CADERNOS DE MATEMÁTICA **07**, 155–167 May (2006)
ARTIGO NÚMERO SMA# 260

$\mathcal{A}_e$-codimension of germs of analytic monomial curves

Marcelo Escudeiro Hernandes *

Departamento de Matemática, Universidade Estadual de Maringá, Av. Colombo 5790 Maringá-PR 87020-900, Brazil
E-mail: mehernandes@uem.br

Maria Elenice Rodrigues Hernandes †

Departamento de Matemática, Universidade Estadual de Maringá, Av. Colombo 5790 Maringá-PR 87020-900, Brazil
E-mail: merhernandes@uem.br

Maria Aparecida Soares Ruas ‡

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: maasruas@icmc.usp.br

This paper deals with the local study of invariants of analytic irreducible curve singularities. We obtain an algebraic description of the $\mathcal{A}_e$-codimension of monomial curves in terms of classical invariants of the theory of curves, like the delta invariant, the Tjurina number, and the Cohen-Macaulay type of the local ring of the curve. May, 2006 ICMC-USP

1. INTRODUCTION

In the study of complete intersection analytic curves, the dimension τ of the base of a miniversal deformation, called the Tjurina number, plays a similar role to the $\mathcal{A}_e$-codimension of parameterized curve singularities. There are many results relating the Tjurina number with classical analytic invariants, like the Milnor number, the Cohen-Macaulay type and others, see [7] and [8] for example. On the other hand, recent results in singularity theory relate the $\mathcal{A}_e$-codimension of an analytic map-germ $\phi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ and the number of 0-dimensional singularities appearing in a stable perturbation ϕ_t of ϕ (see [5, 10, 11]). The number of isolated stable singularities of a given type of ϕ_t is an ana-

* Research Partially Supported by CAPES/PROCAD and CNPq, Brazil
† Research Supported by CNPq, Brazil
‡ Research Partially Supported by CAPES/PROCAD, CNPq and FAPESP, Brazil

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

lytic invariant of the germ ϕ, named by D. Mond, 0-*stable invariant.* In [11], he studies the topology of a stable perturbation of an $\mathcal{A}$-finitely determined germ $\phi : \mathbb{C}^n, S \to \mathbb{C}^{n+1}, 0$ where S is a finite set of points, and its relation with the $\mathcal{A}_e$-codimension of ϕ. When $n = 1$ there is only one 0-stable invariant, namely, the invariant δ, measuring the number of double points in a stable perturbation of ϕ. In this case, Mond proves that if ϕ is $1-1$ outside S then $\mathcal{A}_e\mathrm{cod}(\phi) \leq \delta - r + 1$, where r is the number of branches of the curve, and the equality holds if and only if the map-germ ϕ is quasihomogeneous (see [10]). This geometric approach does not hold for curves $\phi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$ with $n \geq 2$, since in these dimensions ϕ is stable if and only if it has no singularities.

Our approach is algebraic: we relate the $\mathcal{A}_e$-tangent space of the germ of an analytic irreducible curve parameterized by $\phi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$, $n \geq 1$, with the module of the Kähler differentials of the local ring of the curve and thus we obtain a formula for the $\mathcal{A}_e$-codimension of ϕ, when ϕ is monomial, in terms of the delta invariant of the curve and its Cohen-Macaulay type. As a consequence of this result, we obtain a characterization of Gorensteinness in terms of the $\mathcal{A}_e$-codimension and its relation with the Tjurina number. In particular, for monomial curves in $\mathbb{C}^3$, the property of complete intersection follows from the parity of the $\mathcal{A}_e$-codimension of its parametrization.

2. NOTATIONS

We denote by $\mathbb{C}\{X_0, \ldots, X_n\}$ the ring of absolutely convergent power series in some neighborhood of the origin. Let $f_1, \ldots, f_r \in \mathbb{C}\{X_0, \ldots, X_n\}$ such that $f_i(0) = 0$ for $i = 1, \ldots, r$ and $\langle f_1, \ldots, f_r\rangle$ is a prime ideal of $\mathbb{C}\{X_0, \ldots, X_n\}$.

An *analytic irreducible curve* in $\mathbb{C}^{n+1}$, defined by $f_1, \ldots, f_r$ is the set $\mathcal{C} = \{X \in V \subset \mathbb{C}^{n+1};\ f_1(X) = \cdots = f_r(X) = 0\}$ in some neighborhood V of the origin, such that the ring

$$\mathcal{O} = \frac{\mathbb{C}\{X_0, \ldots, X_n\}}{\langle f_1, \ldots, f_r\rangle}$$

is a domain whose Krull dimension is 1. The ring $\mathcal{O}$ is called the *local ring* of the curve $\mathcal{C}$ and its maximal ideal will be denoted by $\mathcal{M}$. Notice that $\mathcal{O} = \mathbb{C}\{x_0, \ldots, x_n\}$, where $x_i = X_i + \langle f_1, \ldots, f_r\rangle$. In what follows, the origin in $\mathbb{C}^{n+1}$ is an isolated singularity of the curve and we will consider germs of analytic curves.

