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Well posedness, asymptotics and regularity of solutions to semilinear strongly damped wave equations in the Banach spaces $W_0^{1,p}(\Omega) \times L^p(\Omega)$

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In this paper we consider

$$u_{tt} + 2\eta A^{\frac{1}{2}} u_t + au_t + Au = f(u)$$

in the spaces $W_0^{1,p}(\Omega) \times L^p(\Omega)$, where A is the Dirichlet Laplacian in a bounded smooth domain Ω. We show local well posedness for continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(s) - f(t)| \leq C|s-t|(1 + |s|^{\frac{2p}{N-p}} + |t|^{\frac{2p}{N-p}})$ $(N > p)$ and prove that the solutions are classical. If f is dissipative and $\lim_{|s|\to\infty} f'(s)|s|^{\frac{-2}{N-2}} = 0$ $(N \geq 3)$, we show that the semigroup associate to the above equation has a global attractor in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ which is bounded in $W_p^2(\Omega) \cap W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ and coincide with the attractor $\boldsymbol{A}_{\eta,2}$ for the associated semigroup in $H_0^1(\Omega) \times L^2(\Omega)$. We also obtain that $\boldsymbol{A}_{\eta,2}$ is a compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ and attracts bounded subsets of $H_0^1(\Omega) \times L^2(\Omega)$ in the norm of $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for each $\mu \in (0,1)$. If, for each $p \geq 2$, the linear strongly damped wave operator $-\mathcal{A}_\eta$ generates an exponentially decaying analytic semigroup in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ for all $\eta \in (0,1]$ and all these semigroups are of the same type in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ we prove that the attractor $\boldsymbol{A}_0$ corresponding to the associated semigroup with $\eta = 0$ is a bounded subset of $C^{1+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega})$ for every $\mu \in (0,1)$. October, 2005 ICMC-USP

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1. INTRODUCTION

In this paper we study a family of damped wave equations

$$\begin{cases} u_{tt} + 2\eta A^{\frac{1}{2}} u_t + a u_t + A u = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, where A denote a negative Laplacian with homogeneous Dirichlet boundary condition in $X_p = L^p(\Omega)$, $p \in (1, \infty)$, $a > 0$, and $f : \mathbb{R} \to \mathbb{R}$ fulfils appropriate regularity and growth conditions.

The strongly damped wave equations have been already intensively studied in the Hilbert setting $H_0^1(\Omega) \times L^2(\Omega)$ and much progress have been achieved. In [8], [9] analyticity of the semigroup generated by the strongly damped wave operator $-\mathcal{A}_\eta = \left[\begin{smallmatrix} 0 & I \\ -A & -2\eta A^{\frac{1}{2}} - aI \end{smallmatrix}\right]$ in $H_0^1(\Omega) \times L^2(\Omega)$ was proved and characterization of the domains of fractional powers of $\mathcal{A}_\eta$ in terms of the domains of fractional powers of A above $L^2(\Omega)$ has also been given. In [6] the problems like (1.1) have been locally well posed in $H_0^1(\Omega) \times L^2(\Omega)$ allowing the critical growth of f; namely $f(u) \sim |u|^{\frac{N+2}{N-2}}$ $(N \geq 3)$. The results of [6] included even stronger damping terms $A^\theta u_t$, $\theta \in [\frac{1}{2}, 1]$, as well as the dependence of f on u_t, and were based on the consideration of the ε-regular solutions introduced in [3]. These mentioned ideas were further exploited in [6] where (1.1) with $f(u)$ dissipative and growing like $u^{\frac{N+2}{N-2}-}$ was shown to possess a compact global attractor in $H_0^1(\Omega) \times L^2(\Omega)$.

In [7] the existence of global attractors was obtained for the case when the nonlinearity satisfies (3) with $\rho = \frac{N+2}{N-2}$, $(N \geq 3)$.

In [4] it was proved the upper and lower semicontinuity of the family of attractors $\boldsymbol{A}_\eta$ for the semigroup associated to (1.1) in $H_0^1(\Omega) \times L^2(\Omega)$ when the nonlinearity f satisfies (1) and (3) with $\rho = \frac{N}{N-2}$, $(N \geq 3)$. A point to stress here is the fact that the growth is limited to that needed to obtain existence of global attractors for the problem (1.1) with $\eta = 0$ and to that needed to show uniform bounds (with respect to η) for the attractor in the space $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

In this paper we consider the problem (1.1) in the spaces $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in (1, \infty)$, and take advantage of the smoothing action of the solutions to abstract parabolic equations to obtain appropriate knowledge about regularity of the solutions to (1.1). We show that local existence of solutions can be obtained for non-linearities satisfying (3.7) with $\rho = \frac{N+p}{N-p}$ $(N > p)$ as well as that these solutions are classical. In case when f satisfies (3) with $\rho = \frac{N+2}{N-2}$ $(N \geq 3)$ we show that the attractors $\boldsymbol{A}_{\eta,p}$ for the semigroup associated to (1.1) in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ are bounded in $W_p^2(\Omega) \cap W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ and coincide with the attractor $\boldsymbol{A}_{\eta,2}$ obtained for the semigroup associated to (1.1) in $H_0^1(\Omega) \times L^2(\Omega)$. From this it follows that $\boldsymbol{A}_{\eta,2}$ is a compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ and attracts bounded subsets of $H_0^1(\Omega) \times L^2(\Omega)$ in the norm of $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for each $\mu \in (0, 1)$.

All our reasoning is based on the fact that $\mathcal{A}_\eta$ generates an analytic semigroup in $W_0^{1,p}(\Omega) \times L^p(\Omega)$. Though we are able to prove that this is the case for η in an open interval around $\eta = 1$ we have not yet been able to establish it in the whole interval $(0, 1]$.

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Nevertheless, if latter is true and, for all $\eta\in(0,1]$, the linear semigroups associated to $\mathcal{A}_\eta$ are exponentially decaying semigroups of the same type in $W_0^{1,p}(\Omega)\times L^p(\Omega)$, then the attractors $\boldsymbol{A}_{\eta,2}$ are bounded in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ uniformly for $\eta\in(0,1]$ provided that f is dissipative and satisfies (3) with $\rho=\frac{N}{N-2}$ $(N\geq 3)$. If the latter is true for all $p\geq 2$, from this and Sobolev embeddings we obtain that $\cup_{\eta\in(0,1]}\boldsymbol{A}_{\eta,2}$ is bounded in $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$. Using the fact that, in this case, $\boldsymbol{A}_{\eta,2}$ is upper and lower semicontinuous at $\eta=0$ we obtain that $\boldsymbol{A}_0$ is a bounded subset of $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$ for all $\mu\in(0,1)$.

This paper is organized in the following way. In Section 2 we prove that for a certain range of η the operators $\mathcal{A}_\eta$ are sectorial in $W_0^{1,p}(\Omega)\times L^p(\Omega)$ for each $p>1$. In Section 3 we prove embedding properties of fractional power spaces associated to $\mathcal{A}_\eta$. In Sections 4 and 5 we prove local well posedness results for (1.1) in $W_0^{1,p}(\Omega)\times L^p(\Omega)$ and present a bootstrapping procedure for these equations. In Sections 6 and 7 we prove the existence of global solutions as well as existence and regularity of attractors. In Section 8 we conjecture that, for each $p\geq 2$, $\mathcal{A}_\eta$ is sectorial in $W_0^{1,p}(\Omega)\times L^p(\Omega)$ for all $\eta\in(0,1]$ and that the semigroups associated to them are exponentially decaying semigroups of the same type. We then prove that $\boldsymbol{A}_0$ is a bounded subset of $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$ for all $\mu\in(0,1)$.

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2. LINEAR DAMPED WAVE EQUATIONS AND ANALYTIC SEMIGROUPS IN $X_P^{\frac12}\times X_P$

Let Ω be a bounded subdomain in $\mathbb{R}^N$, with C^2 boundary $\partial\Omega$ and let A denote a negative Laplacian in $X_p=L^p(\Omega)$, $p\in(1,\infty)$, with homogeneous Dirichlet boundary condition and domain $D(A)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Let X_p^α, $\alpha\geq 0$, $p\in(1,\infty)$, be a fractional power scale generated by (A,X_p). Since A in $L^p(\Omega)$, $p>1$, has bounded imaginary powers (see [15]) spaces X_p^α, $\alpha\in(0,1)$ can be characterized as complex interpolation spaces $[L^p(\Omega),W^{2,p}(\Omega)\cap W^{1,p}(\Omega)]_\alpha$ so that we have $X_p^\alpha=H^{2\alpha}_{p,\{I\}}(\Omega)$, the latter space being equal to $\{\phi\in H_p^{2\alpha}(\Omega):\ \phi_{|\partial\Omega}=0\}$ for $2\alpha p>1$ and to $H_p^{2\alpha}(\Omega)$ for $2\alpha p<1$. We refer to [16] for the definition of spaces $H_p^{2\alpha}(\Omega)$ as well as for the description of the intermediate space in the case when $2\alpha p=1$.

