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Uniform Exponential Dichotomy and Continuity of attractors for singularly perturbed damped wave equations

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Damped wave equations $u_{tt}+\eta\Lambda^{\frac{1}{2}}u_t+au_t+\Lambda u = f(u)$, $t > 0$, $x \in \Omega \subset \mathbb{R}^N$, $\eta \geq 0$, where Λ denotes negative Laplacian in $L^2(\Omega)$ with Dirichlet boundary condition, are considered in the phase space $H_0^1(\Omega) \times L^2(\Omega)$ chosen according to the energy functional. Existence and properties of the family of attractors $\boldsymbol{A}_\eta$, $\eta \geq 0$, are discussed under suitable assumptions on f and this family is shown to behave continuously with respect to the parameter η as $\eta \to 0^+$.
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1. INTRODUCTION

For $\eta \geq 0$ consider a family of damped wave equations

$$\begin{cases} u_{tt} + \eta \Lambda^{\frac{1}{2}} u_t + a u_t + \Lambda u = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, where $\Lambda : D(\Lambda) \subset L^2(\Omega) \to L^2(\Omega)$, $D(\Lambda) = H^2(\Omega) \cap H^1_0(\Omega)$, denotes the negative Laplacian in $L^2(\Omega)$ with homogeneous Dirichlet condition, $a > 0$, and $f : \mathbb{R} \to \mathbb{R}$.

Our main objective in this paper will be to investigate the relationship between the nonlinear asymptotic dynamics when the type of equation changes at $\eta = 0$, namely the family of attractors for (1) behave upper and lower semicontinuously at $\eta = 0$. Note that this kind of information is of independent interest since it may allow us to view a damped hyperbolic equation, in the context of approximation, as a problem with parabolic structure.

Attractors for damped wave equations have been considered by many authors, see [1, 3, 4, 5, 6, 11, 14, 19] and references there in. In [9] the authors prove that, if $f = 0$, the linear semigroup associated to (1) is analytic. The local well posedness of (1) has been considered in [5]. The existence of attractors was considered in [6] for a more general class of problems which include (1) when Ω is a bounded smooth domain. When $\eta = 0$ the existence of attractors was considered in [2]. The continuity of attractors for related problems have been considered by many other authors, see [1, 8] and references there in.

In [15] the authors consider the continuity of attractors for a family of damped hyperbolic problems which degenerate to a parabolic problem as the parameter tends to zero. Here we aim to approximate a damped hyperbolic problem by *parabolic problems.* To illustrate the sort of difficulties that one faces investigating the dynamics of (1) note that in the linear case ($f \equiv 0$), the problems $(54)_{\eta>0}$ generate on $Y = H^1_0(\Omega) \times L^2(\Omega)$ a compact analytic semigroup $\{e^{-A_\eta t}\}$ whereas the semigroup $\{e^{-A_0 t}\}$ defined on Y by the linear hyperbolic problem $(54)_{\eta=0}$ ($f \equiv 0$) is of the class C^0 and is neither compact nor analytic. This implies in particular that the uniform exponential dichotomy (see Definition 2.1) for the semigroups corresponding to the linearized problems is much more difficult to prove and can hardly be obtained as a consequence of the general theory.

The main result of the paper is contained in

Theorem 1.1. *Suppose* (58), (55), (56) *and let* $\boldsymbol{A}_\eta$, $\eta \geq 0$, *be a family of attractors corresponding to* (1) *in* Y. *Then these attractors are upper semicontinuous with respect to* η *as* $\eta \to 0^+$. *If in addition condition* (67) *is also satisfied, then the family* $\boldsymbol{A}_\eta$, $\eta \geq 0$ *is continuous at* $\eta = 0$*; that is*

$$\lim_{\eta \to 0^+} dist(\boldsymbol{A}_\eta, \boldsymbol{A}_0) = 0 \tag{2}$$

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in the sense of Hausdorff distance of sets

$$dist(\boldsymbol{A}_\eta, \boldsymbol{A}_0) = \sup_{a_0 \in A_0} \inf_{a_\eta \in A_\eta} \|a_0 - a_\eta\|_Y + \sup_{a_\eta \in A_\eta} \inf_{a_0 \in A_0} \|a_0 - a_\eta\|_Y.$$

Section 2 is devoted to the study of the continuity properties of the semigroups generated by the linear strongly damped wave operators when the strong damping approaches zero. In Section 3 the continuity properties of the corresponding nonlinear semigroups are proved and uniform bounds on the attractors are obtained. In Section 4 we prove the upper semicontinuity of attractors using the results of Section 3, we also obtain the lower semicontinuity of attractors using the continuity properties of the local unstable manifolds of equilibrium solutions. Finally, in the Appendix we prove the existence and continuity of the local unstable manifolds.

Additionally, at the end of Section 4, we also address regularity of the attractor $\boldsymbol{A}_0$ corresponding to a 'limit' hyperbolic problem and obtain that

Corollary 1.1. *Under the assumptions of Theorem 1.1 the global attractor $\boldsymbol{A}_0$ for the semigroup governed by the damped wave equation ((1) with $\eta = 0$) in $H^1_0(\Omega) \times L^2(\Omega)$ is a bounded subset of $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$.*

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2. PROPERTIES OF THE LINEAR PROBLEMS

2.1. Linear damped wave operators and corresponding semigroups

For $\eta \geq 0$ consider a family of the linear damped wave operators,

$$A_\eta : D(A_\eta) \subset Y \to Y,\ A_\eta \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -I \\ \Lambda & \eta\Lambda^{\frac{1}{2}} + aI \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $Y = H^1_0(\Omega) \times L^2(\Omega)$, $a > 0$, and

$$D(A_\eta) = Y^1 = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega),\ \eta \geq 0.$$

We remark that Y is a Hilbert space with the product

$$\langle \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \begin{bmatrix} \tilde\phi \\ \tilde\psi \end{bmatrix} \rangle_Y = \langle \Lambda^{\frac{1}{2}}\phi, \Lambda^{\frac{1}{2}}\tilde\phi\rangle_{L^2(\Omega)} + \langle \psi, \tilde\psi\rangle_{L^2(\Omega)},\ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y,$$

which defines in Y a norm equivalent to $H^1_0(\Omega) \times L^2(\Omega)$ norm and also to the norm

$$\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_Y = \|\Lambda^{\frac{1}{2}}\phi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}$$

that we use in this paper.

PROPOSITION 2.1. *The following conditions hold.*
i) A_η, $\eta \geq 0$, *is a maximal accretive operator and* $\{e^{-A_\eta t}\}$ *is a* C^0*-semigroup of contractions on* Y.
ii) $0 \in \rho(A_\eta)$ *and* A_η *has compact resolvent for each* $\eta \geq 0$.
iii) For $\eta > 0$ *semigroups* $\{e^{-A_\eta t}\}$ *are analytic and compact.*
iv) For each $\eta_0 > 0$ *there is certain* $d > 0$ *such that*

$$d^{-1}\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1} \leq \|A_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_Y \leq d\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1},\ \eta \in [0,\eta_0],\ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y^1. \tag{3}$$

Proof: For i) we first note that

$$\begin{aligned}\langle A_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix}\rangle_Y &= -\langle \Lambda^{\frac12}\psi, \Lambda^{\frac12}\phi\rangle_{L^2(\Omega)} + \langle \Lambda\phi + \eta\Lambda^{\frac12}\psi + a\psi, \psi\rangle_{L^2(\Omega)} \\ &= -2\mathrm{i}\mathcal{I}m\langle \Lambda^{\frac12}\psi, \Lambda^{\frac12}\phi\rangle_{L^2(\Omega)} + \eta\langle \Lambda^{\frac14}\psi, \Lambda^{\frac14}\psi\rangle_{L^2(\Omega)} + a\langle\psi,\psi\rangle_{L^2(\Omega)}\end{aligned}$$

and hence

$$Re\langle A_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix}\rangle_Y \geqslant 0,\ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y^1.$$

Furthermore, the equation

$$(I + A_\eta) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \tilde\phi \\ \tilde\psi \end{bmatrix}$$

possesses for each $\begin{bmatrix} \tilde\phi \\ \tilde\psi \end{bmatrix} \in Y$ a unique solution[1]

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \left((1+a)I+\eta\Lambda^{\frac12}+\Lambda\right)^{-1}\left((1+a)\tilde\phi+\eta\Lambda^{\frac12}\tilde\phi+\tilde\psi\right) \\ \left((1+a)I+\eta\Lambda^{\frac12}+\Lambda\right)^{-1}\left((1+a)\tilde\phi+\eta\Lambda^{\frac12}\tilde\phi+\tilde\psi\right)-\tilde\phi \end{bmatrix} \in Y^1. \tag{4}$$

Hence the remaining part of the proof follows from the Lumer-Phillips theorem (see [18]).

Concerning ii) we recall that there exists bounded inverse operator $A_\eta^{-1} : Y \to Y$

$$A_\eta^{-1} = \begin{bmatrix} \eta\Lambda^{-\frac12}+a\Lambda^{-1} & \Lambda^{-1} \\ -I & 0 \end{bmatrix},\ \eta \geq 0, \tag{5}$$

which takes bounded subsets of Y into bounded subsets of Y^1, the latter space being compactly embedded in Y.

The property iii) that $-A_\eta$, $\eta > 0$, generates a C^0 analytic semigroup in Y follows from [9, Theorem 1.1]. Compactness of $\{e^{-A_\eta t}\}$, $\eta > 0$ is then a consequence of the compactness of the resolvent of A_η.

[1]Operator $\left((1+a)I + \eta\Lambda^{\frac12} + \Lambda\right)$ appearing in (4) is selfadjoint in $L^2(\Omega)$ (see e.g. [17, p. 287]) and bounded from below.

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Concerning iv), the right hand side inequality in (3) is a consequence of the estimate

$$\begin{aligned}&\|A_\eta \begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y = \|\begin{bmatrix}-\psi\\ \Lambda\phi+\eta\Lambda^{\frac12}\psi+a\psi\end{bmatrix}\|_Y\\ &\leqslant \|\Lambda^{\frac12}\psi\|_{L^2(\Omega)} + \|\Lambda\phi\|_{L^2(\Omega)} + \eta\|\Lambda^{\frac12}\psi\|_{L^2(\Omega)} + a\|\psi\|_{L^2(\Omega)}\\ &\leqslant (1+\eta_0+a\lambda_1^{-\frac12})\left(\|\Lambda^{\frac12}\psi\|_{L^2(\Omega)} + \|\Lambda\phi\|_{L^2(\Omega)}\right) = d\ \|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y^1},\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in Y^1,\end{aligned}$$

where λ_1 is the first positive eigenvalue of Λ in $L^2(\Omega)$. To justify the left hand side inequality we consider A_η^{-1} as in (5) for which we have

$$\begin{aligned}&\|A_\eta^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y^1} = \|\begin{bmatrix}\eta\Lambda^{-\frac12}\phi+a\Lambda^{-1}\phi+\Lambda^{-1}\psi\\ -\phi\end{bmatrix}\|_{Y^1}\\ &\leqslant \eta\|\Lambda^{\frac12}\phi\|_{L^2(\Omega)} + a\|\phi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)} + \|\Lambda^{\frac12}\phi\|_{L^2(\Omega)}\\ &\leqslant (1+\eta_0+a\lambda_1^{-\frac12})\left(\|\Lambda^{\frac12}\phi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\right) = d\ \|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y,\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in Y.\end{aligned} \tag{6}$$

The proof of Proposition 2.1 is thus complete. □

From the point of view of further applications it is reasonable to consider further in this section the family of linear problems

$$\tfrac{d}{dt}\begin{bmatrix}u\\ v\end{bmatrix} + A_\eta\begin{bmatrix}u\\ v\end{bmatrix} - B\begin{bmatrix}u\\ v\end{bmatrix} = 0,\ t>0,\ \begin{bmatrix}u\\ v\end{bmatrix}_{t=0} = \begin{bmatrix}u_0\\ v_0\end{bmatrix}\in Y, \tag{7}$$

where $B \in L(Y)$. Proposition 2.1 extends then to the result below.

PROPOSITION 2.2. *The following conditions hold.*
i) For each $\eta \geq 0$,

$$\tilde{A}_\eta = A_\eta - B \text{ with domain } D(\tilde{A}_\eta) = D(A_\eta)$$

is a closed (unbounded) operator in Y with compact resolvent.
ii) $-\tilde{A}_\eta$, $\eta \geq 0$, generates on Y a C^0-semigroup $\{e^{-\tilde{A}_\eta t}\}$ such that

$$\|e^{-\tilde{A}_\eta t}\|_{L(Y)} \leqslant e^{\|B\|_{L(Y)}t},\ t\geq 0,\ \eta\geq 0. \tag{8}$$

iii) $\{e^{-\tilde{A}_\eta t}\}$, $\eta > 0$, are analytic and compact semigroups in Y.

2.2. Convergence of resolvents and convergence of semigroups

LEMMA 2.1. *For a complex number $\lambda_0 \in \boldsymbol{C}$ and $r_1 > 0$ for which $\{\lambda \in \boldsymbol{C} : |\lambda - \lambda_0| \leqslant r_1\} \subset \rho(-\tilde{A}_0)$ there exist $\eta_1 > 0$ such that if $|\lambda - \lambda_0| \leqslant r_1$ and $\eta < \eta_1$ then λ cannot be the eigenvalue of $-\tilde{A}_\eta$.*

Proof: For

$$\mathcal{R}_\eta = \begin{bmatrix}0 & 0\\ 0 & \eta\Lambda^{\frac12}\end{bmatrix} \tag{9}$$

and $|\lambda - \lambda_0| \leqslant r_1$ we have the relation

$$\lambda I + \tilde{A}_\eta = (\lambda I + \tilde{A}_0)(I + (\lambda I + \tilde{A}_0)^{-1}\mathcal{R}_\eta). \tag{10}$$

Also

$$\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1} = \|\Lambda\phi\|_{L^2(\Omega)} + \|\Lambda^{\frac{1}{2}}\psi\|_{L^2(\Omega)} = 1 \quad \text{and} \quad (\lambda I + \tilde{A}_\eta) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = 0$$

imply

$$(I + (\lambda I + \tilde{A}_0)^{-1}\mathcal{R}_\eta) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = 0 \quad \text{and} \quad 1 = \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1} = \|(\lambda I + \tilde{A}_0)^{-1}\mathcal{R}_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1}.$$

Applying next (3) (with constant d chosen to $\eta_0 = 1$) we obtain the estimate

$$\begin{aligned} 1 &\leqslant d\|A_0(\lambda I + \tilde{A}_0)^{-1}\mathcal{R}_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_Y = d\|(I + (-\lambda I + B)(\lambda I + \tilde{A}_0)^{-1})\mathcal{R}_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_Y \\ &\leqslant \eta d \sup_{|\lambda-\lambda_0|\leqslant r_1} \|(I + (-\lambda I + B)(\lambda I + \tilde{A}_0)^{-1})\|_{L(Y)}. \end{aligned}$$

Therefore, the main assertion of the lemma holds with

$$\eta_1 = \min\{1, (d \sup_{|\lambda-\lambda_0|\leqslant r_1} \|(I + (-\lambda I + B)(\lambda I + \tilde{A}_0)^{-1})\|_{L(Y)})^{-1}\}.$$

LEMMA 2.2. *For any compact set $K \subset \rho(-\tilde{A}_0)$ there exists $\hat{\eta} > 0$ and $C_K > 0$ such that*

$$K \subset \rho(-\tilde{A}_\eta), \ \eta \in (0, \hat{\eta}) \tag{11}$$

and

$$\|(\lambda + \tilde{A}_0)^{-1} - (\lambda + \tilde{A}_\eta)^{-1}\|_{L(Y)} \leqslant \eta C_K d, \ \eta \in (0, \hat{\eta}). \tag{12}$$

Furthermore, the corresponding semigroups converge; namely

$$e^{-\tilde{A}_\eta t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \xrightarrow{Y} e^{-\tilde{A}_0 t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \quad as \quad \eta \to 0^+ \quad (thus\ also \quad e^{-A_\eta t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \xrightarrow{Y} e^{-A_0 t} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \quad as \quad \eta \to 0^+) \tag{13}$$

on bounded time intervals uniformly with respect to $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ varying in arbitrarily fixed compact subset $\mathcal{J}$ of Y.

Proof: Since K is compact, property (11) follows from Lemma 2.1. Applying (10) we then have

$$\begin{aligned} &\|(\lambda I + \tilde{A}_0)^{-1} - (\lambda I + \tilde{A}_\eta)^{-1}\|_{L(Y)} \leq \|(\lambda I + \tilde{A}_0)^{-1}\mathcal{R}_\eta(\lambda I + \tilde{A}_\eta)^{-1}\|_{L(Y)} \\ &\leq \|(\lambda I + \tilde{A}_0)^{-1}\|_{L(Y)}\|\mathcal{R}_\eta\|_{L(Y^1,Y)}\|(\lambda I + \tilde{A}_\eta)^{-1}\|_{L(Y,Y^1)}, \ \lambda \in K. \end{aligned}$$

Note that

$$\|\mathcal{R}_\eta\|_{L(Y^1,Y)} = \| \begin{bmatrix} 0 & 0 \\ 0 & \eta\Lambda^{\frac{1}{2}} \end{bmatrix} \|_{L(Y^1,Y)} \leqslant \eta, \ \eta > 0,$$

whereas, by (3),

$$\|(\lambda I+\tilde{A}_\eta)^{-1}\|_{L(Y,Y^1)} \leq d\|A_\eta(\lambda I+\tilde{A}_\eta)^{-1}\|_{L(Y)}$$
$$= d\|I-(\lambda I-B)(\lambda I+\tilde{A}_\eta)^{-1}\|_{L(Y)},\ \eta\in(0,\hat{\eta}).$$

What was said above ensures validity of (12), where

$$C_K=\sup_{\lambda\in K}\|(\lambda I+\tilde{A}_0)^{-1}\|_{L(Y)}\|I-(\lambda I-B)(\lambda I+\tilde{A}_\eta)^{-1}\|_{L(Y)}.$$

Convergence of the semigroups on bounded time intervals is a consequence of the Trotter-Kato theorem. To prove that this convergence is actually uniform on compact subsets of Y fix $\delta>0$, $\tau>0$, choose $\epsilon=\frac{\delta}{4}e^{-\tau\|B\|_{L(Y)}}$ and cover a compact set $\mathcal{J}\subset Y$ with ϵ-balls whose centers run through the points of $\mathcal{J}$. Then $\mathcal{J}$ will be contained in a union of finite number of the latter balls $\mathcal{B}^1_\epsilon,\dots,\mathcal{B}^n_\epsilon$ and if $\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]$ $(i=1,\dots,n)$ are their centers there exists a positive number $\eta_\delta=\eta(\delta)$ such that

$$\|e^{-A_\eta t}\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]-e^{-A_0 t}\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]\|_Y<\frac{\delta}{2}\ \text{ for all }\ t\in[0,\tau],\ \eta\in(0,\eta_\delta),\ i=1,\dots,n. \tag{14}$$

Since the semigroups are of the same type (see (8)), choosing now together with an arbitrary point $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in\mathcal{J}$ an appropriate ball $\mathcal{B}^i_\epsilon$ we conclude that

$$\sup_{t\in[0,\tau]}\|e^{-A_\eta t}\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]-e^{-A_0 t}\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_Y$$
$$\leqslant\sup_{t\in[0,\tau]}\|e^{-A_\eta t}(\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]-\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right])\|_Y+\sup_{t\in[0,\tau]}\|(e^{-A_\eta t}-e^{-A_0 t})\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]\|_Y$$
$$+\sup_{t\in[0,\tau]}\|e^{-A_0 t}(\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]-\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right])\|_Y<2e^{-\tau\|B\|_{L(Y)}}\varepsilon+\frac{\delta}{2}=\delta$$

whenever $\eta\in(0,\eta_\delta)$. □

2.3. Projected spaces and spectral properties

Lemma 2.3. *The following conditions hold.*
i) For every compact contour $\tilde{\Gamma}\subset\rho(-\tilde{A}_0)$ there exists certain $\tilde{\eta}>0$ such that

- $\sup_{\eta\in[0,\tilde{\eta})}\|(\lambda-\tilde{A}_\eta)^{-1}\|_{L(Y)}<\infty$,
- $\tilde{\Gamma}\subset\rho(\tilde{A}_\eta)$ *for each* $\eta\in[0,\tilde{\eta})$,
- $\tilde{\Gamma}$ *encloses finitely many points of* $\sigma(A_\eta)$,
- *operators* $\tilde{Q}_\eta:Y\to Y$ *are projections on* Y

$$\tilde{Q}_\eta=\frac{1}{2\pi\mathrm{i}}\int_{\tilde{\Gamma}}(\lambda I-\tilde{A}_\eta)^{-1}d\lambda,\ \eta\in[0,\tilde{\eta}), \tag{15}$$

•$\tilde{Q}_\eta$ *are convergent to* $\tilde{Q}_0$ *in the uniform norm; namely*

$$\|\tilde{Q}_\eta - \tilde{Q}_0\|_{L(Y)} \leqslant (2\pi)^{-1}\eta d C_{\tilde{\Gamma}},\ \eta \in [0,\tilde{\eta}). \tag{16}$$

ii) For arbitrarily fixed $\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix} \in \tilde{Q}_0 Y$ *any sequence of the form* $\{\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\}$, *where* $\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix} = \tilde{Q}_{\eta_n}\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}$ *and* $\eta_n \to 0^+$, *is convergent to* $\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}$.
iii) For each bounded set $\subset Y$

$$\{\tilde{Q}_\eta B,\ \eta \in [0,\tilde{\eta})\}\ \text{ is a precompact subset of } Y; \tag{17}$$

in particular $\tilde{Q}_\eta B$ *is finite dimensional for every* $\eta \in [0,\tilde{\eta})$.