Let K be the quotient field of $\mathcal{O}$ and $\overline{\mathcal{O}}$ the integral closure of $\mathcal{O}$ in K. It is known that $\overline{\mathcal{O}} \simeq \mathbb{C}\{t\}$. From the inclusion of $\mathcal{O}$ in $\overline{\mathcal{O}}$, we obtain a monomorphism of $\mathbb{C}$-algebras, $\varphi : \mathcal{O} \to \overline{\mathcal{O}}$ with $x_i \mapsto \phi_i(t)$ where $\phi_i(t) \in \mathcal{M}_t \setminus \{0\}$, for $i = 0, 1, \ldots, n$ and $\mathcal{M}_t$ is the maximal ideal of $\mathbb{C}\{t\}$. Hence $\phi(t) = (\phi_0(t), \phi_1(t), \ldots, \phi_n(t))$ is a parametrization of the curve $\mathcal{C}$ in some neighborhood of 0. Using a particular change of coordinates, we obtain a parametrization of the form

$$\phi : \begin{cases} x_0 &= t^{v_0} \\ x_1 &= t^{v_1} + \sum_{i>v_1} a_{i1}t^i \\ &\vdots \\ x_n &= t^{v_n} + \sum_{i>v_n} a_{in}t^i, \end{cases}$$

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with $v_0 < v_1 < \cdots < v_n$, where we identify x_i with its image $\varphi(x_i) = \phi_i(t)$. The integer v_0 is the *multiplicity* of the curve. We clearly have $\mathcal{O} \simeq \varphi(\mathcal{O}) = \mathbb{C}\{t^{v_0}, \phi_1(t), \ldots, \phi_n(t)\} \subset \mathbb{C}\{t\}$. In what follows we consider only primitive parametrizations.

We now fix some further notation. Let $\mathcal{O}(n,p)$ be the space of analytic map-germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p$. We denote $\mathcal{O}(n,1) = \mathcal{O}_n$, which is a local ring, with maximal ideal $\mathcal{M}_n$.

For the Mather groups $\mathcal{A}$ and $\mathcal{K}$ (of left-right and of contact equivalence, respectively) acting on the space of smooth map-germs we can define the tangent spaces to the $\mathcal{A}$ and $\mathcal{K}$-orbits as follows.

Let $\theta(f) = \{\sigma : \mathbb{C}^n, 0 \to T\mathbb{C}^p;$ germs of vector fields along $f\}$. Similarly we define $\theta(n)$ and $\theta(p)$ as the sets of germs of vector fields along the identity in $\mathbb{C}^n$ and $\mathbb{C}^p$, respectively.

Let $tf : \theta(n) \to \theta(f)$ be such that $tf(s) = df(s)$, where df is the differential of f and $wf : \theta(p) \to \theta(f)$ such that $wf(\eta) = \eta \circ f$. The *extended tangent space* to the orbit of a germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ by the action of the group $\mathcal{A}$ (respectively, $\mathcal{K}$) is defined by $T\mathcal{A}_e(f) = tf(\theta(n)) + wf(\theta(p))$ (respectively, $T\mathcal{K}_e(f) = tf(\theta(n)) + f^*(\mathcal{M}_p)\theta(f)$), where the map $f^* : \mathcal{O}_p \to \mathcal{O}_n$ is such that $f^*(\eta) = \eta \circ f$.

For $\mathcal{G} = \mathcal{A}$ or $\mathcal{K}$, we define the *$\mathcal{G}_e$-codimension of* f, denoted by $\mathcal{G}_e\mathrm{cod}(f)$, as the $\mathbb{C}$-dimension of the quotient $\theta(f)/T\mathcal{G}_e(f)$.

A basic reference for these concepts is the survey on determinacy [15] by Wall.

2.1. Invariants

Let $\mathcal{C}$ be the germ of a reduced curve in $\mathbb{C}^n$ with isolated singularity at the origin. The *Tjurina number* τ is the $\mathbb{C}$-dimension of the space T^1 of first order infinitesimal deformations of $\mathcal{C}$. If $V_0 = f^{-1}(0)$ is an analytic subvariety of complete intersection, then $\tau = \mathcal{K}_e\mathrm{cod}(f)$ (see [14]). In this case, the Tjurina number is the minimum number of parameters in a $\mathcal{K}_e$-versal deformation of f. Similarly the *$\mathcal{A}_e$-codimension* of a parametrization $\phi : \mathbb{C}, 0 \to \mathbb{C}^n, 0$ is the minimum number of parameters in an $\mathcal{A}_e$-versal unfolding of ϕ (see [9]).

Bruce and Gaffney in [2] obtain some relations between the parametrization $\phi : \mathbb{C}, 0 \to \mathbb{C}^2, 0$ of germs of irreducible analytic plane curves and the defining equation $f = 0$ of these curves.

The *semigroup of values* of an analytic irreducible curve $\mathcal{C}$ is defined by

$$\Gamma = \{\nu(g);\ g \in \mathcal{O} \setminus \{0\}\},$$

where $\nu(g) = \nu(g(t))$, $g(t) = \varphi(g)$ and ν is the natural valuation of $\mathbb{C}\{t\}$. The semigroup Γ has a conductor, that is, there exists $c \in \Gamma$ such that $c - 1 \notin \Gamma$ and if $n \in \mathbb{N}$ and $n \geq c$ then $n \in \Gamma$. Therefore, the set $L = \mathbb{N} \setminus \Gamma$ is finite and it is called the set of *gaps of the semigroup* Γ. Since the semigroup Γ has a conductor, it is finitely generated, and in this case, if $\{v_0, v_1, \ldots, v_g\}$ is a set of generators of Γ, we write $\Gamma = \langle v_0, v_1, \ldots v_g \rangle$.

