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Zeta invariants for Dirichlet series

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We introduce a general method to obtain the main zeta invariants for a class of double series of Dirichlet type and we apply it to the case of homogeneous quadratic and linear double series. October, 2005 ICMC-USP

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Let n be a 2-dimensional integer vector in $\mathbb{N}_0^2$, where $\mathbb{N}_0 = \{1, 2, 3, \dots\}$, and A a (real) symmetric matrix of rank 2 whose associated quadratic form is positive definite on $\mathbb{N}_0$. Then, consider the zeta function of the complex variable s defined by the series of Dirichlet type

$$\zeta_2(s; A) = \sum_{n\in\mathbb{N}_0^2} (n^T An)^{-s},$$

when $\mathrm{Re}(s) > s_0$, for some s_0 (depending on the matrix A), and by analytic continuation elsewhere. The aim of this work is to obtain the first coefficients in the Kronecker limit formula for $\zeta_2(s; A)$ near $s = 0$. Our result is stated in Theorem 3.1 of Section 3, where we shows that the analytic continuation of $\zeta_2(s; A)$ is regular at $s = 0$, and we give the values of $\zeta_2(0; A)$ and $\zeta_2'(0; A)$. The proof of Theorem 3.1 is based on a more general lemma, Lemma 2.1 of Section 2, that provides the first coefficients in the Kronecker limit formula at $s = 0$ for the zeta function associated to a more general class of double series of Dirichlet type, whenever some coefficients in the asymptotic expansions of some associated spectral functions are known. We conclude this section with a brief overview on this subject and on the motivations for our analysis.

Let $S = \{\lambda_n\}_{n\in\mathbb{N}_0}$ be sequence of non vanishing complex numbers with unique accumulation point at infinity and with finite exponent s_0. Then we define the associated zeta function by the sum

$$\zeta(s; S) = \sum_{n=1}^{\infty} \lambda_n^{-s},$$

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for $\mathrm{Re}(s) > s_0$, and by analytic extension elsewhere [2]. Information on the analytic properties of this function, usually called zeta invariants, like poles, residua and particular values, are important in many field, from number theory to mathematical physics, and different techniques have been defined and applied in different contexts in order to compute such invariants. In particular, we are interested in the value of the derivative of the zeta function at $s = 0$, that is to say in the coefficient of the linear term in the Kronecker first limit formula at $s = 0$ [23] [15]. By a Mellin transform, the zeta function can be immediately associated to a series of theta type $\sum \mathrm{e}^{-\lambda_n t}$, and many properties of the zeta function can be easily deduced from the properties of the theta series. Poles, residua and particular values can be obtained whenever the asymptotic expansion of the theta series for small t is available [14] [17]; unfortunately, this approach fails when the coefficients of the positive powers of s are involved, as for $\zeta'(0; S)$. A particularly interesting case is when λ_n is quadratic in one or more integer variables $\lambda_n = n^T A n$, where n is an integer vector in $\mathbb{Z}^k$, and A a k-square real symmetric matrix (zeta functions of this type are called Epstein zeta functions since they were first considered by Epstein in [12] and [13]). Multi dimensional theta series have been deeply studied in the literature. In particular, the Poisson summation formula, namely the Fourier expansion of the theta function, has suitable generalizations to the multi dimensional case (see for example [5] XI.2, 3). Using these formulas and properties of special functions, it is possible to compute the main zeta invariants for multiple series of Epstein type, also called multiple Eisenstein zeta functions (see [23] [16] or [11] and references thereby). Next, moving to the zeta functions associated to series of Dirichlet type (namely when the sums are over $\mathbb{N}_0^k$) the main difficulty is precisely the lack of a formula of Poisson type. Consequently, it is hard to find general results, and different techniques have been introduced to deal with the specific cases (see for example [3] [6] [7] [8] [19] and [20]). In particular, homogeneous linear series of Dirichlet type, defined (for $\mathrm{Re}(s) > k$) by the sum

$$\zeta(s; a) = \sum_{n \in \mathbb{N}_0^k} [a_1 n_1 + \ldots a_k n_k]^{-s},$$

were first introduced by Barnes in his work on multiple Gamma functions, where, however, he just considered the case of equal coefficients ($a_i = a_j$). For these functions it is quite easy to find out a decomposition in a finite sum of Riemann zeta functions, and thus all zeta invariants can be computed (see also [22]). More recently, Actor studied in [1] the case of unequal coefficients. Considering for simplicity the case $k = 2$, $\mathrm{Re}(s) > 2$,

$$\zeta_1(s; a, b) = \sum_{n,k=1}^{\infty} [an + bk]^{-s},$$

and expanding the binomial in one of the coefficients, a decomposition can be obtained, but an infinite sum of Riemann zeta functions multiplied by the binomial coefficient $\binom{-s}{i}$ appears (see equation (2) of [1]). This is the reason why this decomposition can be used to get the analytic extension of $\zeta_1(s; a, b)$ and to compute residua at the poles and particular values, but not to deal with the derivative. Beside, a formula for $\zeta_1'(0; a, 1)$ (that immediately gives that for $\zeta_1'(0; a, b)$) was given in [20], using the Plana Theorem. Notice also the

