^*(v),\ v \in \mathbb{R}^r \right\},$$

where $\mathcal{S}_\epsilon^* : \mathbb{R}^r \to (I - Q_\epsilon^+)X_\epsilon^{\frac{1}{2}}$ is bounded and Lipschitz continuous. Furthermore, for any $R > 0$,

$$\sup_{v \in B_{\mathbb{R}^r}(0,R)} \left\| \mathcal{S}_\epsilon^*(v) - \mathcal{S}_0^*(v) \right\|_{X_\epsilon^{\frac{1}{2}}} \xrightarrow{\epsilon \to 0} 0.$$

Proposition 5.2. *Let u_* be a hyperbolic equilibrium of* (2). *Then by proposition 4.3,* (1) *has only one equilibrium solution, u_*^ϵ, near u_*. For $D > 0$, $\Delta > 0$, $0 < \vartheta < 1$ given, let $\rho_0 > 0$ be such that, for all $0 < \rho \le \rho_0$,*

$$\begin{aligned} &\rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \leqslant D \\ &\rho M^2 (1 + \Delta) \left(\frac{\beta}{2}\right)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \leqslant \Delta \\ &\frac{\beta}{2} \leqslant 2\beta - \rho M (1 + \Delta) \leqslant 2\beta \\ &\rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \left[1 + \frac{\rho M (1 + \Delta)}{2\beta - \rho M (1 + \Delta)} \right] \leqslant \vartheta < 1 \\ &\gamma := \beta - \left[\rho M + \frac{\rho^2 M^2 (1 + \Delta)(1 + M)}{2\beta - \rho M (1 + \Delta)} \right] > 0 \\ &\rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) + \left[\rho^2 M^2 \beta^{-1} (1 + \Delta) \left(2\beta - \rho M (1 + \Delta) \right)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \right] \leqslant \frac{1}{2}. \end{aligned} \tag{6}$$

For the above choice of ρ, assume that H_ϵ and G_ϵ satisfy (4) *for all* $(v,z) \in \mathbb{R}^r \times (I - Q_\epsilon^+)X_\epsilon^{\frac{1}{2}}$. *Then, there exists* $\mathcal{S}_\epsilon^* : W \to (I - Q_\epsilon^+)X_\epsilon^{\frac{1}{2}}$ *such that*

$$\|\!|\mathcal{S}_\epsilon^*\|\!| := \sup_{v\in\mathbb{R}^r} \|\mathcal{S}_\epsilon^*(v)\|_{X_\epsilon^{\frac{1}{2}}} \leqslant D$$
$$\|\mathcal{S}_\epsilon^*(v) - \mathcal{S}_\epsilon^*(\tilde{v})\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \Delta \|v - \tilde{v}\|_{R^r}$$

and

$$\mathrm{W}_\epsilon^u = \{(v,z) : z = \mathcal{S}_\epsilon^*(v),\ v \in \mathbb{R}^r\}.$$

Furthermore, for any $R > 0$,

$$\sup_{v\in B_{\mathbb{R}^r}(0,R)} \|\mathcal{S}_\epsilon^*(v) - \mathcal{S}_0^*(v)\|_{X_\epsilon^{\frac{1}{2}}} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Proof: For the proof of existence and exponentially attracting unstable manifold we follow closely the proofs of existence of exponentially attracting invariant manifolds in [9], Chapter 6. For the continuity of unstable manifolds we follow [1]. Let $v^\epsilon(t) = \psi(t,\tau,\eta,\mathcal{S}_\epsilon)$ be the solution of

$$\dot{v^\epsilon} + \bar{A}_\epsilon^+ v^\epsilon = H_\epsilon(v^\epsilon, \mathcal{S}_\epsilon(v^\epsilon)),\ \text{for } t < \tau,\ v^\epsilon(\tau) = \eta. \tag{7}$$

Consider

$$\mathcal{X} = \left\{ \mathcal{S} : \mathbb{R}^r \to (I - Q_\epsilon^+)X_\epsilon^{\frac{1}{2}} : \|\!|\mathcal{S}\|\!| \leqslant D,\ \|\mathcal{S}_\epsilon(v) - \mathcal{S}_\epsilon(\tilde{v})\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \Delta \|v - \tilde{v}\|_{\mathbb{R}^r} \right\}.$$

It is easy to see that $(\mathcal{X}, \|\!|\cdot\|\!|)$ is a complete metric space.

