CADERNOS DE MATEMÁTICA **06**, 173–179 October (2005)
ARTIGO NÚMERO SMA#231

$\mathcal{K}$-bi-Lipschitz equivalence of real function-germs

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In this paper we prove that the set of equivalence classes of germs of real polynomials of degree less than or equal to k, with respect to the $\mathcal{K}$-bi-Lipschitz equivalence, is finite. October, 2005 ICMC-USP

Mathematical Subject Classification 2000: 32S15, 32S05
Key words and phrases: $\mathcal{K}$-bi-Lipschitz equivalence, finiteness theorem, metric theory.

1. INTRODUCTION

Finiteness theorems of different kinds appear in modern development of Singularity Theory. When one considers a classification problem, it is important to know if the problem is "tame" or not. In other words, how difficult the problem is and if there is any hope to develop a complete classification. For the problem of topological classification of polynomial function-germs, a finiteness result was conjectured by R. Thom ([12]) and proved by Fukuda ([4]). He proved that the number of equivalence classes of polynomial function-germs of degree less than or equal to some k, with respect to the topological equivalence,

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is finite. Note that R. Thom also discovered that this finiteness results does not take place for polynomial map-germs $P : \mathbb{R}^3 \to \mathbb{R}^3$ ([12]). Finiteness theorems for polynomial map-germs, in real and in complex case, with respect to a topological equivalence were a subject of investigation of various authors (see for example ([11]), ([9]), ([2]), ([3])) and many interesting results were obtained in this direction.

Mostowski ([8]) and Parusinski ([10]) proved that the set of equivalence classes of semi-algebraic sets with a complexity bounded from below by some k is finite.

A finiteness result does not hold for polynomial function-germs with respect to bi-Lipschitz equivalence. Henry and Parusinski ([5]) showed that this problem is not tame - has "moduli".

Here we consider the problem of $\mathcal{K}$-bi-Lipschitz classification of polynomial function-germs ($\mathcal{K}$-equivalence is the contact equivalence defined by Mather ([6])). We show that this problem is still tame. The main idea of the proof is the following. First, we consider Lipschitz functions "of the same contact". Namely, f and g are of the same contact if $\frac{f}{g}$ is positive and bounded away from zero and infinity. We show that two functions of the same contact are $\mathcal{K}$-bi-Lipschitz equivalent. The next step is related to the geometry of contact equivalence. Recall that two function-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are $\mathcal{C}^\infty$-contact equivalent if there exists a $\mathcal{C}^\infty$-diffeomorphism in the product space $(\mathbb{R}^n \times \mathbb{R}, 0)$ which leaves $\mathbb{R}^n$ invariant and maps the graph(f) to the graph(g). This definition is due to Mather ([6]) for map-germs $f, g : \mathbb{R}^n \to \mathbb{R}^p$. Montaldi extended this notion introducing a purely geometrical definition of contact: two pairs of submanifolds of $\mathbb{R}^n$ have the same contact type if there is a diffeomorphism of $\mathbb{R}^n$ taking one pair to the other, and relating this with the $\mathcal{K}$-equivalence of convenient map-germs. This geometrical interpretation also exists for a topological version of the $\mathcal{K}$-equivalence (cf. [1]).

In this paper, we give a definition of the Montaldi's construction for the bi-Lipschitz case and show that the existence of a bi-Lipschitz analogue to Montaldi's construction, for two germs f and g, implies that $f \circ h$ and g are of the same contact, for some bi-Lipschitz map-germ h. Finally, the finiteness result follows from the Montaldi's construction and Mostowski-Parusinski's ([10]) theorem on Lipschitz stratifications.

2. BASIC DEFINITIONS AND RESULTS

Definition 2.1. Two function-germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ are called *$\mathcal{K}$-bi-Lipschitz equivalent* (or *contact bi-Lipschitz equivalent*) if there exist two germs of bi-Lipschitz homeomorphisms $h : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(id, f)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ h \downarrow & & H \downarrow & & h \downarrow \\ (\mathbb{R}^n, 0) & \xrightarrow{(id, g)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

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where $id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map and $\pi_n : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is the canonical projection.

The function-germs f and g are called *$\mathcal{C}$-bi-Lipschitz equivalent* if $h = id$.

In other words, two function-germs f and g are $\mathcal{K}$-bi-Lipschitz equivalent if there exists a germ of a bi-Lipschitz map $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(x, y)$ can be written in the form $H(x, y) = (h(x), \tilde{H}(x, y))$, where h is a bi-Lipschitz map-germ, $\tilde{H}(x, 0) = 0$ and H maps the germ of the graph (f) onto the graph (g).