The *delta invariant* of $\mathcal{C}$ is defined by $\delta = \dim_{\mathbb{C}} \dfrac{\overline{\mathcal{O}}}{\mathcal{O}}$, and is exactly the number of the gaps of the semigroup Γ.

3. $\mathcal{A}_E$-CODIMENSION OF PARAMETERIZED CURVES

D. Mond in [10] proves that if $\phi : \mathbb{C}, S \to \mathbb{C}^2, 0$ is $1-1$ outside S, where S is a finite set, then $\mathcal{A}_e\mathrm{cod}(\phi) \leq \delta - r + 1$, where r is the number of branches of the curve, and the equality holds if and only if ϕ is quasihomogeneous. In this section we present an upper bound for the $\mathcal{A}_e\mathrm{cod}(\phi)$ where $\phi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$ is an irreducible parametrization of the curve $\mathcal{C}$ and we give a relation between the elements of the tangent space to the $\mathcal{A}$-orbit of ϕ and some Kähler differentials.

Notice that $(w_0, w_1, \ldots, w_n) \in T\mathcal{A}_e(\phi)$ if and only if there exists ϵ in $\mathcal{O}_1$ and η_i in $\mathcal{O}_{n+1}$ for $i = 0, 1, \ldots, n$ such that

$$\left\{ \begin{array}{l} w_0(t) = dx_0(t) \cdot \epsilon(t) + \eta_0(\phi(t)) \\ w_1(t) = dx_1(t) \cdot \epsilon(t) + \eta_1(\phi(t)) \\ \qquad \vdots \\ w_n(t) = dx_n(t) \cdot \epsilon(t) + \eta_n(\phi(t)). \end{array} \right. \tag{1}$$

Let $(\phi_0, \psi) = (\phi_0, \phi_1, \ldots, \phi_n) = \phi$ where $\psi = (\phi_1, \ldots, \phi_n)$; it is easy to see that $\theta(\phi) \simeq \theta(\phi_0) \oplus \theta(\psi)$. We consider the following subspaces

$$T_0 \;=\; \{\ w_0 \in \theta(\phi_0);\quad w_0 = dx_0 \cdot \epsilon + \eta_0(\phi),\ \epsilon \in \mathcal{O}_1,\ \eta_0 \in \mathcal{O}_{n+1}\ \} \quad \text{and}$$

$$T \;=\; \{\ (w_1, \ldots, w_n) \in \theta(\psi);\quad (0, w_1, \ldots, w_n) \in T\mathcal{A}_e(\phi)\ \}.$$

The codimensions of T_0 and T are defined by cod $T_0 = \dim_{\mathbb{C}} \dfrac{\theta(\phi_0)}{T_0}$ and cod $T = \dim_{\mathbb{C}} \dfrac{\theta(\psi)}{T}$, respectively.

By Mather's infinitesimal criterion for finite determinacy (see [15]), ϕ is $\mathcal{A}$-finitely determined, since $\mathcal{M}_1^k\theta(\phi) \subset T\mathcal{A}_e(\phi)$, for all $k \geq c$, where c is the conductor of the semigroup of $\mathcal{C}$. If $(w_0, w_1, \ldots, w_n) \in \theta(\phi)$, we denote by $[(w_0, w_1, \ldots, w_n)]$ the class of this element in the quotient $\theta(\phi)/T\mathcal{A}_e(\phi)$, similarly for $\theta(\phi_0)$ and $\theta(\psi)$.

Proposition 3.1. *If $\mathcal{C}$ is an analytic irreducible curve in $\mathbb{C}^{n+1}$ given by a parametrization ϕ, then $\mathcal{A}_e\mathrm{cod}(\phi) = \mathrm{cod}\ T_0 + \mathrm{cod}\ T$.*

Proof: In fact, the sequence

$$0 \longrightarrow \frac{\theta(\psi)}{T} \xrightarrow{i^*} \frac{\theta(\phi)}{T\mathcal{A}_e(\phi)} \xrightarrow{\pi^*} \frac{\theta(\phi_0)}{T_0} \longrightarrow 0,$$

is exact, where i^* and π^* are defined by $i^*([(w_1, \ldots, w_n)]) = [(0, w_1, \ldots, w_n)]$ and $\pi^*([(w_0, w_1, \ldots, w_n)]) = [w_0]$. □

Remark 3. 1. Notice that cod $T_0 = v_0 - 2$, where v_0 is the multiplicity of the curve $\mathcal{C}$. Moreover, $\theta(\psi) \simeq \bigoplus_{i=1}^n \overline{\mathcal{O}}$, thus $\theta(\psi)/T$ is isomorphic to a submodule of $\bigoplus_{i=1}^n \frac{\overline{\mathcal{O}}}{\mathcal{O}}$. Therefore cod $T \leq n\delta$.