Consider a family of the damped wave operators

$$\mathcal{A}_\eta: D(A_\eta)\subset X_p^{\frac12}\times X_p\to X_p^{\frac12}\times X_p,\ \mathcal{A}_\eta=\begin{bmatrix}0 & -I\\ A & 2\eta A^{\frac12}+aI\end{bmatrix},$$

depending on parameter $\eta\geq 0$, where $a>0$ and

$$D_p(\mathcal{A}_\eta)=X_p^1\times X_p^{\frac12},\ \eta\geq 0,\ p\in(1,\infty).$$

Consider further in $X_p^{\frac12}\times X_p$ a norm $\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^{\frac12}\times X_p}=\|\phi\|_{X_p^{\frac12}}+\|\psi\|_{X_p}$, which is equivalent to the 'usual' $W_0^{1,p}(\Omega)\times L^p(\Omega)$ norm. If $p=2$ then $\mathcal{A}_\eta$ is a sectorial positive operator in a Hilbert space $H^1_0(\Omega)\times L^2(\Omega)$ possessing compact resolvent and bounded imaginary powers. In this case fractional power spaces associated to $\mathcal{A}_\eta$ can be viewed as complex interpolation spaces and its embedding properties are satisfactorily described (see [8, 9, 5]). If $p\neq 2$, then analogous results in the Banach spaces are so far unknown and - although expected - cannot be found in the existing references.

Our first concern is thus to fill this gap and obtain basic resolvent estimates that ensure $-\mathcal{A}_\eta$ generates an analytic semigroup in $X_p^{\frac12}\times X_p$ for each $p>1$.

Lemma 2.1. *The following conditions hold.*
i) There exists an interval $J\subset(0,\infty)$ such that if $\eta\in J$ then for every $p\in(1,\infty)$ operator $-\mathcal{A}_\eta$ generates in $X_p^{\frac12}\times X_p$ a C^0 analytic semigroup of bounded linear operators $\{e^{-\mathcal{A}_\eta t}\}$.
ii) $0\in\rho(\mathcal{A}_\eta)$ and $\mathcal{A}_\eta$ in $X_p^{\frac12}\times X_p$ has compact resolvent for each $\eta\geq 0$ and $p\in(1,\infty)$.
iii) the spectrum of $\mathcal{A}_\eta$ in $X_p^{\frac12}\times X_p$, $p\in(1,\infty)$, consists entirely of isolated eigenvalues which are given by the formula

$$\lambda_n^{\pm}=\frac12\big(a+2\eta\sqrt{\mu_n}\pm\sqrt{(a+2\eta\sqrt{\mu_n})^2-4\mu_n}\ \big),\ \mu_n\in\sigma(A).$$

iv) For each $\eta_0>0$ there is certain $d>0$ such that

$$d^{-1}\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^1\times X_p^{\frac12}}\leq\|\mathcal{A}_\eta\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^{\frac12}\times X_p}\leq d\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^1\times X_p^{\frac12}},\ \eta\in[0,\eta_0],\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in X_p^1\times X_p^{\frac12}. \tag{2.1}$$

Proof: In part i) let us consider first the case $\eta=1$. Since $\left[\begin{smallmatrix}0&0\\0&aI\end{smallmatrix}\right]$ defines a bounded operator in $X_p^{\frac12}\times X_p$ we will assume in this proof that $a=0$ in $\mathcal{A}_\eta$ and consider $\hat{\mathcal{A}}_\eta=\begin{bmatrix}0&-I\\ A&2\eta A^{\frac12}\end{bmatrix}$. It is then convenient to fix any sector $\mathcal{S}_{\theta,\lambda_1}$ of analicity of $A^{\frac12}$ such that $\lambda_1>0$ and

$$\|(\lambda I-A^{\frac12})^{-1}\|_{L(X_p)}\leq\frac{M}{|\lambda-\lambda_1|},\quad\text{for}\quad\lambda\in\mathcal{S}_{\theta,\lambda_1}. \tag{2.2}$$

Keeping in memory that $a=0$ and choosing arbitrary $\tilde\lambda_1\in(0,\lambda_1)$ we have the formula

$$\begin{aligned}(\lambda I-\hat{\mathcal{A}}_1)^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}&=\begin{bmatrix}\lambda I& I\\ -A&\lambda I-2A^{\frac12}\end{bmatrix}^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\\&=\begin{bmatrix}(\lambda I-2A^{\frac12})(\lambda I-A^{\frac12})^{-2}&-(\lambda I-A^{\frac12})^{-2}\\ A(\lambda I-A^{\frac12})^{-2}&\lambda(\lambda I-A^{\frac12})^{-2}\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix},\ \lambda\in\mathcal{S}_{\theta,\tilde\lambda_1},\end{aligned}$$

in which the components are estimated as follows:

$$\begin{aligned}&\|\lambda(\lambda I-A^{\frac12})^{-2}\phi\|_{X_p^{\frac12}}=\|\lambda(\lambda I-A^{\frac12})^{-2}A^{\frac12}\phi\|_{X_p}\\&\leq\frac{M^2|\lambda|}{|\lambda-\lambda_1|^2}\|A^{\frac12}\phi\|_X\leq\frac{C}{|\lambda-\lambda_1|}\|\phi\|_{X_p^{\frac12}},\ \lambda\in\mathcal{S}_{\theta,\tilde\lambda_1},\end{aligned} \tag{2.3}$$

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$$\begin{aligned}&\|2A^{\frac12}(\lambda I-A^{\frac12})^{-2}\phi\|_{X_p^{\frac12}}=2\|(-I+\lambda(\lambda I-A^{\frac12})^{-1})(\lambda I-A^{\frac12})^{-1}A^{\frac12}\phi\|_{X_p}\\&\le 2M\frac{1+\frac{M|\lambda|}{|\lambda-\lambda_1|}}{|\lambda-\lambda_1|}\|A^{\frac12}\phi\|_{X_p}\le\frac{2C_1}{|\lambda-\lambda_1|}\|\phi\|_{X_p^{\frac12}},\ \lambda\in\mathcal{S}_{\theta,\tilde\lambda_1},\end{aligned}\tag{2.4}$$

$$\begin{aligned}&\|(\lambda I-A^{\frac12})^{-2}\psi\|_{X_p^{\frac12}}=\|A^{\frac12}(\lambda I-A^{\frac12})^{-2}\psi\|_{X_p}\\&=\|(-I+\lambda(\lambda I-A^{\frac12})^{-1})(\lambda I-A^{\frac12})^{-1}\psi\|_{X_p}\le\frac{C_1}{|\lambda-\lambda_1|}\|\psi\|_{X_p},\ \lambda\in\mathcal{S}_{\theta,\tilde\lambda_1},\end{aligned}\tag{2.5}$$

$$\begin{aligned}&\|A(\lambda I-A^{\frac12})^{-2}\phi\|_{X_p}=\|(A^{\frac12}(\lambda I-A^{\frac12})^{-2})A^{\frac12}\phi\|_{X_p}\\&\le\frac{C_1}{|\lambda-\lambda_1|}\|A^{\frac12}\phi\|_{X_p}=\frac{C_1}{|\lambda-\lambda_1|}\|\phi\|_{X_p^{\frac12}},\ \lambda\in\mathcal{S}_{\theta,\tilde\lambda_1},\end{aligned}\tag{2.6}$$

$$\|\lambda(\lambda I-A^{\frac12})^{-2}\phi\|_{X_p}\le\frac{C}{|\lambda-\lambda_1|}\|\phi\|_{X_p},\ \lambda\in\mathcal{S}_{\theta,\tilde\lambda_1}.\tag{2.7}$$

Connecting the above estimates we obtain that

$$\|(\lambda I-\hat{\mathcal{A}}_1)^{-1}\|_{L(Y_p)}\le\frac{\tilde M}{|\lambda-\lambda_1|},\quad\text{for}\quad\lambda\in\mathcal{S}_{\theta,\tilde\lambda_1}.\tag{2.8}$$

which completes the proof of i) in case $\eta=1$. We can then prove analyticity for $\eta\in(1-\epsilon,1+\epsilon)$ with sufficiently small $\epsilon>0$.

To prove ii) note that there exists bounded inverse operator $A_\eta^{-1}:Y\to Y$

$$\mathcal{A}_\eta^{-1}=\begin{bmatrix}2\eta A^{-\frac12}+aA^{-1} & A^{-1}\\ -I & 0\end{bmatrix},\ \eta\ge0,\tag{2.9}$$

which takes bounded subsets of $X_p^{\frac12}\times X_p$ into bounded subsets of $X_p^1\times X^{\frac12}$, the latter space being compactly embedded in $X_p^{\frac12}\times X_p$ for every $p\in(1,\infty)$.