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related works of Matsumoto, where asymptotic expansions are given for non homogeneous linear series $\zeta_1(s;a,1,q)=\sum_{n,k=1}^{\infty}[an+k+q]^{-s}$, for large a. Although these formulas do not give finite results for the zeta invariants, it should be investigated the possibility of coupling them with available finite formulas (like the ones given in Theorem 3.1) for the homogeneous case in order to get the zeta invariants of the non homogeneous case. The case of homogeneous quadratic series of Dirichlet type, defined (for $\mathrm{Re}(s)>1$) by the sum

$$\zeta_1(s;a,b)=\sum_{n,k=1}^{\infty}[an^2+2cnk+bk^2]^{-s},$$

is much harder. These zeta functions with integer coefficients appear when dealing with the zeta functions of a narrow ideal class for a real quadratic field as shown by Zagier in [24], where he also computed the values at non positive integers (the same result was obtained by Shitani in [18] by different methods). Also the values at negative half integers are obtained in [10]. On the other side, no results for the derivative $\zeta_2'(0;a,b,c)$ are known, and this is the main motivation for this work.

The technique we develop in order to obtain $\zeta_2'(0;a,b,c)$ is to find a suitable representation of $\zeta_2(s;a,b,c)$, in terms of some complex integral, that allows to use known information, contained in the asymptotic expansion of some spectral functions associated to $\zeta_2(s;a,b,c)$, to compute $\zeta_2'(0;a,b,c)$ (Theorem 3.1). This method automatically gives also the value of $\zeta_2(0;a,b,c)$ for all real positive values of the constants a,b and c, thus extending one of the result of [24]. Our formula also cover the linear case $\zeta_1(s;a,b)$ (Section 4.2), for all real positive a and b, extending one of the result of [10]. Furthermore, our technique is not defined ad hoc for the case under study in this paper, but can be applied to a larger class of double series of Dirichlet type with polynomial general term or even generalized without much effort to series whose general term is not polynomial, and there are works in progress in different directions. This is the other motivation for this work.

2. A LEMMA TO DEAL WITH A CLASS OF DOUBLE SERIES OF DIRICHLET TYPE

The lemma we are going to prove is a general version of a technique introduced in [21] to treat the zeta function associated to the Laplacian on a cone. Let $P(x,y)$ be a polynomial in two variables with real coefficients. Consider the double sequence $S=\{\lambda_{n,k}=P(n^p,k^q)\}$, $n,k\in\mathbb{N}_0$, where p and q are some positive real numbers. Assume[1] that S is positive definite (namely $\lambda_{n,k}>0$ for all n,k), has the unique accumulation point at infinity and is a sequence of finite exponent s_0 (see [9] 7.43). Then, there exist finite α and β such that $\lambda_{n,k}$ behaves like n^α for fixed k, and like k^β for fixed n, and the series

$$\zeta(s)=\sum_{n,k=1}^{\infty}\lambda_{n,k}^{-s},$$

[1] For convenience also assume the numbers $\lambda_{n,k}$ to be ordered by increasing values, i.e. $\lambda_{1,1}\leq\lambda_{1,2}\leq\lambda_{2,1}\leq\lambda_{1,3}\leq\dots$.

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converges absolutely and locally uniformly when $\mathrm{Re}(s) > s_0$. In this situation, we can prove the following lemma.

LEMMA 2.1. **Spectral decomposition lemma (SDL)**. *Let $\{\lambda_{n,k}\}_{n,k\in\mathbb{N}_0}$ and $\zeta(s)$ be as above, then we have the following analytic representation for $\zeta(s)$*

$$\zeta(s) = \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_\epsilon} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} T(\lambda, s) d\lambda dt,$$

where the contour is defined below, and

$$T(\lambda, s) = \sum_{n=1}^\infty n^{-\alpha s} t_n(\lambda),$$

$$t_n(\lambda) = -\log \prod_{k=1}^\infty \left[1 + \frac{n^\alpha(-\lambda)}{\lambda_{n,k}}\right] \mathrm{e}^{\sum_{j=1}^{[1/\beta]} \frac{n^{\alpha j}\lambda^j}{j\lambda_{n,k}^j}}.$$