Let $\mathcal{S}_\epsilon \in \mathcal{X}$ and define $\Phi : \mathcal{X} \to \mathcal{X}$ by

$$\Phi(\mathcal{S}_\epsilon)(\eta) = \int_{-\infty}^{\tau} e^{-\bar{A}_\epsilon^-(\tau - s)} G_\epsilon(v^\epsilon(s), \mathcal{S}_\epsilon(v^\epsilon(s)))ds. \tag{8}$$

It is not difficult to see that

$$\|\Phi(\mathcal{S}_\epsilon)(\cdot)\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \leqslant D. \tag{9}$$

and that, whenever $\eta, \tilde{\eta} \in \mathbb{R}^r$, $\mathcal{S}_\epsilon, \tilde{\mathcal{S}}_\epsilon \in \mathcal{X}$

$$\left\|\Phi(\mathcal{S}_\epsilon)(\eta) - \Phi(\tilde{\mathcal{S}}_\epsilon)(\tilde{\eta})\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \left[1 + \frac{\rho M(1+\Delta)}{2\beta - \rho M(1+\Delta)}\right] \|\!|\mathcal{S}_\epsilon - \tilde{\mathcal{S}}_\epsilon\|\!|$$
$$+ \rho M^2 (1+\Delta) \left(\frac{\beta}{2}\right)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \|\eta - \tilde{\eta}\|_{\mathbb{R}^r}.$$

From our hypothesis on ρ, we have that

$$\left\|\Phi(\mathcal{S}_\epsilon)(\eta) - \Phi(\tilde{\mathcal{S}}_\epsilon)(\tilde{\eta})\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \Delta \|\eta - \tilde{\eta}\|_{\mathbb{R}^r} + \vartheta \, |\!|\!| \mathcal{S}_\epsilon - \tilde{\mathcal{S}}_\epsilon |\!|\!| \, . \tag{10}$$

From inequalities (9) and (10) it follows that Φ is a contraction map from $\mathcal{X}$ into itself. Hence, there is only one fixed point $\mathcal{S}_\epsilon^* = \Phi(\mathcal{S}_\epsilon^*)$ in $\mathcal{X}$.

It remains to show that $\mathrm{W}_\epsilon^u = \{(v, \mathcal{S}_\epsilon^*(v)) : v \in \mathbb{R}^r\}$ is an invariant manifold for (5). Let $(v_0^\epsilon, z_0^\epsilon) \in \mathrm{W}_\epsilon^u$, $z_0^\epsilon = \mathcal{S}_\epsilon^*(v_0^\epsilon)$. Denote by $v_*^\epsilon(t)$ the solution of the initial value problem

$$\begin{cases} \dot{v}^\epsilon + \bar{A}_\epsilon^+ v^\epsilon = H_\epsilon(v^\epsilon, \mathcal{S}_\epsilon^*(v^\epsilon)) \\ v^\epsilon(0) = v_0^\epsilon. \end{cases}$$

This defines a curve $(v_\epsilon^*(t), \mathcal{S}_\epsilon^*(v_\epsilon^*(t))) \in \mathrm{W}_\epsilon^u$, $t \in \mathbb{R}$. However, the only solution of the equation

$$\dot{z}^\epsilon + \bar{A}_\epsilon^- z^\epsilon = G_\epsilon(v_*^\epsilon(t), \mathcal{S}_\epsilon^*(v_*^\epsilon(t)))$$

which remains bounded as $t \to -\infty$ is

$$z_*^\epsilon(t) = \int_{-\infty}^t e^{-\bar{A}_\epsilon^-(t-s)} G_\epsilon(v_*^\epsilon(s), \mathcal{S}_\epsilon^*(v_*^\epsilon(s)) ds = \mathcal{S}_\epsilon^*(v_*^\epsilon(t)).$$

Therefore, $(v_\epsilon^*(t), \mathcal{S}_\epsilon^*(v_\epsilon^*(t)))$ is a solution of (5) through $(v_0^\epsilon, z_0^\epsilon)$, and it proves the invariance.