Part iii) follows from ii) and [8, Lemma A.1]. Part iv) can be proved exactly as Proposition 2.1 iv) in [4]. ▌

Corollary 2.1. *Suppose that $J\subset(0,\infty)$ is any bounded interval such that, for $\eta\in J$ operator $-\mathcal{A}_\eta$ generates in $X_p^{\frac12}\times X_p$, $p>1$, analytic semigroup $\{e^{-\mathcal{A}_\eta t}\}$. Then all these semigroups $\{e^{-\mathcal{A}_\eta t}\}$, $\eta\in J$, are compact and have the same decay rate which is independent of $p\in(1,\infty)$, $\eta\in J$; namely there exists $\epsilon>0$ such that*

$$\|A_\eta^\alpha e^{-\mathcal{A}_\eta t}\|_{L(Y_p)}\le c_{\alpha,\eta,p}t^{-\alpha}e^{-\epsilon t},\ t>0,\ \alpha\ge0,\ \eta\in J,\ p\in(1,\infty).\tag{2.10}$$

Proof: Since, if $n \in \mathbb{N}$ is fixed, each eigenvalue $\lambda_n^{\pm}$ decreases with respect to η the set $\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{A}_\eta),\ \eta \in J \subset (0,\infty)\}$ is bounded from below by a positive number $\epsilon = \epsilon(J)$ whenever J is a bounded set and the result follows easily. ▮

3. LOCAL WELL POSEDNESS RESULT IN $X_P^{\frac{1}{2}} \times X_P$

Define the Nemytskiĭ map $F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 \\ f(u) \end{smallmatrix}\right]$ and consider semilinear damped wave equations

$$\tfrac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right),\ t>0,\ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X_p^{\frac{1}{2}} \times X_p. \tag{3.1}$$

Lemma 3.1. *Let $p \in (1,\infty)$ and let Y_p denote the extrapolation space of $X_p^{\frac{1}{2}} \times X_p$ generated by $\mathcal{A}_\eta$. The following equality holds:*

$$Y_p = X_p \times X_p^{-\frac{1}{2}}. \tag{3.2}$$

Proof: Recall first that Y_p is the completion of the normed space $(Y_p, \|\mathcal{A}_\eta^{-1}\cdot\|_{Y_p})$. Since

$$\|\mathcal{A}_\eta^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y_p} \le c\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p\times X_p^{-\frac{1}{2}}} \le c'\|\mathcal{A}_\eta^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y_p},\ \begin{bmatrix}\phi\\ \psi\end{bmatrix} \in Y_p,$$

the completions of $(Y_p, \|\mathcal{A}_\eta^{-1}\cdot\|_{Y_p})$ and $(Y_p, \|\cdot\|_{X_p\times X_p^{-\frac{1}{2}}})$ coincide. ▮

Note that $\mathcal{A}_\eta$ can be extended to its closed Y_p-realization (see [2, p. 262]), which we will still denote by the same symbol so that $\mathcal{A}_\eta$ considered in Y_p is then sectorial positive operator (see [5]). Our next concern will be to obtain embeddings of the spaces from the fractional power scale Y_p^α, $\alpha \ge 0$, generated by $(\mathcal{A}_\eta, Y_p)$.

Lemma 3.2. *For each $p \in (1,\infty)$ and $0 \le \theta < \sigma \le 1$ we have that*

$$Y_p{}^\theta \hookrightarrow X_p^{\frac{\sigma}{2}} \times X_p^{\frac{\sigma-1}{2}}. \tag{3.3}$$

Proof: Fix $p \in (1,\infty)$, $0<\theta<\sigma<1$, and note that the matrix $\left[\begin{smallmatrix} A^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{smallmatrix}\right]$ represents a closed linear operator in Y_p whose domain is contained in Y_p. We also have

$$\begin{aligned}
&\|\mathcal{A}_\eta{}^\theta \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p} \le c\|\mathcal{A}_\eta \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^\theta \|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^{1-\theta} = c\|\left[\begin{smallmatrix}-\psi\\ A\phi+2\eta A^{\frac{1}{2}}\psi+a\psi\end{smallmatrix}\right]\|_{X_p\times X_p^{-\frac{1}{2}}}^\theta \|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^{1-\theta}\\
&\le c(\|\psi\|_{X_p} + \|A^{\frac{1}{2}}\phi + 2\eta\psi + aA^{-\frac{1}{2}}\psi\|_{X_p})^\theta \|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^{1-\theta}\\
&\le c(1+2\eta+a\lambda_1^{-1})^\theta (\|A^{\frac{1}{2}}\phi\|_{X^p} + \|\psi\|_{X^p})^\theta \|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^{1-\theta}\\
&= c(1+2\eta+a\lambda_1^{-1})^\theta \|\left[\begin{smallmatrix} A^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{smallmatrix}\right]\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^\theta \|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y_p}^{1-\theta},\ \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \in X_p^1 \times X_p^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

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From (3.4) and [14, Exercise 1.4.11], [11, Lemma 17.1] we obtain the estimate

$$\begin{aligned}&\|\mathcal{A}_\eta{}^\theta \begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y_p} \le \| \begin{bmatrix} A^{\frac{\sigma}{2}} & 0\\ 0 & A^{\frac{\sigma}{2}}\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y_p}\\ &= \|A^{\frac{\sigma}{2}}\phi\|_{X_p} + \|A^{\frac{\sigma-1}{2}}\psi\|_{X_p} = \|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X^{\frac{\sigma}{2}}\times X^{\frac{\sigma-1}{2}}},\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in X_p^1\times X_p^{\frac12},\end{aligned}$$

which leads to (3.3). ▮

In the similar manner we prove that

Lemma 3.3. *For each $p\in(1,\infty)$ and $0\le\alpha<\gamma\le 1$ we have*

$$Y_p{}^{1+\gamma}\hookrightarrow X^{\frac{1+\alpha}{2}}\times X^{\frac{\alpha}{2}}. \tag{3.5}$$

Proof: Fix $p\in(1,\infty)$, $0<\alpha<\gamma<1$, and note that matrix $\begin{bmatrix} A^{\frac{\alpha}{2}} & 0\\ 0 & A^{\frac{\alpha}{2}}\end{bmatrix}$ represents a closed linear operator in $X_p^{\frac12}\times X_p$ whose domain contains $X_p^1\times X_p^{\frac12}$. We then obtain the subordination condition

$$\begin{aligned}&\|\begin{bmatrix} A^{\frac{\alpha}{2}} & 0\\ 0 & A^{\frac{\alpha}{2}}\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^{\frac12}\times X_p} = \|A^{\frac{1+\alpha}{2}}\phi\|_{X_p} + \|A^{\frac{\alpha}{2}}\psi\|_{X_p} = \|(A^{\frac12})^\alpha A^{\frac12}\phi\|_{X_p} + \|(A^{\frac12})^\alpha\psi\|_{X_p}\\ &\le \|(A^{\frac12})^\alpha(A^{\frac12}\phi+2\eta\psi+aA^{-\frac12}\psi)\|_{X_p} + (1+2\eta+a\lambda_1^{-1})\|(A^{\frac12})^\alpha\psi\|_{X_p}\\ &\le c\|A^{\frac12}(A^{\frac12}\phi+2\eta\psi+aA^{-\frac12}\psi)\|_{X_p}^\alpha \|(A^{\frac12}\phi+2\eta\psi+aA^{-\frac12}\psi)\|_{X_p}^{1-\alpha}\\ &\qquad + (1+2\eta+a\lambda_1^{-1})c\|A^{\frac12}\psi\|_{X_p}^\alpha\|\psi\|_{X_p}^{1-\alpha}\\ &\le c\|\begin{bmatrix} 0 & -I\\ A & 2\eta A^{\frac12}+aI\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|^\alpha_{X_p^{\frac12}\times X_p}(\|A^{\frac12}\phi\|_{X_p}+(2\eta+a\lambda_1^{-1})\|\psi\|_{X_p})^{1-\alpha}\\ &\qquad + (1+2\eta+a\lambda_1^{-1})c\|\begin{bmatrix} 0 & -I\\ A & 2\eta A^{\frac12}+aI\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|^\alpha_{X_p^{\frac12}\times X_p}\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|^{1-\alpha}_{X_p^{\frac12}\times X_p}\\ &\le 2c(1+2\eta+a\lambda_1^{-1})\|\mathcal{A}_\eta\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|^\alpha_{X_p^{\frac12}\times X_p}\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|^{1-\alpha}_{X_p^{\frac12}\times X_p},\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in X_p^1\times X_p^{\frac12}.\end{aligned} \tag{3.6}$$

As a consequence of (3.6) and [14, Exercise 1.4.11] (see also [11, Lemma 17.1]) we obtain relation

$$\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X^{\frac{\alpha+1}{2}}\times X^{\frac{\alpha}{2}}} = \|\begin{bmatrix} A^{\frac{\alpha}{2}} & 0\\ 0 & A^{\frac{\alpha}{2}}\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^{\frac12}\times X_p} \le \|\mathcal{A}_\eta{}^\gamma\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{X_p^{\frac12}\times X_p},\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in X_p^1\times X_p^{\frac12},$$

and thus also embedding (3.5). ▮

Lemma 3.4. *Let J be as in Corollary 2.1 and $\eta\in J$ be arbitrarily fixed.*

Problem (3.1) *is locally well posed in $X_p^{\frac12}\times X_p$, $p\in(1,\infty)$, provided that*
i) if $p>N$ then $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz continuous function,
ii) if $p<N$ then f satisfies condition

$$|f(s_1)-f(s_2)|\leqslant c|s_1-s_2|(1+|s_1|^{\rho-1}+|s_2|^{\rho-1}),\ s_1,s_2\in\mathbb{R}^1, \tag{3.7}$$

with certain $1 < \rho < \frac{N+p}{N-p}$,
iii) if $p = N$ *then* (3.7) *holds with arbitrarily fixed* $\rho \in (1, \infty)$.