Proof: Part i) can be immediately infered from (11) and (12). We remark than because of the compactness of resolvents any compact contour can enclose merely finite number of eigenvalues of $-\tilde{A}_\eta$.
Part ii) is a consequence of (16). For the final part iii) note that because the resolvent operators are uniformly bounded on $[0,\tilde{\eta})$ then $\|\tilde{A}_\eta\tilde{Q}_\eta\|_Y$ possess the same property. Recalling (3) we obtain boundedness of the set considered in (17) in the norm of Y^1. Its compactness and finite dimensionality of $rg\tilde{Q}_\eta$ is thus evident (see [17, §III.4.1]). □

Lemma 2.4. *For any* $\lambda_0 \in \sigma(-A_0)$, *there exists a sequence* $\eta_n \to 0$ *and a sequence of eigenvalues* $\lambda_{\eta_n} \in \sigma(-A_{\eta_n})$, $n \in \mathbb{N}$, *such that* $\lambda_{\eta_n} \to \lambda_0$ *as* $n \to \infty$.

Proof: Suppose that there exists $r_0 > 0$ and $\eta_0 > 0$ such that for each $\eta \in (0,\eta_0)$ the spectrum $\sigma(-\tilde{A}_\eta)$ is disoint with $\{\lambda \in \boldsymbol{C}; |\lambda - \lambda_0| \leqslant r_0\}$. Let $\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}$ be an eigenvector corresponding to λ_0 and consider projections

$$P_\eta := \frac{1}{2\pi \mathrm{i}}\int_{\{|\lambda-\lambda_0|=r_0\}} (\lambda I - \tilde{A}_\eta)^{-1} d\lambda,\ \eta \in (0,\eta_0).$$

Then $0 = \tilde{P}_\eta\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix} \to P_0\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix} = \begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}$, which is impossible. □
The following 'upper semicontinuity' result is a direct consequence of Lemma 2.1.

Corollary 2.1. *If* $\eta_n \to 0$ *and* λ_{η_n} *is an eigenvalue of* $-\tilde{A}_{\eta_n}$, *then the set consisting of the limits of convergent subsequences of* $\{\lambda_{\eta_n}\}$ *is contained in* $\sigma(-\tilde{A}_0)$.

Lemma 2.5. *Consider arbitrary compact contour* $\tilde{\Gamma} \subset \rho(-\tilde{A}_0)$ *and let* $\tilde{Q}_\eta$ *be defined for* $\eta \in [0,\eta_0)$ *as in* (15). *If* $\eta_n \to 0^+$, $\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix} \in \tilde{Q}_{\eta_n} Y$ *for* $n \in \mathbb{N}$, *and sequence* $\{\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\}$ *lies in a bounded set* $B \subset Y$ *then* $\{\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\}$ *is precompact sequence in* Y *and a limit of any of its convergent subsequence belongs to* $\tilde{Q}_0 Y$.

Proof: Property (17) ensures that the set $\{\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\}$ is precompact as a subset of $\{\tilde{Q}_\eta B,\ \eta\in[0,\tilde{\eta})\}$. If $\{\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\}$ is a limit of any convergent subsequence of $\{\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\} = \{\tilde{Q}_{\eta_n}\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\}$, then $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] = \lim_{n\to\infty}\tilde{Q}_{\eta_n}\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right] = \tilde{Q}_0\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]$ as a consequence of (16). □

COROLLARY 2.2. *Consider arbitrary compact contour $\tilde{\Gamma}\subset\rho(-\tilde{A}_0)$ enclosing exactly one eigenvalue of $-\tilde{A}_0$ and let $\tilde{Q}_\eta$ be defined for $\eta\in[0,\eta_0)$ as in (15).*
Then, there exists $\hat{\eta}>0$

$$\dim\tilde{Q}_\eta Y = \dim\tilde{Q}_0 Y \ \textit{for all}\ \eta\in[0,\hat{\eta}). \tag{18}$$

Proof: Suppose it is possible to choose a sequence $\eta_n\to 0^+$ such that $\dim\tilde{Q}_{\eta_n}Y > \dim\tilde{Q}_0 Y$ for each $n\in\mathbb{N}$. Then, for each $n\in\mathbb{N}$, there exists $\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\in\tilde{Q}_\eta Y$ with $\|\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\|_Y = 1$ for which $\text{dist}_Y(\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right], Q_0 Y) = 1$ (see [17, Lemma IV.2.3]). Since in the light of Lemma 2.5 it is impossible, we obtain that

$$\dim\tilde{Q}_\eta Y \leqslant \dim\tilde{Q}_0 Y \ \text{for all}\ \eta\in[0,\eta_1).$$

We remark that

$$\dim\tilde{Q}_\eta Y \geqslant \dim\tilde{Q}_0 Y \ \text{for all}\ \eta\in[0,\eta_2),$$

because each $\tilde{Q}_\eta$ takes basis of $\tilde{Q}_0 Y$ into a system linearly independent in $\tilde{Q}_\eta Y$. □

From now on we will assume that

$$B\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ b(x)u\end{smallmatrix}\right],\ \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\in Y,$$

where $b:\Omega\to\mathbb{R}$ is a measurable function such that $b(x)I\in L(H_0^1(\Omega), L^2(\Omega))$ and

$$\tilde{\Lambda} := \Lambda - b(x) \ \text{with domain}\ D(\tilde{\Lambda}) = D(\Lambda)$$

is a selfadjoint operator in $L^2(\Omega)$. Since our concern here is to develop functional analytic tools useful for further description of the continuous dynamics of the nonlinear problems (1) we will simply suppose throughout the rest of this section that $b\in L^\infty(\Omega)$.

LEMMA 2.6. *If $\eta\geq 0$ and λ_η is an eigenvalue of $-\tilde{A}_\eta$ with non-zero imaginary part, then $Re\lambda_\eta\leq -\frac{a}{2}$.*

Proof: Suppose $\lambda_\eta = c + d\mathrm{i}$ where $d\neq 0$ and $c > -\frac{a}{2}$. Let $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\in Y^1$ be an eigenvector of $-\tilde{A}_\eta$ corresponding to $c+\mathrm{i}d$ and $\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_Y = 1$. Then,

$$\left(\begin{bmatrix}(c+\mathrm{i}d)I & 0\\ 0 & (c+\mathrm{i}d)I\end{bmatrix} + \begin{bmatrix}0 & -I\\ \Lambda - b(x) & \eta\Lambda^{\frac{1}{2}} + aI\end{bmatrix}\right)\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \tag{19}$$

and consequently

$$(-\Lambda + b(x) - c^2 + d^2 - ac - \eta c\Lambda^{\frac{1}{2}})\phi = \mathrm{i}d(2c + a + \eta\Lambda^{\frac{1}{2}})\phi. \tag{20}$$

Multiply both sides of (20) by ϕ in $L^2(\Omega)$ to get:

$$\frac{-\|\Lambda^{\frac{1}{2}}\phi\|_{L^2(\Omega)} + \int_\Omega b(x)|\phi(x)|^2dx - (c^2 - d^2 + ac)\|\phi\|^2_{L^2(\Omega)} - \eta c\|\Lambda^{\frac{1}{4}}\phi\|^2_{L^2(\Omega)}}{d((2c+a)\|\phi\|^2_{L^2(\Omega)} + \eta\|\Lambda^{\frac{1}{4}}\phi\|^2_{L^2(\Omega)})} = \mathrm{i}. \tag{21}$$

Since the left hand side is a real number this is impossible and the proof of (22) is thus complete. □

Corollary 2.3. *Let $\hat{\lambda}_1, \hat{\lambda}_2$ be two eigenvalues of $-\tilde{A}_0$ such that $-\frac{a}{2} < \hat{\lambda}_1 < \hat{\lambda}_2$ and no other eigenvalue of $-\tilde{A}_0$ lies between them.*

Consider any two real numbers r_1, r_2 such that $\hat{\lambda}_1 < r_1 < r_2 < \hat{\lambda}_2$. Then,

$$\forall_{\hat{\lambda}_1 < r_1 < r_2 < \hat{\lambda}_2}\ \exists_{\eta_0 > 0}\ \forall_{\eta \in [0,\eta_0]}\ \sigma(-\tilde{A}_\eta) \cap \{\lambda \in \mathbb{C} : Re\lambda \in [r_1, r_2]\} = \emptyset. \tag{22}$$

Proof: If the assertion fails then there exists sequence $\eta_n \to 0$ and a corresponding sequence of complex numbers $\lambda_n = c_n + \mathrm{i}d_n$, where λ_n is an eigenvalue of $-\tilde{A}_{\eta_n}$ and $c_n \in [r_1, r_2]$. From Lemma 2.6 we infer that the sequence $\{\lambda_n\}$ consists of real numbers and thus has a subsequence convergent to certain number $r \in [r_1, r_2]$. From Corollary 2.1 we infer that r is an eigenvalue of $\tilde{A}_0$ which is impossible. □

Corollary 2.4. *Suppose that*

$$b \in L^\infty(\Omega) \quad and \quad 0 \notin \sigma(\tilde{\Lambda}). \tag{23}$$

Then (22) holds with $\hat{\lambda}_2$ being the first positive eigenvalue of $-\tilde{A}_0$ and $\hat{\lambda}_1 = \max\limits_{\substack{\lambda \in \sigma(\tilde{A}_0) \\ Re\lambda < 0}} \{Re\lambda\}$.

2.4. Exponential dichotomy and continuity of projected semigroups

Definition 2.1. Let $\mathcal{Z}$ be a Banach space, η be a non-negative parameter and consider the family of C^0-semigroups $\{e^{-\mathcal{A}_\eta t} : t \geqslant 0\} \subset L(\mathcal{Z})$, with generators $-\mathcal{A}_\eta$. We say that this family satisfies uniform exponential dichotomy condition (see [16, 12]) iff there exists $\eta_0 > 0$ such that $\mathcal{Z}$ can be decomposed for each $\eta \in [0, \eta_0)$ as a direct sum $\mathcal{Z} = \mathcal{Z}^-_\eta \oplus \mathcal{Z}^+_\eta$ of two closed subspaces, invariant under $\mathcal{A}_\eta$, with the property that $\mathcal{A}^+_\eta = \mathcal{A}_{\eta|_{\mathcal{Z}^+_\eta}} \in L(\mathcal{Z}^+_\eta)$ and there are constants $\epsilon > 0$, $M \geq 1$ such that

$$\|e^{-\mathcal{A}^-_\eta t}\|_{L(\mathcal{Z}^-_\eta)} \leqslant Me^{-\epsilon t},\ t \geqslant 0,\ \eta \in [0, \eta_0),$$
$$\|e^{-\mathcal{A}^+_\eta t}\|_{L(\mathcal{Z}^+_\eta)} \leqslant Me^{\epsilon t},\ t \leqslant 0,\ \eta \in [0, \eta_0),$$

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where $\mathcal{A}_\eta^- = \mathcal{A}_{\eta|_{\mathcal{Z}_\eta^-}}$.

Our main concern in this subsection will be to show that

THEOREM 2.1. *If* (23) *holds then the family of semigroups* $\{e^{-\tilde{A}_\eta t}\}$, $\eta \geq 0$, *satisfies uniform exponential dichotomy condition.*

The proof of the above theorem will be a consequence of Corollary 2.5 and Lemmas 2.9, 2.10.

Before we proceed with the proof it is important to recall that the eigenvalues of $-\tilde{A}_0$ can be written as

$$\lambda_n^{\pm} = \tfrac{1}{2}(-a \pm \sqrt{a^2 - 4\mu_n}),\ \mu_n \in \sigma(\tilde{\Lambda}),\ n \in \mathbb{N}, \tag{24}$$

(see e.g. [3, 9]). Assuming (23) consider a finite set $\sigma^-(\tilde{\Lambda})$ consisting of all negative eigenvalues of $\tilde{\Lambda}$ and the set $\dot{\sigma}(-\tilde{A}_0)$;

$$\dot{\sigma}(-\tilde{A}_0) = \{\tfrac{1}{2}(-a - \sqrt{a^2 - \mu}),\ \mu \in \sigma^-(\tilde{\Lambda})\}. \tag{25}$$

Let γ be any circumference enclosing $\sigma^-(\tilde{\Lambda})$ and define the projection $P : L^2(\Omega) \to L^2(\Omega)$,

$$P = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - \tilde{\Lambda})^{-1} d\lambda.$$

Then $\tilde{\Lambda}_{|_{kerP}}$ is a selfadjoint positive operator on $kerP$ and the norms $\|(\tilde{\Lambda}_{|_{kerP}})^{\frac{1}{2}}(\cdot)\|_{L^2(\Omega)}$ and $\|\Lambda^{\frac{1}{2}}(\cdot)\|_{L^2(\Omega)}$ are equivalent on $kerP$.

Let $\dot{\Gamma} \subset \rho(-\tilde{A}_0)$ be a circumference lying to the left of imaginary axis surrounding $\dot{\sigma}(-\tilde{A}_0)$. Similarily, let $\Gamma \subset \rho(-\tilde{A}_0)$ be a circumference lying on the right hand side of imaginary axis and enclosing those finite number of elements from $\sigma(-\tilde{A}_0)$ whose real part is positive. Note that Γ can be choosen such that for each $\eta \geq 0$ all eigenvalues of $-A_\eta$ with positive real parts[2] are 'inside' Γ. On the other hand, by Corollary 2.1, there exists $\eta_0 > 0$ such that $\dot{\Gamma} \subset \rho(-\tilde{A}_\eta)$ for every $\eta \in (0, \eta_0)$. Furthermore, compactness of the resolvents ensure that 'inside' $\dot{\Gamma}$ there are merely finitely many eigenvalues of $-\tilde{A}_\eta$. Considering projections,

$$\dot{Q}_\eta : Y \to Y,\ \dot{Q}_\eta = \frac{1}{2\pi \mathrm{i}} \int_{\dot{\Gamma}} (\lambda I - \tilde{A}_\eta)^{-1} d\lambda,$$

$$Q_\eta : Y \to Y,\ Q_\eta = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\lambda I - \tilde{A}_\eta)^{-1} d\lambda,$$

we then obtain decomposition of the 'base' space Y

$$Y = (I - \dot{Q}_\eta - Q_\eta)Y \oplus \dot{Q}_\eta Y \oplus Q_\eta Y,\ \eta \in [0, \eta_0),$$

[2]We remark that by Lemma 2.6 eigenvalues of $-A_\eta$ with positive real parts are positive real numbers and by (8) they cannot exceed $\|B\|_{L(Y)}$.

as well as the corresponding decomposition of operators $-\tilde{A}_\eta$, $\eta \in [0,\eta_0)$ (see [17, §III.6.4] for details). We thus restrict our further considerations to $\eta \in [0,\eta_0)$.

PROPOSITION 2.3. *Space $(I-\dot{Q}_0-Q_0)Y$ is spanned by*

$$\left\{\begin{bmatrix} e_n \\ \frac{1}{2}(-a\pm\sqrt{a^2-4\mu_n})e_n\end{bmatrix}; (\mu_n,e_n)\in\mathcal{M}\right\}.$$

In particular $\phi,\psi \in kerP$ whenever $\begin{bmatrix}\phi\\ \psi\end{bmatrix} \in (I-\dot{Q}_0-Q_0)Y$.

Proof: It is evident that $(I-\dot{Q}_0-Q_0)Y$ contains the space spanned by the system of eigenvectors

$$\left\{\begin{bmatrix} e_n \\ \frac{1}{2}(-a\pm\sqrt{a^2-4\mu_n})e_n\end{bmatrix}; (\mu_n,e_n)\in\mathcal{M}\right\},$$

where $\mathcal{M}$ consists of all eigenpairs (μ_n,e_n) of $\tilde{\Lambda}$ with $\mu_n>0$. On the other hand if $\begin{bmatrix}\phi\\ \psi\end{bmatrix} \in (I-\dot{Q}_0-Q_0)Y$ then $\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \sum_{n=1}^{\infty} c_n^{\pm}\begin{bmatrix} e_n \\ \frac{1}{2}(-a\pm\sqrt{a^2-4\mu_n})e_n\end{bmatrix}$ (see [9, Lemma A.1]) and consequently

$$\begin{bmatrix}\phi\\ \psi\end{bmatrix} = (I-\dot{Q}_0-Q_0)\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \sum_{(\mu_n,e_n)\in\mathcal{M}}^{\infty} c_n^{\pm}\begin{bmatrix} e_n \\ \frac{1}{2}(-a\pm\sqrt{a^2-4\mu_n})e_n\end{bmatrix},$$

which completes the proof. □

It is useful to introduce in Y a 'new' scalar product

$$\begin{aligned}\langle\begin{bmatrix}\phi_1\\ \psi_1\end{bmatrix},\begin{bmatrix}\phi_2\\ \psi_2\end{bmatrix}\rangle_Y^{new} &= \langle\tilde{\Lambda}^{\frac{1}{2}}_{|kerP}(I-P)\phi_1,\tilde{\Lambda}^{\frac{1}{2}}_{|kerP}(I-P)\phi_2\rangle_{L^2(\Omega)} + \langle\Lambda^{\frac{1}{2}}P\phi_1,\Lambda^{\frac{1}{2}}P\phi_2\rangle_{L^2(\Omega)}\\ &+\delta(a-\delta)\langle\phi_1,\phi_2\rangle_{L^2(\Omega)} + \langle\delta\phi_1+\psi_1,\delta\phi_2+\psi_2\rangle_{L^2(\Omega)}, \quad \delta\in(0,a),\end{aligned} \tag{26}$$

which defines in Y a norm

$$\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y^{new} = \sqrt{\langle\begin{bmatrix}\phi\\ \psi\end{bmatrix},\begin{bmatrix}\phi\\ \psi\end{bmatrix}\rangle_Y^{new}}$$

equivalent to the norm $\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y = \|\Lambda^{\frac{1}{2}}\phi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}$.