In addition, if $(v^\epsilon(t), z^\epsilon(t)) \in W_\epsilon^u$ for all $t \in \mathbb{R}$, then

$$\|z^\epsilon(t) - \mathcal{S}_\epsilon^*(v^\epsilon(t))\|_{X_\epsilon^{\frac{1}{2}}} \leqslant M e^{-\gamma(t-t_0)} \|z^\epsilon(t_0) - \mathcal{S}_\epsilon^*(v^\epsilon(t_0))\|_{X_\epsilon^{\frac{1}{2}}}, \quad t_0 \leqslant t. \tag{11}$$

Consequently, making $t_0 \to -\infty$, $z^\epsilon(t) = \mathcal{S}_\epsilon^*(v^\epsilon(t))$ for all $t \in \mathbb{R}$. This shows that $W_\epsilon^u = \{(v, \mathcal{S}_\epsilon^*(v)) : v \in \mathbb{R}^r\}$.

Next we show that the fixed points $\mathcal{S}_\epsilon^*$ depend continuously upon ϵ at $\epsilon = 0$, that is, if $0 < \epsilon \leqslant \bar{\epsilon}$ is such that the unstable manifold os given by the graph of $\mathcal{S}_\epsilon^*$, $0 \leqslant \epsilon \leqslant \bar{\epsilon}$, we want to show that

$$|\!|\!| \mathcal{S}_\epsilon^* - \mathcal{S}_0^* |\!|\!| := \sup_{\eta \in B_{\mathbb{R}^r}(0,R)} \|\mathcal{S}_\epsilon^*(\eta) - \mathcal{S}_0^*(\eta)\|_{X_\epsilon^{\frac{1}{2}}} \to 0 \text{ as } \epsilon \to 0.$$

First note that $\mathbb{R}^r \ni v^0 \mapsto G_0(v^0, \mathcal{S}_0^*(v^0)) \in X_0$ takes bounded subsets of $\mathbb{R}^r$ into relatively compact subsets of X_0 since it is a continuous map.

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Therefore,

$$\begin{aligned}\|\mathcal{S}_\epsilon^*(\eta)-\mathcal{S}_0^*(\eta)\|_{X_\epsilon^{\frac{1}{2}}} &\leqslant \int_{-\infty}^{\tau}\left\|e^{-\bar{A}_\epsilon^-(\tau-s)}G_\epsilon(v^\epsilon,\mathcal{S}_\epsilon^*(v^\epsilon))-e^{-\bar{A}_0^-(\tau-s)}G_0(v^0,\mathcal{S}_0^*(v^0))\right\|_{X_\epsilon^{\frac{1}{2}}}ds\\
&\leqslant \int_{-\infty}^{\tau}\left\|e^{-\bar{A}_\epsilon^-(\tau-s)}G_\epsilon(v^\epsilon,\mathcal{S}_\epsilon^*(v^\epsilon))-e^{-\bar{A}_\epsilon^-(\tau-s)}G_\epsilon(v^0,\mathcal{S}_0^*(v^0))\right\|_{X_\epsilon^{\frac{1}{2}}}ds\\
&+\int_{-\infty}^{\tau}\left\|e^{-\bar{A}_\epsilon^-(\tau-s)}G_\epsilon(v^0,\mathcal{S}_0^*(v^0))-e^{-\bar{A}_\epsilon^-(\tau-s)}G_0(v^0,\mathcal{S}_0^*(v^0))\right\|_{X_\epsilon^{\frac{1}{2}}}ds\\
&+\int_{-\infty}^{\tau}\left\|e^{-\bar{A}_\epsilon^-(\tau-s)}G_0(v^0,\mathcal{S}_0^*(v^0))-e^{-\bar{A}_0^-(\tau-s)}G_0(v^0,\mathcal{S}_0^*(v^0))\right\|_{X_\epsilon^{\frac{1}{2}}}ds.\end{aligned}$$

Denoting the last three integrals by I_1, I_2 and I_3, respectively, we have:

$$\begin{aligned}I_1 &\leqslant \int_{-\infty}^{\tau}Me^{-\beta(\tau-s)}(\tau-s)^{-\frac{1}{2}}\rho[(1+\Delta)\left\|v^\epsilon-v^0\right\|_{\mathbb{R}^r}+|||\mathcal{S}_\epsilon^*-\mathcal{S}_0^*|||]\,ds\\
&\leqslant \rho M\beta^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)|||\mathcal{S}_\epsilon^*-\mathcal{S}_0^*|||+\rho M(1+\Delta)\int_{-\infty}^{\tau}e^{-\beta(\tau-s)}(\tau-s)^{-\frac{1}{2}}\left\|v^\epsilon(s)-v^0(s)\right\|_{\mathbb{R}^r}ds.\end{aligned}$$