Assume that exist an asymptotic expansion of $t_n(\lambda)$ for large n, uniformly in λ for λ in some (unbounded) domain D of the complex plane, disjoint from S and containing the origin $\lambda = 0$, and an asymptotic expansion of $t_n(\lambda)$, for each fixed n, in powers of $-\lambda$ and powers of $-\lambda$ times $\log(-\lambda)$ (like the one considered in [4]) for large λ in D. Then, $\zeta(s)$ can be analytically extended to the whole complex plane up to a set of poles. Moreover, $s = 0$ is a regular point and we can compute the main zeta invariants as follows. Let $f(\lambda)$ be the coefficient of the $\frac{1}{n}$ term in the asymptotic expansion of $t_n(\lambda)$ for large n, and define

$$p_n(\lambda) = t_n(\lambda) - \frac{1}{n} f(\lambda),$$

$$P(\lambda, s) = \sum_{n=1}^\infty n^{-\alpha s} p_n(\lambda);$$

then

$$\zeta(0) = -A(0) + \frac{1}{\alpha}\mathrm{Res}_1\Big(F(s), s = 0\Big),$$

$$\zeta'(0) = -A'(0) - B(0) +$$

$$+ \frac{1}{\alpha}\mathrm{Res}_0\Big(F(s), s = 0\Big) + \gamma\left(1 + \frac{1}{\alpha}\right)\mathrm{Res}_1\Big(F(s), s = 0\Big),$$

where

$$F(s) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_\epsilon} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} f(\lambda) d\lambda dt,$$

and

$$A(s)=\sum_{n=1}^{\infty} n^{-\alpha s} a_n, \qquad B(s)=\sum_{n=1}^{\infty} n^{-\alpha s} b_n,$$

where a_n and b_n are the logarithmic and constant term in the asymptotic expansion of $p_n(\lambda)$ for large λ and fixed n.

Proof. The key argument in the proof is the following one. We know how to deal with a simple series [22], say the sum over k, and we can rearrange the sums in such a way to decompose the double series as a sum over n of sums over k. The problem in this decomposition is that we have to regularize also the external sum in n. For that purpose, it is enough to collect an opportune power of n. This will give a suitable analytic representation and will keep track of all contributions in the regularization process. As a first step, we provide an analytic representation for $\zeta(s)$. Notice that all the series and the products appearing in the following calculations converge absolutely and uniformly, due to the assumptions on the sequence $\{\lambda_{n,k}\}$. We can write

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-\alpha s}\sum_{k=1}^{\infty}\left(\frac{\lambda_{n,k}}{n^{\alpha}}\right)^{-s}=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\sum_{n=1}^{\infty} n^{-\alpha s}\sum_{k=1}^{\infty} \mathrm{e}^{-\frac{\lambda_{n,k}}{n^{\alpha}}t}dt=$$

$$=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\sum_{n=1}^{\infty} n^{-\alpha s}\frac{1}{2\pi i}\int_{\Lambda_\epsilon}\mathrm{e}^{-\lambda t}\sum_{k=1}^{\infty}\frac{1}{\lambda-\frac{\lambda_{n,k}}{n^{\alpha}}}d\lambda dt=$$

where $\Lambda_\epsilon=\partial D$ can be deformed to the contour $\left\{\lambda\in\mathbb{C}\mid |\arg(\lambda-\epsilon)|=\frac{\theta}{2}\right\}$, oriented counter clockwise, with some non negative real $\epsilon<\lambda_{1,1}$ and $0<\theta<\pi$,

$$=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_\epsilon}\mathrm{e}^{-\lambda t}R(\lambda,s)d\lambda dt,$$

where

$$R(\lambda,s)=\sum_{n=1}^{\infty} n^{-\alpha s} r_n(\lambda),$$

$$r_n(\lambda)=-\sum_{k=1}^{\infty}\left[\frac{n^{\alpha}}{n^{\alpha}(-\lambda)+\lambda_{n,k}}-\sum_{j=0}^{[1/\beta]-1}\frac{n^{j\alpha}\lambda^j}{\lambda_{n,k}^{j+1}}\right].$$

We can now introduce a primitive function for the resolvent $R(\lambda,s)$, namely let

$$T(\lambda,s)=\sum_{n=1}^{\infty} n^{-\alpha s} t_n(\lambda),$$

$$t_n(\lambda) = -\log \prod_{k=1}^{\infty} \left[1 + \frac{n^\alpha(-\lambda)}{\lambda_{n,k}}\right] \mathrm{e}^{\sum_{j=1}^{[1/\beta]} \frac{n^{\alpha j} \lambda^j}{j \lambda_{n,k}^j}},$$

then

$$-\frac{d}{d\lambda} T(\lambda, s) = R(\lambda, s),$$

and again the canonical product converges; moreover, notice that the exponential factor gives no contribution for the quantities we are going to compute, since it adds terms to $R(\lambda, s)$ that vanish after λ integration (see [21]). Also notice that, since $T(\lambda, s)$ is defined up to a constant, we can always chose it such that $T(0, s) = 0$, and so we do. Integrating by parts first in λ and then in t we get

$$\zeta(s) = \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_\epsilon} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} T(\lambda, s) d\lambda dt.$$