LEMMA 2.7. *For some $\varepsilon>0$, $(\varepsilon I-\tilde{A}_0)_{|(I-\dot{Q}_0-Q_0)Y}$ is dissipative in $(I-\dot{Q}_0-Q_0)Y$. In particular the semigroup $\{e^{\varepsilon t}e^{-\tilde{A}_0 t}{}_{|(I-\dot{Q}_0-Q_0)Y}\}$ is bounded on $(I-\dot{Q}_0-Q_0)Y$.*

Proof: Using (26) and denoting by $\tilde{\lambda}_1$ the first positive eigenvalue of $\tilde{\Lambda}$ in $kerP$ we have

$$\begin{aligned}Re\langle -\tilde{A}_0\begin{bmatrix}\phi\\ \psi\end{bmatrix},\begin{bmatrix}\phi\\ \psi\end{bmatrix}\rangle_Y^{new} &= -\delta\|\tilde{\Lambda}^{\frac{1}{2}}_{|kerP}\phi\|^2_{L^2(\Omega)} - (a-\delta)\|\psi\|^2_{L^2(\Omega)}\\ &\leqslant -\frac{\delta}{2}\langle\begin{bmatrix}\phi\\ \psi\end{bmatrix},\begin{bmatrix}\phi\\ \psi\end{bmatrix}\rangle_Y^{new},\ \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in(I-\dot{Q}_0-Q_0)Y\cap Y^1,\ \delta\in(0,\delta_0),\end{aligned} \tag{27}$$

where $\delta_0 = \min\{\frac{a}{2}, \frac{-a+\sqrt{a^2+4\tilde{\lambda}_1}}{2}\}$. Therefore the result holds with $\varepsilon = \frac{\delta}{2}$ and arbitrarily fixed $\delta \in (0, \delta_0)$. □

We will next show that

LEMMA 2.8. *For each $\delta \in (0,\delta_0)$ there exists $\eta_\delta > 0$ such for each $\eta \in (0,\eta_\delta)$*

$$Re\langle -\tilde{A}_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix}\rangle_Y^{new} \leqslant -\frac{\delta}{4}\langle \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix}\rangle_Y^{new}, \tag{28}$$

whenever $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in (I - \dot{Q}_0 - Q_0)Y \cap Y^1$.

Proof: Consider $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in (I - \dot{Q}_\eta - Q_\eta)Y \cap Y^1$ with $\| \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \|_Y^{new} = 1$ and use the decomposition of the space Y induced by the projection $\dot{Q}_0 + Q_0$, so that

$$\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} = (\dot{Q}_0 + Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} + (I - \dot{Q}_0 - Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}.$$

Since the value of $-\tilde{A}_\eta \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$ is given by $-\tilde{A}_0 \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} - \mathcal{R}_\eta \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$ (see (9)) we then have

$$\begin{aligned} Re\langle -\tilde{A}_\eta \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\rangle_Y^{new} &= Re\langle -\tilde{A}_0 (I - \dot{Q}_0 - Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, (I - \dot{Q}_0 - Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\rangle_Y^{new} \\ &+ Re\langle -\tilde{A}_0 (\dot{Q}_0 + Q_0)(\dot{Q}_0 + Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, (\dot{Q}_0 + Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\rangle_Y^{new} \\ &+ Re\langle -\tilde{A}_0 (\dot{Q}_0 + Q_0)(\dot{Q}_0 + Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, (I - \dot{Q}_0 - Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\rangle_Y^{new} \\ &+ Re\langle -\tilde{A}_0 (I - \dot{Q}_0 - Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, (\dot{Q}_0 + Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\rangle_Y^{new} \\ &- \eta Re\langle \begin{bmatrix} 0 \\ \Lambda^{\frac{1}{2}}\psi_\eta \end{bmatrix}, \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\rangle_Y^{new} = I_1 + \ldots + I_5, \end{aligned} \tag{29}$$

where the components appearing on the right hand side are estimated as follows.

First, via (27), we obtain that $I_1 \leqslant -\frac{\delta}{2}$. Choosing constant $c > 0$ such that $\frac{1}{c}\|\cdot\|_Y \leqslant \|\cdot\|_Y^{new} \leqslant c\|\cdot\|_Y$ we have the estimates

$$\|(I - \dot{Q}_0 - Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y^{new} \leqslant c^2\|(I - \dot{Q}_0 - Q_0)\|_{L(Y)} =: d_0, \tag{30}$$

and

$$\begin{aligned} \|(\dot{Q}_0 + Q_0)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y^{new} &\leqslant c^2\|(\dot{Q}_0 + Q_0)(I - \dot{Q}_\eta - Q_\eta)\|_{L(Y)} \\ &= c^2\|(\dot{Q}_0 + Q_0)(I - \dot{Q}_0 - Q_0) + (\dot{Q}_0 + Q_0)(\dot{Q}_0 + Q_0 - \dot{Q}_\eta - Q_\eta)\|_{L(Y)} \\ &= c^2\|(\dot{Q}_0 + Q_0)(\dot{Q}_0 + Q_0 - \dot{Q}_\eta - Q_\eta)\|_{L(Y)} \\ &\leqslant c^2\|(\dot{Q}_0 + Q_0)\|_{L(Y)}(\|(\dot{Q}_0 - \dot{Q}_\eta\|_{L(Y)} + \|Q_0 - Q_\eta)\|_{L(Y)}). \end{aligned} \tag{31}$$

From (31) and (16) we next get

$$\|(\dot{Q}_0+Q_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y^{new} \leqslant (2\pi)^{-1}\eta c^2 d(C_\Gamma|\Gamma|+C_{\dot{\Gamma}}|\dot{\Gamma}|)\|(\dot{Q}_0+Q_0)\|_{L(Y)} =: \eta D, \tag{32}$$

which together with (30) leads to the relations

$$I_2 \leqslant \eta^2 c^2 D^2 \|\tilde{A}_0(\dot{Q}_0+Q_0)\|_{L(Y)}, \tag{33}$$

$$I_3 \leqslant \eta c^2 d_0 D \|\tilde{A}_0(\dot{Q}_0+Q_0)\|_{L(Y)}. \tag{34}$$

We remark that $Y^1 \subset D(-\tilde{A}_0{}^*)$ and

$$-\tilde{A}_0{}^*\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix}0 & -I+\Lambda^{-1}b(x)\\ \Lambda & -aI\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix} \quad \text{for} \quad \begin{bmatrix}\phi\\ \psi\end{bmatrix}\in Y^1.$$

Since $b\in L^\infty(\Omega)$ and $\|\Lambda^{-\frac{1}{2}}\psi\|_{L^2(\Omega)} \leqslant \frac{1}{\sqrt{\lambda_1}}\|\psi\|_{L^2(\Omega)}$ then

$$\begin{aligned}\|\begin{bmatrix}0 & -I+\Lambda^{-1}b(x)\\ \Lambda & -aI\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_Y &\leqslant (1+\frac{a}{\sqrt{\lambda_1}})\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y^1} + \|\Lambda^{-\frac{1}{2}}b(x)\psi\|_{L^2(\Omega)}\\ &\leqslant \big(1+\frac{a}{\sqrt{\lambda_1}}+\frac{\|b(x)\|_{L^\infty(\Omega)}}{\sqrt{\lambda_1}}\big)\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\|_{Y^1},\end{aligned} \tag{35}$$

which ensures that $-\tilde{A}_0{}^* \in L(Y^1,Y)$. With the latter property and (30), (6), (32) we obtain

$$\begin{aligned}I_4 &= Re\langle (I-\dot{Q}_0-Q_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}, -\tilde{A}_0{}^*(\dot{Q}_0+Q_0)(\dot{Q}_0+Q_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\rangle_Y^{new}\\ &\leqslant d_0c^2\|-\tilde{A}_0{}^*(\dot{Q}_0+Q_0)\|_{L(Y)}\|(\dot{Q}_0+Q_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y^{new}\\ &\leqslant c^2d_0\|-\tilde{A}_0{}^*(-\tilde{A}_0)^{-1}\|_{L(Y)}\|-\tilde{A}_0(\dot{Q}_0+Q_0)\|_{L(Y)}\|(\dot{Q}_0+Q_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y^{new}\\ &\leqslant \eta c^2 dDd\|-\tilde{A}_0{}^*\|_{L(Y^1,Y)}\|-\tilde{A}_0(\dot{Q}_0+Q_0)\|_{L(Y)}.\end{aligned} \tag{36}$$

$$I_5 = -\eta\delta Re\langle \Lambda^{\frac{1}{4}}\psi_\eta, \Lambda^{\frac{1}{4}}\phi_\eta\rangle_{L^2(\Omega)} - \eta\|\Lambda^{\frac{1}{4}}\psi_\eta\|^2_{L^2(\Omega)} \leqslant \frac{\eta\delta^2}{4\lambda_1^{\frac{1}{4}}}\|\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y^2 \leqslant \eta\delta^2 c. \tag{37}$$

Since each quantity $I_2,\ldots,I_5$ contains a multiple of η the proof of condition (28) is complete. □

In the light of the Lumer-Phillips theorem condition (28) ensures that

Corollary 2.5. *For arbitrarily fixed* $\delta\in(0,\delta_0)$, $\delta_0=\min\{\frac{a}{2},\frac{-a+\sqrt{a^2+4\lambda_1}}{2}\}$,

$$\|e^{\frac{\delta}{4}t}e^{-\tilde{A}_\eta t}|_{(I-\dot{Q}_\eta-Q_\eta)Y}\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y^{new} \leqslant \|\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y^{new},\ \begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\in(I-\dot{Q}_\eta-Q_\eta)Y,$$

for each $\eta \in [0, \eta_\delta)$.

In particular, if $b \equiv 0$ then $(I - \dot{Q}_\eta - Q_\eta)Y = Y$ and we obtain that

COROLLARY 2.6. *For each* $\eta_0 > 0$ *there exist constants* $c \geq 1$, $\omega < 0$ *for which*

$$\|e^{-A_\eta t}\|_{L(Y)} \leqslant ce^{\omega t},\ t \geq 0,\ \eta \in [0, \eta_0]. \tag{38}$$

Remark 2. 1. By [13, Corollary IV.3.12],

$$\inf\{Re(\lambda) : \lambda \in \sigma(A_\eta)\} = -\lim_{t\to\infty} \frac{\ln \|e^{-A_\eta t}\|_{L(Y)}}{t}.$$

The latter fact and (38) justify that for each $\eta_0 > 0$ there exists $\omega < 0$ such that

$$Re\sigma(A_\eta) \geqslant -\omega > 0 \text{ for } \eta \in (0, \eta_0].$$

Therefore semigroups $\{e^{-A_\eta t}\}$, $\eta > 0$, which are analytic in Y, satisfy the estimates

$$\|A_\eta^\alpha e^{-A_\eta t}\|_{L(Y,Y)} \leqslant c_{\eta,\alpha} t^{-\alpha} e^{\omega_{\eta_0} t},\ t > 0,\ \alpha \geq 0,\ \eta \in [0, \eta_0], \tag{39}$$

where constant $\omega_{\eta_0} < 0$ appearing under the exponent in (39) does not depend on $\eta \in [0, \eta_0]$.

In the next lemma we will obtain the uniform bound for $\{e^{\varepsilon t} e^{-\tilde{A}_\eta t}|_{\dot{Q}_\eta Y}\}$.

LEMMA 2.9. *For each* $\eta \in (0, \eta^*)$ *the following estimate holds*

$$\|e^{\varepsilon t} e^{-\tilde{A}_\eta t} \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \|_Y \leqslant e^{\varepsilon t^* + \|B\|_{L(Y)} t^*},\ t \geq 0,\ \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \dot{Q}_\eta Y,\ \|\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y = 1. \tag{40}$$

Proof: Consider $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \dot{Q}_\eta Y$ with $\|\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y = 1$ and take advantage of the decomposition of the space Y induced by the projection $\dot{Q}_0$ to get

$$\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} = \dot{Q}_0 \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} + (I - \dot{Q}_0) \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$$

where as a consequence of (16)

$$\begin{aligned} \|(I - \dot{Q}_0) \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y &= \|(I - \dot{Q}_0)\dot{Q}_\eta \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y \\ &= \|(I - \dot{Q}_0)(\dot{Q}_\eta - \dot{Q}_0 + \dot{Q}_0) \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\|_Y \\ &\leqslant \|(I - \dot{Q}_0\|_{L(Y)} \|\dot{Q}_0 - \dot{Q}_\eta\|_{L(Y)} \\ &\leqslant \eta(2\pi)^{-1} dC_{\dot{\Gamma}} |\dot{\Gamma}| \|I - \dot{Q}_0\|_{L(Y)} =: \eta\dot{\mathcal{D}}. \end{aligned} \tag{41}$$

We then have

$$\begin{aligned}\|e^{\varepsilon t}e^{-\tilde{A}_\eta t}\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y &\leqslant \|e^{\varepsilon t}e^{-\tilde{A}_\eta t}\dot{Q}_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y + \|e^{\varepsilon t}e^{-\tilde{A}_\eta t}(I-\dot{Q}_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y\\ &\leqslant \|e^{\varepsilon t}(e^{-\tilde{A}_\eta t}-e^{-\tilde{A}_0 t})\dot{Q}_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y + \|e^{\varepsilon t}e^{-\tilde{A}_0 t}\dot{Q}_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y\\ &+ \|e^{\varepsilon t}e^{-\tilde{A}_\eta t}(I-\dot{Q}_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y\\ &\le J_1(t)+J_2(t)+J_3(t)\end{aligned} \tag{42}$$

and, since $\{e^{-\tilde{A}_\eta t}|_{\dot{Q}_\eta Y}\}$ is a C^0 semigroup in $\dot{Q}_\eta Y$ generated by a bounded operator $-\tilde{A}_\eta|_{\dot{Q}_\eta Y}$ with all (finitely many) eigenvalues being negative numbers strictly less than $-a$, there exists constant M_0 such that

$$J_2 \leqslant \|e^{\varepsilon t}e^{-\tilde{A}_0 t}\dot{Q}_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y \leqslant M_0 e^{-(a-\varepsilon)t}\|\dot{Q}_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y,\ t\ge 0.$$

Note that $\|\dot{Q}_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y \leqslant \|\dot{Q}_0\|_{L(Y)}$ and define $t^*>0$ such that

$$J_2(t^*) \leqslant \frac{1}{4}. \tag{43}$$

All $\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}$ lie on the unit sphere in Y, which is taken by $\dot{Q}_\eta$ into a precompact subset of Y. As a consequence of (13) there exists $\eta_1>0$ such that

$$J_1(t^*) \leqslant \frac{1}{4} \quad \text{for all} \quad \eta\in(0,\eta_1). \tag{44}$$

Finally, applying (41) and the estimate for the perturbed semigroups we find the estimate

$$J_3(t^*) \leqslant \eta\dot{\mathcal{D}}e^{\varepsilon t^*+\|B\|_{L(Y)}t^*} \leqslant \frac{1}{4} \quad \text{for all} \quad \eta\in(0,\eta_2). \tag{45}$$

We have thus shown that there exists $t^*>0$ and $\eta^*>0$ for which

$$\|e^{\varepsilon t^*}e^{-\tilde{A}_\eta t^*}\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y \leqslant 1,\ \begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\in\dot{Q}_\eta Y,\ \|\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}\|_Y=1$$

uniformly with respect to $\eta\in(0,\eta^*)$. Decomposing next $t=nt^*+r$ and using the estimate for the perturbed semigroups to find the estimate on an interval $[0,t^*]$ we derive the required bound (40) uniform with respect to $\eta\in(0,\eta^*)$. □

Remark 2. 2. If ${Y_\eta}^- := (I-Q_\eta)Y$ then, because of the relation $\dot{Q}_\eta Q_\eta = 0$ (see [17, p. 179]), we have the equalities

$$ker\dot{Q}_\eta|_{{Y_\eta}^-} = (I-\dot{Q}_\eta|_{{Y_\eta}^-}){Y_\eta}^- = (I-\dot{Q}_\eta|_{{Y_\eta}^-})(I-Q_\eta)Y = (I-\dot{Q}_\eta-Q_\eta)Y,$$

$$rg\dot{Q}_{\eta|_{Y_\eta{}^-}} = \dot{Q}_{\eta|_{Y_\eta{}^-}} Y_\eta{}^- = \dot{Q}_\eta(I-Q_\eta)Y = \dot{Q}_\eta(I-Q_\eta+Q_\eta)Y = \dot{Q}_\eta Y.$$

Thus $\dot{Q}_{\eta|_{Y_\eta{}^-}}$ is a projection on Y_η^- whereas in our further consideration we will make use of the decomposition

$$Y = Y_\eta^- \oplus Y_\eta^+ \ \text{ with } \ Y_\eta^- := (I-Q_\eta)Y \ \text{ and } \ Y_\eta^+ = Q_\eta Y, \ \eta \in [0,\eta_0). \tag{46}$$

Using the notation of spaces as in (46) and denoting by $\{e^{\tilde{A}_\eta^+ t}\}$, $\{e^{-\tilde{A}_\eta^+ t}\}$ the semigroups generated by $\pm\tilde{A}_\eta^+ = \pm\tilde{A}_{\eta|_{Y_\eta^+}}$ we obtain from Corollary 2.5 and Lemma 2.9 that for all η sufficiently small space Y can be decomposed as a direct sum $Y = Y_\eta^- \oplus Y_\eta^+$ of two closed subspaces, invariant under $\{e^{-\tilde{A}_\eta t}\}$, for which the estimate

$$\|e^{-\tilde{A}_\eta^- t}\|_{L(Y_\eta^-)} \leqslant Me^{-\epsilon t}, \ t \geq 0,$$

holds with constants $\epsilon > 0$, $M \geq 1$ independent of η.

