CADERNOS DE MATEMÁTICA **06**, 237–263 October (2005)
ARTIGO NÚMERO SMA#241

Some Results on Reversible-Equivariant Vector Fields

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The aim of this work is to explore the behaviour of a differential system under the simultaneous action of symmetries and reversing symmetries. We analyse the general structure of mappings that commute and anti-commute with the action of a compact Lie group, illustrating typical differences to equivariant mappings with a series of examples in the plane. The results presented here are the basis for the sistematic study of reversible-equivariant vector fields. They are also fundamental to the singularity theory that shall be applied, in a forthcoming paper, to the steady-state bifurcation of systems of differential equations in the reversible-equivariant context. October, 2005 ICMC-USP

1. INTRODUCTION

Many natural phenomena possess symmetry properties. In particular, symmetries and reversing symmetries occur simultaneously in many dynamical systems physically motivated. The presence of symmetries in natural phenomena has been taken into account in their mathematical formulation, which can simplify significantly the interpretation of such phenomena. Reversing symmetries are equally explored in natural science and appears, for example, in classical mechanics, quantum mechanics and thermodynamic. Such occurrence can be analysed and its effect can also be investigated in a systematic way by means of a mathematical model.

The conventional notion of presence of symmetry, or equivariance, in differential equations consists of phase space transformations that leave the equations of motion invariant. Such symmetry transformations map trajectories of a dynamical system to other trajectories of the same system. In Figure 1(a), for example, all rotations about the origin are symmetries and in Figure 1(b) the mirror in the $x-$axis is a symmetry. We now discuss the presence of reversing symmetries in these figures.

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Reversing symmetries in a dynamical system are transformations of the phase space that map trajectories onto other trajectories reverting direction in time. Obviously, in the phase space this reversion corresponds to an inversion of the arrows. In Figure 1(a) the mirror in the $x-$axis is a reversing symmetry and in Figure 1(b) the rotation of angle $\frac{\pi}{2}$ about the origin is a reversing symmetry.

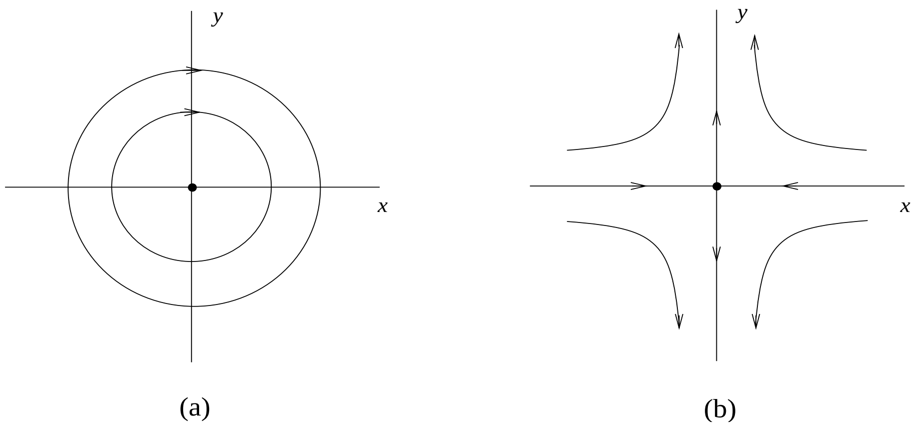

FIG. 1. Schematic phase portraits of two type of flows of vector fields in $\mathbb{R}^2$ possessing symmetries and reversing symmetries.

It is well known that the symmetry properties of a system affect the genericity of the occurrence of local bifurcations. The influence of symmetries and reversing symmetries in steady-state bifurcations has been extensively studied in the context of equivariant dynamical systems (for example, [12], [13], [22], [23], [25], [30]) as well as in the reversible case (for example, [5], [8], [17], [24], [27], [29]). Although the bifurcation theory has had important results for both purely equivariant and purely reversible systems, little has been done on the mixed case (for example, [2], [3], [6], [19]). A dynamical system is said to be reversible-equivariant if it has both symmetries (or spatio symmetries) and reversing symmetries (or time-reversal symmetries).

In the reversible-equivariant context linear systems have been classified in terms of the representation group theory, and steady-state bifurcations have been analysed using equivariant transversality theory ([3], [16] and [19]). Moreover, local bifurcation problems in reversible systems can also be studied via equivariant singularity theory after performing a Liapunov-Schmidt reduction ([11], ([15] and [30]). In this way, despite the dynamical differences between reversible and equivariant systems, some techniques developed for the equivariant context carry over to the reversible one.

In this work we introduce the ideas that motivate the study of certain reversible-equivariant differential equations under the action of a compact Lie group, showing that some general results developed for equations in the equivariant context can be used to the reversible-equivariant context. This paper is organized as follows. In Section 2 we introduce some basic notations and definitions. In Section 3 we develop a theory related

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to reversible-equivariant mappings that is presented in three parts: in Subsection 3.1 we show why the equivariant versions of the Hilbert-Weyl theorem and Schwarz's theorem [12] can be used in the reversible-equivariant case, illustrating with two important examples; in Subsection 3.2 we obtain general forms of reversible-equivariant vector fields on $\mathbb{R}^2$ under the action of different groups and in Subsection 3.3 we treat reversible-equivariant vector fields for the linear case. In Section 4 we adapt the Liapunov-Schmidt reduction presented in [11] to the reversible-equivariant case. In other works the Liapunov-Schmidt reduction without symmetry has been adapted in different contexts, namely in the equivariant case [11], with hidden symmetries [21] and reversible Hopf bifurcation cases [6]. Finally, in Section 5 we introduce the ideas that motivate the study of the bifurcation theory for reversible-equivariant mappings analysing, particularly, an one parameter bifurcation problem on $\mathbb{R}^2$ under the action of the dihedral group $\mathbf{D_4}$.

2. PRELIMINARIES

In this section we present the definitions, notations and basic results we shall use throughout this paper.

2.1. Reversible and equivariant theory

Let Γ be a compact Lie group acting linearly on a finite-dimensional vector space V. Using the Haar integral, we can construct a $\Gamma-$invariant inner product $\langle\ ,\ \rangle_\Gamma$ on V (that is, $\langle \gamma v, \gamma w\rangle_\Gamma = \langle v, w\rangle_\Gamma$, for all $\gamma \in \Gamma$) such that we may identify Γ with a closed subgroup of the orthogonal group $\mathbf{O}(n)$ ([12], Proposition XII. 1.3). Therefore, we can assume Γ acting orthogonally on V.

Let

$$\dot{x} + g(x) = 0 \tag{1}$$

be a system of ordinary differential equations defined in a neighbourhood of a point $x_0 \in V$. In this system, $g : (V, x_0) \to V$ is a germ of smooth (C^∞) mapping.

We start our discussion defining an *invariant* function under the action of the group Γ, or a $\Gamma-$invariant function, as a real-valued function $f : V \to \mathbb{R}$ such that

$$f(\gamma x) = f(x)$$

for all $\gamma \in \Gamma$ and $x \in V$. We denote by $\mathcal{E}_n$ the ring of C^∞ germs $(\mathbb{R}^n, 0) \mapsto \mathbb{R}$, by $\mathcal{P}(\Gamma)$ the ring of $\Gamma-$invariant polynomials and by $\mathcal{E}(\Gamma)$ the ring of $\Gamma-$invariant function germs (at the origin).

We also treat equivariant and reversible mappings under the action of Γ :

Definition 2.1. We say that g commutes with Γ, or g is $\Gamma-$equivariant, if

$$g(\gamma x) = \gamma g(x), \tag{2}$$

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for all $\gamma \in \Gamma$ and $x \in V$.

Definition 2.2. We say that g anti-commutes with Γ, or g is $\Gamma-$reversible, if

$$g(\gamma x) = -\gamma g(x), \tag{3}$$

for all $\gamma \in \Gamma$ and $x \in V$.

When g in (1) is $\Gamma-$equivariant, it follows that $x(t)$ is a solution of (1) if and only if $\gamma x(t)$ is. In this case, γ is called a *symmetry* of (1) and we denote by Γ_+ the set containing all the symmetries of (1). When g in (1) is $\Gamma-$reversible, we have the following: if $x(t)$ is a solution of (1) then so is $\gamma x(-t)$. In this case, γ is called a *reversing symmetry* of (1) and we denote by Γ_- the set containing all the reversing symmetries of (1). Therefore, the solution set of $g = 0$ is preserved by any symmetry $\gamma \in \Gamma_+$ in the first case and by any reversing symmetry $\gamma \in \Gamma_-$ in the second case.

Notice that Γ_+ forms a group under composition. However, Γ_- is not a group, since it is not closed under composition. We call a group of symmetries and reversing symmetries of a dynamical system a *reversing symmetry group* Γ and we define it as $\Gamma_+ \cup \Gamma_-$. In other words, Γ_- is the complement of Γ_+ in Γ. From the above definitions, we have the following properties:

(i) the composition of two symmetries is a symmetry;
(ii) the composition of two reversing symmetries is a symmetry;
(iii) the composition of a symmetry and a reversing symmetry is a reversing symmetry.

The identity element of Γ is denoted by I. Note that Γ_+ forms a normal subgroup of Γ. Moreover, when $\Gamma_+ \neq \Gamma$, that is, Γ contains reversing symmetries, then Γ_+ is a subgroup of index 2,

$$\Gamma/\Gamma_+ \simeq \mathbb{Z}_2.$$

Notice also that Γ can be written as the semi-direct product $\Gamma \simeq \Gamma_+ \dot{+} \mathbb{Z}_2$ if and only if Γ/Γ_+ contains an involution (see definition of involution in Subsection 3.3).