Since $Q_\epsilon^+ \overset{\epsilon\to 0}{\longrightarrow} Q_0^+$ uniformly on compact sets, we have that $G_\epsilon \overset{\epsilon\to 0}{\longrightarrow} G_0$ uniformly for η in bounded subsets of $\mathbb{R}^r$ and I_2 goes to 0 as $\epsilon\to 0$ uniformly for η in bounded subsets of $\mathbb{R}^r$. Moreover, it follows from Theorem 2.4 that I_3 is also $o(1)$ as $\epsilon\to 0$ uniformly for η in bounded subsets of $\mathbb{R}^r$. Therefore,

$$\begin{aligned}\|\mathcal{S}_\epsilon^*(\eta)-\mathcal{S}_0^*(\eta)\|_{X_\epsilon^{\frac{1}{2}}} &\leqslant o(1)+\rho M\beta^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)|||\mathcal{S}_\epsilon^*-\mathcal{S}_0^*|||\\
&+\rho M(1+\Delta)\int_{-\infty}^{\tau}e^{-\beta(\tau-s)}(\tau-s)^{-\frac{1}{2}}\left\|v^\epsilon(s)-v^0(s)\right\|_{\mathbb{R}^r}ds.\end{aligned}$$

Thus, it is enough to estimate $\left\|v^\epsilon(t)-v^0(t)\right\|_{\mathbb{R}^r}$. Since $\|\eta\|_{\mathbb{R}^r}\leqslant R$, working as above we obtain

$$\begin{aligned}\left\|v^\epsilon(t)-v^0(t)\right\|_{\mathbb{R}^r} &\leqslant \left\|e^{-\bar{A}_\epsilon^+(t-\tau)}\eta-e^{-\bar{A}_0^+(t-\tau)}\eta\right\|_{\mathbb{R}^r}\\
&+\int_t^{\tau}\left\|e^{-\bar{A}_\epsilon^+(t-s)}H_\epsilon(v^\epsilon,\mathcal{S}_\epsilon^*(v^\epsilon))-e^{-\bar{A}_\epsilon^+(t-s)}H_\epsilon(v^0,\mathcal{S}_0^*(v^0))\right\|_{\mathbb{R}^r}ds\\
&+\int_t^{\tau}\left\|e^{-\bar{A}_\epsilon^+(t-s)}H_\epsilon(v^0,\mathcal{S}_0^*(v^0))-e^{-\bar{A}_\epsilon^+(t-s)}H_0(v^0,\mathcal{S}_0^*(v^0))\right\|_{\mathbb{R}^r}ds\\
&+\int_t^{\tau}\left\|e^{-\bar{A}_\epsilon^+(t-s)}H_0(v^0,\mathcal{S}_0^*(v^0))-e^{-\bar{A}_0^+(t-s)}H_0(v^0,\mathcal{S}_0^*(v^0))\right\|_{\mathbb{R}^r}ds\\
&\leqslant o(1)+\rho M\int_t^{\tau}e^{\beta(t-s)}ds|||\mathcal{S}_\epsilon^*-\mathcal{S}_0^*|||\\
&+\rho M(1+\Delta)\int_t^{\tau}e^{\beta(t-s)}\left\|v^\epsilon-v^0\right\|_{\mathbb{R}^r}ds.\end{aligned}$$

Let $\phi(t) = e^{\beta(\tau-t)} \left\| v^\epsilon(t) - v^0(t) \right\|_{\mathbb{R}^r}$. Then

$$\phi(t) \leqslant o(1) + \rho M \int_t^\tau e^{\beta(\tau-s)} ds |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| + \rho M(1+\Delta) \int_t^\tau \phi(s) ds,$$

and we obtain

$$\left\| v^\epsilon(t) - v^0(t) \right\|_{\mathbb{R}^r} \leqslant \left[o(1) + \rho M \beta^{-1} \, |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| \right] e^{[\rho M(1+\Delta)-\beta](\tau-t)}.$$