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In this way we have the following result:

PROPOSITION 3.2. *If $\mathcal{C}$ is an analytic irreducible curve in $\mathbb{C}^{n+1}$, with multiplicity v_0 and ϕ is a parametrization of $\mathcal{C}$ then*

$$\mathcal{A}_e\mathrm{cod}(\phi) \;\leq\; n\delta + v_0 - 2.$$

3.1. $T\mathcal{A}_e(\phi)$ and the Kähler differentials

Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ with local ring $\mathcal{O}$. The *module of Kähler differentials* associated to the curve $\mathcal{C}$ is the $\mathcal{O}$-module generated by the differentials $dx_0, \ldots, dx_n$ and denoted by $\mathcal{O}d\mathcal{O} = \mathcal{O}dx_0 + \mathcal{O}dx_1 + \cdots + \mathcal{O}dx_n$. Let Ψ be the homomorphism of $\mathcal{O}$-modules

$$\begin{array}{rcl} \mathcal{O}dx_0 + \cdots + \mathcal{O}dx_n & \longrightarrow & \overline{\mathcal{O}} \;\simeq\; \mathbb{C}\{t\} \\ g_0dx_0 + \cdots + g_ndx_n & \longmapsto & g_0(t)\frac{dx_0(t)}{dt} + \cdots + g_n(t)\frac{dx_n(t)}{dt}, \end{array}$$

where $g_i \in \mathcal{O}$, for $i = 0, 1, \ldots, n$. It follows that $\mathcal{T} = Ker(\Psi)$ where $\mathcal{T}$ is the torsion submodule of $\mathcal{O}d\mathcal{O}$. We will consider the $\mathcal{O}d\mathcal{O}/\mathcal{T}$ and in what follows we identify the module $\mathcal{O}d\mathcal{O}$ with its image in $\mathbb{C}\{t\}$ by Ψ.

We write $\mathcal{O}dx_0 + \mathcal{O}dx_i = \mathcal{O}_{x_i}$ with $i = 1, \ldots, n$, the $\mathcal{O}$-submodule of $\mathcal{O}d\mathcal{O}$, generated by dx_0 and dx_i.

PROPOSITION 3.3. *Let $\phi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$ be a parametrization of an analytic irreducible curve $\mathcal{C}$.*

1.If $(0,\ f_1,\ f_2, \ldots,\ f_n) \in T\mathcal{A}_e(\phi)$ then for all $j = 1, \ldots, n$ we have $f_j = (\eta_j dx_0 - \eta_0 dx_j)/dx_0$ for some $\eta_0 \in \mathcal{M}$ and $\eta_j \in \mathcal{O}$;

2.if there exists $f_j = (\eta_j dx_0 - \eta_0 dx_j)/dx_0$ with $\eta_0 \in \mathcal{M}$ and $\eta_j \in \mathcal{O}$ then the element $(0, f_1, f_2, \ldots, f_j, \ldots, f_n) \in T\mathcal{A}_e(\phi)$, where $f_i = (\eta_i dx_0 - \eta_0 dx_i)/dx_0$ with $\eta_i \in \mathcal{O}$ for all $i = 1, \ldots, n$.

Proof: If $(0,\ f_1, \ldots, f_n) \;\in\; T\mathcal{A}_e(\phi)$ then the system

$$\begin{cases} dx_0(t)\cdot\epsilon(t) + \eta_0(\phi(t)) = \;\; 0 \\ dx_1(t)\cdot\epsilon(t) + \eta_1(\phi(t)) = f_1(t) \\ \qquad\vdots \\ dx_n(t)\cdot\epsilon(t) + \eta_n(\phi(t)) = f_n(t), \end{cases}$$

has a solution, for some $\epsilon \in \overline{\mathcal{O}}$ and $\eta_i \in \mathcal{O}_{n+1}$ for all $i = 0, 1, \ldots, n$. We denote $\eta_i(\phi(t))$ by η_i. If $\eta_0 = 0$ we have that $f_i = \eta_i \in \mathcal{O} \subset \frac{\mathcal{O}_{x_j}}{dx_0}$ for all $i = 1, \ldots, n$. On the other hand, if $\eta_0 \neq 0$ since $\epsilon = -\frac{\eta_0}{dx_0} \in \overline{\mathcal{O}}$ then $\eta_0 \in \mathcal{M} \setminus \{0\}$ and $f_i = \frac{\eta_i dx_0 - \eta_0 dx_i}{dx_0}$ for all $i = 1, \ldots, n$. The item (2) follows similarly. ▯

4. MONOMIAL CURVES

In this section, we determine the $\mathcal{A}_e$-codimension of monomial curves in terms of classical analytic invariants. Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ admitting a monomial parametrization $\phi(t) = (t^{v_0}, t^{v_1}, \ldots, t^{v_n})$ and suppose that the semigroup Γ of the curve $\mathcal{C}$ is generated by $\{v_0, v_1, \ldots, v_n\}$ with $v_0 < v_1 < \cdots < v_n$ and let L be the set of gaps of Γ.