If g in (1) is reversible and does not possess nontrivial symmetries, that is, $\Gamma \simeq \mathbb{Z}_2$ and $\Gamma_+ = I$, we call *g purely reversible.*

In both purely equivariant and purely reversible cases, for each $\gamma \in \Gamma$ the phase space of (1) is symmetric with respect to the fixed point subspace

$$Fix(\gamma) = \{x \in V : \gamma x = x\},$$

with direction of the arrows preserved if $\gamma \in \Gamma_+$ and reversed if $\gamma \in \Gamma_-$. Moreover, if $\gamma \in \Gamma_+$, then $Fix(\gamma)$ is setwise invariant under the dynamics of the system. However, if $\gamma \in \Gamma_-$, $Fix(\gamma)$ is usually not setwise invariant under the dynamics and has the property that if a regular orbit ρ intercepts $Fix(\gamma)$ in two different points then ρ is a closed and periodic orbit [17]. In this work we consider mappings that possess symmetries and reversing symmetries:

DEFINITION 2.3. We say that g is $\Gamma-$reversible-equivariant if there is a homomorphism $\sigma : \Gamma \to \{\pm 1\}$ such that for all $\gamma \in \Gamma$,

$$g(\gamma x) = \sigma(\gamma)\gamma g(x), \tag{4}$$

where $\sigma(\gamma) = 1$ if $\gamma \in \Gamma_+$ and $\sigma(\gamma) = -1$ if $\gamma \in \Gamma_-$.

When g in (1) satisfies (4), we have that $x(t)$ is a solution of (1) if and only if $\gamma x(\sigma(\gamma)t)$ is a solution of (1), that is, the differential equation is invariant under the transformation $(x, t) \mapsto (\gamma x, \sigma(\gamma)t)$.

We denote by $\vec{\mathcal{Q}}(\Gamma)$ the space of Γ-reversible-equivariant polynomial mappings from V into V and by $\vec{\mathcal{F}}(\Gamma)$ the space of Γ-reversible-equivariant germs (at the origin) of C^∞ mappings from $(V, 0)$ into V. It is straightfoward that $\vec{\mathcal{Q}}(\Gamma)$ is a module over $\mathcal{P}(\Gamma)$ and $\vec{\mathcal{F}}(\Gamma)$ is a module over $\mathcal{E}(\Gamma)$.

2.2. Invariant theory

This subsection is devoted to the presentation of two essential results concerning the invariant theory of compact Lie groups. We present here the Hilbert-Weyl theorem, which states that the set of the invariant polynomial functions is generated by a finite set of elements and the Schwarz's theorem, that extends the Hilbert-Weyl theorem to invariant germs (at the origin) of C^∞ mappings.

There is a finite subset of invariant polynomials $u_1, \ldots, u_s$ such that every invariant polynomial may be written as a polynomial function of $u_1, \ldots, u_s$. This finite set of generators for $\mathcal{P}(\Gamma)$, that is not necessarily unique, is called *Hilbert basis* for $\mathcal{P}(\Gamma)$. The two theorems are stated as follows (see [12]):

THEOREM 2.1. **Hilbert-Weyl Theorem** *Let Γ be a compact Lie group acting on V. Then there exists a finite Hilbert basis for $\mathcal{P}(\Gamma)$.*

THEOREM 2.2. **Schwarz's Theorem** *Let Γ be a compact Lie group acting on V. Let $u_1, \ldots, u_s$ be a Hilbert basis for the $\Gamma-$invariant polynomials $\mathcal{P}(\Gamma)$. Let $f \in \mathcal{E}(\Gamma)$. Then there exists a smooth germ $h \in \mathcal{E}_s$ such that $f(x) = h(u_1(x), \ldots, u_s(x))$.*

3. REVERSIBLE-EQUIVARIANT MAPPINGS

In [12] the authors present an efficient way to describe germs of mappings that commute with an action of a compact Lie group Γ, that is, $\Gamma-$equivariant mappings. In this section, we show how such theory can be used in the reversible-equivariant context, when the extra condition of reversibility is introduced in (1), and describe the main properties of (1) in the new context. We are interested in presenting a theoretical basis to treat the presence of symmetries and reversing symmetries in the equations, being able to study their influence in the behaviour of the system.

3.1. General results in the reversible-equivariant context

In this subsection, we see the techniques developed in the study of equivariant systems extended to the reversible-equivariant case. We present the $\Gamma-$reversible-equivariant versions of Theorems 2.1 and 2.2, which provide a theoretical basis for many of the calculations performed in this paper. These theorems shall be particularly of great use in the sistematic analysis of bifurcations in the reversible-equivariant context. This study will appear in a forthcoming paper where we develop the general Singularity theory for the classification of such bifurcations. In fact, the results presented here are the first step to find the normal forms in the classification process.

We start by remarking that given a $\Gamma-$reversible-equivariant mapping $g : V \to V$, it is possible to interpret it as a $\Gamma-$equivariant mapping with different actions of Γ on the source and target. In fact, just consider the action on the source by

$$(\gamma, x) \mapsto \gamma x \tag{5}$$

and on the target by

$$(\gamma, y) \mapsto \sigma(\gamma)\gamma y. \tag{6}$$

Therefore, we may view a $\Gamma-$reversible-equivariant mapping from V into V as a $\Gamma-$equivariant mapping, with the actions of Γ on source and target given by (5) and (6), respectively. For this reason, many results for the equivariant case can be used in our context.

Definition 3.1. We say that the $\Gamma-$reversible-equivariant polynomial mappings $g_1, \dots, g_r$ generate the module $\overrightarrow{\mathcal{Q}}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$ if any Γ-reversible-equivariant g can be written as

$$g = f_1 g_1 + \dots + f_r g_r$$

for $\Gamma-$invariant polynomials $f_1, \dots, f_r$.

A similar definition can be given for $\overrightarrow{\mathcal{F}}(\Gamma)$. The following theorem is the reversible-equivariant version of Theorems 2.1 and 2.2. The well known Poènaru's Theorem [12] follow from item (b) by assuming that the actions of Γ on source and target are identical.

Theorem 3.1. *Let Γ be a compact Lie group acting on V. Then*

(a)The module $\overrightarrow{\mathcal{Q}}(\Gamma)$ is finitely generated over the ring of invariant polynomials $\mathcal{P}(\Gamma)$.

(b)Let $g_1, \dots, g_s$ be generators for the module $\overrightarrow{\mathcal{Q}}(\Gamma)$. Then $g_1, \dots, g_s$ are also generators for the module $\overrightarrow{\mathcal{F}}(\Gamma)$ over the ring $\mathcal{E}(\Gamma)$.

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Proof: Just use [12], XII, Theorem 6.8, setting $W = V$ and assuming that the actions of Γ on V and W are those actions given by (5) and (6), respectively. □

Finally, we discuss when the representation of a Γ-reversible-equivariant germ g in terms of $g_1, \ldots, g_s$ is unique.

DEFINITION 3.2. We say that $g_1, \ldots, g_s$ freely generate the module $\vec{\mathcal{F}}(\Gamma)$ over $\mathcal{E}(\Gamma)$ if the relation $f_1 g_1 + \ldots + f_s g_s \equiv 0$, where $f_j \in \mathcal{E}(\Gamma)$, $j = 1, \ldots, s$, implies that $f_1 \equiv \cdots \equiv f_s \equiv 0$.

Note that if $g_1, \ldots g_s$ freely generate $\vec{\mathcal{F}}(\Gamma)$, then every $g \in \vec{\mathcal{F}}(\Gamma)$ has a unique decomposition $g = f_1 g_1 + \ldots + f_s g_s$ where $f_j \in \mathcal{E}(\Gamma)$, for $j = 1, \ldots, s$.

We end this subsection with two examples.

EXAMPLE 3.1. Consider a reversible-equivariant vector field g on $\mathbb{R}^2$ under the standard action of the dihedral group $\mathbf{D}_n$ generated by the reflection and rotation

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix}.$$

We consider $\Gamma_+ = \mathbb{Z}_n(S)$, so S is a symmetry and R is a reversing symmetry. Hence, the vector field g satisfies

$$gR = -Rg \quad \text{and} \quad gS = Sg.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, $z = x + iy \equiv (x, y)$, we have $Rz = \bar{z}$ and $Sz = e^{i\frac{2\pi}{n}} z$. We shall prove that g has the form

$$g(z) = p(u, v) iz + q(u, v) i\bar{z}^{n-1}, \tag{7}$$

where $u = z\bar{z}$ and $v = z^n + \bar{z}^n$ are the generators of $\mathbf{D}_n$-invariant germs from $\mathbb{C}$ into $\mathbb{R}$ ([12], Example XII, 4.1(c)).

In fact, g is of the form

$$g(z) = \sum b_{jk} z^j \bar{z}^k, \tag{8}$$

where $b_{jk} \in \mathbb{C}$.

We first obtain restrictions on the $b_{jk}{}'s$ by using the reversibility of g with respect to R. Since $\overline{g(\bar{z})} = \sum \overline{b_{jk}} z^j \bar{z}^k$, the equality $\overline{g(\bar{z})} = -g(z)$ implies that $b_{jk}{}'s$ are purely imaginary, that is, $b_{jk} = a_{jk} i$, with $a_{jk} \in \mathbb{R}$. Now, the equivariance of g with respect to S implies that

$$g(z) = e^{-\frac{2\pi}{n} i} g(e^{\frac{2\pi}{n} i} z) = \sum b_{jk} e^{(j-k-1)\frac{2\pi}{n} i} z^j \bar{z}^k.$$

Hence $b_{jk} = 0$ unless $j \equiv k + 1 (\text{mod } n)$.

We now show that iz and $i\bar{z}^{n-1}$ generate the module $\vec{\mathcal{Q}}(\mathbf{D}_n)$ over $\mathcal{P}(\mathbf{D}_n)$. So far we have

$$g(z) = \sum a_{jk} i z^j \bar{z}^k, \tag{9}$$

with $j \equiv k + 1 (\mathrm{mod}\ n)$, $a_{jk} \in \mathbb{R}$.

In each term in (9) we can factor out powers of $z\bar{z}$, which are $\mathbf{D}_n-$invariants (for example, either $z^j \bar{z}^k = (z\bar{z})^j \bar{z}^{k-j}$ if $k \geq j$ or $z^j \bar{z}^k = (z\bar{z})^k z^{j-k}$ if $j \geq k$), until we are left either with $j = 0$ or $k = 0$. Since $j \equiv k + 1(\mathrm{mod}\ n)$, the terms iz^{ln+1} e $i\bar{z}^{ln+n-1} = i\bar{z}^{(l+1)n-1}$, $l = 0, 1, 2, \dots$, generate the module $\overrightarrow{\mathcal{Q}}(\mathbf{D}_n)$. However, the identities

(a) $iz^{(l+2)n+1} = (z^n + \bar{z}^n) iz^{(l+1)n+1} - (z\bar{z})^n iz^{ln+1}$,
(b) $i\bar{z}^{(l+3)n-1} = (z^n + \bar{z}^n) i\bar{z}^{(l+2)n-1} - (z\bar{z})^n i\bar{z}^{(l+1)n-1}$

show that the generators iz^{ln+1} and $i\bar{z}^{(l+1)n-1}$ are redundant for $l \geq 2$.