Now we need to derive appropriate estimate of $\{e^{\varepsilon t}e^{\tilde{A}_\eta^+ t}\}$. Recall that via (17) projection Q_η is compact and therefore has a finite rank. Following (46) we denote $Y_\eta^+ = Q_\eta Y$, $0 \leqslant \eta \leqslant \eta_0$ and observe via Corollary 2.2 that the map $\mathcal{S}_\eta : Y_0^+ \to Y_\eta^+$ defined by $\mathcal{S}_\eta \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} = Q_\eta \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$ is an isomorphism. We will next prove that

LEMMA 2.10. *The inverse operators $\mathcal{S}_\eta^{-1} : Y_\eta^+ \to Y_0^+$ satisfy*

$$\|\mathcal{S}_\eta^{-1} - Q_0\|_{L(Y_\eta^+, Y_0^+)} \stackrel{\eta\to 0}{\longrightarrow} 0 \tag{47}$$

and

$$\forall_{\tau>0} \sup_{t\in[0,\tau]} \|e^{\pm\tilde{A}_\eta^+ t}Q_\eta - e^{\pm\tilde{A}_0^+ t}Q_0\|_{L(Y)} \stackrel{\eta\to 0}{\longrightarrow} 0. \tag{48}$$

In particular, there exist constants $\tilde{\epsilon} > 0$, $m \geq 1$ (independent of η) such that

$$\|e^{\tilde{A}_\eta^+ t}\|_{L(Y_\eta^+, Y_\eta^+)} \leqslant me^{-\epsilon t}, \ t \geq 0. \tag{49}$$

Proof: We first ensure that

$$\|\mathcal{S}_\eta^{-1}\|_{L(Y_\eta^+, Y_0^+)} \text{ is bounded independently of } \eta. \tag{50}$$

Suppose that there is a sequence $\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}$ with $\| \begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \|_Y = 1$ for which $\|\mathcal{S}_{\eta_n}^{-1} \begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \|_Y \stackrel{\eta_n\to 0}{\longrightarrow} \infty$. Then

$$\frac{\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}}{\|\mathcal{S}_{\eta_n}^{-1} \begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \|_Y} = \mathcal{S}_{\eta_n} \frac{\mathcal{S}_{\eta_n}^{-1} \begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}}{\|\mathcal{S}_{\eta_n}^{-1} \begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \|_Y} \tag{51}$$

and $\{\frac{\mathcal{S}_{\eta_n}^{-1}\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}}{\|\mathcal{S}_{\eta_n}^{-1}\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\|_Y}\} \subset Y_0^+$ has a convergent subsequence (for which we use the same notation) converging to some $\begin{bmatrix}\phi\\ \psi\end{bmatrix} \in Y_0^+$, $\| \begin{bmatrix}\phi\\ \psi\end{bmatrix} \|_Y = 1$. It follows from (51) and (16) that

$$0 \longleftarrow \mathcal{S}_{\eta_n} \frac{\mathcal{S}_{\eta_n}^{-1}\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}}{\|\mathcal{S}_{\eta_n}^{-1}\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\|_Y} = Q_{\eta_n} \frac{\mathcal{S}_{\eta_n}^{-1}\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}}{\|\mathcal{S}_{\eta_n}^{-1}\begin{bmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{bmatrix}\|_Y} \stackrel{\eta_n \to 0}{\longrightarrow} Q_0 \begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix}\phi\\ \psi\end{bmatrix},$$

which is an absurd.

Having property (50) note further that for any $\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix} \in Y_\eta^+$

$$\mathcal{S}_\eta^{-1}\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix} - Q_0\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix} = \mathcal{S}_\eta^{-1} Q_\eta (I - Q_0)\begin{bmatrix}\phi_\eta\\ \psi_\eta\end{bmatrix}.$$

Thus (47) follows from the fact that $Q_\eta(I-Q_0) = (Q_\eta - Q_0)(I - Q_0) \to 0$ in $L(Y)$ (see (16)).

From this we can prove required convergence of the semigroups. In fact, since

$$\begin{aligned} e^{\pm\tilde{A}_\eta^+ t} Q_\eta \ &= \ \mathcal{S}_\eta^{-1} e^{\pm\tilde{A}_\eta^+ t} Q_\eta Q_0 + (Q_0 - \mathcal{S}_\eta^{-1}) e^{\pm\tilde{A}_\eta^+ t} Q_\eta Q_0 \\ &+ \ (I - Q_0) Q_\eta e^{\pm\tilde{A}_\eta^+ t} Q_\eta Q_0 + e^{\pm\tilde{A}_\eta^+ t} Q_\eta (I - Q_0), \end{aligned}$$

we have

$$\sup_{t\in[0,\tau]} \|e^{\pm\tilde{A}_\eta^+ t} Q_\eta - \mathcal{S}_\eta^{-1} e^{\pm\tilde{A}_\eta^+ t} Q_\eta Q_0\|_{L(Y)} \stackrel{\eta\to 0}{\longrightarrow} 0,$$

and it becomes sufficient to show the condition

$$\sup_{t\in[0,\tau]} \|\mathcal{S}_\eta^{-1} e^{\pm\tilde{A}_\eta^+ t} Q_\eta - e^{\pm\tilde{A}_0^+ t}\|_{L(Y_0^+)} \stackrel{\eta\to 0}{\longrightarrow} 0.$$

The latter however follows from the fact that

$$\mathcal{S}_\eta^{-1} e^{\pm\tilde{A}_\eta^+ t} Q_\eta = \mathcal{S}_\eta^{-1} e^{\pm\tilde{A}_\eta^+ t} \mathcal{S}_\eta = e^{\mathcal{S}_\eta^{-1}(\pm\tilde{A}_\eta^+)\mathcal{S}_\eta t} \ \text{ on } \ Y_0^+$$

and from relation

$$\begin{aligned} &\|\mathcal{S}_\eta^{-1}(\pm\tilde{A}_\eta^+)\mathcal{S}_\eta - (\pm\tilde{A}_0^+)\|_{L(Y_0^+)} \\ &= \|\mathcal{S}_\eta^{-1}[(\pm\tilde{A}_\eta^+)Q_\eta - (\pm\tilde{A}_0^+)Q_0 + (Q_0 - Q_\eta)(\pm\tilde{A}_0^+)]\|_{L(Y_0^+)} \stackrel{\eta\to 0}{\longrightarrow} 0, \end{aligned}$$

in which we used (47), (16) and a consequence of (12):

$$\|(\pm\tilde{A}_\eta)Q_\eta - (\pm\tilde{A}_0)Q_0\|_{L(Y)} = \|\frac{\pm\lambda}{2\pi \mathrm{i}} \int_\Gamma ((\lambda I - \tilde{A}_\eta)^{-1} - (\lambda I - \tilde{A}_0)^{-1}) d\lambda\|_{L(Y)} \stackrel{\eta\to 0}{\longrightarrow} 0.$$

Since $Re\sigma(-\tilde{A}_0{}^{+}) \geq \hat{\lambda}^{+} > \tilde{r} > r > 0$ ($\hat{\lambda}^{+}$ being the first positive eigenvalue of $-\tilde{A}_0$) then

$$\|e^{rt}e^{\tilde{A}_0^{+}\hat{t}}\begin{bmatrix}\phi\\\psi\end{bmatrix}\|_Y \leqslant me^{-(\tilde{r}-r)t}\|\begin{bmatrix}\phi\\\psi\end{bmatrix}\|_Y,\ t\geqslant 0,\ \begin{bmatrix}\phi\\\psi\end{bmatrix}\in Y_0^{+}$$

and in particular there exists $\hat{t} > 0$ such that

$$\|e^{r\hat{t}}e^{\tilde{A}_0^{+}\hat{t}}\begin{bmatrix}\phi\\\psi\end{bmatrix}\|_Y \leqslant \frac{1}{2}\|\begin{bmatrix}\phi\\\psi\end{bmatrix}\|_Y,\ \begin{bmatrix}\phi\\\psi\end{bmatrix}\in Y_0^{+},$$

which can be rewritten as

$$\|e^{r\hat{t}}e^{\tilde{A}_0^{+}\hat{t}}Q_0\begin{bmatrix}\phi\\\psi\end{bmatrix}\|_Y \leqslant \frac{1}{2}\|Q_0\begin{bmatrix}\phi\\\psi\end{bmatrix}\|_Y,\ \begin{bmatrix}\phi\\\psi\end{bmatrix}\in Y, \tag{52}$$

Applying (52) and (48), for $\delta = \frac{1}{2}e^{-r\hat{t}}$ and appropriately chosen $\eta_\delta > 0$ we have

$$\begin{aligned}\|e^{r\hat{t}}e^{\tilde{A}_\eta^{+}\hat{t}}\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_Y &\leqslant e^{r\hat{t}}\|\big(e^{\tilde{A}_\eta^{+}\hat{t}}Q_\eta - e^{\tilde{A}_0^{+}\hat{t}}Q_0\big)\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_Y\\ &+ \|e^{r\hat{t}}e^{\tilde{A}_0^{+}\hat{t}}Q_0\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_Y\\ &\leqslant (e^{r\hat{t}}\delta + \frac{1}{2})\|\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_Y \leqslant \|\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_Y,\ \begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\in Y_\eta^{+},\ \eta\in(0,\eta_\delta).\end{aligned} \tag{53}$$

In addition, since

$$\tilde{A}_\eta^{+} = \tilde{A}_\eta Q_\eta = \frac{1}{2\pi \mathrm{i}}\int_\Gamma (-I + \lambda(\lambda I - \tilde{A}_\eta))^{-1}d\lambda$$

and resolvents are convergent then $\tilde{A}_\eta^{+}$ are estimated in a uniform topology uniformly with respect to η sufficiently small. Inequality

$$\|e^{\tilde{A}_\eta^{+}\hat{t}}\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_{L(Y_\eta^{+})} \leq e^{\|\tilde{A}_\eta^{+}\|_{L(Y_\eta^{+})}|t|},\ t\in\mathbb{R},$$

allows us next observe that the norms $\|e^{rt}e^{\tilde{A}_\eta^{+}t}\begin{bmatrix}\phi_\eta\\\psi_\eta\end{bmatrix}\|_{L(Y_\eta^{+})}$ remain bounded on compact time intervals uniformly with respect to all η sufficiently small. Estimate (49) can be thus deduced from (53) via decomposition $t = n_t\hat{t} + r_t$. □

3. PROPERTIES OF THE NONLINEAR SEMIGROUPS

3.1. Local solutions to semilinear damped wave equations

Problems (1) will be viewed in the abstract form

$$\tfrac{d}{dt}\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] + A_\eta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\right),\ t>0,\ \left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right] \tag{54}$$

where

$$\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y = H_0^1(\Omega)\times L^2(\Omega)\quad\text{and}\quad F\left(\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}0\\f(u)\end{smallmatrix}\right].$$

LEMMA 3.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function which satisfies*

$$|f'(s)| \leqslant c(1 + |s|^{\rho}), \ s \in \mathbb{R}, \tag{55}$$

where $0 \leqslant \rho \leqslant \frac{2}{N-2}$ if $N \geqslant 3$ and $\rho \in [0, \infty)$ when $N = 1, 2$.

Then,

i) $F : Y \to Y$ is a continuously differentiable map which is Lipschitz continuous on bounded sets. If f also satisfies

$$\lim_{|s|\to\infty} \frac{|f'(s)|}{s^{\frac{2}{N-2}}} = 0, \ N \geqslant 3, \tag{56}$$

then $F : Y \to Y$ is compact.

ii) The problem (54) is locally well posed in Y for each $\eta \geq 0$. Furthermore, if $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in D(A_\eta)$, then the corresponding solution to (54) is a classical solution

iii) If $T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, $\eta \geq 0$, denotes local mild solution (see [18, p. 146]) to (54) on Y, then

$$\sup_{t\in[0,\tau]} \|T_{\eta_n}(t)\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - T_0(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \to 0, \ \eta_n \to 0^+,$$

provided that $\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y$, $\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] \xrightarrow{Y} \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, the solutions $T_{\eta_n}(t)\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]$, $T_0(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ exist and remain bounded in Y-norm uniformly with respect to $t \in [0, \tau]$ and $n \in \mathbb{N}$.

Proof: Suppose that $\rho \leq \frac{2}{N-2}$. The proof that F is a Lipschitz continuous map from Y into Y follows easily from Sobolev type embeddings and Hölder inequality. The continuous differentiability of the map $F : Y \to Y$ follows from the continuous differentiability of the map $f^e : L^{\frac{N}{N-2}}(\Omega) \to L^2(\Omega)$ given by $f^e(u)(x) = f(u(x))$ which can be obtained using the Dominated Convergence Theorem and a Converse on the Dominated Convergence Theorem.

Assume now that (56) holds. Of course F is a continuously differentiable map which is Lipschitz continuous in bounded sets. To obtain that $F : Y \to Y$ is a compact map we decompose f as $f = \epsilon g_\epsilon + h_\epsilon$ where $g_\epsilon : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function satisfying $|g_\epsilon(s)| \leq |s|^{\frac{2}{N-2}}$ and $h_\epsilon : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with compact support. If B is a bounded subset of Y it is easy to see that the measure of non-compactness of the set $F(B)$ is zero.

The existence of a local mild solution to (54) in Y follows then from the Banach contraction principle (see e.g. [18, Theorems 6.1.4, 6.1.5]), which proves ii).

As for iii) we first use the Cauchy integral formula to get

$$\begin{aligned} \|T_{\eta_n}(t)\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - T_0(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \leq{}& \|\left(e^{-A_{\eta_n}t} - e^{-A_0 t}\right)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y + \|e^{-A_{\eta_n}t}\left(\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\right)\|_Y \\ &+ \int_0^t \|\left(e^{-A_{\eta_n}(t-s)} - e^{-A_0(t-s)}\right) F(T_0(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\|_Y ds \\ &\int_0^t \|e^{-A_{\eta_n}(t-s)}\left(F(T_{\eta_n}(s)\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]) - F(T_0(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])\right)\|_Y ds \quad \text{for} \ \ t \in [0, \tau]. \end{aligned}$$

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By Lemma 2.2 the first term on the right hand side converges to zero. Moreover, thanks to the uniform bound $\|e^{-A_\eta t}\|_{L(Y)} \leqslant 1$, the second term tends to zero as well. Furthermore, convergence of both these terms is uniform with respect to $t \in [0,\tau]$.

The third term is bounded by

$$\tau \sup \left\{ \left\| (e^{-A_{\eta_n} t}(r) - e^{-A_0 t}(r)) \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right\|_Y : \begin{bmatrix} \psi \\ \phi \end{bmatrix} \in \mathcal{J} \text{ and } r \in [0,\tau] \right\}$$

where $\mathcal{J} = \{F(T_0(s) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]),\ s \in [0,\tau]\}$ is a compact set. Thus, as a consequence of Lemma 2.2 i), the third term tends to zero uniformy with respect to $t \in [0,\tau]$.

The fourth term can be estimated with the aid of the Lipschitz continuity of F and (55) so that, recalling that $\|T_{\eta_n}(s) \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\|_Y \leqslant C$ and $\|T_0(s) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \leqslant C$ for $s \in [0,\tau]$, we get

$$\int_0^t \|e^{-A_{\eta_n}(t-s)}(F(T_{\eta_n}(s) \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]) - F(T_0(s) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]))\|_Y ds$$
$$\leqslant c(1+C(N)) \int_0^t \|T_{\eta_n}(s) \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - T_0(s) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y ds.$$

What was said above ensures that for arbitrarily fixed $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\|T_{\eta_n}(t) \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - T_0(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \leqslant \varepsilon e^{-c(1+C(N))\tau} + c(1+C(N)) \int_0^t \|T_{\eta_n}(s) \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - T_0(s) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y ds$$

whenever $n \geqslant N_\varepsilon$ and $t \in [0,\tau]$. Applying Gronwall's Lemma we thus get the estimate

$$\|T_{\eta_n}(t) \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] - T_0(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_Y \leqslant \varepsilon,\ t \in [0,\tau],\ n \geqslant N_\varepsilon,$$

which completes the proof. □

Let us consider resulting from (1) equation for $v = \dot{u}$ and multiply it by v in $L^2(\Omega)$. Since the negative Laplacian with Dirichlet boundary condition is self-adjoint we obtain, for $\eta \geq 0$,

$$\frac{d}{dt}\left(\frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\Omega)} - \int_\Omega \int_0^u f(s) ds dx\right) = -a\|v\|^2_{L^2(\Omega)} - \eta\|\Lambda^{\frac{1}{4}} v\|_{L^2(\Omega)},$$

which shows that the function $\mathcal{L}$,

$$\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) = \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Lambda^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} - \int_\Omega \int_0^{w_1} f(s) ds dx,\ \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in Y, \tag{57}$$

is nonincreasing along the solutions. If the dissipativeness condition holds

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \tag{58}$$

then via the Poincaré inequality

$$\lambda_1 \|u\|^2_{L^2(\Omega)} \leqslant \|\Lambda^{\frac{1}{2}} u\|^2_{L^2(\Omega)}$$

we get the estimate

$$\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t, u_0, v_0)\|_Y \leqslant c\sqrt{1 + \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])}, \tag{59}$$

where $c > 0$ is independent of $\eta \geq 0$. Indeed, thanks to condition (58)

$$-\int_\Omega \int_0^{w_1} f(s) ds dx \geqslant -\frac{\lambda_1 - \varepsilon}{2} \|w_1\|^2_{L^2(\Omega)} - M|\Omega|,$$

where $\varepsilon > 0$ and $M = M(\varepsilon)$ is a positive constant. Hence the Lyapunov function $\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right])$ is bounded from below, i.e.

$$\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) \geqslant \frac{\varepsilon}{2\lambda_1} \|\Lambda^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} - M|\Omega|$$

and also

$$\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) \to \infty \quad \text{as} \quad \| \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \|_Y \to \infty.$$

Since we simultaneusly consider the hyperbolic case $\eta = 0$, note that the above calculations are valid for classical solutions corresponding to initial conditions chosen from Y^1. Therefore, classical solutions to (54) exist globally in time (see [18, Theorem 1.4]) and are bounded uniformly with respect to $t \in [0, \infty)$, $\eta \geq 0$, and $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ varying in bounded subsets of Y. The mentioned properties are then extended to all initial data in Y based on the continuous dependence of mild solutions on initial conditions, which result follows from the Gronwall's Lemma.

3.2. Family of global attractors

Consideration of the previous subsection leads to the following proposition.

Proposition 3.1. *Suppose that* (58) *and* (55) *hold. Then for each* $\eta \geq 0$ *problem* (54) *defines on* Y *a* C^0*-semigroup of global mild solutions with bounded orbits of bounded sets. If, in addition,* $\eta > 0$ *or* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^1$ *then*

$$T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in C([0, \infty), Y) \cap C^1(0, \infty), Y) \cap C((0, \infty), Y^1)$$

and $T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ *is a classical solution to* (54).

For $\eta \geq 0$ denote by $\mathcal{E}_\eta$ the set of all equilibria corresponding to $\{T_\eta(t)\}$ in Y. Let $\omega_\eta(B)$ be an ω-limit set of $B \subset Y$ with respect to $\{T_\eta(t)\}$. If $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ is such that there is a complete orbit $\gamma_\eta(\begin{bmatrix} \phi \\ \psi \end{bmatrix})$ passing through $\begin{bmatrix} \phi \\ \psi \end{bmatrix}$ let $\alpha_{\gamma_\eta}(\begin{bmatrix} \phi \\ \psi \end{bmatrix})$ be a corresponding α-limit set.

PROPOSITION 3.2. *Suppose that* (58) *and* (55) *hold. Then*
i) $\mathcal{E} = \{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y: \ u_0 = \Lambda^{-1} f(u_0), \ v_0 = 0\}$ *is a bounded subset of* Y^1,
ii) The equilibria of $\{T_\eta(t)\}$ *are smooth and* $\mathcal{E}_\eta = \mathcal{E}$ *for* $\eta \geqslant 0$,
iii) $\omega_\eta(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) \subset \mathcal{E}$ *and* $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) \subset \mathcal{E}$ *for* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y$ *and each* $\eta \geq 0$.