We could now use the above representation to compute both the value at $s = 0$ and the derivative, if we could provide an expansion for $T(\lambda, s)$ for large λ [20]. The problem is that more care is necessary due to the possible appearance of a singularity in the sum over n. The unique possible pole in such a sum comes from a term behaving like $\frac{1}{n}$ in the expansion of t_n for large n uniformly in λ, so we can overcome this problem as follows. Let $f(\lambda)$ be the coefficient of the term in $\frac{1}{n}$ in the uniform expansion of $t_n(\lambda)$ for large n, and define

$$p_n(\lambda) = t_n(\lambda) - \frac{1}{n} f(\lambda),$$

$$P(\lambda, s) = \sum_{n=1}^{\infty} n^{-\alpha s} p_n(\lambda),$$

then

$$\zeta(s) = \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_\epsilon} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} P(\lambda, s) d\lambda dt +$$

$$+ \frac{s}{\Gamma(s)} \zeta_R(\alpha s + 1) \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_\epsilon} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} f(\lambda) d\lambda dt = z_1(s) + z_2(s),$$

and the singular contribution is isolated in the second term. We can deal with the two terms as follows. For the first, the assumed expansion of $t_n(\lambda)$ for large λ in D and fixed n, gives the following expansion of $p_n(\lambda)$,

$$p_n(\lambda) = t_n(\lambda) - \frac{1}{n} f(\lambda) = \sum_{\gamma} c_{\gamma,n} (-\lambda)^{\gamma_0 - \gamma} + \sum_{\delta} d_{\delta,n} (-\lambda)^{\delta_0 - \delta} \log(-\lambda) + u_n(\lambda),$$

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where γ and δ run through some finite sets of real positive numbers with $\gamma-\gamma_0, \delta-\delta_0 < M$, for some positive M, γ_0 and δ_0 [4], and the reminder is such that for $\lambda \in D$,

$$\lim_{\lambda\to\infty} \frac{u_n(\lambda)}{(-\lambda)^{-M}} = 0.$$

Moreover, due to the uniformity of the expansion of $t_n(\lambda)$ for large n, we know that the coefficients $c_{\gamma,n}$, $d_{\delta,n}$ and the function $u_n(\lambda)$ have an expansion for large n without terms of the form $\frac{1}{n}$ and such that the series $\sum_{n=1}^{\infty} c_{\gamma,n} n^{-\alpha s}$, $\sum_{n=1}^{\infty} d_{\delta,n} n^{-\alpha s}$ and $\sum_{n=1}^{\infty} u_n(\lambda) n^{-\alpha s}$ converge uniformly for $\mathrm{Re}(s) > s_0$ and $\lambda \in D$, and have analytic extensions regular at $s = 0$. This means that we can use classical methods to deal with the analytic extension of functional zeta functions (see [14] or [17]). In particular we need the constant and logarithmic terms, hence let's write

$$p_n(\lambda) = \cdots + a_n \log(-\lambda) + b_n + \ldots;$$

this gives

$$P(\lambda, s) = \cdots + A(s)\log(-\lambda) + B(s) + \ldots,$$

with

$$A(s) = \sum_{n=1}^{\infty} n^{-\alpha s} a_n, \qquad B(s) = \sum_{n=1}^{\infty} n^{-\alpha s} b_n.$$

Following the standard approach (see for example [14]), we split the t integral at $t = 1$. The $t > 1$ part defines a regular function of s near $s = 0$, while in the $t < 1$ part we must change the contour of the λ integral (here C_ϵ is a circle around the origin of ray ϵ) and then we can re-scale λ by t and use the expansion above to obtain (see [22])

$$z_1(s) = \frac{s}{\Gamma(s)} \int_0^1 t^{s-1} \left[\frac{1}{2\pi i} \int_{\Lambda_{-\epsilon}} \frac{\mathrm{e}^{-\lambda}}{-\lambda} P(\lambda/t, s) d\lambda dt - \frac{1}{2\pi i} \int_{C_\epsilon} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} P(\lambda, s) d\lambda dt \right] +$$

$$+ s^2 h_1(s) =$$

$$= \frac{s}{\Gamma(s)} \int_0^1 t^{s-1} \frac{1}{2\pi i} \left[\int_{\Lambda_{-\epsilon}} \frac{\mathrm{e}^{-\lambda}}{-\lambda} \left[A(s) \log\frac{-\lambda}{t} + B(s) \right] d\lambda dt - P(0, s) dt \right] +$$

$$+ s^2 h_2(s),$$

where $h_i(s)$ are regular functions of s near $s = 0$. At this point notice that $P(0, s) = 0$ for all s, since $\lambda = 0$ is assumed to belong to the domain D. After some computations we obtain

$$z_1(s) = \frac{s}{\Gamma(s+1)} \left[\gamma A(s) - \frac{A(s)}{s} - B(s) \right] + s^2 h_2(s).$$