It only remains to analyse the cases $l = 0$ and $l = 1$. For $l = 1$, we have

(c) $iz^{n+1} = (z^n + \bar{z}^n) iz - (z\bar{z}) i\bar{z}^{n-1}$,
(d) $i\bar{z}^{2n-1} = (z^n + \bar{z}^n) i\bar{z}^{n-1} - (z\bar{z})^{n-1} iz$.

Hence, the generators iz^{n+1} and $i\bar{z}^{2n-1}$ are redundant. Thus, only the terms iz and $i\bar{z}^{n-1}$, corresponding to $l = 0$, generate the module $\overrightarrow{\mathcal{Q}}(\mathbf{D}_n)$ and, therefore, g has the form

$$g(z) = p(u, v) iz + q(u, v) i\bar{z}^{n-1},$$

where $u = z\bar{z}$ and $v = z^n + \bar{z}^n$.

Remark 3. 1.

1. The example above also is presented in [6], where it is stated that all $\mathbf{D}_n-$reversible-equivariant vector field f has the form $f(z) = izF(z\bar{z}, z^n + \bar{z}^n)$, where F is a real function. As we have seen in this example, terms of the form $i\bar{z}^{n-1} G(z\bar{z}, z^n + \bar{z}^n)$, with G a real function, is lacking in the field f.

2. Note that the generators of $\overrightarrow{\mathcal{Q}}(\mathbf{D}_n)$ and, hence, the generators of $\overrightarrow{\mathcal{F}}(\mathbf{D}_n)$, are equal to the generators of $\overrightarrow{\mathcal{P}}(\mathbf{D}_n)$ multiplied by i, which represents a rotation over the angle $\frac{\pi}{2}$. For the group $\Gamma = \mathbf{O}(2)$ in its standard action on $\mathbb{C}$ we have the same fact because every $\mathbf{O}(2)$-reversible-equivariant mapping g has the form

$$g(z) = p(z\bar{z}) iz.$$

The $\mathbb{O}-$equivariant version corresponding to the next example appears in [25].

Example 3.2. We consider an $\mathbb{O}-$reversible-equivariant vector field $g : \mathbb{R}^3 \to \mathbb{R}^3$, that is, with reversing symmetry group given by the octahedral group $\mathbb{O}$ of the symmetries of the cube. $\mathbb{O}$ has 48 elements and is generated by

$$\kappa_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

These elements represent reflection in the $x = 0$ plane, which we consider a reversing symmetry, and rotations of $\frac{\pi}{2}$ about the $x-$axis and $y-$axis, acting as symmetries. The subgroup $\mathcal{S}_3$ of $\mathbb{O}$ of permutations is a reversing symmetry group in this case. In the computations below we also use the reversing symmetries κ_y, κ_z and the symmetry R_z, with the obvious matrix representations.

Using the following lemma we obtain the general form of $g \in \overrightarrow{\mathcal{F}}(\mathbb{O})$:

Lemma 3.1. *For any $f \in \mathcal{E}(\mathbb{O})$ there exists $P \in \mathcal{E}_3$ such that $f(x, y, z) = P(u, v, w)$, where $u = x^2 + y^2 + z^2$, $v = x^2y^2 + y^2z^2 + x^2z^2$ e $w = x^2y^2z^2$.*

Proof: See [25], Lemma A.1 . □

For $a = x^2$, $b = y^2$ e $c = z^2$, we use the notation

$$\langle \phi(a, b, c) \rangle = \begin{pmatrix} \phi(a, b, c)yz \\ \phi(b, c, a)xz \\ \phi(c, a, b)xy \end{pmatrix}.$$

Applying Theorem 3.1 we restrict our attention to polynomials. We show that an $\mathbb{O}-$reversible-equivariant polynomial g is characterized by the form

$$g(x, y, z) = \sum_{ijk} g_{ijk} \langle a^i (b^j c^k - c^j b^k) \rangle, \quad g_{ijk} \in \mathbb{R}. \tag{10}$$

Observe that such a mapping satisfies $g(\kappa_x(x, y, z)) = -\kappa_x g(x, y, z)$, $g(R_x(x, y, z)) = R_x g(x, y, z)$ e $g(R_y(x, y, z)) = R_y g(x, y, z)$, and hence it is $\mathbb{O}-$reversible-equivariant. We now show that every $\mathbb{O}-$reversible-equivariant polynomial has the form (10).

Suppose $g = (g^1, g^2, g^3)$ an $\mathbb{O}-$reversible-equivariant polynomial mapping. Using the reversing symmetries κ_x, κ_y e κ_z we find that g^1 is even in x, odd in y and z; g^2 is even in y, odd in x and z; g^3 is even in z, odd in x and y. Hence, we can write

$$g(x, y, z) = \sum_{ijk} a^i b^j c^k \begin{pmatrix} g^1_{ijk} yz \\ g^2_{ijk} xz \\ g^3_{ijk} xy \end{pmatrix}, \tag{11}$$

where $g^l_{ijk} \in \mathbb{R}$, for $l = 1, 2, 3$.

Now we use the transpositions $\kappa_x \circ R_z$, $\kappa_y \circ R_x$ and $\kappa_z \circ R_y$, which are reversing symmetries of g, to find that

$$g^1_{ikj} = -g^1_{ijk}, \quad g^2_{kji} = -g^2_{ijk}, \quad g^3_{jik} = -g^3_{ijk}.$$

Thus (11) becomes

$$g(x,y,z) = \sum_{ijk} \begin{pmatrix} g^1_{ijk} a^i (b^j c^k - c^j b^k) yz \\ g^2_{ijk} b^i (a^j c^k - a^k c^j) xz \\ g^3_{ijk} c^i (a^k b^j - a^j b^k) xy \end{pmatrix}. \tag{12}$$

Using again the reversing symmetries $\kappa_x \circ R_z$ and $\kappa_z \circ R_y$ in (12) we obtain

$$g^2_{ijk} = g^3_{ijk} = -g^1_{ijk} = -g_{ijk}, \quad \forall i,j,k.$$

Thus (12) becomes

$$g(x,y,z) = \sum_{ijk} g_{ijk} \begin{pmatrix} a^i (b^j c^k - c^j b^k) yz \\ b^i (c^j a^k - a^j c^k) xz \\ c^i (a^j b^k - b^j a^k) xy \end{pmatrix} = \sum_{ijk} g_{ijk} \langle a^i (b^j c^k - c^j b^k) \rangle. \tag{13}$$

□

In the $\mathbb{O}-$equivariant case we observe the existence of nontrivial linear terms in the generators of $\overrightarrow{\mathcal{E}}(\mathbb{O})$. In our context, every $g \in \overrightarrow{\mathcal{F}}(\mathbb{O})$ possesses terms of at least order two and, therefore, the generators of $\overrightarrow{\mathcal{F}}(\mathbb{O})$ must possess terms of at least order two. So, imposing that κ_x is a reversing symmetry instead of a symmetry, as in the equivariant case, implies zero linear terms. Therefore, reversibility together with equivariance forces a more degenerate situation, that is, multiple zero eigenvalues may be expected.

Motivated by the two previous examples, we explore in the next subsection the structure of vector fields on $\mathbb{R}^2$ that are reversible-equivariant with respect to an action of a class of compact Lie groups Γ and present a way to describe them.

3.2. General form of reversible-equivariant vector fields

In this subsection we discuss the structure of systems that are reversible with respect to a reversing symmetry R and equivariant with respect to a linear representation of a compact Lie group Γ_+, in such a way that the group Γ generated by Γ_+ and R is a compact Lie group.

We obtain general forms of reversible-equivariant vector fields on $\mathbb{R}^2$ under the action of different groups. From now on we denote by $\mathcal{R}_\theta$ the rotation around the origin of angle θ.

Case 1. Vector fields on $\mathbb{R}^2$ under the action of $\Gamma = \mathbf{Z}_n(R)$, where $R = \mathcal{R}_{\frac{2\pi}{n}}$ is a reversing symmetry.

To describe $\mathbf{Z}_n(R)-$reversible-equivariant vector fields, we identify $\mathbb{R}^2 \equiv \mathbb{C}$ and start with a polynomial $g : \mathbb{C} \to \mathbb{C}$ of the general form

$$g(z) = \sum b_{jk} z^j \bar{z}^k, \tag{14}$$

where $b_{jk} \in \mathbb{C}$. The reversibility of g with respect to R implies that

$$g(z) = -e^{-\frac{2\pi}{n}i} g(e^{\frac{2\pi}{n}i} z) = -\sum b_{jk} e^{(j-k-1)\frac{2\pi}{n}i} z^j \bar{z}^k,$$

so $e^{(j-k-1)\frac{2\pi}{n}i} = -1$. Thus, n must be even because $b_{jk} = 0$ unless $j \equiv k + 1 + \frac{n}{2} (\text{mod } n)$. Notice that this restriction implies that $\mathbf{Z}_n(R)-$reversible-equivariant vector fields do not exist if n is odd. We observe that $\mathbf{Z}_n(R)-$equivariance does not impose such restriction on n, and there are $\mathbf{Z}_n(R)-$equivariant vector fields on $\mathbb{R}^2$ for all $n \geq 2$.

It is easy to conclude that every $\mathbf{Z}_n(R)-$reversible-equivariant vector field takes the form

$$g(z) = p(u, v) z^{\frac{n}{2}+1} + q(u, v) \bar{z}^{\frac{n}{2}-1}, \tag{15}$$

where $u = z\bar{z}$ and $v = Re(z^n)$ are the generators of $\mathbf{Z}_n$-invariant germs from $\mathbb{C}$ into $\mathbb{R}$ ([9], p. 206) and p, q are complex-valued polynomials.