Proof: If $u_0 \in H^1_0(\Omega)$ and $u_0 = \Lambda^{-1} f(u_0)$ then (58) ensures the estimate

$$\|\Lambda^{\frac{1}{2}} u_0\|_{L^2(\Omega)} \leqslant M_0^{\mathcal{E}}. \tag{60}$$

Since f takes bounded subsets of $H^1_0(\Omega)$ into bounded subsets of $L^2(\Omega)$ we also get the estimate

$$\|\Lambda u_0\|_{L^2(\Omega)} = \|f(u_0)\|_{L^2(\Omega)} \leqslant \sup_{\|\Lambda^{\frac{1}{2}}\phi\|_{L^2(\Omega)} \leqslant M_0^{\mathcal{E}}} \|f(\phi)\|_{L^2(\Omega)} = M_1^{\mathcal{E}}, \tag{61}$$

which proves i).

Concerning ii) observe that $\mathcal{E} \subset \mathcal{E}_\eta$. Next, if $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \mathcal{E}_\eta$, i.e. $T_\eta(t) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, then through the Cauchy integral formula we get

$$\frac{1}{t}\left(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] - e^{-A_\eta t}\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\right) = \frac{1}{t}\int_0^t \left(e^{-A_\eta(t-s)}\left[\begin{smallmatrix} 0 \\ f(u_0) \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ f(u_0) \end{smallmatrix}\right]\right) ds + \left[\begin{smallmatrix} 0 \\ f(u_0) \end{smallmatrix}\right].$$

Since the first right hand side term converges to zero as $t \to 0^+$ we obtain that $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in D(A_\eta)$ and

$$A_\eta \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ f(u_0) \end{smallmatrix}\right].$$

Hence, $v_0 = 0$ and $\Lambda u_0 = f(u_0)$.

Suppose further that $\eta > 0$ and $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \omega_\eta(\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right])$ or $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \alpha_\eta(\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right])$. Then $\mathcal{L}\left(T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\right) \equiv$ *const*, consequently its time derivate is zero, so that $v \equiv 0$. Therefore, $T_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]$ and $u_0 = \Lambda^{-1} f(u_0)$, which proves iii). □

LEMMA 3.2. *Suppose* (56) *and dissipativeness condition* (58) *hold.*
Then, for each $\eta \geq 0$, *problem* (54) *defines on* Y *a* C^0*-semigroup of global solutions* $\{T_\eta(t)\}$ *which possesses a global attractor* $\boldsymbol{A}_\eta$. *Furthermore,* $\bigcup_{\eta>0} \boldsymbol{A}_\eta$ *is bounded in* Y. *In particular,* $\bigcup_{\eta>0} \boldsymbol{A}_\eta$ *is contained in a ball in* Y *of radius* $r_{\mathcal{E}} := \sup_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \mathcal{E}} c\sqrt{1 + \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])}$.

Proof: The existence of attractors follows easily from the fact that (54) is a gradient system, Proposition 3.1, Proposition 3.2 and Lemma 3.1.

If $\eta > 0$ and $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \in \boldsymbol{A}_\eta$, then there exists a precompact complete orbit passing through $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$. Denoting it by $\gamma_\eta = \{T_\eta(t)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right], \ t \in \mathbb{R}\}$ we have that $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] = T_\eta(k) T_\eta(-k)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$ for arbitrary $k \in \mathbb{N}$. Sequence $\{T_\eta(-k)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ possesses a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ convergent to certain element $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ of $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$. This together with (59), (60) and Proposition 3.2 iii) ensures that for each k_l

$$\|\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_Y = \|T_\eta(k_l) T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_Y \leqslant \sqrt{1 + \mathcal{L}(T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])}$$

and, therefore,

$$\| \left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \|_Y \leqslant \lim_{k_l\to\infty} \sqrt{1+\mathcal{L}(T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])} \leq \sqrt{1+\mathcal{L}(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])} \leqslant r_{\mathcal{E}}.$$

The proof is thus complete. □

Remark 3. 1. The existence of attractors and their uniform bounds in Y can be obtained under some more general growth conditions (see [2] for $\eta = 0$, [6] for $\eta > 0$). Nevertheless, the uniform bounds in Y^1 below require the stronger restriction (56).

LEMMA 3.3. *Suppose that* (56) *and* (58) *hold. For each $\eta > 0$ any complete orbit lying on $\boldsymbol{A}_\eta$ is precompact in Y^1 and $\bigcup_{\eta\in(0,\eta_0)} \boldsymbol{A}_\eta$ is bounded in Y^1 for each $\eta_0 > 0$.*

Proof: We first prove that $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ is a precompact subset of Y^1 for each $\eta > 0$. Fix $\eta > 0$, choose $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \in \boldsymbol{A}_\eta$ and consider any complete orbit $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) = \{T_\eta(t)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\ t \in \mathbb{R}\}$ passing through $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$ and lying on $\boldsymbol{A}_\eta$. By the smoothing properties of the semigroups generated by abstract parabolic equations it is known (see [10, Lemma 3.2.1]) that $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ is bounded in Y^1; i.e.

$$\sup_{\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\in\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])} \| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \|_{Y^1} \leqslant M_\eta. \tag{62}$$

Note that

$$\|\nabla[f(\phi)-f(0)]\|_{L^2(\Omega)} = \|f'(\phi)\nabla\phi\|_{L^2(\Omega)} \leqslant \|u_\eta\|^{\frac{2}{N-2}}_{L^{\frac{2N}{N-2}}} \|\nabla u_\eta\|_{L^{\frac{2N}{N-2}}}.$$

Therefore $F - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])$ takes bounded sets of Y^1 into bounded sets of Y^1, and based on (62) we then have

$$\sup_{\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\in\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])} \|F(\begin{bmatrix} \phi \\ \psi \end{bmatrix}) - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_{Y^1} \leqslant M^F_\eta.$$

Using the Cauchy integral formula and (39) for $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$, $t \geq 0$, we obtain[3] for $\frac{1}{2}^- < \frac{1}{2}$

$$\begin{aligned} \|A_\eta^{1+\frac{1}{2}^-} T_\eta(t)\begin{bmatrix} \phi \\ \psi \end{bmatrix}\|_Y &\leqslant c_\eta \frac{e^{\omega_\eta t}}{t^{\frac{1}{2}^-}} \|A_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix}\|_Y + \|A_\eta^{\frac{1}{2}^-}\left(I - e^{-A_\eta t}\right) F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y \\ &+ \int_0^t c_\eta \frac{e^{\omega_\eta (t-s)}}{(t-s)^{\frac{1}{2}^-}} \|A_\eta \left(F(T_\eta(s)\begin{bmatrix} \phi \\ \psi \end{bmatrix}) - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\right)\|_Y ds. \end{aligned}$$

[3]We observe that $F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]) \in D(A_\eta^{\frac{1}{2}^-})$ but does not belongs to $D(A_\eta^\alpha)$ for $\alpha \geq \frac{1}{2}$ unless $f(0) = 0$.

Since the orbit $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ is invariant under the semigroup $\{T_\eta(t)\}$, this and (3) leads to the relation

$$\sup_{\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in \gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])} \|A_\eta^{1+\frac{1}{2}^-} \begin{bmatrix} \phi \\ \psi \end{bmatrix}\|_Y = \sup_{\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in \gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])} \|A_\eta^{1+\frac{1}{2}^-} T_\eta(1) \begin{bmatrix} \phi \\ \psi \end{bmatrix}\|_Y$$
$$\leqslant c_\eta e^{\omega_\eta} M_\eta + \|A_\eta^{\frac{1}{2}^-} \left(I - e^{-A_\eta}\right) F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y + c_\eta d M_\eta^F c_\eta \int_0^1 \frac{e^{\omega_\eta(1-s)}}{(1-s)^{\frac{1}{2}}} ds.$$

Hence $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ is precompact in Y^1.

Now let us prove that $\bigcup_{\eta \in (0,\eta_0)} \boldsymbol{A}_\eta$ is bounded in Y^1 for each $\eta_0 > 0$.

From the fact that $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ is a precompact subset of Y^1, there exists a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ convergent in Y^1 to an element of the α-limit set $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \subset \mathcal{E}$.

Since $\mathcal{E}$ is bounded in Y^1 we are able to obtain Y^1-bound on the attractors, uniform with respect to $\eta \in (0, \eta_0)$ for any $\eta_0 > 0$. In fact we will get the Y^1 estimate of complete orbits $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ lying on the attractors based on the estimates of the solutions with initial conditions $T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$ chosen arbitrarily close in Y^1-metric to the set $\mathcal{E}$.

Following this scheme, we have

$$T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] = e^{-A_\eta t} T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] + \int_0^t e^{-A_\eta(t-s)} F(T_\eta(s)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) ds$$

and, choosing $\varepsilon > 0$, we find certain $N_\varepsilon \in \mathbb{N}$ such that for all $k_l \geqslant N_\varepsilon$

$$\|T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_{Y^1} \leqslant M_1^{\mathcal{E}} + \epsilon$$

where $M_1^{\mathcal{E}}$ is defined in (61). From this we have that

$$\begin{aligned} &\|A_\eta T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_Y \\ &\qquad \leqslant \|e^{-A_\eta t}\|_{L(Y)} \|A_\eta T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_Y + \|\left(I - e^{-A_\eta t}\right) F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y \\ &\qquad + \int_0^t \|e^{-A_\eta(t-s)}\|_{L(Y)} \|A_\eta \left(F(T_\eta(s)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\right)\|_Y ds \\ &\qquad \leq c e^{\omega t}(M_1^{\mathcal{E}} + \varepsilon) + (1 + c e^{\omega t}) \|F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y \\ &\qquad + \int_0^t c e^{\omega(t-s)} \|A_\eta \left(F(T_\eta(s)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) - F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\right)\|_Y ds. \end{aligned} \tag{63}$$

If we let $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] =: \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]$ and use (56), which reads

$$\forall_{\nu>0} \exists_{C_\nu>0} \, \forall_{s\in\mathbb{R}^1} \; |f'(s)| \leqslant \nu |s|^{\frac{2}{n-2}} + C_\nu$$

then, for each $\eta \in (0, \eta_0)$, we have

$$\begin{aligned}
&\|A_\eta\big(F\left(\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\right) - F\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right)\big)\|_Y = \|A_\eta \left[\begin{smallmatrix} 0 \\ f(u_\eta)-f(0) \end{smallmatrix}\right]\|_Y \\
&\leqslant C_{\eta_0}\|\nabla(f(u_\eta) - f(0))\|_{L^2(\Omega)} + a\|f(u_\eta) - f(0)\|_{L^2(\Omega)} \\
&= C_{\eta_0}\|f'(u_\eta)\nabla u_\eta\|_{L^2(\Omega)} + a\|f(u_\eta) - f(0)\|_{L^2(\Omega)} \\
&\leqslant \nu C_{\eta_0}\|u_\eta\|^{\frac{2}{N-2}}_{L^{\frac{2N}{N-2}}(\Omega)}\|\nabla u_\eta\|_{L^{\frac{2N}{N-2}}(\Omega)} + C_{\eta_0}C_\nu\|\nabla u_\eta\|_{L^2(\Omega)} + a\|f(u_\eta) - f(0)\|_{L^2(\Omega)} \\
&\leqslant \bar{C}_{\eta_0}(\nu\|A_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_Y + 1),
\end{aligned} \tag{64}$$

where we have used the estimate of Lemma 3.2 and (3). From (63) and (64) we obtain

$$\begin{aligned}
\|A_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_Y &\leqslant ce^{\omega t}(M_1^{\mathcal{E}} + \varepsilon) + (1 + ce^{\omega t})\|F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y \\
&\quad + \int_0^t ce^{\omega(t-s)}\bar{C}_{\eta_0}(\nu\|A_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_Y + 1)ds \\
&\leqslant c(M_1^{\mathcal{E}} + \varepsilon) + (1 + c)\|F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y - \frac{c}{\omega}\bar{C}_{\eta_0}(\nu \sup_{t\geq 0}\|A_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_Y + 1)ds,
\end{aligned}$$

where $\omega < 0$ is as in (38). Adding supremum and estimating, we find that

$$\left(1 + \nu\frac{c\bar{C}_{\eta_0}}{\omega}\right)\sup_{t\in[0,\infty)}\|A_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_Y \leqslant c(M_1^{\mathcal{E}} + \varepsilon) + (1 + c)\|F(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])\|_Y - \frac{c\bar{C}_{\eta_0}}{\omega},$$

which for sufficiently small $\nu > 0$ provides the estimate

$$\sup_{t\in[0,\infty)}\|A_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]\|_Y \leqslant M,$$

where $M > 0$ is independent $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \in \mathbf{A}_\eta$ and of $\eta \in (0, \eta_0)$. Since $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]$, the above considerations ensure that for any $\eta \in (0, \eta_0)$ and arbitrarily chosen $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \in \boldsymbol{A}_\eta$ we have the uniform bound

$$\|A_\eta \left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_Y \leqslant M.$$

By Proposition 2.1 iv) we thus conclude that

$$\sup_{\eta\in(0,\eta_0)}\ \sup_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\in\boldsymbol{A}_\eta}\|\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_{Y^1} \leqslant d^{-1}M,$$

which completes the proof.□

Corollary 3.1. *Under the assumptions of Lemma 3.3, for each $\eta_0 > 0$, $\bigcup_{\eta\in(0,\eta_0)}\boldsymbol{A}_\eta$ is precompact in Y.*

4. UPPER AND LOWER SEMICONTINUITY OF THE ATTRACTORS

In the light of the results obtained in Section 3 the attractors $\boldsymbol{A}_\eta$, $\eta \geq 0$, are upper semicontinuous at $\eta = 0$. Indeed, if this is not true then there exist $\eta_n \to 0^+$ and $\left[\begin{smallmatrix} u_{0\eta_n} \\ v_{0\eta_n} \end{smallmatrix}\right] \in \boldsymbol{A}_{\eta_n}$ such that $\inf_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in \boldsymbol{A}_0} \| \left[\begin{smallmatrix} u_{0\eta_n} \\ v_{0\eta_n} \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \|_Y > \varepsilon$. Based on Corollary 3.1 one can choose a subsequence $\{\left[\begin{smallmatrix} u_{0\eta_{n_k}} \\ v_{0\eta_{n_k}} \end{smallmatrix}\right]\}$ convergent in Y to certain $\left[\begin{smallmatrix} \hat{u}_0 \\ \hat{v}_0 \end{smallmatrix}\right] \in Y$. Lemma 3.1 allows then to show that there is a complete bounded orbit of $\{T_0(t)\}$ passing through $\left[\begin{smallmatrix} \hat{u}_0 \\ \hat{v}_0 \end{smallmatrix}\right]$ and lying on $\boldsymbol{A}_0$, which leads to a contradiction.

For the lower semicontinuity note first that, under assumptions (58), (55) the semigroups $\{T_\eta(t)\}$, $\eta \geq 0$, are *gradient systems* (see [14, Definition 3.8.1] or [15]). Suppose further that

$$\text{equation } u = \Lambda^{-1} f(u) \text{ has only finitely many solutions } u_{0_1}, \ldots, u_{0_k} \in H_0^1(\Omega). \tag{65}$$

Under the above assumption attractors $\boldsymbol{A}_\eta$ corresponding to $\{T_\eta(t)\}$, $\eta \geq 0$, can be described as

$$\boldsymbol{A}_\eta = \bigcup_{i=1}^{k} W_\eta^u(\left[\begin{smallmatrix} u_{0_i} \\ 0 \end{smallmatrix}\right]), \ \eta \geq 0, \tag{66}$$

where $W_\eta^u(\left[\begin{smallmatrix} u_{0_i} \\ 0 \end{smallmatrix}\right])$ is the unstable set of $\left[\begin{smallmatrix} u_{0_i} \\ 0 \end{smallmatrix}\right]$; i.e.

$$W_\eta^u(\left[\begin{smallmatrix} u_{0_i} \\ 0 \end{smallmatrix}\right]) = \{\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in Y; T_\eta(-t)\begin{bmatrix} \phi \\ \psi \end{bmatrix} \text{ exists for } t \geq 0 \text{ and } T_\eta(-t)\begin{bmatrix} \phi \\ \psi \end{bmatrix} \to \left[\begin{smallmatrix} u_{0_i} \\ 0 \end{smallmatrix}\right] \text{ as } t \to \infty\}.$$

Proposition 4.1. *Suppose* (58), (55) *are satisfied. If*

$$0 \notin \sigma(-\Lambda + f'(u_0)) \quad \text{whenever} \quad u_0 \in H_0^1(\Omega) \quad \text{and} \quad u_0 = \Lambda^{-1} f(u_0), \tag{67}$$

then (65), (66) *hold.*

Proof: Note that if $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y$ is an equilibrium of $\{T_0(t)\}$ then $v_0 = 0$, $u_0 = \Lambda^{-1} f(u_0)$ and $0 \notin \sigma(-\Lambda + f'(u_0))$. Consequently $I - \Lambda^{-1} f'(u_0) : H_0^1(\Omega) \to H_0^1(\Omega)$ is a linear isomorphism. Since $\Lambda^{-1} f : H_0^1(\Omega) \to H_0^1(\Omega)$ is differentiable we obtain that

$$\begin{aligned} &\|\phi - \Lambda^{-1} f(\phi)\|_{H_0^1(\Omega)} = \|\phi - u_0 + \Lambda^{-1} f(u_0) - \Lambda^{-1} f(\phi)\|_{H_0^1(\Omega)} \\ &\|\phi - u_0 - (\Lambda^{-1} f)'(u_0)(\phi - u_0) + r(u_0, \phi - u_0)\|_{H_0^1(\Omega)} \\ &\geqslant \|(I - \Lambda^{-1} f'(u_0))(\phi - u_0)\|_{H_0^1(\Omega)} - \|r(u_0, \phi - u_0)\|_{H_0^1(\Omega)}, \end{aligned} \tag{68}$$

where $\|r(u_0, \phi - u_0)\|_{H_0^1(\Omega)} \|\phi - u_0\|_{H_0^1(\Omega)}^{-1} \to 0$ as $\|\phi - u_0\|_{H_0^1(\Omega)} \to 0$. Fix $\frac{1}{\epsilon_0}$ bigger than the norm of $(I - \Lambda^{-1} f'(u_0))^{-1}$ and choose arbitrary $\epsilon \in (0, \epsilon_0)$. Then the first term on the right hand side of (68) is bounded from below by $\epsilon \|\phi - u_0\|_{H_0^1(\Omega)}$ for all $\phi \in H_0^1(\Omega)$. The second one becomes less then $\frac{1}{2}\epsilon \|\phi - u_0\|_{H_0^1(\Omega)}$ whenever ϕ belongs to certain neighborhood of u_0 in $H_0^1(\Omega)$. Therefore $\|\phi - \Lambda^{-1} f(\phi)\|_{H_0^1(\Omega)} \geqslant \frac{1}{2}\epsilon \|\phi - u_0\|_{H_0^1(\Omega)}$ in the latter neighborhood, which shows that the equilibria of $\{T_0(t)\}$ are isolated points of $H^1_0(\Omega)$. Since all these equilibria belong to the compact attractor $\boldsymbol{A}_0$ their number needs to be finite. □

Recall that the proof of upper semicontinuity of $\{\boldsymbol{A}_\eta, \eta \geq 0\}$ at $\eta = 0$ was based on the fact that from each sequence $\{\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}\}$ of elements $\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \in \boldsymbol{A}_{\eta_n}$ we were able to choose a subsequence convergent in Y as $\eta_n \to 0^+$ to certain $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} \in \boldsymbol{A}_0$. To justify lower semicontinuity we will ensure that each element $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} \in \boldsymbol{A}_0$ is a limit in Y of a sequence $\{\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}\}$ as $\eta_n \to 0^+$, where $\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \in \boldsymbol{A}_{\eta_n}$.