Proof: The result in this lemma follows easily from Hölder's inequality and Sobolev type embeddings. Similar results can be found in [3]. We only include some sketch of a proof to indicate how these apply to our case.

If condition i) is satisfied then the Nemytskiĭ map F is Lipschitz continuous on bounded sets from $X_p^{\frac{1}{2}} \times X_p$ into $X_p^{\frac{1}{2}} \times X_p$ which is a consequence of the embedding $X_p^{\frac{1}{2}} \hookrightarrow L^\infty(\Omega)$. The same becomes true in the case of iii) as a result of the embedding $X_p^{\frac{1}{2}} \hookrightarrow L^q(\Omega)$ valid with any $q \in [1, \infty)$.

In case when ii) holds, F is Lipschitz continuous on bounded sets from $X_p^{\frac{1}{2}} \times X_p$ into $X_p^\sigma \times X_p^{\frac{\sigma-1}{2}} \subset Y_p^\theta$ whenever $0 \le \theta < \sigma \le \tilde{\sigma}$ and $\tilde{\sigma} = \frac{N}{p} + 1 - \rho(\frac{N}{p} - 1)$. Indeed, for such choice of the parameters, if B is a bounded subset of Y_p^1 and $\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \in B$, we obtain that

$$\begin{aligned}
&\|F\left(\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}\right) - F\left(\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\right)\|_{Y_p^\theta} \le c\|F\left(\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}\right) - F\left(\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\right)\|_{X^{\frac{\sigma}{2}} \times X^{\frac{\sigma-1}{2}}} \\
&= c\|f(\phi_1) - f(\phi_2)\|_{X^{\frac{\sigma-1}{2}}} \le c_1\|f(\phi_1) - f(\phi_2)\|_{L^{\frac{Np}{N+(1-\sigma)p}}(\Omega)} \\
&\le c_2\|\phi_1 - \phi_2\|_{X_p^{\frac{1}{2}}}\|1 + |\phi_1|^{\rho-1} + |\phi_2|^{\rho-1}\|_{L^{\frac{Np}{(1-\sigma)p+p}}(\Omega)} \\
&\le c_2\|\phi_1 - \phi_2\|_{X_p^{\frac{1}{2}}}(\|1\|_{L^{\frac{Np}{(1-\sigma)p+p}}(\Omega)} + \|\phi_1\|^{\rho-1}_{L^{\frac{Np(\rho-1)}{(1-\sigma)p+p}}(\Omega)} + \|\phi_2\|^{\rho-1}_{L^{\frac{Np(\rho-1)}{(1-\sigma)p+p}}(\Omega)}).
\end{aligned} \tag{3.8}$$

We merely remark that if $\sigma \in [0, \sigma(\rho)]$ then as a consequence of the Sobolev embeddings the right hand side of (3.8) is bounded by $c(B)\|\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\|_{Y_p^1}$. ∎

4. BOOTSTRAPPING: ADDITIONAL REGULARITY OF SOLUTIONS

In what follows we apply the smoothing properties of parabolic equations to obtain additional regularity of solutions to (1.1). First recall that

$$\begin{aligned}
&Y_p^0 = L^p(\Omega) \times W^{-1,p}(\Omega), \quad Y_p^1 = W_0^{1,p}(\Omega) \times L^p(\Omega), \\
&Y_p^2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times L^p(\Omega), \quad p \in (1, \infty),
\end{aligned}$$

and rewrite the embedding in (3.3), (3.3) as

$$\begin{aligned}
&Y_p^{1+\gamma} \hookrightarrow W^{1+\gamma^-,p}(\Omega) \times W^{\gamma^-,p}(\Omega), \ 0 < \gamma^- < \gamma < 1, \ 1 < p < \infty, \\
&Y_p^{1+\gamma} \hookleftarrow W^{1+\gamma^+,p}(\Omega) \times W^{\gamma^+,p}(\Omega), \ -1 < \gamma < \gamma^+ < 0, \ 1 < p < \infty.
\end{aligned}$$

For arbitrarily fixed $\eta \in J$ and $p \in (1, \infty)$ consider next the solution

$$[0, \tau_0) \ni t \longrightarrow \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in Y_p^1$$

to (1.1) starting at $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_p^1 = W_0^{1,p}(\Omega)\times L^p(\Omega)$. Note that if $N>p$ then $F: Y_p^1 \to Y_p^\alpha$ for $\alpha\in(0,1]$ satisfying $\alpha<\min\tilde\sigma_p$, where $\tilde\sigma_p = \frac{N}{p}+1-\rho(\frac{N}{p}-1)$ (see the proof of Lemma 3.4).

It follows from the regularity properties of the solutions that, for any $t>0$ and $0<\beta<\alpha$,

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in Y_p^{1+\alpha} \hookrightarrow W^{1+\beta,p}(\Omega)\cap W_0^{1,p}(\Omega)\times W^{\beta,p}(\Omega) \hookrightarrow Y_q^1 = W_0^{1,q}(\Omega)\times L^q(\Omega), \quad q=\frac{Np}{N-\beta p}.$$

Also, for suitably chosen $\beta<\alpha^-<\alpha$, we have that

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t(t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in Y_p^{1+\alpha^-} \hookrightarrow W^{1+\beta,p}(\Omega)\cap W_0^{1,p}(\Omega)\times W^{\beta,p}(\Omega) \hookrightarrow Y_q^1 = W_0^{1,q}(\Omega)\times L^q(\Omega), \quad q=\frac{Np}{N-\beta p}.$$

Now, if $N>\beta q$ we observe in particular that $F: Y_q^1\to Y_q^\alpha$, where parameter α can stay the same as in the preceding step since $\tilde\sigma_p\le\tilde\sigma_q=\frac{N}{q}+1-\rho(\frac{N}{q}-1)$. Thus, restarting the solution at any positive instant of time and taking advantage of the solvability of (1.1) in Y_q^1 we obtain

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in Y_q^{1+\alpha} \hookrightarrow W^{1+\beta,q}(\Omega)\cap W_0^{1,q}(\Omega)\times W^{\beta,q}(\Omega) \hookrightarrow Y_{\bar q}^1 = W_0^{1,\bar q}\times L^{\bar q}(\Omega), \quad \bar q=\frac{Nq}{N-\beta q}=\frac{Np}{N-2\beta p},$$

and

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t(t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in Y_q^{1+\alpha^-} \hookrightarrow W^{1+\beta,q}(\Omega)\cap W_0^{1,q}(\Omega)\times W^{\beta,q}(\Omega) \hookrightarrow Y_{\bar q}^1 = W_0^{1,\bar q}(\Omega)\times L^{\bar q}(\Omega), \quad \bar q=\frac{Nq}{N-\beta q}=\frac{Np}{N-2\beta p}.$$

Proceeding by induction we observe that in a certain number of steps we will arrive at the embedding

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta),\ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t(t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in Y_{\hat q}^{1+\alpha^-} \hookrightarrow W^{1+\beta,\hat q}(\Omega)\cap W_0^{1,\hat q}(\Omega)\times W^{\beta,\hat q}(\Omega) \hookrightarrow Y_{\tilde q}^1 = W_0^{1,\tilde q}\times L^{\tilde q}(\Omega) \text{ where } \tilde q \text{ can be chosen arbitrarily large.}$$

Of course, for such $\tilde q$ we obtain that $F: Y_{\tilde q}^1\to Y_{\tilde q}^1$ which in turn gives us that

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in W^{2,\tilde q}(\Omega)\cap W_0^{1,\tilde q}(\Omega)\times W_0^{1,\tilde q}(\Omega)$$

as well as

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t(t,\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],\eta) \in W^{2^-,\tilde q}(\Omega)\cap W_0^{1,\tilde q}(\Omega)\times W_0^{1^-,\tilde q}(\Omega) \text{ with any } 2^-<2,\ 1^-<1.$$

Embedding of the Sobolev spaces into spaces of Hölder functions ensure further relation

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t (t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta), \ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in C^{1+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega}) \quad \text{for all} \ \ 0 < \mu < 1.$$

Reading finally (1.1) as an elliptic equation $\Delta u = u_{tt} + \eta(-\Delta)^{\frac{1}{2}} u_t + a u_t - f(u)$ and using Schauder type estimate[1] (see [12, Theorem 9.19]) we obtain that

$$u\,(t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in C^{2+\mu}(\bar{\Omega}) \ \text{ for arbitrary } \mu \in (0,1).$$

The above procedure leads to the following regularity result.