To illustrate, consider the $\mathbf{Z}_4(R)-$reversible-equivariant vector field

$$g(z) = p(u, v) z^3 + q(u, v) \bar{z} \tag{16}$$

for $p(u, v) = q(u, v) = i$. In real coordinates, it takes the form

$$g(x, y) = (y - 3x^2 y + y^3, x + x^3 - 3xy^2).$$

The equilibrium points are $(0, 0)$ and $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$. In this case, $(0, 0)$ is of saddle type; $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ are stable nodes and $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ are unstable nodes. A numerical simulation using the software $P4$ [1] gives the phase portrait as in Figure 2.

For $p(u, v) = iu$ and $q(u, v) = 1$ in (16) we have

$$g(x, y) = (x - 3x^4 y - 2x^2 y^3 + y^5, -y + x^5 - 2x^3 y^2 - 3xy^4).$$

In Figure 3 we can verify the nature of equilibrium points of this vector field:

$$\left(\frac{\sqrt{-\sqrt{2}+2}(\sqrt{2}+1)}{2}, \frac{\sqrt{-\sqrt{2}+2}}{2} \right) \quad \text{and} \quad \left(-\frac{\sqrt{-\sqrt{2}+2}(\sqrt{2}+1)}{2}, -\frac{\sqrt{-\sqrt{2}+2}}{2} \right)$$

are asymptotically stable focus,

$$\left(\frac{\sqrt{\sqrt{2}+2}(\sqrt{2}-1)}{2}, -\frac{\sqrt{\sqrt{2}+2}}{2} \right) \quad \text{and} \quad \left(-\frac{\sqrt{\sqrt{2}+2}(\sqrt{2}-1)}{2}, \frac{\sqrt{\sqrt{2}+2}}{2} \right)$$

are asymptotically unstable focus and the origin is an equilibrium point of saddle type.

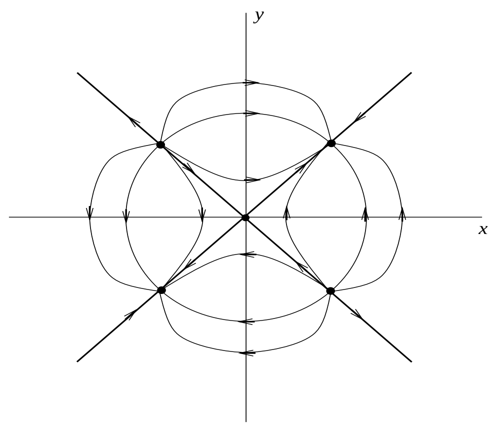

FIG. 2. $\mathbf{Z}_4(R)-$reversible-equivariant vector field (16) for $p(u,v) = q(u,v) = i$.

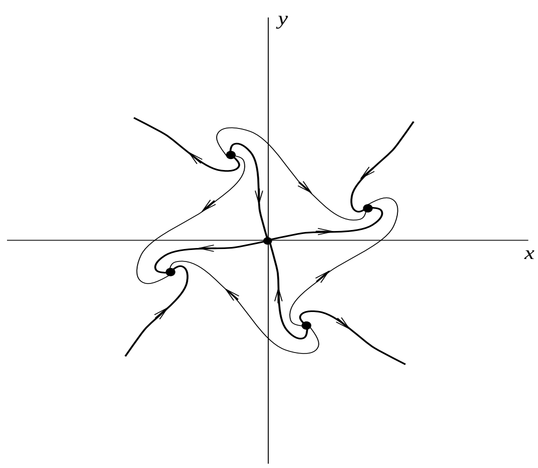

FIG. 3. $\mathbf{Z}_4(R)-$reversible-equivariant vector field (16) for $p(u,v) = iu$ and $q(u,v) = 1$.

We observe in the two last examples that the origin is generically an equilibrium point of saddle type. That is not a coincidence. In fact, in the $\mathbf{Z}_4(R)-$reversible-equivariant case (and only for the case $n = 4$) this fact always holds. We notice here that in the corresponding equivariant context the origin is not an equilibrium point of saddle type for any n.

Case 2. Vector fields on $\mathbb{R}^2$ under the action of $\mathbf{D}_n$ where the generator $R = \mathcal{R}_{\frac{2\pi}{n}}$ is a reversing symmetry and the generator $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a symmetry. Note that in Example 3.1 we set S as a reversing symmetry and R as a symmetry.

We first recall that since every $\mathbf{D}_n-$reversible-equivariant vector field is also $\mathbf{Z}_n(R)-$reversible-equivariant and there are no $\mathbf{Z}_n(R)-$reversible-equivariant vector fields for n odd, then the same goes for $\mathbf{D}_n-$reversible-equivariants.

Furthermore, every $\mathbf{D}_n-$reversible-equivariant vector field for n even takes the form

$$g(z) = p(u,v)z^{\frac{n}{2}+1} + q(u,v)\bar{z}^{\frac{n}{2}-1}, \qquad (17)$$

where $u = z\bar{z}$ and $v = z^n + \bar{z}^n$ are the generators of the $\mathbf{D}_n$-invariants and p, q are real-valued functions. Note that if we set p and q real in (15), we obtain the general form of $\mathbf{D}_n-$reversible-equivariants.

We analyse below two cases of the action of $\mathbf{D}_n$ on $\mathbb{R}^2$ for different elements of the group acting as reversing symmetries.

(a) Consider the $\mathbf{D}_n-$reversible-equivariant vector field (17) with $\mathcal{R}_\pi$ as a reversing symmetry. Note that n is necessarily even, otherwise $\mathcal{R}_\pi$ is not an element of $\mathbf{D}_n$. So we can write $n = 2m$, $m \in \mathbb{N}$. Moreover, we have $R^m = \mathcal{R}_\pi$ and for $\mathcal{R}_\pi$ to be a reversing symmetry it is necessary to have the generator R a reversing symmetry and m odd, since composition of an even number of reversing symmetries is a symmetry. Hence, there are $\mathbf{D}_n-$reversible-equivariant vector fields in presence of the reversing symmetry $\mathcal{R}_\pi$ only for $n = 4p+2$, $p \in \mathbb{N}$, and such vector fields have the form

$$g(z) = p(u,v)z^{2(p+1)} + q(u,v)\bar{z}^{2p},$$

where $u = z\bar{z}$ e $v = z^n + \bar{z}^n$ are the generators of the $\mathbf{D}_n$-invariants.

To illustrate, consider the $\mathbf{D}_6-$reversible-equivariant vector field in presence of the reversing symmetry $\mathcal{R}_\pi$ with $p(u,v) = u$ and $q(u,v) = 1$. In this case,

$$g(z) = z^5\bar{z} + \bar{z}^2, \qquad (18)$$

which in real coordinates becomes

$$g(x,y) = (x^6 - 5x^4y^2 - 5x^2y^4 + y^6 + x^2 - y^2, 4x^5y - 4xy^5 - 2xy).$$

The origin is a non-hyperbolic equilibrium point and the six nontrivial equilibrium points are $(0, \pm 1)$ e $(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$. We again use the program $P4$ [1] to simulate the phase portrait (Figure 4).

The point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the fixed-point subspace of a reversing symmetry, namely $Fix(\mathcal{R}_{\frac{\pi}{3}} \circ S)$. Recall that an orbit nearby that intercepts this subspace twice must be

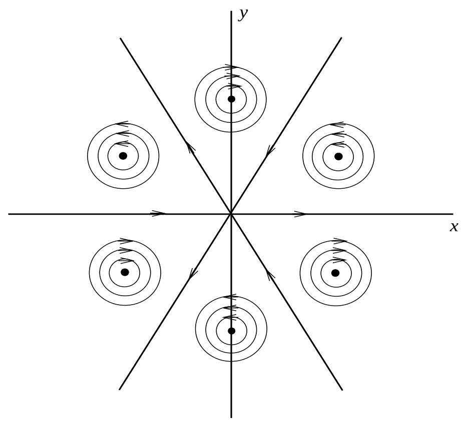

FIG. 4. $\mathbf{D}_6(R)-$reversible-equivariant (18) vector field.

a periodic orbit. Hence, we have proved that this equilibrium is in fact a centre. All the other five nonzero equilibria obtained by symmetry are also of centre type.

If we now take $p(u, v) = 1$ and $q(u, v) = u$, we have the $\mathbf{D}_6-$reversible-equivariant vector field with $\mathcal{R}_\pi$ as a reversing symmetry given in real coordinates by

$$g(x, y) = (2x^4 - 6x^2y^2, 2x^3y - 6xy^3).$$

In this case, the only equilibrium point is the origin and it is an unstable node.

(b) Consider the $\mathbf{D}_n-$reversible-equivariant vector field (17) with $\mathcal{R}_{\frac{\pi}{2}}$ as a reversing symmetry.

We write $n = 2m$, with $m \in \mathbb{N}$ and note that $R^{\frac{m}{2}} = \mathcal{R}_{\frac{\pi}{2}}$. Thus, m must be even. We can write $n = 4(\frac{m}{2})$ with $m \in \mathbb{N}$ even. In addition, for $\mathcal{R}_{\frac{\pi}{2}}$ to be a reversing symmetry, the generator R has to be a reversing symmetry and $\frac{m}{2}$ has to be odd. As a result, n is of the form $8p + 4$ with $p \in \mathbb{N}$. So every $\mathbf{D}_n-$reversible-equivariant vector field in presence of reversing symmetry $\mathcal{R}_{\frac{\pi}{2}}$ takes the form

$$g(z) = p(u, v)z^{4p+3} + q(u, v)\bar{z}^{4p+1},$$

for $n = 8p + 4$, $p \in \mathbb{N}$.

To illustrate, we set $n = 4$ and

$$g(z) = p(u, v)z^3 + q(u, v)\bar{z}, \tag{19}$$

where $u = z\bar{z}$ and $v = z^4 + \bar{z}^4$ are the generators of the $\mathbf{D}_4$-invariants.

If we take $p(u, v) = q(u, v) = 1$, we have the vector field in real coordinates

$$g(x, y) = (x + x^3 - 3xy^2, -y + 3x^2y - y^3),$$

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with five equilibrium points: $(0,0)$ e $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$.