The above expression allows to deal with both $\zeta(0)$ and $\zeta'(0)$. In fact,

$$z_1(0) = -A(0),$$

while taking the derivative near $s = 0$ and using the known expansion for the inverse of the Gamma function we obtain

$$z_1'(0) = -A'(0) - B(0).$$

The second term, $z_2(s)$, can be treated by using the expansions for the different factors as functions of s. Writing

$$z_2(s) = \frac{s}{\Gamma(s)}\zeta_R(\alpha s + 1)F(s),$$

with

$$F(s) = \int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_\epsilon}\frac{e^{-\lambda t}}{-\lambda}f(\lambda)d\lambda dt,$$

we obtain

$$z_2(s) = \frac{F_1}{\alpha} + \frac{1}{\alpha}[F_0 + (\alpha+1)\gamma F_1]s + O(s^2),$$

and hence

$$z_2(0) = \frac{1}{\alpha}\mathrm{Res}_1(F(s), s=0),$$

and

$$z_2'(0) = \frac{1}{\alpha}\left[\mathrm{Res}_0(F(s), s=0) + \gamma(\alpha+1)\mathrm{Res}_1(F(s), s=0)\right].$$

∎

Notice that the above proposition gives a theoretical result that becomes effective up to availability of the concrete expansion for $t_n(\lambda)$; this is actually the case in many interesting examples, as the ones we will deal with in the next sections or the one studied in [21].

3. THE HOMOGENEOUS QUADRATIC CASE

In this section, we consider the quadratic case, namely the function defined by the series

$$\zeta_2(s; a, b, c) = \sum_{n\in\mathbb{N}_0^2}(n^T A n)^{-s} = \sum_{n,k=1}^{\infty}[an^2 + bk^2 + 2cnk]^{-s},$$

when $\mathrm{Re}(s) > 1$, and by analytic continuation elsewhere. Here $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a real symmetric matrix with positive definite associated quadratic form on $\mathbb{N}_0$. We introduce

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the following notation: $\Delta = c^2 - ab = -\det A$, and we further assume $ab \neq 0$. To compute the zeta invariants at $s = 0$ we apply the SDL. Since $ab \neq 0$, we have $\alpha = \beta = 2$, and

$$\zeta_2(s; a, b, c) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_\epsilon} \frac{e^{-\lambda t}}{-\lambda} T(\lambda, s) d\lambda,$$

where

$$T(\lambda, s) = \sum_{n=1}^\infty n^{-2s} t_n(\lambda),$$

$$t_n(\lambda) = -\log \prod_{k=1}^\infty \left(1 + \frac{n^2(-\lambda)}{an^2 + 2cnk + bk^2}\right);$$

since $an^2 + 2cnk + bk^2 = b\left(k + \frac{c}{b}n\right)^2 - \frac{\Delta}{b}n^2$, we obtain

$$t_n(\lambda) = -\log \prod_{k=1}^\infty \left(1 + \frac{\frac{-\lambda}{b}n^2 - \frac{\Delta}{b^2}n^2}{\left(k + \frac{c}{b}n\right)^2}\right) + \log \prod_{k=1}^\infty \left(1 + \frac{-\frac{\Delta}{b^2}n^2}{\left(k + \frac{c}{b}n\right)^2}\right).$$

Using the product definition of the Gamma function

$$t_n(\lambda) = \log \Gamma\left(\frac{c + \sqrt{\Delta - b(-\lambda)}}{b} n\right) + \log \Gamma\left(\frac{c - \sqrt{\Delta - b(-\lambda)}}{b} n\right) +$$

$$-\log \Gamma\left(\frac{c + \sqrt{\Delta}}{b} n\right) - \log \Gamma\left(\frac{c - \sqrt{\Delta}}{b} n\right) + \log\left(1 + \frac{-\lambda}{a}\right).$$

The expansion for large n is

$$\log \Gamma((x+y)n) + \log \Gamma((x-y)n) =$$

$$= \left((x+y)n - \frac{1}{2}\right) \log(x+y)n - (x+y)n + \frac{1}{2}\log 2\pi + \frac{1}{12}\frac{1}{x+y}\frac{1}{n} +$$

$$+ \left((x-y)n - \frac{1}{2}\right) \log(x-y)n - (x-y)n + \frac{1}{2}\log 2\pi + \frac{1}{12}\frac{1}{x-y}\frac{1}{n} + O(n^{-3}) =$$

$$= \cdots + \frac{1}{6}\frac{x}{x^2 - y^2}\frac{1}{n} + \ldots,$$

with $x = \frac{c}{b}$ and $y_\lambda = \frac{\sqrt{\Delta - b(-\lambda)}}{b}$. Notice that $\lambda = 0$ belongs to the domain where the expansion is uniform. The expansion of the whole $t_n(\lambda)$ will be

$$t_n(\lambda) = \cdots + \frac{1}{6}\frac{x}{x^2 - y_\lambda^2} - \frac{1}{6}\frac{x}{x^2 - y_0^2} + \ldots,$$

that gives

$$f(\lambda)=\frac{c}{6}\left(\frac{1}{a-\lambda}-\frac{1}{a}\right).$$

Therefore,

$$F(s)=\frac{c}{6}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_\epsilon}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\left(\frac{1}{a-\lambda}-\frac{1}{a}\right)d\lambda dt=$$