THEOREM 4.4.1. *Consider the set J as in Corollary 2.1 and fix arbitrary number $\eta \in J$. Then*
i) local solutions to (3.1) *resulting from Lemma 3.4 are classical solutions;*
ii) if $p \in (1, \infty)$ and B is a bounded subset of Y^1_p then there exists certain number $\tau = \tau(B, \eta) > 0$ such that local solutions $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ to (3.1) *originating at $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$ exist until the time τ and for any choice of $\mu \in (0,1)$ the set $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (\tau, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta);\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$ is a bounded subset of $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$.*

Proof: Part i) is a consequence of the smoothing action of the semigroup described at the beginning of this section.

For the proof of part ii) fix $\eta \in J$, $p \in (1, \infty)$ and consider a bounded subset B of Y^1_p. Let $\alpha \in (0,1]$ be such that $F : Y^1_p \to Y^\alpha_p$ is Lipschitz continuous on bounded sets (see the proof of Lemma 3.4). From Theorem 8.8.1 there exist $\tau_0 = \tau_0(B, \eta) > 0$ such that sets

$$B_{\tau_0} = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (\tau_0, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta);\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\} \ \text{ and } \ \dot{B}_{\tau_0} = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t (\tau_0, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta); \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$$

are bounded in $Y^{1+\alpha^-}_p$. As a consequence of Sobolev type embedding both these sets are bounded in $W^{1+\beta,p}(\Omega) \cap W^{1,p}_0(\Omega) \times W^{\beta,p}(\Omega)$ and thus also in $Y^1_{p_1} = W^{1,p_1}_0(\Omega) \times L^{p_1}(\Omega)$ whenever $p_1 = \frac{Np}{N - \beta p}$ and $\beta \in (0, \alpha)$.

If $N > \beta p_1$ we observe (similarily as at the beginning of this section) that F takes Y^1_q into Y^α_q and is Lipschitz continuous on bounded sets, where parameter α can stay the same as was chosen above. Thus, applying Theorem 8.8.1 to the solutions of (1.1) in Y^1_q we can find certain $\tau_1 > 0$ for which we have that

$$\begin{aligned} B_{\tau_0+\tau_1} &= \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (\tau_0 + \tau_1, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta);\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\} \ \text{ and} \\ \dot{B}_{\tau_0+\tau_1} &= \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t (\tau_0 + \tau_1, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta);\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\} \end{aligned}$$

are bounded subsets of $Y^{1+\alpha^-}_{p_2}$ and thus of $Y^1_{p_2} = W^{1,p_2}_0(\Omega) \times L^{p_2}(\Omega)$ with $p_2 = \frac{Np_1}{N - \beta p_1} = \frac{Np}{N - 2\beta p}$.

---

[1] We remark that since u_t belongs to $X^{1^-}_{\tilde{q}}$, then $A^{\frac{1}{2}} u_t$ belongs to $X^{\frac{1}{2}^-}_{\tilde{q}}$ and that the latter space is embedded in $C^{\mu}(\bar{\Omega})$ for large $\tilde{q} > N$.

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It is clear from the above procedure that after a finite number of steps we reach a number $p_k > 1$ such that both

$$B_{\tau_0+\ldots+\tau_k} = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\tau_0 + \tau_1 + \ldots + \tau_k, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)\,; \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$$

and

$$\dot{B}_{\tau_0+\ldots+\tau_k} = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t(\tau_0 + \tau_1 + \ldots + \tau_k, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)\,;\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$$

will be bounded in $Y_{p_k}^{1+\alpha^-} \hookrightarrow W^{1+\beta,p_k}(\Omega) \cap W_0^{1,p_k}(\Omega) \times W^{\beta,p_k}(\Omega)$ and simultaneously $N \leq \beta p_k$ so that both of these sets will also be bounded in

$$Y_{\tilde{q}}^1 = W_0^{1,\tilde{q}} \times L^{\tilde{q}}(\Omega) \text{ with arbitrarily large } \tilde{q} \in (N, \infty).$$

For each choice of $\tilde{q} \in (N, \infty)$ map $F : Y_{\tilde{q}}^1 \to Y_{\tilde{q}}^1$ is Lipschitz continuous on bounded sets and using again Theorem 8.8.1 we conclude that sets

$$B_{\tau_0+\ldots+\tau_k+\tau_{\tilde{q}}} = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\tau_0 + \tau_1 + \ldots + \tau_k + \tau_{\tilde{q}}, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)\,; \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$$

and

$$\dot{B}_{\tau_0+\ldots+\tau_k+\tau_{\tilde{q}}} = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t(\tau_0 + \tau_1 + \ldots + \tau_k + \tau_{\tilde{q}}, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)\,;\ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$$

are now bounded in $W^{2^-,\tilde{q}}(\Omega) \cap W_0^{1,\tilde{q}}(\Omega) \times W_0^{1^-,\tilde{q}}(\Omega)$ and thus in $C^{1+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega})$ for certain $0 < \mu < 1$. Recalling finally that $\Delta u = u_{tt} + \eta(-\Delta)^{\frac{1}{2}} u_t + a u_t - f(u)$ and using Schauder type estimate[2] we conclude that $\{u(\tau_0 + \tau_1 + \ldots + \tau_k + \tau_{\tilde{q}}, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta), \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$ is also bounded in the norm of $C^{2+\mu}(\bar{\Omega})$ (see [12, Theorems 9.19, 6.6]).

Note that μ can be in fact arbitrarily close to 1 since the number $\tilde{q}$ considered in the above procedure can be chosen as big as it is required. Furthermore, 'time' $\tau := \tau_0 + \tau_1 + \ldots + \tau_k + \tau_{\tilde{q}}$ can be made arbitrarily close to zero, since τ is a sum of finitely many positive numbers which can be chosen as close to zero as it is needed. ▮

5. GLOBAL SOLUTIONS AND A FAMILY OF GLOBAL ATTRACTORS

Throughout this section we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies (3.7) with $\rho < \frac{N+2}{N-2}$ $(N \geq 3)$ and

$$\limsup_{|s| \to \infty} \frac{f(s)}{s} < \lambda_1, \tag{1}$$

where λ_1 is the first positive eigenvalue of A.

[2] We remark that since the set $\dot{B}_{\tau_0+\ldots+\tau_k+\tau_{\tilde{q}}}$ is bounded in a fractional power space $X_{\tilde{q}}^{1^-} \times X_{\tilde{q}}^{\frac{1}{2}^-}$, then $\{A^{\frac{1}{2}} u_t \ \ \tau_0 + \tau_1 + \ldots + \tau_k + \tau_{\tilde{q}}, \ \begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \ , \eta \ , \ \begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \ \in B\}$ is bounded in $X_{\tilde{q}}^{\frac{1}{2}^-}$. For large $\tilde{q} > N$ the latter set is thus bounded in $C^{\mu}(\bar{\Omega})$.

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Theorem 5.5.1. *Let J be as in Corollary 2.1 and $\eta \in J$ be arbitrarily fixed. If $p \geq 2$, then*
i) problem (3.1) *is globally well posed in Y^1_p and for any $\mu \in (0,1)$ positive obits of bounded subsets of Y^1_p are eventually bounded in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$,*
ii) if $\mathcal{E}$ is the set of stationary solutions to (3.1) *and $\mathcal{E}_1 = \{u_0 : \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right] \in \mathcal{E}\}$, for any $\mu \in (0,1)$ the set $\mathcal{E}_1$ (which is independent of η) is bounded in $C^{2+\mu}(\bar{\Omega})$.*

Proof: Let $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ be a local solution to (3.1) corresponding to $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^1_p$ with certain $p \geq 2$ and let $[0, \tau_{u_0,v_0})$ be its maximal interval of existence. Then in the light of [4] the set $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta),\ t \in [0, \tau_{u_0,v_0})\}$ remains bounded in Y^1_2. As a consequence of Theorem 4.4.1 there is a positive number $\tau < \tau_{u_0,v_0}$ such that $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta),\ t \in [\tau, \tau_{u_0,v_0})\}$ is bounded in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$, which proves global well posedness of (3.1) in Y^1_p. Since orbits of bounded subsets of Y^1_p are bounded in Y^2_p the remaining assertion of part i) also follows from Theorem 4.4.1. The proof of ii) is now obvious. ▮

Theorem 5.5.2. *Suppose that $\eta \in J$. Then, for every $p \geq 2$,*
i) problem (3.1) *defines in Y^1_p a C^0 semigroup $\{T_\eta(t)\}$ of global solutions which possesses a global attractor $\boldsymbol{A}_{\eta,p}$,*
ii) $\boldsymbol{A}_\eta$ is compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for any $\mu \in (0,1)$,
iii) the attractors $\boldsymbol{A}_{\eta,p}$ do not depend on $p \geq 2$; namely

$$\boldsymbol{A}_{\eta,p} = \boldsymbol{A}_{\eta,2} =: \boldsymbol{A}_\eta \quad \textit{for each} \quad p \geq 2,$$

iv) any set B bounded in Y^1_2 is attracted by $\boldsymbol{A}_\eta$ in the sense of Hausdorff semidistance in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ with arbitrarily chosen $\mu \in (0,1)$.