The origin is of saddle type and the four remaining equilibria are of centre type. This result follows analogously as in the previous example.

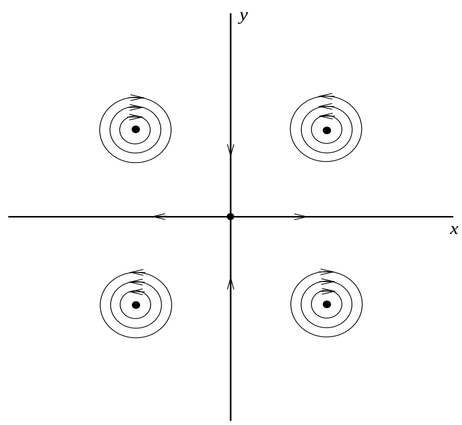

FIG. 5. $\mathbf{D}_4(R)-$reversible-equivariant vector field (19) for $p(u,v) = q(u,v) = 1$.

As well as in $\mathbf{Z}_4(R)-$reversible-equivariant case, in $\mathbf{D}_4(R)-$reversible-equivariant case the origin is generically an equilibrium point of saddle type. More specifically, only for the case $n = 4$ we have, under certain conditions, the origin as a hyperbolic equilibrium point. In the corresponding equivariant context, the origin is generically hyperbolic for all $n \in \mathbb{N}$. The following example shows that if $q(0,0) = 0$, then a $\mathbf{D}_4(R)-$reversible-equivariant vector field does not have the origin as a hyperbolic equilibrium: set $p(u,v) = 1$ e $q(u,v) = u$, then we have

$$g(x,y) = (2x^3 - 2xy^2, 2x^2y - 2y^3),$$

with the origin as the only equilibrium point, being an unstable node.

In Section 5, we obtain the solution branches for the non-degenerate $\mathbf{D}_4-$reversible-equivariant bifurcation problem on the plane. This is the first case that shall appear in the classification of such bifurcations by Singularity methods which we present in a forthcoming paper.

Case 3. Vector fields on $\mathbb{R}^2$ under the action of the orthogonal group $\mathbf{O}(2)$ where the mirror $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a reversing symmetry and the rotations $\mathcal{R}_\theta$ are symmetries, $0 \le \theta < 2\pi$.

In this case, every $\mathbf{O}(2)-$reversible-equivariant vector field has the form

$$g(z) = p(z\bar{z})iz,$$

where p is a real-valued polynomial, and we can write our differential equation as

$$(\dot{x}, \dot{y}) = \sum_j a_j (x^2 + y^2)^j (-y, x).$$

Since $\sum_j a_j (x^2 + y^2)^j$ is a common factor, we simulate its dynamics via the vector field

$$(\dot{x}, \dot{y}) = (-y, x),$$

that has the origin as the only equilibrium, which is a centre.

Notice that there are no $\mathbf{O}(2)-$reversible-equivariant vector fields with $\mathcal{R}_\theta$ as a reversing symmetry and $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as a symmetry.

By the general forms obtained in all these examples, we observe that the reversibility condition in place of equivariance condition implies more degenerate vector fields. This is what we expect to see when we treat bifurcation theory, that is, as well as presence of symmetries leads to more degenerate bifurcation problems than absence of symmetries, we may find in reversible-equivariant context that the presence of reversing symmetries leads to more degenerate bifurcation problems than presence of symmetries.

3.3. Reversible-equivariant linear vector fields on the plane

The aim of this subsection is to deal with particular examples illustrating the effect caused by reversing symmetries in linear vector fields on the plane.

The concept of reversibility of a vector field is linked with an involution. An involution is a germ of a C^∞ diffeomorphism at the origin $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ satisfying $\phi^2 = Id$. We consider $\det(d\phi)_0 = -1$.

Definition 3.3. Two vector fields X_1 and X_2 are called $C^\infty-$conjugate if there exists a C^∞ diffeomorphism h which maps trajectories in the phase portrait of X_1 into trajectories in the phase portrait of X_2, preserving orientation and time, that is, $h(\phi_t^{X_1}(x)) = \phi_t^{X_2}(h(x))$ where $\phi_t^{X_1}$ and $\phi_t^{X_2}$ are flows of X_1 and X_2, respectively.

The following result implies that every involution in a neighbourhood of a fixed point such that $\dim Fix(\phi) = 1$ is $C^\infty-$conjugate to $\varphi(x, y) = (x, -y)$.

Proposition 3.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be an involution such that $Fix(\phi)$ is a submanifold of dimension k. Then ϕ is locally $C^\infty-$conjugate (that is, in a neighbourhood of a point $x_0 \in \mathbb{R}^n$) to the linear involution $\varphi(x, y) = (x, -y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$.*

Proof: See [4], p. 29. □

We consider linear differential equations of the form $\dot{x} = g(x)$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is linear and represented by a non-singular $n \times n$ matrix. From Definition 2.3 we have that a

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linear vector field g is reversible when there exists a linear invertible mapping $R : \mathbb{R}^n \to \mathbb{R}^n$ such that $gR = -Rg$. Here we do not demand that R is an involution. In addition, g is $S-$equivariant if it commutes with a linear invertible mapping $S : \mathbb{R}^n \to \mathbb{R}^n$, that is, $gS = Sg$.

The eigenvalues of reversible linear mappings on the plane come in singlets $\lambda = 0$, real doublets $\lambda, -\lambda$, with $\lambda \in \mathbb{R}$ or purely imaginary doublets $\lambda, -\lambda$, with $\lambda \in i\mathbb{R}$ (see [19], Lemma 1.1). Hence, the reversibility has important implications on the eigenvalues of the linear part of reversible-equivariant systems. In the following examples we see that such eigenvalues are either zero, or a pair of real eigenvalues of opposite sign, or a pair of purely imaginary eigenvalues.

Example 3.3. Let $\Gamma = \mathbb{Z}_4$ act on $\mathbb{R}^2$ as in Case 1 of Subsection 3.2. In order to get a $\mathbf{Z}_4(R)-$reversible-equivariant linear vector field we have to impose $p(u, v) = 0$ e $q(u, v) = a + ib$, $a, b \in \mathbb{R}$, in (16). So a general $\mathbf{Z}_4(R)-$reversible-equivariant linear vector field is of the form

$$g(z) = (a + ib)\bar{z}$$

or, in real coordinates,

$$g(x, y) = (ax + by, bx - ay),$$

with $a, b \in \mathbb{R}$. The eigenvalues of the matrix of g are $\lambda_{\pm} = \pm\sqrt{a^2 + b^2}$, that is, always a pair of real eigenvalues of opposite sign. Therefore, the reversibility here forces the equilibrium point to be of saddle type.

In the example above, $\Gamma_+ = \mathbb{Z}_2$. If $\Gamma = \Gamma_+ = \mathbb{Z}_4$ then we have a $\mathbf{Z}_4-$equivariant linear vector field which is of the form

$$g(x, y) = (\alpha x - \beta y, \beta x + \alpha y),$$

with $\alpha, \beta \in \mathbb{R}$. In this case, the eigenvalues of the matrix of g are $\lambda_{\pm} = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$. In contrast to the reversible-equivariant case, where we have always saddle, in this case the origin can be a centre, a node or a focus.

Example 3.4. Let $\Gamma = \mathbf{D}_4$ act as the symmetries of the square in $\mathbb{R}^2$, as in Case 2(b) of Subsection 3.2. In this case, $\Gamma_+ = \mathbf{D}_2$ and in order to get a $\mathbf{D}_4-$reversible-equivariant linear vector field we demand in (19) that $p(u, v) = 0$ and $q(u, v) = a \in \mathbb{R}$. Therefore, the Γ_+-equivariance together with the reversibility imply

$$g(z) = a\bar{z}$$

or, in real coordinates,

$$g(x, y) = (ax, -ay),$$

with $a \in \mathbb{R}$. Thus the eigenvalues of the matrix of g are $\lambda_{\pm} = \pm a$ and, hence, the equilibrium is typically of saddle type.

Example 3.5. From Example 3.1, we consider the $\mathbf{D}_4-$reversible-equivariant vector field $g(z) = p(u,v)iz + q(u,v)i\bar{z}^3$. We have $\Gamma_+ = \mathbb{Z}_4$. In this case, the Γ_+-equivariance with the additional reversibility imply in a $\mathbf{D}_4-$reversible-equivariant linear vector field of the form

$$g(x,y) = (-ay, ax),$$

with $a \in \mathbb{R}$. Thus the eigenvalues of the matrix of g are $\lambda_\pm = \pm ia$ and, hence, the equilibrium is typically of centre type.

The last two examples illustrate a change of the local dynamics due to the difference in nature of the equilibrium points when we change a symmetry by a reversing symmetry and vice-versa.

4. THE LIAPUNOV-SCHMIDT REDUCTION IN REVERSIBLE-EQUIVARIANT BIFURCATION PROBLEMS

The theory of steaty-state bifurcation is the study of changes in the number and/or nature of equilibria of an equation as a parameter varies. More specifically, consider a differential equation

$$\dot{u} + \Phi(u,\lambda) = 0, \tag{20}$$

where $\Phi : X \times \mathbb{R} \to Y$ is a smooth mapping satisfying $\Phi(0,0) = 0$ and $\det(d\Phi)_{0,0} = 0$ and X, Y are Banach spaces. We call u the state variable and λ the bifurcation parameter. The study of the bifurcation problem (20) is the study of its equilibrium solutions when λ varies around $\lambda = 0$. When $\ker(d\Phi)_{0,0}$ is a finite-dimensional vector subspace, this can be simplified by performing in (20) the procedure known as the Liapunov-Schmidt reduction.

The basic idea underlying the Liapunov-Schmidt reduction is to reduce the study of bifurcation of steady-states of (20) to the study of zeros of an associated problem defined on a vector space isomorphic to $\ker(d\Phi)_{0,0}$. More precisely, the Liapunov-Schmidt reduction is a procedure that reduces the bifurcation equations of steaty-state solutions defined on Banach spaces to the bifurcations of zeros of a mapping defined on a finite-dimensional vector space, provided that the linearized operator at the bifurcation point is elliptic and Fredholm of index zero (Definitions 4.1 and 4.2).