Lemma 4.1. *If $k \in L$ is given by the order of an element in $\mathcal{O}_{x_i}/dx_0$, then $k = \alpha + v_i - v_0$, for some $\alpha \in \Gamma$.*

Proof: Notice that, since the curve $\mathcal{C}$ is monomial, if $\eta \in \mathcal{O}$ then $\eta(t) = \sum_i a_i t^{\lambda_i}$ with $a_i \in \mathbb{C}$ and $\lambda_i \in \mathbb{N}$ is such that $a_i t^{\lambda_i} = g_i(t)$ for some $g_i \in \mathcal{O}$.

Let $\omega = \eta_i dx_0 - \eta_0 dx_i$, for some $\eta_0, \eta_i \in \mathcal{O}$ such that $\nu(\frac{\omega}{dx_0}) = k$, where $\eta_i(t) = \sum_l a_l t^{\alpha_l}$ and $\eta_0(t) = \sum_j b_j t^{\beta_j}$, with $a_l, b_j \in \mathbb{C}$ and $\alpha_l, \beta_j \in \mathbb{N}$. Thus

$$\eta_i - \frac{\eta_0 dx_i}{dx_0} = \sum_l a_l t^{\alpha_l} - \left(\sum_j b_j t^{\beta_j}\right) \frac{v_i}{v_0} t^{v_i - v_0}.$$

Since $k \notin \Gamma$, it follows that $k = \beta_j + v_i - v_0$ for some j. By the above argument, we conclude that $b_j t^{\beta_j} = g(t)$ for some $g \in \mathcal{O}$, therefore $k = \nu\left(\frac{g dx_i}{dx_0}\right)$. ▯

Now we introduce the following sets:

$$N_i = \{k \in L;\ \ k \neq \alpha + v_i - v_0,\ \text{with}\ \alpha \in \Gamma \setminus \{0\}\},$$

$$\begin{aligned} M_i \ &= \ \{k \in L,\ k > v_i;\ \text{for every representation}\ k = \alpha + v_i - v_0,\ \alpha \in \Gamma, \\ &\qquad \text{there exists}\ j = 1, \ldots, i-1,\ \text{such that}\ \alpha + v_j - v_0 \in L\}, \end{aligned}$$

$$\begin{aligned} \Delta_i \ &= \ \{k \in L,\ k > v_i;\ \text{there exists a representation}\ k = \alpha + v_i - v_0, \\ &\qquad \alpha \in \Gamma\ \text{and}\ \alpha + v_j - v_0 \in \Gamma,\ \text{for all}\ j = 1, \ldots, i-1\}. \end{aligned}$$

For each $i = 1, \ldots, n$ these sets are pairwise disjoint and

$$L \ = \ N_i \ \dot{\cup} \ M_i \ \dot{\cup} \ \Delta_i. \tag{2}$$

Notice that M_1 is empty. We can determine the $\mathcal{A}_e\mathrm{cod}(\phi)$ in terms of these sets as we show in the following proposition.

PROPOSITION 4.1. *If ϕ is a monomial parametrization of an analytic irreducible curve $\mathcal{C}$ in $\mathbb{C}^{n+1}$ with multiplicity v_0 then*

$$\mathcal{A}_e\mathrm{cod}(\phi) \;=\; v_0 - 2 \;+\; \sum_{i=1}^{n} \sharp N_i \;+\; \sum_{i=2}^{n} \sharp M_i.$$

Proof: By the Remark 3.1, to determine a monomial basis to the normal space $\mathcal{N} = \frac{\theta(\phi)}{T\mathcal{A}_e(\phi)}$ it is sufficient to find elements of the form $(0, 0, \ldots, t^k, \ldots, 0)$ with $k \in L$. If $k \in \Delta_i$ then k is order of an element $(\eta_0 dx_i)/dx_0$ for some $\eta_0 \in \mathcal{M} \setminus \{0\}$. Since $\mathcal{C}$ is a monomial curve it follows by the Proposition 3.3 that $(0, \ldots, 0, t^k, 0, \ldots, 0) \in T\mathcal{A}_e(\phi) + \mathcal{M}_1^{k+1}\theta(\phi)$ where the non zero coordinate appears in the $(i+1)$-position. If $k \in N_i \mathbin{\dot\cup} M_i$ then $(0, \ldots, 0, t^k, 0, \ldots, 0) \notin T\mathcal{A}_e(\phi)$. Hence by the Proposition 3.1,

$$\beta \;=\; \left\{ \begin{pmatrix} t^{k_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^{k_1} \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ t^{k_n} \end{pmatrix}; \;\; \begin{array}{c} k_0 = 1, \ldots, v_0 - 2, \\ k_i \in N_i \mathbin{\dot\cup} M_i \;, \\ i = 1, \ldots, n \end{array} \right\}$$

is a basis for $\mathcal{N}$. □

Adding the number of elements of the n decompositions in (2) the following result follows:

THEOREM 4.1. *If ϕ is a monomial parametrization of an analytic irreducible curve $\mathcal{C}$ in $\mathbb{C}^{n+1}$ with multiplicity v_0 then*

$$\mathcal{A}_e\mathrm{cod}(\phi) \;+\; \sum_{i=1}^{n} \sharp \Delta_i \;=\; n\delta + v_0 - 2.$$

The sets N_i, M_i and Δ_i for $i = 1, \ldots, n$ can be described in terms of the Apéry set of the semigroup Γ of the curve $\mathcal{C}$.