Since under the assumptions (58), (55), (67) we know that the attractors exist and possess the structure stated in (66) the unstable manifold technique can be applied (see Appendix) and the lower semicontinuity result of Theorem 1.1 follows. Indeed, let us fix any nonequilibrium point $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} \in \boldsymbol{A}_0$. Then, there exists a complete orbit $\gamma_0(\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}) = \{T_0(t)\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix},\ t \in \mathbb{R}\}$ through $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}$ such that, $T_0(t)\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} \to \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right] \in \mathcal{E}$ as $t \to -\infty$. Note that u_0 is such that $\Lambda u_0 + f(u_0) = 0$ and that $u_0 \in L^\infty(\Omega)$. Rewrite (54) as

$$\frac{d}{dt}\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] + \tilde{A}_\eta \left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] = h\left(\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]\right), \tag{69}$$

where

$$\tilde{A}_\eta = [A_\eta - F'(\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right])]$$

and

$$h\left(\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]\right) = F\left(\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] + \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\right) - F\left(\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\right) - F'(\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right])\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right].$$

Then, the conditions of Propostition 5.2 hold. As a consequence of that, a point $\begin{bmatrix} \tilde{\phi}_0 \\ \tilde{\psi}_0 \end{bmatrix} \in \gamma_0(\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix})\ \left(T_0(t_0)\begin{bmatrix} \tilde{\phi}_0 \\ \tilde{\psi}_0 \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix},\ t_0 > 0\right)$ can be chosen which can be approximated by a sequence $\{\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}\}$, where $\eta_n \to 0$ and $\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}$ is in the unstable set of $\left[\begin{smallmatrix} u_{0_i} \\ 0 \end{smallmatrix}\right]$. Thus there exists a complete orbit $\gamma_{\eta_n}(\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}) = \{T_{\eta_n}(t)\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}, t \in \mathbb{R}\}$ through each such $\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}$ and it lies on the corresponding attractor $\boldsymbol{A}_{\eta_n}$. As a result of Lemma 3.1 the point $\begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}$ is in fact a limit of a sequence of points $T_{\eta_n}(t_0)\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix}$. Since we ensured that $T_{\eta_n}(t_0)\begin{bmatrix} \phi_{\eta_n} \\ \psi_{\eta_n} \end{bmatrix} \in \boldsymbol{A}_{\eta_n}$ for $n \in \mathbb{N}$, the proof of Theorem 1.1 is complete.

Corollary 1.1 follows as a consequence of lower semicontinuity of attractors and (uniform with respect to the parameter) Y^1-estimate of the attractors reported in Lemma 3.3.

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5. APPENDIX: EXISTENCE AND CONTINUITY OF LOCAL UNSTABLE MANIFOLDS

Let $\mathcal{Z}$ be a complex Banach space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of a strongly continuous semigroup $\{e^{\mathcal{A}t} : t \geqslant 0\}$ of bounded linear operators such that the set $\sigma^+ = \{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\lambda > 0\}$ is compact. If γ is a smooth closed simple curve in $\rho(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ oriented counterclockwise enclosing σ^+ let

$$\mathcal{Q} = \mathcal{Q}(\sigma^+) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - \mathcal{A})^{-1} d\lambda \tag{70}$$

and define $\mathcal{Z}^+ = \mathcal{Q}(\mathcal{Z})$ and $\mathcal{Z}^- = (I - \mathcal{Q})(\mathcal{Z})$, $\mathcal{A}^\pm = \mathcal{A}_{|_{\mathcal{Z}^\pm}}$. It is clear that $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$, that $\mathcal{A}^-$ generates a strongly continuous semigroup of operators and that $\mathcal{A}^+ \in L(\mathcal{Z}^+)$. Assume that

$$\begin{aligned} &\|e^{\mathcal{A}^+ t}\|_{L(\mathcal{Z}^+)} \leqslant M e^{\omega t}, \quad t \leqslant 0, \\ &\|e^{\mathcal{A}^- t}\|_{L(\mathcal{Z}^-)} \leqslant M e^{-\omega t}, \quad t \geqslant 0. \end{aligned} \tag{71}$$

Let $h : \mathcal{Z} \to \mathcal{Z}$ be a continuously differentiable function which satisfies $h(0) = 0$ and $h'(0) = 0 \in L(\mathcal{Z})$ and consider the initial value problem

$$\begin{aligned} &\dot{z} = \mathcal{A}z + h(z), \\ &z(0) = z_0 \in \mathcal{Z}. \end{aligned} \tag{72}$$

In this equation, in a small neighborhood of $z = 0$, the nonlinear part has small Lipschitz constant. Let us consider what happens when we neglect the nonlinearity; that is,

$$\begin{aligned} &\dot{y} = \mathcal{A}y \\ &y(0) = y_0. \end{aligned} \tag{73}$$

For $y_0 \in \mathcal{Z}^+$, the solution $y(t, y_0)$ of (73) exists for all negative time, $y(t, y_0) \overset{t \to -\infty}{\longrightarrow} 0$.

When we perturb (73) with the very small nonlinearity f we should observe solutions of (72) that exist for all negative time and converge to $z = 0$ as $t \to -\infty$. Of course the initial data for which such solutions exist will no longer be in $\mathcal{Z}^+$ but in a nonlinear manifold near it.

Definition 5.1. The unstable manifold of the equilibrium solution $z = 0$ to (72) is the set

$$W^u(0) = \{\, \zeta \in \mathcal{Z} : \text{there is a backward solution } z(t, \zeta) \text{ through } \zeta \text{ such that } \lim_{t \to -\infty} z(t, \zeta) = 0\}$$

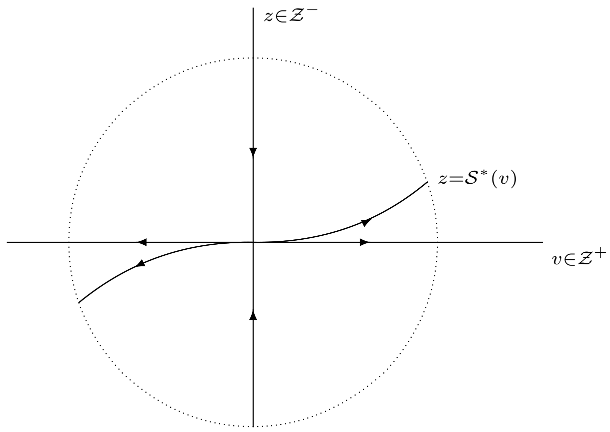

Figure 1

If z is a solution to (72) we write $z^+ = \mathcal{Q}z$ and $z^- = z - z^+$. Thus we have

$$\begin{aligned} \dot{z}^+ &= \mathcal{A}^+ z^+ + H(z^+, z^-), \\ \dot{z}^- &= \mathcal{A}^- z^- + G(z^+, z^-). \end{aligned} \tag{74}$$

where $H(z^+, z^-) = \mathcal{Q}h(z^+ + z^-)$ and $G(z^+, z^-) = (I - \mathcal{Q})h(z^+ + z^-)$.

Since we have that, at $(0, 0)$ the functions H and G are zero with zero derivatives, from the continuous differentiability of H and G we obtain that given $\rho > 0$ there exists $\delta > 0$ such that if $\|z^+\|_{\mathcal{Z}} + \|z^-\|_{\mathcal{Z}} < \delta$, we have

$$\begin{aligned} &\|H(z^+, z^-)\|_{\mathcal{Z}} \leqslant \rho, \\ &\|G(z^+, z^-)\|_{\mathcal{Z}} \leqslant \rho, \\ &\|H(z^+, z^-) - H(\tilde{z}^+, \tilde{z}^-)\|_{\mathcal{Z}} \leqslant \rho(\|z^+ - \tilde{z}^+\|_{\mathcal{Z}} + \|z^- - \tilde{z}^-\|_{\mathcal{Z}}), \\ &\|G(z^+, z^-) - G(\tilde{z}^+, \tilde{z}^-)\|_{\mathcal{Z}} \leqslant \rho(\|z^+ - \tilde{z}^+\|_{\mathcal{Z}} + \|z^- - \tilde{z}^-\|_{\mathcal{Z}}). \end{aligned} \tag{75}$$

We can extend H, G outside ball of radius δ in such a way that the bounds (75) hold for all $z^+ \in \mathcal{Z}^+$, $z^- \in \mathcal{Z}^-$.

Remark 5. 1. The procedure to extend a function g defined in the ball of radius δ in such a way that it becomes globally Lipschitz continuous without changing its Lipschitz constant is the following. Given a function $g : V \times Z \to W$, where V, Z and W are Banach spaces, define

$$g_\delta(z^+, z^-) = \begin{cases} g(z^+, z^-), & \|z^+ + z^-\|_Z \leqslant \delta \\ g\left(\frac{\delta\, z^+}{\|z^+ + z^-\|_Z}, \frac{\delta\, z^-}{\|z^+ + z^-\|_Z}\right), & \|z^+ + z^-\|_Z > \delta \end{cases}$$

The Lipschitz constant g_δ is the Lipshitz constant for g restricted to the ball of radius δ.

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5.1. Existence of unstable manifolds as a graph

With the notation above, the equation (72) can be rewritten in the form (74). Assume that H and G satisfy (75) for all $z^+ \in \mathcal{Z}^+$, $z^- \in \mathcal{Z}^-$ and let $W^u(0)$ be the unstable manifold of equilibrium solution $(z^+, z^-) = (0,0)$ to (74). We will show that for a suitably small $\rho > 0$, there is a function $\Sigma : \mathcal{Z}^+ \to \mathcal{Z}^-$ such that

$$W^u(0) = \{(z^+, z^-) : z^- = \Sigma^*(z^+),\ z^+ \in \mathcal{Z}^+\}$$

where $\Sigma^* : \mathcal{Z}^+ \to \mathcal{Z}^-$ is bounded and Lipschitz continuous.

We observe that we are looking for a function Σ^* such that if $\mathcal{Z} \ni (\zeta, \Sigma^*(\zeta))$ then the solution of (74) starting at $z^+(\tau) = \zeta$, $z^-(\tau) = \Sigma^*(\zeta)$ stays in the graph of Σ^* for all positive and all negative time. This means that $z^-(t) = \Sigma^*(z^+(t))$ and that, for all t and (74) becomes

$$\begin{aligned} \dot{z}^+ &= \mathcal{A}^+ z^+ + H(z^+, \Sigma^*(z^+)) \\ \dot{z}^- &= \mathcal{A}^- z^- + G(z^+, \Sigma^*(z^+)). \end{aligned} \tag{76}$$

The solution $(z^+(t), z^-(t))$ must go to zero as $t \to -\infty$ and in particular it must stay bounded. Since

$$z^-(t) = e^{\mathcal{A}^-(t-t_0)} z^-(t_0) + \int_{t_0}^t e^{\mathcal{A}^-(t-s)} G(z^+(s), \Sigma^*(z^+(s)))ds,$$

making $t_0 \to -\infty$, we have that

$$z^-(t) = \Sigma^*(z^+(t)) = \int_{-\infty}^t e^{\mathcal{A}^-(t-s)} G(z^+(s), \Sigma^*(z^+(s)))ds,$$

in particular

$$\Sigma^*(\zeta) = \Sigma^*(z^+(\tau)) = z^-(\tau) = \int_{-\infty}^\tau e^{\mathcal{A}^-(\tau-s)} G(z^+(s), \Sigma^*(z^+(s)))ds.$$

After proving that the graph of Σ is in the unstable manifold of $z = 0$ we need to verify that any solution which is in the unstable manifold must be in this graph.

Proposition 5.1. *Assume that the above conditions are satisfied. Then, there exist function $\Sigma^* : \mathcal{Z}^+ \to \mathcal{Z}^-$, such that the unstable manifold $W^u(0)$ to (74) is given by*

$$W^u(0) = \{w \in \mathcal{Z} : w = (\mathcal{Q}w, \Sigma^*(\mathcal{Q}w))\}.$$

Proof: In order to show the existence of such function Σ^* we will use the Banach contraction principle. Let $D > 0$, $L > 0$ and $0 < \vartheta < 1$ be given and choose $\rho_0 > 0$ such that

$$\begin{aligned} &\tfrac{\rho M}{\omega} \leqslant D,\\ &\tfrac{\rho M}{\omega}(1+L) \leqslant \vartheta < 1\\ &\frac{\rho M^2(1+L)}{\omega - \rho M(1+L)} \leqslant L,\\ &\rho M + \tfrac{\rho^2 M^2 (1+L)(1+M)}{2\omega - \rho M(1+L)} < \omega. \end{aligned} \tag{77}$$

for all $0 < \rho \leqslant \rho_0$.

For positive constants D and L, let $\mathcal{LB}(D,L)$ be the set of all globally Lipschitz bounded functions $\Sigma : \mathcal{Z}^+ \to \mathcal{Z}^-$ satisfying

$$\sup_{z^+ \in \mathcal{Z}^+} \|\Sigma(z^+)\|_{\mathcal{Z}} \leqslant D, \quad \|\Sigma(z^+) - \Sigma(\tilde{z}^+)\|_{\mathcal{Z}} \leqslant L\|z^+ - \tilde{z}^+\|_{\mathcal{Z}}. \tag{78}$$

For Σ and $\tilde{\Sigma}$ in $\mathcal{LB}(D,L)$ we define their distance $|\!|\!|\Sigma - \tilde{\Sigma}|\!|\!|$ as

$$|\!|\!|\Sigma - \tilde{\Sigma}|\!|\!| := \sup_{z^+ \in \mathcal{Z}^+} \|\Sigma(z^+) - \tilde{\Sigma}(z^+)\|_{\mathcal{Z}}.$$

It is easy to see that with this metric $\mathcal{LB}(D,L)$ is a complete metric space.

If $z^+(t) = \psi(t,\tau,\zeta,\Sigma)$ be the solution of

$$\frac{dz^+}{dt} = \mathcal{A}^+ z^+ + H(z^+, \Sigma(z^+)), \quad \text{for} \quad t < \tau, \; z^+(\tau) = \zeta, \tag{79}$$

we define

$$\Phi(\Sigma)(\zeta) = \int_{-\infty}^{\tau} e^{\mathcal{A}^-(\tau - s)} G(z^+(s), \Sigma(z^+(s)))ds. \tag{80}$$

In what follows we will show that, for $\rho > 0$ satisfying (77), the map Φ takes $\mathcal{LB}(D,L)$ into itself and is a strict contraction. Hence it has a unique fixed point in $\mathcal{LB}(D,L)$.

Note that by (71) one has

$$\|\Phi(\Sigma)(\cdot)\|_{\mathcal{Z}} \leqslant \int_{-\infty}^{\tau} \rho M e^{-\omega(\tau - s)} ds = \frac{\rho M}{\omega}. \tag{81}$$

From the choice of ρ we have that, $\|\Phi(\Sigma)(\cdot)\|_{\mathcal{Z}} \leqslant D$. Next, suppose that Σ and $\tilde{\Sigma}$ are functions satisfying (78), ζ, $\tilde{\zeta} \in \mathcal{Z}^+$ and denote $z^+(t) = \psi(t,\tau,\zeta,\Sigma)$, $\tilde{z}^+(t) = \psi(t,\tau,\tilde{\zeta},\tilde{\Sigma})$. Then,

$$z^+(t) - \tilde{z}^+(t) = e^{\mathcal{A}^+(t-\tau)}(\zeta - \tilde{\zeta}) + \int_{\tau}^{t} e^{\mathcal{A}^+(t-s)}[H(z^+, \Sigma(z^+)) - H(\tilde{z}^+, \tilde{\Sigma}(\tilde{z}^+))]ds.$$

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With some simple and standard computations we obtain

$$\begin{aligned}
&\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Z}}\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+M\int_t^\tau e^{\omega(t-s)}\|H(z^+,\Sigma(z^+))-H(\tilde{z}^+,\tilde{\Sigma}(\tilde{z}^+))\|_{\mathcal{Z}}ds\\
&\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\int_t^\tau e^{-\omega(t-s)}\left(\|\Sigma(z^+)-\tilde{\Sigma}(\tilde{z}^+)\|_{\mathcal{Z}}+\|z^+-\tilde{z}^+\|_{\mathcal{Z}}\right)ds\\
&\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\int_t^\tau e^{\omega(t-s)}\left(\|\Sigma(\tilde{z}^+)-\tilde{\Sigma}(\tilde{z}^+)\|_{\mathcal{Z}}+(1+L)\|z^+-\tilde{z}^+\|_{\mathcal{Z}}\right)ds\\
&\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\int_t^\tau e^{\omega(t-s)}\left((1+L)\|z^+-\tilde{z}^+\|_{\mathcal{Z}}+\|\!|\Sigma-\tilde{\Sigma}|\!\|\right)ds\\
&\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M(1+L)\int_t^\tau e^{\omega(t-s)}\|z^+-\tilde{z}^+\|_{\mathcal{Z}}ds+\rho M\|\!|\Sigma-\tilde{\Sigma}|\!\|\int_t^\tau e^{\omega(t-s)}ds.
\end{aligned}$$

Let $\phi(t)=e^{-\omega\,(t-\tau)}\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Z}}$. Then,

$$\phi(t)\leqslant M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\int_t^\tau e^{\omega(\tau-s)}ds\|\!|\Sigma-\tilde{\Sigma}|\!\|+M\,\rho\,(1+L)\int_t^\tau\phi(s)ds.$$

By Gronwall's inequality

$$\begin{aligned}
\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Z}}&\leqslant[M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}e^{\omega\,(t-\tau)}+\rho M\int_t^\tau e^{\omega(t-s)}ds\|\!|\Sigma-\tilde{\Sigma}|\!\|]e^{-\rho M(1+L)(t-\tau)}\\
&\leqslant[M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\omega^{-1}\|\!|\Sigma-\tilde{\Sigma}|\!\|]e^{-\rho M(1+L)(t-\tau)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Phi(\Sigma)(\zeta)-\Phi(\tilde{\Sigma})(\tilde{\zeta})\|_{\mathcal{Z}}&\leqslant M\int_{-\infty}^\tau e^{-\omega(\tau-s)}\|G(z^+,\Sigma(z^+))-G(\tilde{z}^+,\tilde{\Sigma}(\tilde{z}^+))\|_{\mathcal{Z}}ds\\
&\leqslant\rho M\int_{-\infty}^\tau e^{-\omega(\tau-s)}\left(\|\Sigma(z^+)-\tilde{\Sigma}(\tilde{z}^+)\|_{\mathcal{Z}}+\|z^+-\tilde{z}^+\|_{\mathcal{Z}}\right)ds\\
&\leqslant\rho M\int_{-\infty}^\tau e^{-\omega(\tau-s)}\left[(1+L)\|z^+-\tilde{z}^+\|_{\mathcal{Z}}+\|\!|\Sigma-\tilde{\Sigma}|\!\|\right]ds.
\end{aligned}$$

Using the estimates for $\|z^+-\tilde{z}^+\|_{\mathcal{Z}}$ we obtain

$$\|\Phi(\Sigma)(\zeta)-\Phi(\tilde{\Sigma})(\tilde{\zeta})\|_{\mathcal{Z}}\leqslant\frac{\rho M}{\omega}\left[1+\frac{\rho M(1+L)}{\omega-\rho M(1+L)}\right]\|\!|\Sigma-\tilde{\Sigma}|\!\|+\frac{\rho M^2(1+L)}{\omega-\rho M(1+L)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}.$$

Let

$$I_\Sigma=\frac{\rho M}{\omega}\left[1+\frac{\rho M(1+L)}{\omega-\rho M(1+L)}\right]\leqslant\frac{\rho M}{\omega}(1+L)\quad\text{ and } I_\zeta=\frac{\rho M^2(1+L)}{\omega-\rho M(1+L)}.$$

It follows from our choice or ρ that $I_\Sigma\leqslant\vartheta$ and $I_\zeta\leqslant L$ and

$$\|\Phi(\Sigma)(\zeta)-\Phi(\tilde{\Sigma})(\tilde{\zeta})\|_{\mathcal{Z}}\leqslant L\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\vartheta\|\!|\Sigma-\tilde{\Sigma}|\!\|. \tag{82}$$

The inequalities (81) and (82) imply that Φ is a contraction map from the class of functions that satisfy (78) into itself. Therefore, it has a unique fixed point $\Sigma^* = \Phi(\Sigma^*)$ in this class.