In this section we show that, in the same way as in other symmetry contexts mentioned in Section 1, the Liapunov-Schmidt reduction can also be adapted to the reversible-equivariant context. The proof follows the steps that appear in [11] for the equivariant case, but now X is not necessarily an invariant subspace of Y, as it is required in [11]. Also, recall that $\Gamma-$reversible-equivariant mappings can be viewed as $\Gamma-$equivariant mappings with appropriate distinct actions on source and target. Here we prove that the reduction process can be performed to preserve the symmetry properties for general distinct actions on X and

Y. In particular, this implies that, in the reversible-equivariant case, the reduced equation inherits all the symmetries and reversing symmetries of the original equation.

DEFINITION 4.1. Let $L : X \to Y$ be a bounded linear operator where X, Y are Banach spaces. L is called *Fredholm* if $\ker L$ is a finite-dimensional subspace of X and range L is a closed subspace of Y of finite codimension.

DEFINITION 4.2. The *index* of a Fredholm operator L is the integer

$$i(L) = \dim \ker L - \text{codim range } L.$$

As a first step of the Liapunov-Schmidt reduction, under the action of a compact Lie group Γ we must choose complements to certain subspaces of a Banach space so that they are $\Gamma-$invariant. A subspace W is $\Gamma-$invariant if $\gamma w \in W$, for all $w \in W$ and $\gamma \in \Gamma$. The existence of such complements is discussed below.

PROPOSITION 4.1. *If $L : X \to Y$ is a Fredholm operator, then there exist closed $\Gamma-$invariant subspaces M and N of X and Y, respectively, such that $X = \ker L \oplus M$ and $Y = N \oplus$ range L.*

Proof: See [11], p. 304. □

We now present the reduction process:
Let Γ be a compact Lie group acting linearly on the Banach spaces X and Y. Let

$$\begin{array}{rcl} \Phi \ : X \times \mathbb{R} & \to & Y \\ (u, \lambda) & \mapsto & \Phi(u, \lambda) \end{array}, \quad \Phi(0,0) = 0,$$

be a smooth $\Gamma-$equivariant mapping, that is,

$$\Phi(\gamma u, \lambda) = \gamma \Phi(u, \lambda), \tag{21}$$

where the action of γ on the left-hand side of (21) is that on X and the action of γ on the left-hand side is that on Y. Note that we are assuming that the parameter λ is not affected by the action of Γ. We want to use the Liapunov-Schmidt reduction to solve the equation $\Phi(u, \lambda) = 0$ for u and λ near the origin. Suppose that the differential $L = (d\Phi)_{0,0}$ is a Fredholm operator of index zero.

Step 1 : Decompose X and Y,

$$(a) X = \ker L \oplus M \quad \text{and} \quad (b) Y = N \oplus \text{range } L, \tag{22}$$

where M, N, $\ker L$ and range L are $\Gamma-$invariant subspaces. By Proposition 4.1, such splittings are possible. Moreover,

$$\dim \ker L = \dim N = n < \infty,$$

because L is an operator of index zero.

Step 2 : Let $E : Y \to$ range L be the projection associated with the splitting $(22(b))$ and let I be the identity on Y. The equation $\Phi(u, \lambda) = 0$ can be replaced by the equivalent pair of equations

$$(a) E\Phi(u, \lambda) = 0 \quad \text{and} \quad (b)(I - E)\Phi(u, \lambda) = 0. \tag{23}$$

Step 3 : Any $u \in X$ can be written as $u = v + w$, where $v \in \ker L$ and $w \in M$, and by the Implicit Function Theorem we have that $(23(a))$ may be solved for w as a function of v and λ, that is, there exists a unique $W : \ker L \times \mathbb{R} \to M$ such that

$$E\Phi(v + W(v, \lambda), \lambda) = 0. \tag{24}$$

Step 4 : Define the mapping

$$\begin{array}{rcl} \phi \ : \ker L \times \mathbb{R} & \to & N \\ (v, \lambda) & \mapsto & (I - E)\Phi(v + W(v, \lambda), \lambda). \end{array}$$

Since W satisfies (24), it follows that $\Phi(u, \lambda) = 0$ if and only if $\phi(v, \lambda) = 0$. Moreover, ϕ is $\Gamma-$equivariant, as we prove below.

Step 5 : Choose a basis $\{v_1, \ldots, v_n\}$ for $\ker L$ and a basis $\{v_1^*, \ldots, v_n^*\}$ for N. Define the mapping

$$\begin{array}{rcl} g : \mathbb{R}^n \times \mathbb{R} & \to & \mathbb{R}^n \\ (x, \lambda) & \mapsto & \big(g_1(x, \lambda), \ldots, g_n(x, \lambda)\big) \end{array}$$

by $g_i(x, \lambda) = \langle v_i^*, \phi(x_1 v_1 + \ldots x_n v_n, \lambda)\rangle$, $i = 1, \ldots, n$.

Since $\dim \ker L = \dim N = n < \infty$, both bases $\{v_1, \ldots, v_n\}$ and $\{v_1^*, \ldots, v_n^*\}$ contain in fact the same number of vectors. Finally, notice that $\Phi(u, \lambda) = 0$ if and only if $g(x, \lambda) = 0$. Thus, the zeros of Φ are in one-to-one correspondence with the zeros of g.

We first discuss how to choose such bases for $\ker L$ and N, respectively, in order to define an action of Γ on $\mathbb{R}^n$ such that the mapping g is $\Gamma-$equivariant. This follows from the representation theory of Lie groups.

Let $\{v_1, \ldots, v_n\}$ be an arbitrary basis for $\ker L$. Since Γ acts linearly on $\ker L$, for each $\gamma \in \Gamma$ there is an $n \times n$ matrix $A(\gamma) = (a_{ij}(\gamma))_{i,j}$ such that

$$\gamma . v_i = \sum_{j=1}^{n} a_{ji}(\gamma) v_j. \tag{25}$$

We want to choose a basis $\{v_1^*, \ldots, v_n^*\}$ for N such that, for the same matrix $A(\gamma)$ given above, we have

$$\gamma . v_i^* = \sum_{j=1}^{n} a_{ji}(\gamma) v_j^*. \tag{26}$$

A *consistent choice* of bases $\{v_1, \ldots, v_n\}$ and $\{v_1^*, \ldots, v_n^*\}$ occurs if (25) and (26) hold simultaneously. If we make a consistent choice of bases for $\ker L$ and N, then, using the fact that ϕ is $\Gamma-$equivariant, the reduced equation g satisfies $g(A(\gamma)x, \lambda) = A(\gamma)g(x, \lambda)$, as required.

In applications, making a consistent choice of bases presents no problem. Moreover, in general and, in particular for our context, we can always make a consistent choice of the bases $\{v_1, \ldots, v_n\}$ and $\{v_1^*, \ldots, v_n^*\}$ for $\ker L$ and N, respectively (for details see [11]).

We shall prove now that the mapping ϕ obtained in Step 4 is $\Gamma-$equivariant. Let $E : Y \to$ range L be the projection associated with the splitting $(22(b))$, whose kernel is N. Then E commutes with Γ. In fact, suppose $u = v + w \in Y$, where $v \in$ range L and $w \in N$. By linearity, and since both range L and N are invariant subspaces, we have, for all $\gamma \in \Gamma$,

$$\gamma E(u) = \gamma E(v) = \gamma v = E(\gamma v) = E(\gamma v + \gamma w) = E(\gamma u).$$

Hence, $(I - E)$ also commutes with Γ.

Let $W : \ker L \times \mathbb{R} \to M$ be the function defined by (24). Then $W(\gamma v, \lambda) = \gamma W(v, \lambda)$ for all $\gamma \in \Gamma$. In fact, fix $\gamma \in \Gamma$ and define $W_\gamma(v, \lambda) = \gamma^{-1} W(\gamma v, \lambda)$. Then

$$E\Phi(v + W_\gamma(v, \lambda), \lambda) = E\Phi(\gamma^{-1}(\gamma v + W(\gamma v, \lambda)), \lambda) = \gamma^{-1} E\Phi(\gamma v + W(\gamma v, \lambda)) = 0,$$

since (24) is valid for all v and, in particular, for γv. Thus W_γ also solves the implicit equation $(23(a))$ and $W_\gamma(0,0) = 0$. By the uniqueness of solutions in the Implicit Function Theorem, we conclude that $W_\gamma(u, \lambda) = W(u, \lambda)$ and, then, $W(\gamma u, \lambda) = \gamma W(u, \lambda)$.

Therefore, by the facts that Φ is $\Gamma-$equivariant, $(I - E)$ and W commute with Γ we have

$$\begin{aligned} \phi(\gamma v, \lambda) &= (I - E)\Phi(\gamma v + W(\gamma v, \lambda), \lambda) \\ &= (I - E)\Phi(\gamma(v + W(v, \lambda)), \lambda) \\ &= (I - E)(\gamma \Phi(v + W(v, \lambda), \lambda)) \\ &= \gamma(I - E)\Phi(v + W(v, \lambda), \lambda) \\ &= \gamma \phi(v, \lambda). \end{aligned}$$

Therefore, the reduced equation $g = 0$ inherits the symmetries of the original equation $\Phi = 0$.

Setting $X = Y$ and assuming that the actions of Γ on X and Y are given by (5) and (6), respectively, we obtain the result for the reversible-equivariant context. In this case, the reduced equation $g = 0$ inherits the symmetries and reversing symmetries of the original equation $\Phi = 0$.

5. A $\mathbf{D_4}-$REVERSIBLE-EQUIVARIANT BIFURCATION PROBLEM

In this section we consider steady-state bifurcation problems which have been reduced - via Liapunov-Schmidt procedure - to a two-dimensional representation of $\mathbf{D}_4$. More specifically, we consider the action of the dihedral group $\mathbf{D}_4$ on $\mathbb{C}$ generated by

$$\kappa : z \mapsto \bar{z} \quad \text{and} \quad \xi : z \mapsto e^{i\frac{\pi}{2}} z,$$

where the flip κ is a symmetry and the rotation ξ is a reversing symmetry.