DEFINITION 4.1. Let $\Gamma \subseteq \mathbb{N}$ be a semigroup with conductor and $q \in \Gamma$ with $q \neq 0$. Let $a_0 = 0$ and

$$a_i = \min \left\{ \Gamma \setminus \bigcup_{j=0}^{i-1} (a_j + q\mathbb{N}) \right\}$$

for $i = 1, \ldots, q-1$. The sequence $a_0 < a_1 < \cdots < a_{q-1}$ is called the *Apéry sequence of* Γ *with respect to* q and the set $A_p = \{a_0, a_1, \ldots, a_{q-1}\}$ is the *Apéry set.*

In this work we consider the Apéry sequence with respect to the multiplicity v_0 of the curve. It is a known fact that the set of gaps L of the semigroup Γ is given by $L = \{a_j - lv_0 > 0 \;;\; a_j \in A_p \setminus \{0\}, \text{ and } l \in \mathbb{N} \setminus \{0\}\}$.

Example 4.1. Let $\mathcal{C}$ be a curve given by the monomial parametrization $\phi(t) = (t^6, t^9, t^{17}, t^{25})$. The semigroup of $\mathcal{C}$ is $\Gamma = \langle 6, 9, 17, 25 \rangle$ and its conductor is $c = 29$.

A_p	L				
0					
$v_1 = 9$	3				
$v_2 = 17$	11	5			
$v_3 = 25$	19	13	7	1	
$v_1 + v_2 = 26$	**20**	14	8	2	
$v_1 + v_3 = 2v_2 = 34$	**28**	22	16	10	4

We can determine the sets Δ_i's analyzing the gaps of the first column of the table above. Notice that $20 = v_1+v_2-v_0$ and $28 = v_1+v_3-v_0 = 2v_2-v_0$, in this way it is easy to note that $\Delta_1 = \{20, 28\}$, $\Delta_2 = \{20\}$ and $\Delta_3 = \emptyset$. Hence $\mathcal{A}_e\mathrm{cod}(\phi) = 3\delta + v_0 - 2 - \sharp\Delta_1 - \sharp\Delta_2 - \sharp\Delta_3 = 48 + 4 - 2 - 1 = 49$, and the following set is a basis for the normal space $\mathcal{N}$:

$$\beta = \left\{ \begin{pmatrix} t^{k_0} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^{k_1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t^{k_2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^{k_3} \end{pmatrix}; \begin{array}{c} k_0 = 1, \ldots, 4, \\ k_i \in L \setminus \Delta_i, \\ i = 1, 2, 3 \end{array} \right\}.$$

Notice that the representation of a gap in terms of the Apéry set is important to determine the sets Δ_i's and consequently the $\mathcal{A}_e\mathrm{cod}(\phi)$.

5. FORMULA FOR THE $\mathcal{A}_e\mathrm{cod}(\phi)$

In this section we express the $\mathcal{A}$-invariant $\sum_{i=1}^{n} \sharp\Delta_i$ in terms of the multiplicity of the curve and of the Cohen-Macaulay type of its local ring. Notice that every element in Δ_i can be expressed by $a_j - v_0$ with $a_j \in A_p \setminus \{0\}$, thus we can omit $-v_0$ in the description of the elements in this set, because we need to determine the cardinality of Δ_i. We use the same notation to denote this set. Then,

$$\Delta_i = \{\alpha + v_i \in A_p; \quad \alpha > 0 \ \text{ and } \ \alpha + v_j \notin A_p, \ \forall\, j = 1, \ldots, i-1\}.$$

For each $i = 1, \ldots, n$, let

$$\overline{\Delta}_i = \{a \in A_p; \ a > 0, \ \ a + v_j \notin A_p, \ \forall\, j < i \text{ and } a + v_i \in A_p\}.$$

It is easy to check that the sets $\overline{\Delta}_i$ are pairwise disjoint and there exists a bijection between $\overline{\Delta}_i$ and Δ_i. In fact, it is sufficient to consider $\Phi_i : \Delta_i \to \overline{\Delta}_i$ defined by $\Phi_i(x) = x - v_i$. We also consider the following subset of A_p,

$$\widetilde{A}_p \ = \ \{b \in A_p \ ; \ \ b \neq 0, \ b + a_i \notin A_p, \ \forall \ a_i \in A_p \setminus \{0\}\}.$$

This is a non-empty set because the biggest element of A_p, that is a_{v_0-1}, belongs to it. The number of elements of $\widetilde{A}_p$ is denoted by t_p and it is an $\mathcal{A}$-invariant that admits an interesting interpretation in the case of monomial curves.

Definition 5.1. Let $(A, \mathcal{M}, k)$ be a d-dimensional Cohen-Macaulay local ring with maximal ideal $\mathcal{M}$ and k a field. The **type of** A is the number $r(A) = \dim_k \ Ext^d_A(k, A)$. If $r(A) = 1$ then A is a Gorenstein ring.