$$=\frac{c}{6a}a^{-s}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_\epsilon}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\left(\frac{1}{1-\lambda}-1\right)d\lambda dt=$$

$$=\frac{c}{6a}a^{-s}\int_0^\infty t^{s-1}\Gamma(1,t)dt=\frac{c}{6a}a^{-s}\Gamma(s),$$

where we have used the definition of the incomplete Gamma function to compute the last integral and get the last equality. Notice that in the integral defining $F(s)$, the constant term gives no contribution and $F(s)$ is not regular at $s=0$. Expanding

$$F(s)=\frac{c}{6a}\left(\frac{1}{s}-\gamma-\log a+O(s)\right),$$

and hence

$$\mathrm{Res}_0(F(s),s=0)=-\frac{c}{6a}(\gamma+\log a),\qquad \mathrm{Res}_1(F(s),s=0)=\frac{c}{6a}.$$

Next, we compute the expansion of $p_n(\lambda)$ for large λ,

$$t_n(\lambda)=\left(xn-\frac{1}{2}\right)\log(x^2-y_\lambda^2)n^2+ny_\lambda\log\frac{x+y_\lambda}{x-y_\lambda}+ln2\pi-2xn+$$

$$-\log\Gamma((x+y_0)n)\Gamma((x-y_0)n)+\log(-\lambda)-\log a+O((-\lambda)^{-1}).$$

with $y_\lambda=\frac{\sqrt{\Delta-b(-\lambda)}}{b}$. Setting $y_\lambda=iu_\lambda$, $u_\lambda=\frac{\sqrt{b(-\lambda)-\Delta}}{b}\sim\sqrt{-\lambda}$, we can expand

$$ny_\lambda\log\frac{x+y_\lambda}{x-y_\lambda}=inu_\lambda\log\frac{x+iu_\lambda}{x-iu_\lambda}=-2nu_\lambda\arctan\frac{u_\lambda}{x}n=$$

$$=-\pi nu_\lambda+2xn+O(u_\lambda^{-2})=-\pi n\sqrt{-\lambda}+2xn+O((-\lambda)^{-1}),$$

$$\log(x^2-y_\lambda^2)n^2=\log(-\lambda)+\log\frac{n^2}{b}+O((-\lambda)^{-1}),$$

$$f(\lambda)=-\frac{c}{6a}+O((-\lambda)^{-1}),$$

to obtain

$$p_n(\lambda) = \left(xn - \frac{1}{2}\right)\left(\log(-\lambda) + \log\frac{n^2}{b}\right) - \pi n\sqrt{-\lambda} + \log(-\lambda) - \log a + \log 2\pi +$$

$$-\log\Gamma((x+y_0)n)\Gamma((x-y_0)n) + \frac{c}{6an} + O((-\lambda)^{-1}).$$

This means that

$$a_n = \frac{c}{b}n + \frac{1}{2},$$

$$b_n = -\log\Gamma\left(\frac{c+\sqrt{\Delta}}{b}n\right)\Gamma\left(\frac{c-\sqrt{\Delta}}{b}n\right) + \log 2\pi + \left(\frac{c}{b}n - \frac{1}{2}\right)\log\frac{n^2}{b} - \log a + \frac{c}{6a}\frac{1}{n}.$$

and gives

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-\alpha s} = \frac{c}{b}\zeta(2s-1) + \frac{1}{2}\zeta(2s),$$

$$A(0) = -\frac{c}{12b} - \frac{1}{4}, \qquad A'(0) = -\frac{1}{2}\log 2\pi + \frac{2c}{b}\zeta'(-1).$$

We have obtained

$$\zeta_2(0; a, b, c) = \frac{1}{4} + \frac{c}{12a} + \frac{c}{12b}.$$

Next

$$B(s) = \sum_{n=1}^{\infty} n^{-2s} b_n = \sum_{n=1}^{\infty}\left[-\log\Gamma\left(\frac{c+\sqrt{\Delta}}{b}n\right)\Gamma\left(\frac{c-\sqrt{\Delta}}{b}n\right) + \right.$$

$$\left. + \log 2\pi - \log a + \left(\frac{c}{b}n - \frac{1}{2}\right)\log\frac{n^2}{b} + \frac{c}{6a}\frac{1}{n}\right] n^{-2s}.$$

By Lemma 2.1, $B(s)$ has an analytic extension regular at $s = 0$. We provide two different representations. Let

$$\log\Gamma\left(\frac{c+\sqrt{\Delta}}{b}n\right)\Gamma\left(\frac{c-\sqrt{\Delta}}{b}n\right) =$$

$$= \left(\frac{c}{b}n - \frac{1}{2}\right)\log\frac{an^2}{b} + \frac{\sqrt{\Delta}}{b}n\log\frac{c+\sqrt{\Delta}}{c-\sqrt{\Delta}} + \log 2\pi - \frac{2cn}{b} + K,$$

where

$$K=\frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{k=1}^{\infty}\left[\frac{1}{\left(k+\frac{c+\sqrt{\Delta}}{b}n\right)^{m+1}}+\frac{1}{\left(k+\frac{c-\sqrt{\Delta}}{b}n\right)^{m+1}}\right]. \tag{1}$$

or

$$K=2\int_0^{\infty}\left[\arctan\frac{bt}{(c+\sqrt{\Delta})n}+\arctan\frac{bt}{(c-\sqrt{\Delta})n}\right]\frac{dt}{\mathrm{e}^{2\pi t}-1}. \tag{2}$$