We obtain steady-state solution branches of the differential equation

$$\dot{z} + g(z, \lambda) = 0, \tag{27}$$

where $g : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is a germ of smooth mapping $\mathbf{D}_4-$reversible-equivariant, that is,

$$g(z, \lambda) = p(u, v, \lambda) z^3 + q(u, v, \lambda) \bar{z}, \tag{28}$$

with $u = z\bar{z}$ and $v = z^4 + \bar{z}^4$ generators of the $\mathbf{D}_4$-invariants. We assume that the bifurcation parameter is not affected by the symmetries.

We suppose that $z = 0$ is an equilibrium for all $\lambda \in \mathbb{R}$ and a bifurcation takes place when λ crosses the origin. So we have $g(0, \lambda) = 0$, for all $\lambda \in \mathbb{R}$ and $(dg)_{0,0} = 0$.

Since

$$(dg)_{0,0} = \begin{pmatrix} 0 & q(0,0,0) \\ q(0,0,0) & 0 \end{pmatrix},$$

we impose $q(0, 0, 0) = 0$. We also assume the non-degeneracy conditions $p(0, 0, 0) \neq 0$, $q_\lambda(0, 0, 0) \neq 0$ and $q_u(0, 0, 0) \neq 0$. The last assumption prevents purely imaginary eigenvalues.

Recall that the orbit of z under the action of $\mathbf{D}_4$ is the set $\{\gamma z : \gamma \in \mathbf{D}_4\}$ and $\mathbf{D}_4-$reversible-equivariant equations do not distinguish points on the same orbit.

The isotropy subgroup of $z \in \mathbb{C}$ is

$$\Sigma_z = \{\gamma \in \mathbf{D}_4 : \gamma z = z\}.$$

Recall that any two points on the same line through the origin, except the origin, have the same isotropy subgroup and that points on the same orbit have conjugate isotropy subgroups. Therefore, up to conjugacy, we consider Σ_z for points $z = e^{i\theta}$ on the unit circle for $0 \leq \theta \leq \frac{\pi}{4}$ (see [12], XIII, §5(b)). Moreover, the isotropy subgroup of $z = 0$ is $\mathbf{D}_4$ and of $z = e^{i\theta}$ is $\mathbb{Z}_2(\kappa)$ if $\theta = 0$; $\mathbb{Z}_2(\xi\kappa)$ if $\theta = \frac{\pi}{4}$ and $\mathbf{1}$ if $0 < \theta < \frac{\pi}{4}$. Such isotropy subgroups are presented in Table 1, with the respective steady-state solutions of (28) and corresponding equations.

We now deduce the information given in Table 2 concerning the branching equations and corresponding eigenvalues for each isotropy.

Along the trivial solution branch $z = 0$ we have

$$(dg)_{0,\lambda} = \begin{pmatrix} 0 & q(0,0,\lambda) \\ q(0,0,\lambda) & 0 \end{pmatrix}.$$

Thus, $(dg)_{0,\lambda}$ has a pair of real eigenvalues of opposite signs, namely $\pm q(0, 0, \lambda)$. Therefore, the solution $z = 0$ is a hyperbolic equilibrium point and its stability does not change when λ varies around $\lambda = 0$.

Next, we consider $\mathbb{Z}_2(\kappa)$ solutions, that is, solutions with isotropy subgroup $\mathbb{Z}_2(\kappa)$. In this case, $Fix(\mathbb{Z}_2(\kappa)) = \mathbb{R}$ and (28) becomes

$$p(x^2, 2x^4, \lambda)x^2 + q(x^2, 2x^4, \lambda) = 0.$$

We assume $x > 0$ because $\xi^2 = -I$, so that the points z and $-z$ are on the same orbit.

Since the isotropy subgroup $\mathbb{Z}_2(\kappa)$ is generated by the flip $z \mapsto \bar{z}$, at a $\mathbb{Z}_2(\kappa)$ solution (x, λ) the jacobian $(dg)_{x,\lambda}$ must commute with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In fact, this is the matrix of the flip κ as a function of real coordinates $z, \bar{z}$. Thus, $(dg)_{x,\lambda}$ has the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, which has eigenvalues $a \pm b$, with

$$a = g_z(x, \lambda) = (3p + q_u)x^2 + (p_u + 4q_v)x^4 + 4p_v x^6$$

and

$$b = g_{\bar{z}}(x, \lambda) = q + q_u x^2 + (p_u + 4q_v)x^4 + 4p_v x^6,$$

where p, q and their derivatives are calculated at (x, λ). Now, we use that $q = -px^2$ on the $\mathbf{Z}_2(\kappa)$ branch to obtain the eigenvalues as they appear in Table 2.

Now, we obtain the equation of the $\mathbf{Z}_2(\kappa)$ solution branch up to degree two:

$$g(x, \lambda) = 0 \Leftrightarrow p(u, v, \lambda)x^2 + q(u, v, \lambda) = 0 \Leftrightarrow$$

$$\Leftrightarrow (p(0,0,0) + p_u(0,0,0)u + p_\lambda(0,0,0)\lambda + \cdots)x^2 + q(0,0,0) + q_u(0,0,0)u + q_\lambda(0,0,0)\lambda + \cdots = 0$$

$$\Leftrightarrow (p(0,0,0) + q_u(0,0,0))x^2 + q(0,0,0) + q_\lambda(0,0,0)\lambda + o(3) = 0.$$

Since $q(0,0,0) = 0$ we have, up to degree two,

$$\lambda(x) \simeq \frac{-\big(p(0,0,0) + q_u(0,0,0)\big)}{q_\lambda(0,0,0)} x^2. \tag{29}$$

If we also assume that $p(0,0,0) + q_u(0,0,0) \neq 0$, then the direction of branching is determined.

For the isotropy subgroup $\mathbb{Z}_2(\xi\kappa)$ we have the following: $Fix(\mathbb{Z}_2(\xi\kappa)) = \mathbb{R}\{e^{i\frac{\pi}{4}}\}$ and on the $\mathbb{Z}_2(\xi\kappa)$ branch of solutions (28) becomes

$$-p(x^2, -2x^4, \lambda)x^2 + q(x^2, -2x^4, \lambda) = 0,$$

for $x > 0$. Each point (x, λ) on this branch represents the solution $(xe^{i\frac{\pi}{4}}, \lambda)$.

In this case, the isotropy subgroup $\mathbb{Z}_2(\xi\kappa)$ is generated by the reversing symmetry $z \mapsto i\bar{z}$ and, therefore, at a $\mathbb{Z}_2(\xi\kappa)$ solution the jacobian matrix anti-commutes with the matrix

TABLE 1.
Solution of $g = 0$ for $\mathbf{D}_4-$reversible-equivariant g

Solution	Isotropy Subgroup	Equations
Trivial	$\mathbf{D}_4$	$z = 0$
$\mathbb{R}$	$\mathbb{Z}_2(\kappa)$	$p(x^2, 2x^4, \lambda)x^2 + q(x^2, 2x^4, \lambda) = 0$ $x > 0$
$\mathbb{R}e^{i\frac{\pi}{4}}$	$\mathbb{Z}_2(\xi\kappa)$	$-p(2x^2, -2x^4, \lambda)x^2 + q(2x^2, -2x^4, \lambda) = 0$ $x > 0$
$\mathbb{C}$	$\mathbf{1}$	$p = q = 0$ $\mathrm{Im}(z^4) \neq 0$

$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Thus, $(dg)_{xe^{i\frac{\pi}{4}},\lambda}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, which has eigenvalues $\pm\sqrt{a^2+b^2}$, with

$$a = g_z(xe^{i\frac{\pi}{4}}, \lambda) = \Big((3p - q_u)x^2 + (p_u + 4q_v)x^4 - 4p_v x^6\Big)i$$

and

$$b = g_{\bar{z}}(xe^{i\frac{\pi}{4}}, \lambda) = q + q_u x^2 - (p_u + 4q_v)x^4 + 4p_v x^6,$$

where p, q and their derivatives are calculated at $(xe^{i\frac{\pi}{4}}, \lambda)$.

Using that $q = px^2$ on this branch, we obtain the eigenvalues

$$\eta = \pm\sqrt{8}x^2\sqrt{pq_u - p^2 - (pp_u + 4pq_v)x^2 + 4pp_v x^4}.$$

If we assume that $pq_u - p^2 - (pp_u + 4pq_v)x^2 + 4pp_v x^4 > 0$, then $(dg)_{xe^{i\frac{\pi}{4}},\lambda}$ has a pair of real eigenvalues of opposite signs.

Next, we obtain the equation of the $\mathbf{Z}_2(\xi\kappa)$ solution branch:

$$g(xe^{i\frac{\pi}{4}}, \lambda) = 0 \Leftrightarrow -p(u, v, \lambda)x^2 + q(u, v, \lambda) = 0 \Leftrightarrow$$

$$\Leftrightarrow -(p(0,0,0)+p_u(0,0,0)u+p_\lambda(0,0,0)\lambda+\cdots)x^2+q(0,0,0)+q_u(0,0,0)u+q_\lambda(0,0,0)\lambda+\cdots=0$$

$$\Leftrightarrow (-p(0,0,0) + q_u(0,0,0))x^2 + q(0,0,0) + q_\lambda(0,0,0)\lambda + o(3) = 0.$$

Since $q(0,0,0) = 0$ we have, up to degree two,

$$\tilde{\lambda}(x) \simeq \frac{(p(0,0,0) - q_u(0,0,0))}{q_\lambda(0,0,0)}x^2. \tag{30}$$

TABLE 2.

Data on solutions of $\mathbf{D}_4-$reversible-equivariant bifurcation problems

Isotropy Subgroup	Branching Equations	Signs of Eigenvalues
$\mathbf{D}_4$	$z = 0$	$\pm q_\lambda(0,0,0)\lambda$
$\mathbb{Z}_2(\kappa)$	$\lambda(x) \simeq \frac{-\ p(0,0,0)+q_u(0,0,0)}{q_\lambda(0,0,0)} x^2$	$p + q_u + (p_u + 4q_v)u + 4p_v u^2$ p
$\mathbb{Z}_2(\xi\kappa)$	$\tilde{\lambda}(x) \simeq \frac{(p(0,0,0)-q_u(0,0,0))}{q_\lambda(0,0,0)} x^2$	$\pm\sqrt{pq_u - p^2 - (pp_u + 4pq_v)u + 4pp_v u^2}$

If we assume that $p(0,0,0)-q_u(0,0,0) \neq 0$, then the direction of branching is determined.