Cavaliere and Niesi in [4] (Proposition 2.7) proved that if the semigroup Γ generated by $v_0, v_1, \ldots, v_n$ is the semigroup of a monomial curve with local ring $\mathcal{O} = \mathbb{C}\{t^{v_0}, t^{v_1}, \ldots, t^{v_n}\}$ then the Cohen-Macaulay type of $\mathcal{O}$ is given by

$$r(\mathcal{O}) = t_p.$$

Proposition 5.1. *Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ admitting a monomial parametrization ϕ with semigroup $\Gamma = \langle v_0, v_1, \ldots, v_n\rangle$ and $t_p = \sharp\widetilde{A}_p$ then*

$$\sum_{i=1}^{n} \sharp\overline{\Delta}_i \ = \ v_0 - t_p - 1.$$

Proof: Remark that $\left(\bigcup_{i=1}^{n} \overline{\Delta}_i\right) \bigcap \widetilde{A}_p = \emptyset$. Now let $x \in A_p \backslash \{0\}$ and suppose that $x \notin \overline{\Delta_i}$ for all $i = 1, \ldots, n$. Given $a_k \in A_p \setminus \{0\}$, there exists $a_l \in A_p$ such that $a_k = v_i + a_l$ for some v_i. In this way $x + a_k = (x + v_i) + a_l \notin A_p$, otherwise $x + v_i \in A_p$ and this is a contradiction, because $x \notin \overline{\Delta_i}$. Hence, $x \in \widetilde{A}_p$. Since $\sharp A_p = v_0$, then $\sharp A_p - 1 \ = \ \sum_{i=1}^{n} \sharp\overline{\Delta}_i \ + \ \sharp\widetilde{A}_p$, and we conclude that $\sum_{i=1}^{n} \sharp\overline{\Delta}_i = v_0 - t_p - 1$. □

The following theorem follows from Theorem 4.1 and Proposition 5.1:

Theorem 5.1. *Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ and ϕ a monomial parametrization of $\mathcal{C}$ with $t_p = \sharp\widetilde{A}_p$. Then*

$$\mathcal{A}_e\mathrm{cod}(\phi) = n\delta + t_p - 1. \tag{3}$$

Remark 5. 1. Since $t_p \leq v_0 - 1$, then the above result is finer than the estimate for the $\mathcal{A}_e\mathrm{cod}(\phi)$ presented in the Proposition 3.2.

We know that an analytic irreducible curve is Gorenstein if its local ring is Gorenstein or equivalently if its semigroup is symmetric. Thus we have the following corollary:

Corollary 5.1. *Let $\Gamma = \langle v_0, v_1, \ldots, v_n\rangle$ be a semigroup that determines a monomial curve in $\mathbb{C}^{n+1}$, parameterized by ϕ, then Γ is symmetric if and only if $\mathcal{A}_e\text{cod}(\phi) = n\delta$.*

Proof: It is sufficient to notice that $t_p = 1$ if and only if Γ is symmetric. □

For monomial curves in $\mathbb{C}^3$, Cavaliere and Niesi in [3] determined the Cohen-Macaulay type t_p of the local ring $\mathcal{O}$:

Proposition 5.2. ([3]) *Let $\Gamma = \langle v_0, v_1, v_2\rangle$ be a semigroup of a curve, then*

$$t_p = \begin{cases} 1 \text{ if } \Gamma \text{ is symmetric} \\ 2 \text{ if } \Gamma \text{ is not symmetric.} \end{cases}$$

Corollary 5.2. *If $\mathcal{C}$ is an analytic irreducible curve in $\mathbb{C}^3$ given by a monomial parametrization ϕ, where c is the conductor of semigroup of $\mathcal{C}$, then we have:*

1.$\mathcal{C}$ is Gorenstein if and only if $\mathcal{A}_e\text{cod}(\phi) = 2\delta = c$.
2.$\mathcal{C}$ is non-Gorenstein if and only if $\mathcal{A}_e\text{cod}(\phi) = 2\delta + 1$.

An analytic irreducible curve $\mathcal{C}$ in $\mathbb{C}^3$ is Gorenstein if and only if it is a complete intersection (see [13]), then the above corollary shows that the parity of the $\mathcal{A}_e$-codimension of the parametrization of the curve in $\mathbb{C}^3$ determines whether or not the curve is a complete intersection. Using the Corollaries 5.1 and 5.2 we can add, to the classification of simple singularities of germs of irreducible curves, obtained by Gibson and Hobbs [6], the information of which curves in the list are complete intersection.

An interesting question in this context is: what is the relation between the invariants τ and $\mathcal{A}_e\text{cod}(\phi)$?

Greuel ([8], Corollary 2.5) presented the following result:

Proposition 5.3. *Let $\mathcal{C}$ be a reduced smoothable curve in $\mathbb{C}^n$ with isolated singularity at the origin and $r(\mathcal{O})$ the Cohen-Macaulay type.*

1.If $\mathcal{C}$ is quasihomogeneous then $\tau \geq \mu + r(\mathcal{O}) - 1$, where μ is the Milnor number of $\mathcal{C}$.
2.The equality holds if and only if $\mathcal{C}$ is quasihomogeneous and unobstructed.