Therefore

$$\begin{aligned}
\|\mathcal{S}_\epsilon^*(\eta) - \mathcal{S}_0^*(\eta)\|_{X_\epsilon^{\frac{1}{2}}} \leqslant\ & o(1) + \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| \\
& + \rho M(1+\Delta) \int_{-\infty}^\tau e^{[-2\beta+\rho M(1+\Delta)](\tau-s)} (\tau-s)^{-\frac{1}{2}} ds \\
& \quad \times \left[o(1) + \rho M \beta^{-1} |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| \right] \\
\leqslant\ & o(1) + \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| \\
& + \left[\rho M(1+\Delta) \left(2\beta - \rho M(1+\Delta)\right)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \rho M \beta^{-1} \right] |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*|||.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
|||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| \leqslant\ & o(1) \\
& + \left\{ \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) + \left[\rho^2 M^2 \beta^{-1} (1+\Delta) \left(2\beta - \rho M(1+\Delta)\right)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \right] \right\} |||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*|||,
\end{aligned}$$

and $|||\mathcal{S}_\epsilon^* - \mathcal{S}_0^*||| \to 0$ as $\epsilon \to 0$. ∎

COROLLARY 5.1. *There is a $\delta > 0$ such that*

$$\mathrm{W}_{\epsilon,\mathrm{loc}}^u = \{(v,z) \in \mathbb{R}^r \times (I - Q_\epsilon^+) X_\epsilon^{\frac{1}{2}} : (v,z) \in W_\epsilon^u,\ \|v\|_{\mathbb{R}^r} + \|z\|_{X_\epsilon^{\frac{1}{2}}} < \delta\}$$

is given by

$$\mathrm{W}_{\epsilon,\mathrm{loc}}^u = \{(v, \mathcal{S}_\epsilon^*(v)) : v \in \mathbb{R}^r\} \cap \{(v,z) \in \mathbb{R}^r \times (I - Q_\epsilon^+) X_\epsilon^{\frac{1}{2}} : \|v\|_{\mathbb{R}^r} + \|z\|_{X_\epsilon^{\frac{1}{2}}} < \delta\}.$$

5.3. Lower semicontinuity of attractors

As in [4] we see that $\{T_0(t) : t \geqslant 0\}$ is a gradient semigroup and since all of its equilibrium points are hyperbolic there are only a finite number of them and the attractor $\mathcal{A}_0$ is the union of its unstable manifolds.

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THEOREM 5.1. *The family of attractors $\{\mathcal{A}_\epsilon : \epsilon \in (0, \epsilon_0]\}$ is lower semicontinuous at $\epsilon = 0$.*

Proof: Let $u \in \mathcal{A}_0$. As $T_0(t)$ is a gradient system, we have $\mathcal{A}_0 = \bigcup_{w_* \in \mathcal{E}_0} \mathrm{W}^u(w_*)$, and then $u \in \mathrm{W}^u(u_*)$, for some $u_* \in \mathcal{E}_0$. Let $\tau \in \mathbb{R}$ and $v \in \mathrm{W}^u_{\mathrm{loc}}(u_*)$ such that $T_0(\tau)v = u$. Let u_*^ϵ such that $u_*^\epsilon \to u_*$. ¿From convergence of unstable manifolds there is a sequence $\{v^\epsilon\}$, $v^\epsilon \in \mathrm{W}^u_{\mathrm{loc}}(u_*^\epsilon)$ such that $v^\epsilon \to v$ as $\epsilon \to 0$. Finally, from continuity of nonlinear semigroups, we obtain $T_\epsilon(\tau)v^\epsilon \to T_0(\tau)v = u$. To conclude, we observe that if $u^\epsilon = T_\epsilon(\tau)v^\epsilon$, then $u^\epsilon \in \mathcal{A}_\epsilon$, since $v^\epsilon \in \bigcup_{u_*^\epsilon \in \mathcal{E}_\epsilon} \mathrm{W}^u(u_*^\epsilon) = \mathcal{A}_\epsilon$ and $\mathcal{A}_\epsilon$ is invariant. ∎

COROLLARY 5.2. *The family of attractors $\{\mathcal{A}_\epsilon \colon \epsilon \in [0, \epsilon_0]\}$ is continuous at $\epsilon = 0$.*

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