Proof: Recall that the semigroups of global solutions to (3.1) in Y^1_p are compact for every $\eta \in J$ and $p \in (1,\infty)$ and, as a consequence of Theorem 5.5.1 they have bounded orbits of bounded sets whenever $p \geq 2$. Recall also that the set of stationary solutions is bounded in each space Y^1_p we consider here and that there exists a Lyapunov functional

$$\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) = \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|A^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} - \int_\Omega \int_0^{w_1} f(s)dsdx,\ \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in Y^1_p,\ p \geq 2, \tag{2}$$

which is nonincreasing along the solutions. This justifies the existence of compact attractors (see [13]) $\boldsymbol{A}_{\eta,p}$, $p \geq 2$, which - in the light of Theorem 5.5.1 and their invariance with respect to $t \geq 0$ - are bounded in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$, $\mu \in (0,1)$, and thus need to coincide with $\boldsymbol{A}_{\eta,2}$. Since this holds for all $\mu \in (0,1)$ the compactness follows easily from Arzelá-Ascoli Theorem.

Now, if B is bounded in Y^1_2 and μ is any number from $(0,1)$, then the orbit $\gamma(B)$ of B is eventually bounded in $C^{2+\tilde{\mu}}(\bar{\Omega}) \times C^{1+\tilde{\mu}}(\bar{\Omega})$ where $\mu < \tilde{\mu} < 1$. This ensures that $\gamma(B)$ is eventually precompact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ using the Arzelá-Ascoli Theorem. If we thus supposed that $\boldsymbol{A}_\eta$ did not attract B in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$, then there would exist a sequence $\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B$ and a sequence of times $t_n \to \infty$ such that $\{T_\eta(t_n)\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\}$ could not possess a subsequence convergent in the $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$-norm to any point of $\boldsymbol{A}_\eta$. However, there is certainly a subsequence $\{T_\eta(t_{n_k})\left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right]\}$ of $\{T_\eta(t_n)\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\}$ convergent to certain element of $\boldsymbol{A}_\eta$ in Y_2^1-norm. Since $\{T_\eta(t_n)\left[\begin{smallmatrix}u_n\\v_n\end{smallmatrix}\right]\}$ is precompact in $C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$ the supposition was absurd. ∎

6. EQUATIONS IN Y_2^1 WITH ALMOST CRITICAL NONLINEARITES

Throughout this section we assume that $\eta\in J$. Following [3] we consider in this section well posedness, regularity and the existence of attractor in a 'phase space' Y_p^1, $p\in(1,\infty)$ in the more general case when F is an ε-regular map relative to the pair (Y_p^1,Y_p). The latter means that there exist constants $C>0$, $\rho>1$, $\varepsilon\in(0,\frac{1}{\rho})$, and $1>\gamma\geq\rho\varepsilon$ such that

$$\begin{aligned}&\|F(\left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right])\|_{Y_p^\gamma}\\&\quad\leq C\|\left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right]\|_{Y_p^{1+\varepsilon}}\left(1+\|\left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right]\|^{\rho-1}_{Y_p^{1+\varepsilon}}+\|\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right]\|^{\rho-1}_{{Y_p}^{1+\varepsilon}}\right),\ \left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right],\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right]\in Y_p^{1+\varepsilon}.\end{aligned}\tag{1}$$

As in [7] we will single out for special attention a proper subclass of the class of maps ε-regular with respect to (Y_p^1,Y_p). Namely, we will consider F being *almost critical* ε-regular map relative to the pair (Y_p^1,Y_p), which means that there are constants $c>0$, $\rho>1$, $\varepsilon\in(0,\frac{1}{\rho})$ and, for any $\delta>0$, there is $C_\delta>0$ such that

$$\begin{aligned}&\|F(\left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right])\|_{Y_p^{\rho\varepsilon}}\\&\leq C\|\left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right]\|_{Y_p^{1+\varepsilon}}\left(C_\delta+\delta\|\left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right]\|^{\rho-1}_{Y_p^{1+\varepsilon}}+\delta\|\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right]\|^{\rho-1}_{{Y_p}^{1+\varepsilon}}\right),\ \left[\begin{smallmatrix}\phi_1\\\psi_1\end{smallmatrix}\right],\left[\begin{smallmatrix}\phi_2\\\psi_2\end{smallmatrix}\right]\in Y_p^{1+\varepsilon}.\end{aligned}\tag{2}$$

Our motive for considering the above property is connected with the fact that - although the nonlinearity is allowed to be critical in the sense that it does not take Y_p^1 into Y_p^α for any $\alpha\in(0,1]$ - nevertheless, global in time continuation property still follows as a consequence of the Y_2^1 bound on the solution.

It has been proved in [7, Section 3] (see also [6, Theorem 4]) that

Proposition 6.1. *If $N\geq 3$ and $f:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable function satisfying*

$$\lim_{|s|\to\infty}\frac{|f'(s)|}{|s|^{\rho-1}}=0\tag{3}$$

with $\rho=\frac{N+2}{N-2}$, then F is an almost critical $\varepsilon-$regular map relatively to (Y_2^1,Y_2) for each $\varepsilon\in[0,\frac{1}{2\rho}]$.

Consequently, if $\eta>0$, $\left[\begin{smallmatrix}\bar u_0\\\bar v_0\end{smallmatrix}\right]\in Y_2^1$ then, given $r>0$ there is certain $\tau_0>0$ with the property that for each $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in B_{Y_2^1}(\left[\begin{smallmatrix}\bar u_0\\\bar v_0\end{smallmatrix}\right],r)$ there exists a unique ε-regular solution $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](\cdot,u_0,v_0,\eta)$ to (3.1) such that
i) $t^\varsigma\|\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](t,u_0,v_0,\eta)\|_{Y_2^{1+\varsigma}}\to 0$ as $t\to 0^+$, $0<\varsigma<\frac{1}{2}$,

ii) $t^{\varsigma}\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t,u_1,v_1,\eta) - \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t,u_2,v_2,\eta)\|_{Y_2^{1+\varsigma}} \leq C' \| \left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right] \|_{Y_2^1}$ *whenever* $t \in [0,\tau_0]$, $0 \leq \varsigma \leq \varsigma_0 < \frac{1}{2}$, $\left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right] \in B_{Y_2^1}(\left[\begin{smallmatrix} \overline{u}_0 \\ \overline{v}_0 \end{smallmatrix}\right], r)$,

iii) $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot,u_0,v_0,\eta) \in C((0,\tau_0],Y_2^{1+\frac{1}{2}}) \cap C^1((0,\tau_0],Y_2^{1+\varsigma})$ *for* $0 \leq \varsigma < \frac{1}{2}$*; in particular the solution,* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot,u_0,v_0,\eta)$ *satisfies both relations in* (3.1).

iv) $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot,u_0,v_0,\eta)$ *exist for all* $t \geq 0$ *whenever it is bounded in* Y_2^1 *in its maximal interval of existence.*

If, in addition, dissipativeness condition (1) *hold, then* ε*-regular solutions* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot,u_0,v_0,\eta)$ *to* (3.1) *corresponding to* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_2^1$ *exist globally in time,*

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot,u_0,v_0,\eta) \in C\big((0,\infty),Y_2^2\big) \cap C^1\big((0,\infty),Y_2^1\big),$$

and (1.1) *possesses a global attractor* $\boldsymbol{A}_{\eta,2}$ *which consists of the elements of* Y_2^2*.*

Regularity properties of the solutions reported in Proposition 6.1 ensures that the bootstrapping procedure described in Section 4 applies in the case ‘almost critical nonlinearity’ as well. We thus conclude that

Corollary 6.1. *Let J be as in Corollary 2.1 and $\eta \in J$ be arbitrarily fixed. Suppose that* $N \geq 3$, $f : \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable function satisfying* (3) *with* $\rho = \frac{N+2}{N-2}$ *and that* (1) *hold. Then global* ε*-regular solutions* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot,u_0,v_0,\eta)$ *to* (3.1) *starting at* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_2^1$ *resulting from Proposition 6.1 are of the class* $C((0,\infty),C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega}))$, $\mu \in (0,1)$, *and for each* $\mu \in (0,1)$ *the corresponding semigroup* $\{T_\eta(t)\}$ *takes bounded subsets of* Y_2^1 *into precompact subsets of* $C((0,\infty),C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega}))$.

The attractor $\boldsymbol{A}_{\eta,2}$ *for* $\{T_\eta(t)\}$ *is a subset of* $C((0,\infty),C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega}))$ *and it attracts bounded subsets of* Y_2^1 *in the sense of Hausdorff semidistance in* $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$.

In fact, conditions i), ii),iii) of Theorem 5.5.2 hold for every $p \geq 2$.

7. FURTHER CONSIDERATIONS

In this section we will proceed with the following assumptions

I) Assume that for all $\eta \in (0,1]$, A_η is a sectorial operator in $Y_p^1 = W_0^{1,p}(\Omega) \times L^p(\Omega)$.