Finally, for the solutions with trivial isotropy **1** the equations in Table 1 follows from (28) since z^3 and $\bar{z}$ are collinear only when $\mathrm{Im}(z^4) = 0$. Thus when $\mathrm{Im}(z^4) \neq 0$, solving $g = 0$ is equivalent to solving $p = q = 0$. Observe that $\mathrm{Im}(z^4) \neq 0$ precisely when the isotropy subgroup of z is **1**. Therefore, under the hypothesis $p(0,0,0) \neq 0$, it is not possible to find solutions with trivial isotropy subgroup to (28) near the origin.

We now exhibit the bifurcation diagrams, including stabilities, for the $\mathbf{D}_4-$reversible-equivariant bifurcation problem (28) on the plane. The computations needed to obtain such diagrams are performed up to degree two. Moreover, since $Fix(\mathbb{Z}_2(\kappa)) = \mathbb{R}$ and $Fix(\mathbb{Z}_2(\xi\kappa)) = \mathbb{R}\{e^{i\frac{\pi}{4}}\}$, we construct the bifurcation diagrams in the plane λ, x, where the points (x, λ) on the $\mathbf{Z}_2(\xi\kappa)$ solution branch represent the solutions $(xe^{i\frac{\pi}{4}}, \lambda)$.

Recall that along the trivial solution branch $z = 0$ we have a pair of real eigenvalues of opposite signs, namely $\pm q_\lambda(0,0,0)\lambda$.

Along the $\mathbf{Z}_2(\kappa)$ solution branch the eigenvalues to lowest order are $p(0,0,0)+q_u(0,0,0)$ and $p(0,0,0)$ (see Table 2). It follows that if we assume $p(0,0,0)+q_u(0,0,0) \neq 0$, then the signs of the eigenvalues are determined.

On the $\mathbf{Z}_2(\xi\kappa)$ solution branch we obtain the eigenvalues to lowest order $\pm\sqrt{pq_u - p^2}$, where p, q_u are calculated at the origin (see Table 2).

Thus we assume

$$q_\lambda(0,0,0) \neq 0, \quad q_u(0,0,0) \neq 0 \quad \text{and} \quad p(0,0,0) \neq 0$$

such that $p(0,0,0) + q_u(0,0,0) \neq 0$ and $p(0,0,0) - q_u(0,0,0) \neq 0$, to draw the bifurcation diagrams of $g = 0$. There are four possible diagrams with real eigenvalues, depending on the signs of $q_\lambda(0,0,0)$ and $p(0,0,0)$. The diagrams below are constructed from the data in Tables 1 and 2, choosing $q_\lambda(0,0,0) < 0$.

In Figure 6(a), we set $p(0,0,0) > 0$. In order to have real eigenvalues, we consider $q_u(0,0,0)$ such that $p(0,0,0) - q_u(0,0,0) < 0$. Then along the $\mathbb{Z}_2(\kappa)$ solutions we have a pair of positive eigenvalues and along the $\mathbb{Z}_2(\xi\kappa)$ solutions the eigenvalues are real and of opposite sign. In addition, both $\mathbb{Z}_2(\kappa)$ and $\mathbb{Z}_2(\xi\kappa)$ branches are supercritical, that is, $x\lambda'(x) > 0$, for x near the origin.

In Figure 6(b), we set $p(0,0,0) < 0$. Again, in order to have real eigenvalues, we consider $q_u(0,0,0)$ such that $p(0,0,0) - q_u(0,0,0) > 0$. Then along the $\mathbb{Z}_2(\kappa)$ solutions we have a pair of negative eigenvalues and along the $\mathbb{Z}_2(\xi\kappa)$ solutions the eigenvalues are real and of opposite sign. In this case, both $\mathbb{Z}_2(\kappa)$ and $\mathbb{Z}_2(\xi\kappa)$ branches are subcritical, that is, $x\lambda'(x) < 0$, for x near the origin.

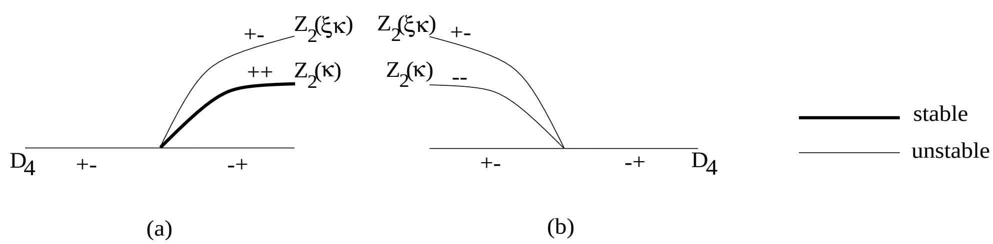

FIG. 6. Bifurcation diagrams for $\mathbf{D}_4-$reversible-equivariant mappings.

A simple analysis shows that in both diagrams the $\mathbb{Z}_2(\kappa)$ solutions always branch off the origin below the $\mathbb{Z}_2(\xi\kappa)$ solutions. We also point out that for such bifurcation, asymptotically stable solutions are obtained only if $p(0,0,0) > 0$ with isotropy subgroup $\mathbb{Z}_2(\kappa)$.

ACKNOWLEDGMENTS

We would like to thank CNPq-Brazil for financial support under the grants 141672/03-0.

REFERENCES

1. Artés, J. C., Dumortier, F., Herssens, C. and Llibre, J. [2000] Computer program P4 to study polynomial differential systems numerically. To be included in a book.
2. Buono, L., Buzzi, C. and Lamb, J.S.W. Reversible equivariant Liapunov centre theorem. In preparation.
3. Buono, L., Lamb, J. S. W. and Roberts, R. M. Branching and bifurcation of equilibria of reversible equivariant vector fields. In preparation.
4. Buzzi, C. A. [1999] *Formas Normais de campos vetoriais reversíveis*, PhD Thesis, Universidade Estadual de Campinas.
5. Buzzi, C. and Teixeira, M.A. [2004] Time-reversible Hamiltonian vector fields with symplectic symmetries. *J. Diff. Eq.* **16**, 559-574.
6. Buzzi, C. A. and Lamb, J. S. W. [2005] Reversible equivariant Hopf bifurcation. *Arch. Ration. Mech. Anal.* **175**, 39-84.
7. Damon, J. [1984] The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$. *Memoirs of the American Mathematical Society* **306** Providence.
8. Devaney, R. L. [1976] Reversible diffeomorphisms and flows, *Trans. Amer. Math. Soc.* **218**, 89-113.

9. Field, M. and Golubitsky, M. [1992] *Symmetry in chaos.* Oxford University Press Inc., NY.

10. Field, M. [1996] *Lectures on bifurcations, dynamics and symmetry*, *Pitman Research Notes in Mathematics* **356**, Longman, Harlow.

11. Golubitsky, M. and Schaeffer, D. [1984] *Singularities and Groups in Bifurcations Theory.* Vol. I, Appl. Math. Sci. **51** Springer-Verlag, NY.

12. Golubitsky, M., Stewart, I. and Schaeffer, D. [1985] *Singularities and Groups in Bifurcation Theory.* Vol. II, Appl. Math. Sci. **69** Springer-Verlag, NY.

13. Golubitsky, M. and Stewart, I. N. [1985] Hopf bifurcation in the presence of symmetry. *Arch. Rational Mech. Anal.* **87**, 107-165.

14. Golubitsky, M., Krupa, M. and Lim, C. [1991] Time-reversibility and particle sedimentation, *SIAM J. Appl. Math.* **51**, 49-72.

15. Golubitsky, M., Marsden, J. E., Stewart, I. and Dellnitz, M. [1995] The constrained Liapunov-Schmidt procedure and periodic orbits. pp. 81-127 of [20].

16. Hoveijn, I., Lamb, J. S. W. and Roberts, R. M. [2003] Linear normal form theory in eigenspaces of involutory (anti-)automorphisms. *J. Diff. Eq.* **190,** 182-213.

17. Lamb, J.S.W. [1992] Reversing symmetries in dynamical systems. *Physica A* **25**, 925-937.

18. Lamb, J.S.W. and Roberts, J.A.G. [1998] Time-reversal symmetry in dynamical systems: a survey. *Physica D* **112**, 1-39.

19. Lamb, J. S. W. and Roberts, R. M. [1999] Reversible equivariant linear systems. *J. Diff. Eq.* **159,** 239-279.

20. Langford, W. and Nagata, W. [1995] Normal Forms and Homoclinic Chaos (Waterloo, ON, 1992). Fields Inst. Commun. **4**. American Mathematical Society, Providence, RI.

21. Manoel, M. [1998] *Hidden symmetries in bifurcation problems: the singularity theory*, PhD Thesis, University of Warwick.

22. Manoel, M. and Stewart, I [1999] Degenerate bifurcations with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetry. *Int. Journal of Bif. and Chaos* **9,** 1653-1667.

23. Manoel, M. and Stewart, I [2000] The classification of bifurcations with hidden symmetries. *Proc. of the London Math. Soc.* **80,** 198-234.

24. Medrado, J. C. R. and Teixeira, M. A. [1998] Symmetric singularities of reversible vector fields in dimension three. *Physica D* **112**, 122-131.

25. Melbourne, I. [1986] A singularity theory analysis of bifurcation problems with octahedral symmetry. *Dynamics and Stability of Systems* **1**, n. **4**.

26. Montgomery, D. and Zippin, L. [1955] *Topological transformations groups* (Interscience, new York).

27. Sevryuk, M. B. [1986] *Reversible systems*, LNM **1211**, Springer, Berlin.

28. Teixeira, M. A. [1977] Generic bifurcation in manifolds with boundary. *J. Diff. Eq.* **25,** 65-89.

29. Teixeira, M. A. [1997] Singularities of reversible vector fields. *Physica D* **100**, 101-118.

30. Vanderbauwhede, A., [1982] Local bifurcation and symmetry. Research Notes in Mathematics, **75**. Pitman, London.

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