We now prove that $\{(z^+, \Sigma^*(z^+)) : z^+ \in \mathcal{Z}^+\}$ is invariant. Let $(z_0^+, z_0^-) \in W^u(0)$, $z_0^- = \Sigma^*(z_0^+)$. Denote by ${z^+}^*(t)$ the solution of the following initial value problem

$$\dot{z}^+ = \mathcal{A}^+ z^+ + H(z^+, \Sigma^*(z^+)), \quad z^+(0) = z_0^+.$$

This defines a curve $({z^+}^*(t), \Sigma^*({z^+}^*(t)))$, $t \in \mathbb{R}$. But the only solution of

$$\dot{z}^- = \mathcal{A}^- z^- + G({z^+}^*(t), \Sigma^*({z^+}^*(t))),$$

which remains bounded as $t \to -\infty$ is

$${z^-}^*(t) = \int_{-\infty}^t e^{\mathcal{A}^-(t-s)} G({z^+}^*(s), \Sigma^*({z^+}^*(s)) ds = \Sigma^*({z^+}^*(t)).$$

Therefore, $({z^+}^*(t), \Sigma^*({z^+}^*(t)))$ is a solution of (74) through (z_0^+, z_0^-) and the invariance is proved.□

In what follows we will prove that, if H and G satisfy (75) for all $(z^+, z^-) \in \mathcal{Z}$ with ρ suitably small and if $(z^+(t), z^-(t))$, $t \in \mathbb{R}$ is a global solution for (74) lying in $W^u(0)$, then $z^-(t) = \Sigma^*(z^+(t))$, for all $t \in \mathbb{R}$. To that end we will show that there are constants $M \geqslant 1$ and $\gamma > 0$ such that

$$\|z^-(t) - \Sigma^*(z^+(t))\|_{\mathcal{Z}} \leqslant M e^{-\gamma(t-t_0)} \|z^-(t_0) - \Sigma^*(z^+(t_0))\|_{\mathcal{Z}}, \quad t_0 \leqslant t. \tag{83}$$

Making $t_0 \to -\infty$ we obtain that $z^-(t) = \Sigma^*(z^+(t))$ for each $t \in \mathbb{R}$.

Let $\xi(t) = z^-(t) - \Sigma^*(z^+(t))$ and $y^+(s,t)$, $s \leqslant t$ be the solution of

$$\begin{aligned} &\dot{y}^+ = \mathcal{A}^+ y^+ + H(y^+, \Sigma^*(y^+)), \quad s \leqslant t \\ &y^+(t,t) = z^+(t). \end{aligned}$$

Hence,

$$\begin{aligned} &\|y^+(s,t) - z^+(s)\|_{\mathcal{Z}} \\ &\quad = \left\| \int_t^s e^{\mathcal{A}^+(s-\theta)} [H(y^+(\theta,t), \Sigma^*(y^+(\theta,t))) - H(z^+(\theta), z^-(\theta))] d\theta \right\|_{\mathcal{Z}} \\ &\quad \leqslant \rho M \int_s^t e^{\omega(s-\theta)} \left[(1+L)\|y^+(\theta,t) - z^+(\theta)\|_{\mathcal{Z}} + \|\Sigma^*(z^+(\theta)) - z^-(\theta)\|_{\mathcal{Z}}\right] d\theta \\ &\quad \leqslant \rho M \int_s^t e^{\omega(s-\theta)} \left[(1+L)\|y^+(\theta,t) - z^+(\theta)\|_{\mathcal{Z}} + \|\xi(\theta)\|_{\mathcal{Z}}\right] d\theta. \end{aligned}$$

If $z(s) = e^{-\omega s}\|y^+(s,t) - z^+(s)\|_{\mathcal{Z}}$,

$$z(s) \leqslant \rho M(1+L) \int_s^t z(\theta) d\theta + \rho M \int_s^t e^{-\omega\theta} \|\xi(\theta)\|_{\mathcal{Z}} d\theta, \quad s \leqslant t.$$

Using Gronwall's Lemma we have

$$\|y^+(s,t) - z^+(s)\|_{\mathcal{Z}} \leqslant \rho M \int_s^t e^{-(\omega-\rho M(1+L))(\theta-s)} \|\xi(\theta)\|_{\mathcal{Z}} d\theta, \quad s \leqslant t. \tag{84}$$

Let $s \leqslant t_0 \leqslant t$. Then,

$$\begin{aligned}
&\|y^+(s,t) - y^+(s,t_0)\|_{\mathcal{Z}} \\
&\quad = \left\| e^{\mathcal{A}^+(s-t_0)}[y^+(t_0,t) - z^+(t_0)] \right\|_{\mathcal{Z}} \\
&\quad + \left\| \int_{t_0}^s e^{\mathcal{A}^+(s-\theta)}[H(y^+(\theta,t), \Sigma^*(y^+(\theta,t))) - H(y^+(\theta,t_0), \Sigma^*(y^+(\theta,t_0)))] d\theta \right\|_{\mathcal{Z}} \\
&\quad \leqslant \rho M^2 e^{\omega(s-t_0)} \int_{t_0}^t e^{-(\omega-\rho M(1+L))(\theta-t_0)} \|\xi(\theta)\|_{\mathcal{Z}} d\theta \\
&\quad + \rho M \int_s^{t_0} e^{\omega(s-\theta)}(1+L) \|y^+(\theta,t) - y^+(\theta,t_0)\|_{\mathcal{Z}} d\theta
\end{aligned}$$

Using Gronwall's Lemma we have

$$\|y^+(s,t) - y^+(s,t_0)\|_{\mathcal{Z}} \leqslant \rho M^2 \int_{t_0}^t e^{-(\omega-\rho M(1+L))(\theta-s)} \|\xi(\theta)\|_{\mathcal{Z}} d\theta \tag{85}$$

In what follows we use this to estimate $\xi(t)$. Note that

$$\begin{aligned}
&\xi(t) - e^{\mathcal{A}^-(t-t_0)}\xi(t_0) = z^-(t) - \Sigma^*(z^+(t)) - e^{\mathcal{A}^-(t-t_0)}[z^-(t_0) - \Sigma^*(z^+(t_0))] \\
&\quad = \int_{t_0}^t e^{\mathcal{A}^-(t-s)} G(z^+(s), z^-(s)) ds - \Sigma^*(z^+(t)) + e^{\mathcal{A}^-(t-t_0)} \Sigma^*(z^+(t_0)) \\
&\quad = \int_{t_0}^t e^{\mathcal{A}^-(t-s)}[G(z^+(s), z^-(s)) - G(y^+(s,t), \Sigma^*(y^+(s,t)))] ds \\
&\quad - \int_{-\infty}^{t_0} e^{\mathcal{A}^-(t-s)}[G(y^+(s,t), \Sigma^*(y^+(s,t))) - G(y^+(s,t_0), \Sigma^*(y^+(s,t_0)))] ds
\end{aligned}$$

Thus, using (84) and (85),

$$
\begin{aligned}
&\|\xi(t)-e^{\mathcal{A}^-(t-t_0)}\xi(t_0)\|_{\mathcal{Z}}\\
&\quad\le\ \rho M\int_{t_0}^{t} e^{-\omega(t-s)}\left[\|z^+(s)-y^+(s,t)\|_{\mathcal{Z}}+\|z^-(s))-\Sigma^*(y^+(s,t))\|_{\mathcal{Z}}\right]ds\\
&\quad+\ \rho M(1+L)\int_{-\infty}^{t_0} e^{-\omega(t-s)}\|y^+(s,t)-y^+(s,t_0)\|_{\mathcal{Z}}ds\\
&\quad\leqslant\ \rho M\int_{t_0}^{t} e^{-\omega(t-s)}\|\xi(s)\|_{\mathcal{Z}}ds+\rho M(1+L)\int_{t_0}^{t} e^{-\omega(t-s)}\|z^+(s)-y^+(s,t)\|_{\mathcal{Z}}ds\\
&\quad+\ \rho M(1+L)\int_{-\infty}^{t_0} e^{-\omega(t-s)}\|y^+(s,t)-y^+(s,t_0)\|_{\mathcal{Z}}ds\\
&\quad\leqslant\ \rho M\int_{t_0}^{t} e^{-\omega(t-s)}\|\xi(s)\|_{\mathcal{Z}}ds\\
&\quad+\ \rho^2M^2(1+L)\int_{t_0}^{t} e^{-\omega(t-s)}\int_{s}^{t} e^{-(\omega-\rho M(1+L))(\theta-s)}\|\xi(\theta)\|_{\mathcal{Z}}d\theta ds\\
&\quad+\ \rho^2M^3(1+L)\int_{-\infty}^{t_0} e^{-\omega(t-s)}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))(\theta-s)}\|\xi(\theta)\|_{\mathcal{Z}}d\theta ds\\
&\quad\leqslant\ \rho M\int_{t_0}^{t} e^{-\omega(t-s)}\|\xi(s)\|_{\mathcal{Z}}ds\\
&\quad+\ \rho^2M^2(1+L)e^{-\omega t}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))\theta}\|\xi(\theta)\|_{\mathcal{Z}}\int_{t_0}^{\theta} e^{(2\omega-\rho M(1+L))s}ds\,d\theta\\
&\quad+\ \rho^2M^3(1+L)e^{-\omega t}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))\theta}\|\xi(\theta)\|_{\mathcal{Z}}\int_{-\infty}^{t_0} e^{(2\omega-\rho M(1+L))s}ds\,d\theta
\end{aligned}
$$

so that

$$
\begin{aligned}
\|\xi(t)-e^{\mathcal{A}^-(t-t_0)}\xi(t_0)\|_{\mathcal{Z}}&\leqslant\left[\rho M+\frac{\rho^2M^2(1+L)}{2\omega-\rho M(1+L)}\right]\int_{t_0}^{t} e^{-\omega(t-s)}\|\xi(s)\|_{\mathcal{Z}}ds\\
&+\ \frac{\rho^2M^3(1+L)}{2\omega-\rho M(1+L)}e^{-\omega(t-t_0)}\int_{t_0}^{t} e^{-(\omega-\rho M(1+L))(\theta-t_0)}\|\xi(\theta)\|_{\mathcal{Z}}d\theta
\end{aligned}
$$

and therefore

$$
\begin{aligned}
e^{\omega(t-t_0)}\|\xi(t)\|_{\mathcal{Z}}\ &\leqslant\ M\|\xi(t_0)\|_{\mathcal{Z}}+\left[\rho M+\frac{\rho^2M^2(1+L)}{2\omega-\rho M(1+L)}\right]\int_{t_0}^{t} e^{\omega(s-t_0)}\|\xi(s)\|_{\mathcal{Z}}ds\\
&+\ \frac{\rho^2M^3(1+L)}{2\omega-\rho M(1+L)}\int_{t_0}^{t} e^{-(2\omega-\rho M(1+L))(s-t_0)}e^{\omega(s-t_0)}\|\xi(s)\|_{\mathcal{Z}}ds\\
&\leqslant\ M\|\xi(t_0)\|_{\mathcal{Z}}+\left[\rho M+\frac{\rho^2M^2(1+L)(1+M)}{2\omega-\rho M(1+L)}\right]\int_{t_0}^{t} e^{\omega(s-t_0)}\|\xi(s)\|_{\mathcal{Z}}ds.
\end{aligned}
$$

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From Gronwall's inequality we have that

$$\|\xi(t)\|_{\mathcal{Z}} \leqslant M\|\xi(t_0)\|_{\mathcal{Z}}\, e^{-\gamma(t-t_0)} \tag{86}$$

where

$$\gamma = \omega - \left[\rho M + \frac{\rho^2 M^2(1+L)(1+M)}{2\omega - \rho M(1+L)}\right].$$

This proves (83) and consequently $z^-(t) = \Sigma^* z^+(t)$ for all $t \in \mathbb{R}$.□

5.2. Continuity of unstable manifolds

Let $\mathcal{Z}$ be a complex Banach space and $\mathcal{A}_\eta : D(\mathcal{A}_\eta) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of a strongly continuous semigroup $\{e^{\mathcal{A}_\eta t} : t \geqslant 0\}$ of bounded linear operators and that there are constants $\bar{M} \geqslant 1$ and $\bar{\omega} > 0$ such that

$$\|e^{\mathcal{A}_\eta t}\|_{L(\mathcal{Z})} \leqslant \bar{M} e^{\bar{\omega} t}. \tag{87}$$

Let $\sigma_\eta^+ = \{\lambda \in \sigma(\mathcal{A}_\eta) : \mathrm{Re}\lambda > 0\}$. If γ is a smooth closed simple curve in $\rho(\mathcal{A}_0) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ oriented counterclockwise enclosing σ_0^+, assume that there exists η_0 such that γ is in $\rho(\mathcal{A}_\eta)$ and encloses σ_η^+ for all $0 \leq \eta \leq \eta_0$. For $0 \leq \eta \leq \eta_0$ let

$$\mathcal{Q}_\eta = \mathcal{Q}_\eta(\sigma_\eta^+) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - \mathcal{A}_\eta)^{-1} d\lambda \tag{88}$$

and define $\mathcal{Z}_\eta^+ = \mathcal{Q}_\eta(\mathcal{Z})$ and $\mathcal{Z}_\eta^- = (I - \mathcal{Q}_\eta)(\mathcal{Z})$, $\mathcal{A}_\eta^\pm = \mathcal{A}|_{\mathcal{Z}_\eta^\pm}$. It is clear that $\mathcal{Z} = \mathcal{Z}_\eta^+ \oplus \mathcal{Z}_\eta^-$, that $\mathcal{A}_\eta^-$ generates a strongly continuous semigroup of operators and that $\mathcal{A}_\eta^+ \in L(\mathcal{Z}_\eta^+)$. Assume that there are constants $M \geqslant 1$ and $\omega > 0$, independent of η, such that

$$\begin{aligned} &\|e^{\mathcal{A}_\eta^+ t}\|_{L(\mathcal{Z}_\eta^+)} \leqslant M e^{\omega t}, \quad t \leqslant 0,\\ &\|e^{\mathcal{A}_\eta^- t}\|_{L(\mathcal{Z}_\eta^-)} \leqslant M e^{-\omega t}, \quad t \geqslant 0. \end{aligned} \tag{89}$$

Assume that $\mathcal{Q}_\eta$ converges to $\mathcal{Q}_0$ in $L(\mathcal{Z})$ and that $e^{\mathcal{A}_\eta t}z$ converges to $e^{\mathcal{A}_0 t}z$ uniformly for t in bounded intervals of $[0, \infty)$ and for z in a compact subset of $\mathcal{Z}$ and that $e^{\mathcal{A}_\eta^+ t}\mathcal{Q}_\eta z$ converges to $e^{\mathcal{A}_0^+ t}\mathcal{Q}_0 z$ uniformly for t in bounded intervals of $(-\infty, 0]$ and for z in a compact subset of $\mathcal{Z}$.

Let $h : \mathcal{Z} \to \mathcal{Z}$ be a compact function which satisfies $h(0) = 0$ and consider the initial value problem

$$\begin{aligned} &\dot{z} = \mathcal{A}_\eta z + h(z),\\ &z(0) = z_0 \in \mathcal{Z}. \end{aligned} \tag{90}$$

Consider the decomposition of a solution to (90) as $z_\eta^+(t) = \mathcal{Q}_\eta(z(t))$ and $z_\eta^-(t) = (I - \mathcal{Q}_\eta)(z(t))$. Then

$$\begin{aligned} \dot{z}_\eta^+ &= \mathcal{A}_\eta^+ z^+ + H_\eta(z_\eta^+, z_\eta^-), \\ \dot{z}_\eta^- &= \mathcal{A}_\eta^- z_\eta^- + G_\eta(z_\eta^+, z_\eta^-). \end{aligned} \tag{91}$$

where $H_\eta(z_\eta^+, z_\eta^-) = \mathcal{Q}_\eta h(z_\eta^+ + z_\eta^-)$ and $G_\eta(z_\eta^+, z_\eta^-) = (I - \mathcal{Q}_\eta)h(z_\eta^+ + z_\eta^-)$. Let $D > 0$, $L > 0$, $0 < \theta < 1$ and $\rho > 0$ satisfying (77). Assume that

$$\begin{aligned} &\|H_\eta(z_\eta^+, z_\eta^-)\|_{\mathcal{Z}} \leqslant \rho, \\ &\|G_\eta(z_\eta^+, z_\eta^-)\|_{\mathcal{Z}} \leqslant \rho, \\ &\|H_\eta(z_\eta^+, z_\eta^-) - H_\eta(\tilde{z}_\eta^+, \tilde{z}_\eta^-)\|_{\mathcal{Z}} \leqslant \rho(\|z_\eta^+ - \tilde{z}_\eta^+\|_{\mathcal{Z}} + \|z_\eta^- - \tilde{z}_\eta^-\|_{\mathcal{Z}}), \\ &\|G_\eta(z_\eta^+, z_\eta^-) - G_\eta(\tilde{z}_\eta^+, \tilde{z}_\eta^-)\|_{\mathcal{Z}} \leqslant \rho(\|z_\eta^+ - \tilde{z}_\eta^+\|_{\mathcal{Z}} + \|z_\eta^- - \tilde{z}_\eta^-\|_{\mathcal{Z}}). \end{aligned} \tag{92}$$

for all $z_\eta^+ \in \mathcal{Z}_\eta^+$ and $z_\eta^- \in \mathcal{Z}_\eta^-$.

Proposition 5.2.