II) Assume that there are constants $c_p \geq 1$ and $\omega_p < 0$ independent of $\eta \in (0,1]$ such that

$$\|e^{-\mathcal{A}_\eta t}\|_{L(Y_p^1)} \leq c_p e^{\omega_p t}, \ t \geq 0. \tag{1}$$

Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies (1) and (3) with $\rho = \frac{N}{N-2}$. It follows from Theorem 5.5.1 that the problem (3.1) is globally well posed in Y_p^1 and that the set of stationary solutions to (3.1) is bounded in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$. From Theorem 5.5.2 we have that for $p \geq 2$ the problem (3.1) has (independent of p) global attractor $\boldsymbol{A}_{\eta,p} = \boldsymbol{A}_{\eta,2} = \boldsymbol{A}_\eta$. In what follows we obtain the uniform bounds for the family of attractors $\{\mathcal{A}_\eta : \eta \in (0,1]\}$ in $Y_p^2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

The proof of our next result follows line by line the proof of Lemma 3.5 in [4].

LEMMA 7.1. *Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies (1) and (3) with $\rho=\frac{N}{N-2}$ ($N\geq 3$). For each $p\geq 2$ suppose that I) and II) hold. Then, for each $\eta>0$, any complete orbit lying on $\boldsymbol{A}_\eta$ is precompact in Y_p^2 and $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is bounded in Y_p^2 for each fixed $p\geq 2$.*

Proof: From Theorem 5.5.2 ii) we have that orbits of points in $\boldsymbol{A}_\eta$ are precompact in Y_p^2.

Now let us prove that $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is bounded in Y_p^2. First note that, from Theorem 5.5.1, the set $\mathcal{E}$ of equilibrium points for (3.1) is a bounded subset of Y_p^2; that is, there is a constant $M_1^{\mathcal{E}}$ such that

$$\sup\left\{\|\left[\begin{smallmatrix}u_0\\0\end{smallmatrix}\right]\|_{Y_p^2}:\left[\begin{smallmatrix}u_0\\0\end{smallmatrix}\right]\in\mathcal{E}\right\}\leq M_1^{\mathcal{E}}.$$

Note that, every solution in the attractor must approach, in Y_p^2 norm, the set $\mathcal{E}$ as time goes to $-\infty$. Following this scheme, we have

$$T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]=e^{-\mathcal{A}_\eta t}T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]+\int_0^t e^{-\mathcal{A}_\eta(t-s)}F(T_\eta(s)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right])ds$$

and, choosing $\varepsilon>0$, we find certain $N_\varepsilon\in\mathbb{N}$ such that for all $k_l\geqslant N_\varepsilon$

$$\|T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]\|_{Y_p^2}\leqslant M_1^{\mathcal{E}}+\epsilon.$$

From this we have that

$$\begin{aligned}
&\|\mathcal{A}_\eta T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]\|_{Y_p^1}\\
&\qquad\leqslant\|e^{-\mathcal{A}_\eta t}\|_{L(Y_p^1)}\|\mathcal{A}_\eta T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]\|_{Y_p^1}+\|\left(I-e^{-\mathcal{A}_\eta t}\right)F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\|_{Y_p^1}\\
&\qquad+\int_0^t\|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_p^1)}\|\mathcal{A}_\eta\left(F(T_\eta(s)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\right)\|_{Y_p^1}ds\\
&\qquad\leq c_pe^{\omega_p t}(M_1^{\mathcal{E}}+\varepsilon)+(1+c_pe^{\omega_p t})\|F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\|_{Y_p^1}\\
&\qquad+\int_0^t c_pe^{\omega_p(t-s)}\|\mathcal{A}_\eta\left(F(T_\eta(s)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\right)\|_{Y_p^1}ds.
\end{aligned}\tag{2}$$

Recall that (3) with $\rho=\frac{N}{N-2}$ implies

$$\forall_{\nu>0}\exists_{C_\nu>0}\;\forall_{s\in\mathbb{R}}\;|f'(s)|\leqslant\nu|s|^{\frac{2}{N-2}}+C_\nu$$

and that $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is bounded in Y_2^2 (see Lemma 3.5 in [4]).

If $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right] =: \left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]$, then we have, for each $\eta \in (0,1]$,

$$\begin{aligned} \|\mathcal{A}_\eta\big(F\left(\left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]\right) - F\left(\left[\begin{smallmatrix} 0 \\ 0\end{smallmatrix}\right]\right)\big)\|_{Y_p^1} &= \|\mathcal{A}_\eta\left[\begin{smallmatrix} 0 \\ f(u_\eta)-f(0)\end{smallmatrix}\right]\|_{Y_p^1} \\ &\leqslant C_{\eta_0}\|\nabla(f(u_\eta)-f(0))\|_{L^p(\Omega)} + a\|f(u_\eta)-f(0)\|_{L^p(\Omega)} \\ &\leqslant \frac{-\omega_p}{2c_p}\nu\|\mathcal{A}_\eta\left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]\|_{Y_p^1} + C_\nu. \end{aligned} \tag{3}$$

From (2) and (3) we obtain

$$\begin{aligned} \|\mathcal{A}_\eta\left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]\|_{Y_p^1} &\leqslant c_p e^{\omega_p t}(M_1^{\mathcal{E}}+\varepsilon) + (1+c_p e^{\omega_p t})\|F(\left[\begin{smallmatrix} 0 \\ 0\end{smallmatrix}\right])\|_{Y_p^1} \\ &\quad + \int_0^t c_p e^{\omega_p(t-s)}(\nu\|\mathcal{A}_\eta\left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]\|_{Y_p^1} + C_\nu)ds \\ &\leqslant c_p(M_1^{\mathcal{E}}+\varepsilon) + (1+c_p)\|F(\left[\begin{smallmatrix} 0 \\ 0\end{smallmatrix}\right])\|_{Y_p^1} + \frac{1}{2}\nu\sup_{t\geq 0}\|\mathcal{A}_\eta\left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]\|_{Y_p^1} - \frac{c_pC_\nu}{\omega_p}. \end{aligned}$$

Taking the supremum and estimating, we find that

$$\sup_{t\in[0,\infty)}\|\mathcal{A}_\eta\left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]\|_{Y_p^1} \leqslant 2c_p(M_1^{\mathcal{E}}+\varepsilon) + 2(1+c_p)\|F(\left[\begin{smallmatrix} 0 \\ 0\end{smallmatrix}\right])\|_{Y_p^1} - 2\frac{c_pC_\nu}{\omega_p} \leqslant M,$$

where $M>0$ is independent of $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right] \in \boldsymbol{A}_\eta$ and of $\eta\in(0,1]$. Since $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right] = \left[\begin{smallmatrix} u_{\eta} \\ v_{\eta}\end{smallmatrix}\right]$, the above considerations ensure that for any $\eta\in(0,1]$ and arbitrarily chosen $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right] \in \boldsymbol{A}_\eta$ we have the uniform bound

$$\|\mathcal{A}_\eta\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right]\|_{Y_p^1} \leqslant M.$$

From this and (2.1) we obtain that

$$\sup_{\eta\in(0,1)}\ \sup_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right]\in\boldsymbol{A}_\eta}\|\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right]\|_{Y_p^2} < \infty, \tag{4}$$

which completes the proof. ▌

Corollary 7.1. *The union of attractors* $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ *is bounded in* $C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)$ *for any* $\mu\in(0,1)$ *and hence it is also precompact in each of these spaces.*

Proof: If $\mu\in(0,1)$ is fixed and $p\geq 2$ is chosen such that $0<\mu<1-\frac{N}{p}$, then using Sobolev embedding and (4) we infer that

$$\sup_{\eta\in(0,1)}\ \sup_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right]\in\boldsymbol{A}_\eta}\|\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta}\end{smallmatrix}\right]\|_{C^{1+\tilde\mu}(\bar\Omega)\times C^{\tilde\mu}(\bar\Omega)} \leq D \ \text{ for arbitrary } \tilde\mu\in[\mu, 1-\frac{N}{p}). \tag{5}$$

Since $C^{1+\tilde\mu}(\bar\Omega)\times C^{\tilde\mu}(\bar\Omega)$ is compactly embedded in $C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)$ for $\mu\in(0,\tilde\mu)$, the proof is complete. ▌

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THEOREM 7.7.1. *Assume that I) and II) hold, $f:\mathbb{R}\to\mathbb{R}$ satisfies (1), and (3) is valid with $\rho=\frac{N}{N-2}$. Suppose also the following hyperbolicity condition*

$$0\notin\sigma(-A+f'(u_0)) \quad whenever \quad u_0\in\mathcal{E}_1. \tag{6}$$

If $\boldsymbol{A}_0$ denotes the attractor for (3.1) with $\eta=0$ in $H_0^1(\Omega)\times L^2(\Omega)$ (see Lemma 3.4 in [4]), then $\boldsymbol{A}_0$ is a bounded subset of $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ for any $p\geq 2$ and consequently a compact subset of $C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)$ for any $\mu\in(0,1)$.

Proof: Note that as shown in [4] the family of attractors $\boldsymbol{A}_\eta$ is lower semicontinuous at $\eta=0$ with respect to the Hausdorff semidistance in Y_2^1. Therefore, if $\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}\in\boldsymbol{A}_0$ then $\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}$ is a limit in Y_2^1 of a sequence $\{\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\}$ as $\eta_n\to 0^+$, where $\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\in\boldsymbol{A}_{\eta_n}$.