The first gives

$$B(s)=\left(\frac{2c}{b}-\frac{c}{b}\log a-\frac{\sqrt{\Delta}}{b}\log\frac{c+\sqrt{\Delta}}{c-\sqrt{\Delta}}\right)\zeta(2s-1)-\frac{1}{2}\log a\zeta(2s)+$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}n^{-2s}\left[\sum_{m=1}^{\infty}\frac{m\left[\zeta\left(m+1,\frac{c+\sqrt{\Delta}}{b}n\right)+\zeta\left(m+1,\frac{c-\sqrt{\Delta}}{b}n\right)\right]}{(m+1)(m+2)}-\frac{c}{3a}\frac{1}{n}\right].$$

Thus,

$$B(0)=-\frac{1}{12}\left(\frac{2c}{b}-\frac{c}{b}\log a-\frac{\sqrt{\Delta}}{b}\log\frac{c+\sqrt{\Delta}}{c-\sqrt{\Delta}}\right)+\frac{1}{4}\log a+$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{m\left[\zeta\left(m+1,\frac{c+\sqrt{\Delta}}{b}n\right)+\zeta\left(m+1,\frac{c-\sqrt{\Delta}}{b}n\right)\right]}{(m+1)(m+2)}-\frac{c}{3a}\frac{1}{n}\right].$$

and

$$\zeta'(0;a,b,c)=\frac{1}{2}\log 2\pi-\left(\frac{1}{4}+\frac{c}{12a}+\frac{c}{12b}\right)\log a-\frac{\sqrt{\Delta}}{12b}\log\frac{c+\sqrt{\Delta}}{c-\sqrt{\Delta}}+\frac{c}{6b}+\gamma\frac{c}{6a}+$$

$$-\frac{2c}{b}\zeta'(-1)+\frac{1}{2}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{m\left[\zeta\left(m+1,\frac{c+\sqrt{\Delta}}{b}n\right)+\zeta\left(m+1,\frac{c-\sqrt{\Delta}}{b}n\right)\right]}{(m+1)(m+2)}-\frac{c}{3a}\frac{1}{n}\right].$$

Notice that to compute the limit for $c\to 0$ of the above expressions is not easy.
Using the second representation for K, equation (2), and assuming a,b and c positive[2]

[2] A simple computation show that this formula holds for real or pure imaginary $\sqrt{\Delta}$ as well. Moreover, the assumption that a,b,c are positive in not necessary; in general, the sign of the number $\mathrm{Re}(z)/(1+\mathrm{Im}(z))$, where $z=b/(c+\sqrt{\Delta})$ will give the sign of the term $-\log\left(1-\mathrm{e}^{-2\pi\sqrt{\frac{a}{b}}n}\right)$ appearing in formula (3).

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$$K = 2\int_0^\infty \arctan\frac{2cnt}{an^2 - bt^2}\frac{dt}{e^{2\pi t}-1} - \log\left(1 - e^{-2\pi\sqrt{\frac{a}{b}}n}\right). \tag{3}$$

This can be used both to see that the sum appearing in $\zeta_2'(0;a,b,c)$ is regular at $s=0$, and to deal with the case $c=0$, as will be done in the last section. For the first point, just consider that, for large n,

$$\int_0^\infty \frac{\arctan\frac{2ctn}{an^2-bt^2}}{e^{2\pi t}-1}dt = \frac{c}{12a}\frac{1}{n} + O(n^{-3}).$$

The complete formula for the derivative with the representation for K given in formula (3) is stated in the following theorem, that generalizes the Kronecker first limit formula to the function $\zeta_2(s;A)$. Notice that the assumption of positivity of the coefficients a, b and c in the theorem can be omitted if we use the representation of K given in formula (1), but then we need to assume the quadratic from associated to the matrix A to be positive definite on $\mathbb{N}_0$.