Recall that graph (f) is the set defined as follows:

$$\text{graph}\,(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = f(x)\}.$$

Definition 2.2. Two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are called *of the same contact at a point* $x_0 \in \mathbb{R}^n$ if there exist a neighbourhood U_{x_0} of x_0 in $\mathbb{R}^n$ and two positive numbers c_1 and c_2 such that, for all $x \in U_{x_0}$, we have

$$c_1 f(x) \leq g(x) \leq c_2 f(x).$$

We use the notation: $f \approx g$.

Remark 2. 1. It is clear that if two function-germs f and g are of the same contact then the germs of their zero-sets are equal.

The main results of the paper are the following:

Theorem 2.1. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two germs of Lipschitz functions. Then, f and g are $\mathcal{C}$-bi-Lipschitz equivalent if and only if one of the two conditions is true:*

i)$f \approx g$,
ii)$f \approx -g$.

Theorem 2.2. *Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of n variables with degree less than or equal to k. Then the set of equivalence classes of the germs at 0 of the polynomials in $\mathcal{P}_k(\mathbb{R}^n)$ with respect to the $\mathcal{K}$-bi-Lipschitz equivalence is finite.*

3. FUNCTIONS OF THE SAME CONTACT

Proof of Theorem 2.1:

Suppose that the germs of Lipschitz functions f and g are $\mathcal{C}$-bi-Lipschitz equivalent. Let $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ be the germ of a bi-Lipschitz homeomorphism satisfying the conditions of Definition 2.1. Let V_+ be the subset of $\mathbb{R}^n \times \mathbb{R}$ of the points (x, y) where $y > 0$, and V_- be the subset of $\mathbb{R}^n \times \mathbb{R}$ where $y < 0$. Clearly, we have one of the following options:

1) $H(V_+) = V_+$ and $H(V_-) = V_-$, or
2) $H(V_+) = V_-$ and $H(V_-) = V_+$.

Let us consider the first possibility. In this case, the functions f and g have the same sign on each connected component of the set $f(x) \neq 0$. Moreover,

$$|g(x)| = \| (x,0)-(x,g(x)) \| = \| H(x,0)-H(x,f(x)) \| \leq c_2 \ \| (x,0)-(x,f(x)) \| = c_2|f(x)|,$$

where c_2 is a positive real number. Using the same argument we can show

$$c_1|f(x)| \leq |g(x)|, \qquad c_1 > 0.$$

Hence, $f \approx g$.

Let us consider the second possibility. Let $\xi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ be a map-germ defined as follows:

$$\xi(x, y) = (x, -y).$$

Applying the same arguments to a map $\xi \circ H$, we will conclude that $f \approx -g$.

Reciprocally, suppose that $f \approx g$. Let us construct a map-germ

$$H : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$$

in the following may:

$$H(x,y) = \begin{cases} (x,0) & \text{se } \ y = 0, \\ (x, \dfrac{g(x)}{f(x)} y) & \text{se } \ 0 \leq |y| \leq |f(x)|, \\ (x, y - f(x) + g(x)) & \text{se } \ 0 \leq |f(x)| \leq |y|, \\ (x,y) & \text{otherwise.} \end{cases} \tag{1}$$

The map $H(x, y) = (x, \tilde{H}(x, y))$ above defined is bi-Lipschitz. In fact, H is injective because, for any fixed $x^\star$, we can show that $\tilde{H}(x^\star, y)$ is a continuous and monotone function. Moreover, H is Lipschitz if $0 \leq |f(x)| \leq |y|$. Let us show that H is Lipschitz if $0 \leq |y| \leq |f(x)|$. It is enough to show that all the partial derivatives $\dfrac{\partial \tilde{H}}{\partial x_i}$ are bounded in this domain, for all $i = 1, \ldots, n$. We have,

$$\frac{\partial \tilde{H}}{\partial x_i} = \frac{(\frac{\partial g}{\partial x_i} f(x) - \frac{\partial f}{\partial x_i} g(x))\, y}{(f(x))^2} = \frac{\partial g}{\partial x_i} \frac{y}{f(x)} - \frac{\partial f}{\partial x_i} \frac{g(x)}{f(x)} \frac{y}{f(x)}.$$

Since $|y| \leq |f(x)|$, then $\dfrac{y}{f(x)}$ is bounded. The expression $\dfrac{g(x)}{f(x)}$ is bounded since $f \approx g$. Moreover, $\dfrac{\partial g}{\partial x_i}$ and $\dfrac{\partial f}{\partial x_i}$ are bounded because f and g are Lipschitz functions.

Since H^{-1} can be constructed in the same form as (1), we conclude that H^{-1} is also Lipschitz and, thus, H is a bi-Lipschitz map. ■

4. MONTALDI'S CONSTRUCTION

DEFINITION 4.1. Two germs of Lipschitz functions are called *$\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent* (or *contact equivalent in the sense of Montaldi*) if there exists a germ of a bi-Lipschitz map $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and $M(\text{graph}\,(f)) = \text{graph}\,(g)$. A map M is called a *Montaldi map*.