As a consequence of Lemma 7.1, for each fixed $p\geq 2$, sequence $\{\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\}$ is in fact bounded in Y_p^2-norm by certain positive constant $D>0$ uniformly with respect to $n\in\mathbb{N}$. Since Y_p^2 is reflexive, the latter ensures that $\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}$ is a weak limit of a subsequence bounded in Y_p^2-norm by the same constant D. This proves the estimate

$$\sup_{\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]\in\boldsymbol{A}_0}\|\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}\|_{Y_p^2}\leq D, \tag{7}$$

from which using Sobolev embedding we infer that

$$\forall_{\mu\in(0,1)}\ \exists_{D>0}\ \sup_{\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]\in\boldsymbol{A}_0}\|\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}\|_{C^{1+\mu}(\bar\Omega)\times C^{\mu}(\bar\Omega)}\leq D. \tag{8}$$

Theorem 7.7.1 is thus proved. ▌

8. APPENDIX: UNIFORM REGULARIZATION FOR LOCAL SOLUTIONS TO ABSTRACT PARABOLIC EQUATIONS

THEOREM 8.8.1. *Let Y be a Banach space, $\mathcal{A}:D(\mathcal{A})\subset Y\to Y$ be a sectorial operator with* Re $\sigma(\mathcal{A})>0$, *$0\leq\alpha<1$ and $f:Y^\alpha\to Y$ be a Lipschitz continuous in bounded sets map. Then, for each bounded set $B\subset Y^\alpha$ there exists $\tau_B>0$ such that the solution to*

$$\begin{aligned}\dot y(t)&=\mathcal{A}y+f(y)\\ y(0)&=y_0\in B\end{aligned}$$

exists for $t\in[0,\tau_B]$ and

$$\sup_{t\in[0,\tau_B]}\sup_{y_0\in B}\|y(t,y_0)\|_{Y^\alpha}<\infty. \tag{1}$$

Furthermore, for any $0 \le \theta < 1$ *and* $0 < t_0 < \tau_B$,

$$\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|\dot{y}(t,y_0)\|_{Y^\theta} < \infty \tag{2}$$

and consequently

$$\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|y(t,y_0)\|_{Y^1} < \infty. \tag{3}$$

If, in addition, f *takes bounded subsets of* Y^1 *into bounded subsets of* Y^β *for some* $\beta > 0$, *then*

$$\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|y(t,y_0)\|_{Y^{1+\gamma}} < \infty \tag{4}$$

for any $0 \le \gamma < \beta$ *and* $0 < t_0 < \tau_B$.

Proof: Proceeding as in [10] we define $\rho = \sup_{y_0\in B}\|u_0\|_{Y^\alpha}$ and consider a ball $B(M\rho+1)$ in Y^α centered at zero and of radius $M\rho+1$. Note that the solutions originating in the points of a bounded set $B \subset Y^\alpha$ - as long as they stay in the mentioned ball - fulfil the estimate

$$y(t) \le M\rho + M(1-\alpha)^{-1}\sup_{z\in B(M\rho+1)}\|f(z)\|_Y\ t^{1-\alpha}.$$

Therefore these solutions will not leave the ball $B(M\rho+1)$ at least until the second term above becomes equal to 1, which proves (1).

Let L be the Lipschitz constant of f in the set $B_\gamma = \gamma_{[0,\tau_B]}(B) = \{y(t,y_0) : 0 \le t \le \tau_B \text{ and } y_0 \in B\}$, $N = \sup_{y\in B_\gamma}\|f(y)\|_Y$ and $M \ge 1$ be such that

$$\|e^{-\mathcal{A}t}\|_{L(Y^{\beta_1},Y^{\beta_2})} \le Mt^{\beta_1-\beta_2} \quad \text{for } t > 0 \text{ and } 0 \le \beta_1 \le \beta_2 \le 1.$$

For $0 \le t \le t+h \le \tau_B$ we then have

$$y(t+h) - y(t) = (e^{-\mathcal{A}h} - I)e^{-\mathcal{A}t}y_0 + \int_0^t e^{-\mathcal{A}(t-s)}(f(y(s+h)) - f(y(s)))ds + \int_0^h e^{-\mathcal{A}(t+h-s)}f(y(s))ds$$

and

$$\begin{aligned}\|y(t+h) - y(t)\|_\alpha &\le \frac{M^2}{\gamma}h^\gamma t^{-\gamma}\|y_0\|_{Y^\alpha} + ML\int_0^t (t-s)^{-\alpha}\|y(s+h) - y(s)\|_{Y^\alpha}ds \\ &\quad + MN\int_0^h (t+h-s)^{-\alpha}ds \\ &\le \frac{M^2}{\gamma}h^\gamma t^{-\gamma}\|y_0\|_{Y^\alpha} + ML\int_0^t (t-s)^{-\alpha}\|y(s+h) - y(s)\|_{Y^\alpha}ds + hMNt^{-\alpha}.\end{aligned}$$

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If $\gamma\in[\alpha,1)$ and $0<t<t+h\le\tau_B<1$ then it follows from Gronwall's inequality that

$$\|y(t+h)-y(t)\|_\alpha\le C(M,N,\sup_{y_0\in B}\|y_0\|_{Y^\alpha},\alpha,\gamma)\,h^\gamma\,t^{-\gamma}=:C_0h^\gamma\,t^{-\gamma}$$

and therefore we get

$$\|f(y(t+h))-f(y(t))\|\le C_0\,L\;h^\gamma\,t^{-\gamma},\ 0<t<t+h\le\tau_B<1. \tag{5}$$

Since

$$\mathcal{A}\int_0^t e^{-\mathcal{A}(t-s)}f(y(s))ds=\int_0^t\mathcal{A}e^{-\mathcal{A}(t-s)}(f(y(s))-f(y(t)))ds+f(y(t))-e^{-\mathcal{A}t}f(y(t))$$

we rewrite time derivative of y in the form

$$\frac{dy}{dt}=-\mathcal{A}y(t)+f(y(t))=-\mathcal{A}e^{-\mathcal{A}t}y_0+\int_0^t\mathcal{A}e^{-\mathcal{A}(t-s)}(f(y(t))-f(y(s)))ds+e^{-\mathcal{A}t}f(y(t)). \tag{6}$$

We can now apply (5) with $t:=s$, $h:=t-s$ to estimate the integral term in (6) as

$$\begin{aligned}\Big\|\int_0^t\mathcal{A}e^{-\mathcal{A}(t-s)}(f(y(t))-f(y(s)))ds\Big\|_{Y^\theta}&\le\int_0^t M(t-s)^{-1-\theta}\|f(y(t))-f(y(s))\|_Yds\\&\le MLC_0\int_0^t(t-s)^{-1-\theta+\gamma}s^{-\gamma}ds=MLC_0\mathcal{B}(1-\gamma,\gamma-\theta)t^{-\theta},\ \alpha<\theta<\gamma<1,\end{aligned} \tag{7}$$

where $\mathcal{B}$ denotes the Beta function; i.e. $\mathcal{B}(a,b)=\int_0^1 r^{a-1}(1-r)^{b-1}dr$ for $a,b>0$.

It thus follows from (6), (7) that

$$\begin{aligned}\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|\dot y(t,y_0)\|_{Y^\theta}&\le Mt_0^{-1-\theta+\alpha}\sup_{y_0\in B}\|y_0\|_{Y^\alpha}\\&\quad+MLC_0\mathcal{B}(1-\gamma,\gamma-\theta)t_0^{-\theta}+MNt_0^{-\theta},\ \alpha<\theta<\gamma<1,\end{aligned}$$

which shows that the estimate of the derivative in (2) holds with $\theta>\alpha$ and the constant that depends only on B, α, θ, M, f, t_0 and arbitrarily chosen $\gamma\in(\theta,1)$. Furthermore, recalling that $\mathcal{A}y=-\dot y+f(y)$ we also observe that similar is true for the Y^1 norm of y in (3); namely

$$\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|y(t,y_0)\|_{Y^1}\le c\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|\dot y(t,y_0)\|_{Y^\theta}+N.$$

Finally, to prove (4) we choose any $0\le\hat t_0<t_0<\tau_B$ and proceed as before to get

$$\begin{aligned}\|y(t)\|_{Y^{1+\gamma}}&=\|e^{-\mathcal{A}(t-\hat t_0)}y(\hat t_0)\|_{Y^{1+\gamma}}+\Big\|\int_{\hat t_0}^t e^{-\mathcal{A}(t-s)}f(y(s))ds\Big\|_{Y^{1+\gamma}}\\&\le M(t-\hat t_0)^{-\gamma}\|y(\hat t_0)\|_{Y^1}+M\sup_{s\in[\hat t_0,\tau_B]}\sup_{y_0\in B}\|F(y(s)\|_{Y^\beta}(\beta-\gamma)^{-1}(t-\hat t_0)^{\beta-\gamma},\end{aligned} \tag{8}$$

whenever $0 \leq \gamma < \beta$. We remark that $\sup_{t\in[\hat{t}_0,\tau_B]} \sup_{y_0\in B} \|y(t,y_0)\|_{Y^1}$ is bounded as shown in (3), so that (4) follows.

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