Theorem 3.1. *The function $\zeta_2(s;A)$ has the following expansion near $s=0$:*

$$\zeta_2(s;A) = \zeta_2(0;A) + \zeta_2'(0;A)s + O(s^2),$$

where $A = \begin{pmatrix} a & c \\ c & b\end{pmatrix}$, with real $a, b, c > 0$, $\Delta = c^2 - ab = -\det A$,

$$\zeta_2(0;A) = \frac{1}{4} + \frac{c}{12a} + \frac{c}{12b}.$$

and

$$\zeta_2'(0;A) = \frac{1}{2}\log 2\pi - \left(\frac{1}{4} + \frac{c}{12a} + \frac{c}{12b}\right)\log a - \frac{\sqrt{\Delta}}{12b}\log\frac{c+\sqrt{\Delta}}{c-\sqrt{\Delta}} + \frac{c}{6b} + \gamma\frac{c}{6a} +$$

$$-\frac{2c}{b}\zeta'(-1) - \frac{\pi}{12}\sqrt{\frac{a}{b}} - \log\eta\left(i\sqrt{\frac{a}{b}}\right) + 2\sum_{n=1}^\infty\left[\int_0^\infty \frac{\arctan\frac{2cnt}{an^2-bt^2}}{e^{2\pi t}-1}dt - \frac{c}{12a}\frac{1}{n}\right].$$

where $\eta(z) = e^{\frac{\pi}{12}iz}\prod_{n=1}^\infty\left(1 - e^{2\pi i n z}\right)$ is the Dedekind eta function [5].

The value of $\zeta_2(0;A)$ is consistent with the one given in [24] for the particular case of matrix A with integer coefficients.

4. PARTICULAR SUBCASES

4.1. Case $c = 0$

The case $c = 0$ can be dealt with directly using the Kronecker first limit formula ([15], [16]) or taking the limit of the result obtained in Section 3, and this shows that Theorem 3.1 reduces continuously to the classical Kronecker formula. We get

$$\zeta_2(0; a, b, 0) = \frac{1}{4},$$

$$\zeta_2'(0; a, b, 0) = \frac{1}{2}\log 2\pi - \frac{1}{4}\ln a - \ln \eta\left(i\sqrt{\frac{a}{b}}\right) =$$

$$= \frac{1}{2}\ln 2\pi - \frac{1}{4}\log a + \frac{\pi}{12}\sqrt{\frac{a}{b}} - \log \prod_{n=1}^{\infty}\left(1 - \mathrm{e}^{-2\pi\sqrt{\frac{a}{b}}n}\right).$$

4.2. Case $\Delta = 0$

When $\Delta = 0$, $c = \sqrt{ab}$, $an^2 + 2cnk + bk^2 = (\sqrt{a}n + \sqrt{b}k)^2$, and thus we reduce to the linear case, namely

$$\zeta_2(s; a, b, \sqrt{ab}) = \zeta_1(2s; \sqrt{a}, \sqrt{b}),$$

where $\zeta_1(s; a, b)$ is defined for real positive a and b by the sum

$$\zeta_1(s; a, b) = \sum_{n,k=1}^{\infty}(an + bk)^{-s},$$

when $\mathrm{Re}(s) > 2$, and by analytic continuation elsewhere. For completeness we give the values of $\zeta_1(0; a, b)$ and $\zeta_1'(0; a, b)$ that can be obtained from Theorem 3.1 or by direct application of Lemma 2.1. We give two equivalent formulas for the derivative.

$$\zeta_1(0; a, b) = \frac{1}{4} + \frac{a}{12b} + \frac{b}{12a},$$

$$\zeta_1'(0; a, b) = \frac{b}{12a}(\gamma - \log a) + \frac{a}{12b}(1 - \log a) + \frac{1}{4}(\log 2\pi - \log a) - \frac{a}{b}\zeta'(-1) +$$

$$+ \begin{cases} \frac{1}{2}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\zeta\left(m+1, \frac{a}{b}n + 1\right) - \frac{b}{6a}\frac{1}{n}\right] \\ 2\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\int_0^{\infty}\arctan\frac{bt}{an}\frac{dt}{\mathrm{e}^{2\pi t}-1} - \frac{b}{24a}\frac{1}{n}\right]. \end{cases}$$

If say $a = 1$ and $0 < b < 2$, the value of $\zeta_1(0; a, b)$ can also be computed by applying equation (19) of [1]. The result is consistent with the one stated here.

4.3. Case $a = 0$

The semi linear case $a = 0$, does not reduce to any other one, and in fact it can be checked that we have a pole at $s = \frac{3}{2}$. Beside, applying the SDL we find out that the behavior at $s = 0$ reduces to the behavior of an appropriate linear case. We have $\alpha = 2$, $\beta = 1$, and we can chose to collect n or k when applying the SDL. In the two cases, we get

$$t_n(\lambda) = -\log \prod_{k=1}^{\infty} \left(1 + \frac{n(-\lambda)}{2cnk + bk^2}\right),$$

$$t_k(\lambda) = -\log \prod_{n=1}^{\infty} \left(1 + \frac{k(-\lambda)}{2cn + bk}\right).$$

Using the first, we find out that the domain of validity of the asymptotic expansion for large n does not contain the origin $\lambda = 0$, and hence we can not apply the SDL properly. Using the second expression, we see that this is the same function appearing in the linear case of $\zeta_1(s; b, 2c)$, and hence

$$\zeta_2(0; 0, b, c) = \zeta_1(0; b, 2c),$$

$$\zeta_2'(0; 0, b, c) = \zeta_1'(0; b, 2c).$$

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