Assume that the above conditions are satisfied and (77). *Then, there exist a function* $\Sigma_\eta^* : \mathcal{Z}_\eta^+ \to \mathcal{Z}_\eta^-$, *such that the unstable manifold* $W_\eta^u(0)$ *of the equilibrium solution* $(z_\eta^+, z_\eta^-) = (0,0)$ *to* (91) *is given by*

$$W_\eta^u(0) = \{w \in \mathcal{Z} : w = (\mathcal{Q}_\eta w, \Sigma_\eta^*(\mathcal{Q}_\eta w))\}$$

and for any $\zeta_\eta \in \mathcal{Z}_\eta^+$, $\eta \geqslant 0$,

$$\Sigma_\eta^*(\zeta_\eta) = \int_{-\infty}^{\tau} e^{\mathcal{A}_\eta^-(\tau - s)} G_\eta(z_\eta^+(s), \Sigma_\eta^*(z_\eta^+(s)))ds.$$

If in addition we suppose that $\rho_0 > 0$ *is such that*

$$\left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2(1+L)}{\omega(2\omega - \rho M(1+L)}\right] \leqslant \frac{1}{2}$$

for all $\rho \leqslant \rho_0$, *then for any* $r > 0$,

$$\sup_{\substack{z \in \mathcal{Z} \\ \|z\|_{\mathcal{Z}} \leqslant r}} \left\{\|\mathcal{Q}_\eta(z) - \mathcal{Q}_0(z)\|_{\mathcal{Z}} + \|\Sigma_\eta^*(\mathcal{Q}_\eta(z)) - \Sigma_0^*(\mathcal{Q}_0(z))\|_{\mathcal{Z}}\right\} \stackrel{\eta \to 0}{\longrightarrow} 0$$

Proof: Note that

$$\begin{aligned} \|\Sigma_\eta^*(\mathcal{Q}_\eta(z)) - \Sigma_0^*(\mathcal{Q}_0(z))\|_{\mathcal{Z}} \leqslant& \|\Sigma_\eta^*(\mathcal{Q}_\eta(z)) - \Sigma_\eta^*(\mathcal{Q}_\eta \mathcal{Q}_0(z))\|_{\mathcal{Z}} \\ &+ \|\Sigma_\eta^*(\mathcal{Q}_\eta \mathcal{Q}_0(z)) - \Sigma_0^*(\mathcal{Q}_0(z))\|_{\mathcal{Z}} \\ &\leqslant L\|\mathcal{Q}_\eta(z) - \mathcal{Q}_0(z)\|_{\mathcal{Z}} + \|\Sigma_\eta^*(\mathcal{Q}_\eta \mathcal{Q}_0(z)) - \Sigma_0^*(\mathcal{Q}_0(z))\|_{\mathcal{Z}} \end{aligned}$$

and from the uniform convergence of resolvents that

$$\sup_{\substack{z \in \mathcal{Z} \\ \|z\|_{\mathcal{Z}} \leqslant r}} \|\mathcal{Q}_\eta(z) - \mathcal{Q}_0(z)\|_{\mathcal{Z}} \stackrel{\eta \to 0}{\longrightarrow} 0.$$

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It remains to prove that

$$\sup_{\substack{z\in\mathcal{Z}_0^+\\ \|z\|_{\mathcal{Z}}\leqslant r}} \|\Sigma^*_\eta(\mathcal{Q}_\eta(z))-\Sigma^*_0(z)\|_{\mathcal{Z}} \stackrel{\eta\to 0}{\longrightarrow} 0.$$

Let $z\in\mathcal{Z}_0^+$ with $\|z\|_{\mathcal{Z}}\leqslant r$, then

$$\begin{aligned}
&\Sigma^*_\eta(\mathcal{Q}_\eta z)-\Sigma^*_0(\mathcal{Q}_0 z)=\int_{-\infty}^{\tau} e^{\mathcal{A}_\eta^-(\tau-s)}G_\eta(z_\eta^+,\Sigma^*_\eta(z^+))ds-\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}G_0(z_0^+,\Sigma^*_0(z_0^+))ds\\
&=\int_{-\infty}^{\tau} e^{\mathcal{A}_\eta^-(\tau-s)}(I-\mathcal{Q}_\eta)h(z_\eta^+,\Sigma^*_\eta(z^+))ds-\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)h(z_0^+,\Sigma^*_0(z_0^+))ds\\
&=\int_{-\infty}^{\tau} e^{\mathcal{A}_\eta^-(\tau-s)}(I-\mathcal{Q}_\eta)h(z_\eta^+,\Sigma^*_\eta(z^+))ds-\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)(I-\mathcal{Q}_\eta)h(z_0^+,\Sigma^*_0(z_0^+))ds\\
&+\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\mathcal{Q}_\eta h(z_0^+,\Sigma^*_0(z_0^+))ds\\
&=\int_{-\infty}^{\tau}\left[e^{\mathcal{A}_\eta^-(\tau-s)}(I-\mathcal{Q}_\eta)-e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\right](I-\mathcal{Q}_\eta)h(z_\eta^+,\Sigma^*_\eta(z^+))ds\\
&+\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\left[(I-\mathcal{Q}_\eta)h(z_\eta^+,\Sigma^*_\eta(z^+))-(I-\mathcal{Q}_\eta)h(z_0^+,\Sigma^*_0(z_0^+))\right]ds\\
&+\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\mathcal{Q}_\eta h(z_0^+,\Sigma^*_0(z_0^+))ds
\end{aligned}$$

Let

$$\begin{aligned}
I_1(\eta)&=\int_{-\infty}^{\tau}\left[e^{\mathcal{A}_\eta^-(\tau-s)}(I-\mathcal{Q}_\eta)-e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\right](I-\mathcal{Q}_\eta)h(z_\eta^+,\Sigma^*_\eta(z^+))ds\\
I_2(\eta)&=\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\left[(I-\mathcal{Q}_\eta)h(z_\eta^+,\Sigma^*_\eta(z^+))-(I-\mathcal{Q}_\eta)h(z_0^+,\Sigma^*_0(z_0^+))\right]ds\\
I_3(\eta)&=\int_{-\infty}^{\tau} e^{\mathcal{A}_0^-(\tau-s)}(I-\mathcal{Q}_0)\mathcal{Q}_\eta h(z_0^+,\Sigma^*_0(z_0^+))ds
\end{aligned}$$

Since f is a bounded map and since $(I-\mathcal{Q}_\eta)\mathcal{Q}_0$ converges to zero in the uniform operator topology we have that I_3 converges to zero uniformly in $z\in\mathcal{Z}$. To see that I_1 converges to zero uniformly for $z\in\mathcal{Z}$ with $\|z\|_{\mathcal{Z}}\leqslant r$ it is enough to note that we have convergence of the linear semigroups uniformly in compact subsets of $\mathcal{Z}$, uniform exponential decay rates and that f is a bounded and compact map. Also note that

$$\begin{aligned}\|I_2\|_{\mathcal{Z}} &\leq \rho M\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\left[\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}+\|\Sigma_\eta^*(z_\eta^+(s))-\Sigma_0^*(z_0^+(s))\|_{\mathcal{Z}}\right]ds\\ &\leq \rho M\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\left[(1+L)\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}+\|\Sigma_\eta^*(\mathcal{Q}_\eta z_0^+(s))-\Sigma_0^*(z_0^+(s))\|_{\mathcal{Z}}\right]ds\\ &\leq \rho M(1+L)\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}ds+\frac{\rho M}{\omega}\|\!|\Sigma_\eta^*-\Sigma_0^*\|\!|_r,\end{aligned}$$

where

$$\|\!|\Sigma_\eta^*-\Sigma_0^*\|\!|_r=\sup_{\substack{z_0^+\in\mathcal{Z}_0^+\\ \|z_0^+\|_{\mathcal{Z}}\leqslant r}}\|\Sigma_\eta^*(\mathcal{Q}_\eta z_0^+)-\Sigma_0^*(z_0^+)\|_{\mathcal{Z}}.$$

After these we obtain that

$$\begin{aligned}\left\|\Sigma_\eta^*(\mathcal{Q}_\eta z)-\Sigma_0^*(\mathcal{Q}_0 z)\right\|_{\mathcal{Z}}&\leqslant o(1)+\tfrac{\rho M}{\omega}\|\!|\Sigma_\eta^*-\Sigma_0^*\|\!|_r\\ &\quad+\rho M(1+L)\int_{-\infty}^{\tau} e^{-\omega(\tau-s)}\|z_\eta^+(s)-z_0^+(s)\|_{\mathcal{Z}}ds\end{aligned}\tag{93}$$

To prove that $I_2(\eta)$ converges to zero uniformly for $z\in\mathcal{Z}$ with $\|z\|_{\mathcal{Z}}\leqslant r$ we need to estimate

$$\sup_{\substack{z\in Z\\ \|z\|_{\mathcal{Z}}\leqslant r}}\sup_{t\leqslant\tau}\|z_\eta^+(t)-z_0^+(t)\|_{\mathcal{Z}}$$

To prove this we first note that

$$z_\eta^+(t)=e^{\mathcal{A}_\eta^+(t-\tau)}\mathcal{Q}_\eta z+\int_\tau^t e^{\mathcal{A}_\eta^+(t-s)}\mathcal{Q}_\eta h(z_\eta^+(s),\Sigma_\eta^*(z_\eta^+(s)))ds.$$

Proceeding as in the proof of (97) we obtain that

$$\|z_\eta^+(t)\|_{\mathcal{Z}}\leqslant Me^{(\omega-\rho M(1+L))(t-\tau)}\|\mathcal{Q}_\eta z\|_{\mathcal{Z}}.\tag{94}$$

When $\eta=0$ we may assume, without loss of generality, that $M=1$.

It remains to estimate $\|z_\eta^+(t)-z_0^+(t)\|_{\mathcal{Z}}$. Note that

$$\begin{aligned}\|z_\eta^+(t)-z_0^+(t)\|_{\mathcal{Z}}&\leqslant\left\|e^{\mathcal{A}_\eta^+(t-\tau)}\mathcal{Q}_\eta z-e^{\mathcal{A}_0^+(t-\tau)}\mathcal{Q}_0 z\right\|_{\mathcal{Z}}\\ &+\left\|\int_\tau^t\left[e^{\mathcal{A}_\eta^+(t-s)}\mathcal{Q}_\eta h(z_\eta^+(s),\Sigma_\eta^*(z_\eta^+(s)))ds-e^{\mathcal{A}_0^+(t-s)}\mathcal{Q}_0 h(z_0^+(s),\Sigma_0^*(z_0^+(s)))\right]ds\right\|_{\mathcal{Z}}\end{aligned}$$

so that

$$
\begin{aligned}
&\|z_\eta^+(t) - z_0^+(t)\|_{\mathcal{Z}} \\
&\quad\le \left\| e^{\mathcal{A}_\eta^+(t-\tau)}\mathcal{Q}_\eta z - e^{\mathcal{A}_0^+(t-\tau)}\mathcal{Q}_0 z \right\|_{\mathcal{Z}} \\
&\quad+ \left\| \int_\tau^t \left[e^{\mathcal{A}_\eta^+(t-s)}\mathcal{Q}_\eta - e^{\mathcal{A}_0^+(t-s)}\mathcal{Q}_0 \right] h(z_0^+(s), \varSigma_0^*(z_0^+(s)))ds \right\|_{\mathcal{Z}} \\
&\quad+ \left\| \int_\tau^t e^{\mathcal{A}_\eta^+(t-s)}\mathcal{Q}_\eta \left[h(z_\eta^+(s), \varSigma_\eta^*(z_\eta^+(s)))ds - h(z_0^+(s), \varSigma_0^*(z_0^+(s))) \right] ds \right\|_{\mathcal{Z}} \\
&\quad\leqslant o(1) + \rho M \int_t^\tau e^{\omega(t-s)} \left[\|z_\eta^+(s) - s_0^+(s)\|_{\mathcal{Z}} + \|\varSigma_\eta^*(z_\eta^+(s)) - \varSigma_0^*(z_0^+(s))\|_{\mathcal{Z}} \right] ds \\
&\quad\leqslant o(1) + \frac{\rho M}{\omega} |\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r + \rho M(1+L) \int_t^\tau e^{\omega(t-s)} \|z_\eta^+(s) - z_0^+(s)\|_{\mathcal{Z}} ds.
\end{aligned}
$$

Using Gronwall's inequality we obtain that

$$
\|z_\eta^+(t) - z_0^+(t)\|_{\mathcal{Z}} \leqslant \left[o(1) + \frac{\rho M}{\omega} |\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r \right] e^{(\omega - \rho M(1+L))(t-\tau)} \tag{95}
$$

Applying (95) to (93) we obtain that

$$
\begin{aligned}
\|\varSigma_\eta^*(\mathcal{Q}_\eta z) \; &- \varSigma_0^*(\mathcal{Q}_0 z)\|_{\mathcal{Z}} \leqslant o(1) + \tfrac{\rho M}{\omega} |\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r \\
&+ \rho M(1+L) \int_{-\infty}^\tau e^{-(2\omega - \rho M(1+L))(\tau - s)} \left[o(1) + \tfrac{\rho M}{\omega} |\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r \right] ds \\
&\leqslant o(1) + \left[\tfrac{\rho M}{\omega} + \tfrac{\rho^2 M^2 (1+L)}{\omega(2\omega - \rho M(1+L))} \right] |\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r
\end{aligned} \tag{96}
$$

From these it follows that

$$
|\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r \leqslant o(1) + \frac{1}{2} |\!|\!| \varSigma_\eta^* - \varSigma_0^* |\!|\!|_r
$$

and the result is proved.∎

5.3. Conclusion

In this section we use the results in Section 5.1 and Section 5.2 to obtain the existence and continuity of local unstable manifolds for the case when h is only a continuously differentiable compact function.

Let $\mathcal{Z}$ be a complex Banach space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of a strongly continuous semigroup $\{e^{\mathcal{A}t} : t \geqslant 0\}$ of bounded linear operators such that the set $\sigma^+ = \{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\lambda > 0\}$ is compact. Let $\mathcal{Q}$ be given by (70), $\mathcal{Z}^+ = \mathcal{Q}(\mathcal{Z})$, $\mathcal{Z}^- = (I - \mathcal{Q})(\mathcal{Z})$ and $\mathcal{A}^\pm = \mathcal{A}_{|_{\mathcal{Z}^\pm}}$. It is clear that $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$, that $\mathcal{A}^-$ generates a strongly continuous semigroup of operators, that $\mathcal{A}^+ \in L(\mathcal{Z}^+)$ and assume (71).

Let $h : \mathcal{Z} \to \mathcal{Z}$ be a continuously differentiable function which satisfies $h(0) = 0$ and $h'(0) = 0 \in L(\mathcal{Z})$ and consider the initial value problem (72)

Proposition 5.3. *Assume that the above conditions are satisfied. Then, there exist a neighborhood V of $z=0$ in $\mathcal{Z}$ and a function $\Sigma^*:\mathcal{Z}^+\to\mathcal{Z}^-$, such that the local unstable manifold $W^u_{\rm loc}(0)$ is given by*

$$W^u_{\rm loc}(0)=W^u(0)\cap V=\{w\in V: w=(\mathcal{Q}w,\Sigma^*(\mathcal{Q}w))\}.$$

Proof: According to Remark 5.1 and Proposition 5.1, we only need to ensure that, given $\delta>0$ there is $0<\delta'\leqslant\delta$ such that any solution $(z^+(t),z^-(t))$ on the unstable manifold which satisfies $\|z^+(t_0)\|_{\mathcal{Z}}+\|z^-(t_0)\|_{\mathcal{Z}}<\delta'$ satisfies $\|z^+(t)\|_{\mathcal{Z}}+\|z^-(t)\|_{\mathcal{Z}}<\delta$, for all $t\leqslant t_0$. It is easy to see from the fact that $z^+(t)$ is the solution of

$$\dot z^+=\mathcal{A}^+z^++H(z^+,\Sigma^*(z^+(t))),\quad t\leqslant t_0,$$

and from the variation of constants formula that

$$\|z^+(t)\|_{\mathcal{Z}}\leqslant Me^{(\omega-\rho M(1+L))(t-t_0)}\|z^+(t_0)\|_{\mathcal{Z}} \tag{97}$$

and

$$\|z^-(t)\|_{\mathcal{Z}}=\|\Sigma^*(z^+(t))\|_{\mathcal{Z}}\leqslant MLe^{(\omega-\rho M(1+L))(t-t_0)}\|z^+(t_0)\|_{\mathcal{Z}}.$$

The proof now follows easily. ∎

Let $\mathcal{Z}$ be as above and $\mathcal{A}_\eta:D(\mathcal{A}_\eta)\subset\mathcal{Z}\to\mathcal{Z}$ be the generator of a strongly continuous semigroup $\{e^{\mathcal{A}_\eta t}:t\geqslant 0\}$ of bounded linear operators satisfying (87), $\eta\geqslant 0$.

Assume also that the set $\sigma^+_\eta=\{\lambda\in\sigma(\mathcal{A}_\eta):\mathrm{Re}\lambda>0\}$ is compact and bounded uniformly in η and that (89) holds. Let $\mathcal{Q}_\eta$ be given by (88), $\mathcal{Z}^+_\eta=\mathcal{Q}_\eta(\mathcal{Z})$, $\mathcal{Z}^-_\eta=(I-\mathcal{Q}_\eta)(\mathcal{Z})$ and $\mathcal{A}^\pm_\eta=\mathcal{A}_{|_{\mathcal{Z}^\pm_\eta}}$. It is clear that $\mathcal{Z}=\mathcal{Z}^+_\eta\oplus\mathcal{Z}^-_\eta$, that $\mathcal{A}^-_\eta$ generates a strongly continuous semigroup of operators and that $\mathcal{A}^+_\eta\in L(\mathcal{Z}^+_\eta)$.

Assume that $\mathcal{Q}_\eta$ converges to $\mathcal{Q}_0$ in $L(\mathcal{Z})$ and that $e^{\mathcal{A}_\eta t}z$ converges to $e^{\mathcal{A}_0 t}z$ uniformly for t in bounded intervals of $[0,\infty)$ and for z in a compact subset of $\mathcal{Z}$ and that $e^{\mathcal{A}^+_\eta t}\mathcal{Q}_\eta z$ converges to $e^{\mathcal{A}^+_0 t}\mathcal{Q}_0 z$ uniformly for t in bounded intervals of $(-\infty,0]$ and for z in a compact subset of $\mathcal{Z}$.

Let $h:\mathcal{Z}\to\mathcal{Z}$ be a continuously differentiable compact function which satisfies $h(0)=0$ and $h'(0)=0\in L(\mathcal{Z})$. If we consider the initial value problem (90) then, as a consequence of Proposition 5.3, the following result holds

Proposition 5.4. *Assume that the above conditions are satisfied and that* (77) *holds. Then, for each $\eta\geqslant 0$, there exist a neighborhood V of $z=0$ in $\mathcal{Z}$ (independent of η) and a function $\Sigma^*_\eta:\mathcal{Z}^+_\eta\to\mathcal{Z}^-_\eta$, such that the local unstable manifold $W^u_{\eta,\rm loc}:=W^u_\eta(0)\cap V$ is given by*

$$W^u_{\eta,\rm loc}(0)=\{w\in V: w=(\mathcal{Q}_\eta w,\Sigma^*_\eta(\mathcal{Q}_\eta w))\}$$

and for any $\zeta_\eta\in\mathcal{Z}^+_\eta$, $\eta\geqslant 0$,

$$\Sigma^*_\eta(\zeta_\eta)=\int_{-\infty}^{\tau}e^{\mathcal{A}^-_\eta(\tau-s)}G_\eta(z^+_\eta(s),\Sigma^*_\eta(z^+_\eta(s)))ds.$$

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If in addition we suppose that $\rho_0 > 0$ *is such that*

$$\left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2(1+L)}{\omega(2\omega - \rho M(1+L)}\right] \leqslant \frac{1}{2}$$

for all $\rho \leqslant \rho_0$*, then for any* $r > 0$*,*

$$\sup_{z\in V} \left\{ \|\mathcal{Q}_\eta(z) - \mathcal{Q}_0(z)\|_{\mathcal{Z}} + \|\Sigma^*_\eta(\mathcal{Q}_\eta(z)) - \Sigma^*_0(\mathcal{Q}_0(z))\|_{\mathcal{Z}} \right\} \overset{\eta\to 0}{\longrightarrow} 0$$

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