A quasihomogeneous curve admits a monomial parametrization, then we can state:

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PROPOSITION 5.4. *Let $\mathcal{C}$ be an analytic irreducible smoothable curve in $\mathbb{C}^n$ with isolated singularity at the origin. If $\mathcal{C}$ is quasihomogeneous and unobstructed then*

$$\begin{cases} \tau \;>\; \mathcal{A}_e\mathrm{cod}(\phi), & \textit{if}\;\; n=2 \\ \tau \;=\; \mathcal{A}_e\mathrm{cod}(\phi), & \textit{if}\;\; n=3 \\ \tau \;<\; \mathcal{A}_e\mathrm{cod}(\phi), & \textit{if}\;\; n>3, \end{cases}$$

where ϕ is a monomial parametrization of $\mathcal{C}$.

Proof: The result follows by Proposition 5.3 and Theorem 5.1. □

It is known that if $\mathcal{C}$ is a quasihomogeneous curve in $\mathbb{C}^3$ then it is smoothable and unobstructed (see [1], for example) and therefore in this case $\tau = \mathcal{A}_e\mathrm{cod}(\phi)$. Moreover complete intersection curves are smoothable and unobstructed and the previous result can be applied in this case as well.

In the special case of plane curves, not necessarily defined by a monomial parametrization, the difference $\delta - \mathcal{A}_e\mathrm{cod}(\phi)$ can be measured in terms of a new invariant defined in terms of Kähler differentials, see [12].

For curves in $\mathbb{C}^{n+1}$, not necessarily monomial, but with *special properties*, we can obtain estimates for the $\mathcal{A}_e\mathrm{cod}(\phi)$. Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ and ϕ a parametrization of $\mathcal{C}$ given by

$$\phi : \begin{cases} x_0 &=& t^{v_0} \\ x_1 &=& t^{v_1} + \sum_{i>v_1} a_{i1}t^i \\ & \vdots & \\ x_n &=& t^{v_n} + \sum_{i>v_n} a_{in}t^i. \end{cases}$$

with $v_0 < v_1 < \cdots < v_n$. We call $\phi_0(t) = (t^{v_0}, t^{v_1}, \ldots, t^{v_n})$ the principal part of the germ ϕ. We say that ϕ has a *good principal part* if ϕ_0 determines a monomial curve in $\mathbb{C}^{n+1}$.

If $\Gamma = \langle v_0, v_1, \ldots, v_g\rangle$ is the semigroup of values associated of ϕ, then the semigroup associated of ϕ_0 is given by $\Gamma_0 = \langle v_0, v_1, \ldots, v_n\rangle \subset \Gamma$, with $n \leq g$. An important property of the invariant $\mathcal{A}_e\mathrm{cod}(\phi)$ is that it is upper-semicontinuous and thus $\mathcal{A}_e\mathrm{cod}(\phi) \;\leq\; \mathcal{A}_e\mathrm{cod}(\phi_0)$. If Γ, A_p and δ indicate, respectively the semigroup of values, the Apéry set, and the delta invariant for the curve $\mathcal{C}$ given by ϕ, then we will denote by Γ_0, A_{p_0} and δ_0 the same objects to the curve given by ϕ_0. Moreover, we will denote by t_{p_0} the Cohen-Macaulay type of the local ring of ϕ_0, where $t_{p_0} = \sharp\widetilde{A}_{p_0}$. In this way the following result is immediate.

PROPOSITION 5.5. *Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ and ϕ a parametrization of $\mathcal{C}$ with good principal part ϕ_0. Then*

$$\mathcal{A}_e\mathrm{cod}(\phi) \;\leq\; n\delta_0 + t_{p_0} - 1.$$

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COROLLARY 5.3. *Let $\mathcal{C}$ be an analytic irreducible curve in $\mathbb{C}^{n+1}$ and ϕ a parametrization of $\mathcal{C}$ with good principal part ϕ_0. If $\Gamma = \Gamma_0$ then*

$$\mathcal{A}_e\mathrm{cod}(\phi) \ \leq \ n\delta + t_p - 1.$$

We finish with the following example:

EXAMPLE 5.1. **(Curves with maximal embedding dimension)** A curve $\mathcal{C}$ in $\mathbb{C}^{n+1}$ is called (a curve) with maximal embedding dimension, if it admits a parametrization

$$\phi : \begin{cases} x_0 &= t^{n+1} \\ x_1 &= t^{v_1} + \sum_{i>v_1} a_{i1}t^i \\ & \vdots \\ x_n &= t^{v_n} + \sum_{i>v_n} a_{in}t^i, \end{cases}$$

such that its semigroup of values is $\Gamma = \langle n+1, v_1, \ldots, v_n\rangle$. We can check by an easy direct calculation that such curves satisfy the following properties:

1. if $n \geq 2$ then $\mathcal{C}$ is non-Gorenstein;
2. the Apéry set is given by $A_p = \{0, v_1, \ldots, v_n\}$. Therefore,

$$t_p = \sharp\widetilde{A}_p = \sharp A_p \setminus \{0\} = n;$$

3. ϕ has good principal part ϕ_0 .

Notice that the semigroup of ϕ is equal to the semigroup of ϕ_0, thus

$$\mathcal{A}_e\mathrm{cod}(\phi) \ \leq \ n\delta + t_p - 1 \ = \ n\delta + n - 1.$$

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