THEOREM 4.1. *Two germs of Lipschitz functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent if and only if they are $\mathcal{K}$-bi-Lipschitz equivalent.*

Proof. It is clear that a $\mathcal{K}$-bi-Lipschitz equivalence implies a $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence. To prove the converse, let f and g be $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent, then

$$M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\} \quad \text{and} \quad M(\text{graph}\,(f)) = \text{graph}\,(g).$$

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be defined by $h(x) = \pi_n(M(x, f(x)))$.

Claim 1. h is a bi-Lipschitz map-germ.

Proof of Claim 1.

Since g is a Lipschitz function, the projection $\pi_{n_{|\text{graph}\,(g)}}$ is a bi-Lipschitz map. By the same argument, a map $x \mapsto (x, f(x))$ is bi-Lipschitz. A map M is bi-Lipschitz by Definition 4.1 and Claim follows.

Claim 2. One of the following assertions is true:

i) $f \approx g \circ h$,
ii) $f \approx -(g \circ h)$.

Proof of Claim 2.

Since M is a bi-Lipschitz map, it follows that there exist two positive numbers c_1 and c_2, such that

$$c_1\,|f(x)| \leq \| M(x, f(x)) - M(x, 0) \| \leq c_2|f(x)|.$$

By the above construction

$$\| M(x, f(x)) - M(x, 0) \| = \| (h(x), g(h(x))) - M(x, 0) \| \geq |g(h(x))|.$$

Therefore, $|g(h(x))| \leq c_2|f(x)|$.

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Using the same procedure for the map M^{-1}, we obtain that

$$c_1|f(x)| \leq |g(h(x))|.$$

Since M is a homeomorphism and $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$, the same argument as in Theorem 2.1 implies that, for all $x \in \mathbb{R}^n$,

$$\operatorname{sign} f(x) = \operatorname{sign} g(h(x)),$$

or, for all $x \in \mathbb{R}^n$,

$$\operatorname{sign} f(x) = -\operatorname{sign} g(h(x)).$$

Hence, Claim 2 is proved.

End of the proof of Theorem 4.1.
Using Claim 2 we obtain, by Theorem 2.1, that f and $g \circ h$ are $\mathcal{C}$-bi-Lipschitz equivalent. By Claim 1, f and g are $\mathcal{K}$-bi-Lipschitz equivalent. ■

5. FINITENESS THEOREM

This section is devoted to a proof of a finiteness theorem (Theorem 2.2).

Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of n variables with degree less than or equal to k. Let $f \in \mathcal{P}_k(\mathbb{R}^n)$. Let X_f be the germ of the set $\mathbb{R}^n \times \{0\} \cup \operatorname{graph}(f)$. By Parusinski's Theorem ([10]), there exists a finite semialgebraic stratification $\{\mathcal{S}_i\}_{i=1}^{p(k)}$ of $\mathcal{P}_k(\mathbb{R}^n)$ satisfying the following conditions:

i) Each $\mathcal{S}_i$ is a semialgebraic connected manifold.

ii) For each S_i and for each $f \in \mathcal{S}_i$, there exists a neighbourhood U_f such that $U_f \cap \mathcal{S}_i$ is arcwise connected and for any $\tilde{f} \in U_f \cap \mathcal{S}_i$ and for every arc $\gamma : [0, 1] \to U_f \cap \mathcal{S}_i$ connecting f and $\tilde{f}$ (that is, $\gamma(0) = f$ and $\gamma(1) = \tilde{f}$), there exists a family of germs of bi-Lipschitz maps $\rho_t : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, where $\rho_0 = id_{\mathbb{R}^n \times \mathbb{R}}$ and $\rho_t(X_f) = X_{\gamma(t)}$.

The condition ii) implies that, for small t, ρ_t is a Montaldi map.

Since $\mathcal{S}_i$ is a connected set, then, for any two polynomials $f_1, f_2 \in \mathcal{S}_i$, there exists a Montaldi map $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $M(X_{f_1}) = X_{f_2}$.

Note, that any equivalence class of the $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence in $\mathcal{P}_k(\mathbb{R}^n)$ is a finite union of some strata of the stratification $\{S_i\}_{i=1}^{p(k)}$. By Theorem 4.1, the set of equivalence classes with respect to the $\mathcal{K}$-equivalence is also finite. ■

Acknowledgements.
The first named author was supported by CNPq grant # 300985/93-2, the second named author was supported by FAPESP grant # 01/14577-0 and the fourth named author was supported by FAPESP grant # 03/03107-9 and by CNPq grants # 146427/1999-8 and # 300066/88-0.

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