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Path formulation for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems

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M. Manoel and I. Stewart ([10]) classify $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems up to codimension 3, using the classical techniques of singularity theory of Golubistky and Schaeffer [8]. In this paper we classify these same problems using an alternative form: the path formulation (Theorem 6.1). One of the advantages of this method is that the calculates to obtain the normal forms are easier. Furthermore, in our classification we observe the presence of only one modal parameter in the generic core. It differs from the classical classification where the core has 2 modal parameters (see [8] or [10]). We finish this work comparing our classification to the one obtained in [10]. October, 2005 ICMC-USP

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1. INTRODUCTION

The symmetry group of a rectangle, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, appears in many bifurcation problems: in PDE problems with rectangular domains, like in the buckling of a rectangular plate [14], or in some Hopf-Hopf mode interaction [8]. More generally, when there is an interaction of two modes in symmetric systems where the normalizer of the isotropy subgroups of the

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two modes have Weyl group $\mathbb{Z}_2$ and are "independent". We refer to the introduction of the paper of Manoel and Stewart [10] for a substantive list of references with such equivariant problems. The classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems of corank two up to topological codimension two appears in [10] using the classical techniques of singularity theory of Golubistky and Schaeffer [8]. In this paper we classify these problems using an alternative form: the path formulation. The basic idea of path formulation was suggested by Golubitsky and Schaeffer in [7] where they related bifurcation problems in one state variable (without symmetry) with a path through a miniversal unfolding of the cuspoid x^{m+1}. This idea has been extended and applied to more complex situations (cf. [4], [12], [1]). The main idea is to consider the bifurcation problem $g(x, \lambda) = 0$ as an unfolding (perturbation) with parameter λ of the core $g_0(x) = g(x, 0)$. If g_0 is of finite codimension, with respect to some equivalence of maps relevant to our problem (cf. Section 2), we have a miniversal unfolding G_0 (with parameter $\alpha \in \mathbb{R}^k$ where k is the codimension of g_0) of g_0 such that g is equivalent (in the previous sense) to a pull-back $\bar{\alpha}^* G_0$ where $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$ is the path associated with g_0 (given the core g_0). Then we can compare paths and determine their miniversal unfoldings (cf. Section 3 thereafter for more precision). The path formulation differentiates between the singular behavior attributable to the core and to the paths. Moreover we can discuss efficiently multiparameter situations and forced symmetry breaking (cf. [4]).

The main goal of this paper is to use the path formulation as an alternative method to obtain the classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems: Theorem 6.1. To do this, we first classify the normal forms to the cores: Theorem 4.1 and Theorem 4.2. We observe the presence of only one modal parameter in the generic core. It differs from the classical classification where the core has two modal parameters (cf. [8], [10]). As a byproduct of our approach we get a set-up that can be easily generalized to multiple bifurcation parameters, even with some additional complex internal structure (cf. Section 3). Moreover we also get new information on the structure of vector fields liftable over the projection onto the parameter space of a miniversal unfolding of singularities in the equivariant case (cf. Sections 5 and 6).

We assume that the reader has some familiarity with Mather's approach to singularity theory as generalized in Damon [3]. In Section 2 we recall the basic ingredients and results we need to classify the bifurcation problems and their perturbations modulo changes of coordinates following the now classical approach of Golubitsky and Schaeffer [8]. In Section 3 we introduce the basic results of the alternative approach of path formulation theory. Then we apply those ideas to classify $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems in corank 2 using this formulation. To that effect we discuss the cores up to codimension 3 in Section 4, followed, in Section 5, by the main technical ingredient: the module of the vector fields liftable over the projection onto the parameter space of their miniversal unfoldings. We can then achieve the classification in Section 6 before finishing this work with some comments and comparing our classification to the one obtained in [10].

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2. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-EQUIVARIANT BIFURCATION PROBLEMS AND THEIR EQUIVALENCE

We consider the usual action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on the plane given by $(\epsilon, \delta) \cdot (x, y) = (\epsilon x, \delta y)$ with $\epsilon^2 = \delta^2 = 1$. The action on the bifurcation and other parameters is always trivial. We denote (x, y) by z and by $GL_l(\mathbb{R})$ the set of real invertible $l{\times}l$-matrices with identity denoted by I_l. Derivatives are denoted by subscripts, for example, f_x is $\frac{\partial f}{\partial x}$, and the superscript ^{o} denotes the value of any function at the origin, for example $f^o = f(0)$, $f_x^o = f_x(0)$. For any variable, or set of variables, $a \in \mathbb{R}^n$, we denote by $\mathcal{E}_a$ the ring of germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ and by $\mathcal{M}_a$ its maximal ideal. For $b \in \mathbb{R}^m$, we denote by $\vec{\mathcal{E}}_{a,b}$ the $\mathcal{E}_a$-module of smooth germs $g : (\mathbb{R}^n, 0) \to \mathbb{R}^m$ and by $\vec{\mathcal{M}}_{a,b}$ the $\mathcal{E}_a$-submodule of germs vanishing at the origin. When b is clear from the context, we denote $\vec{\mathcal{E}}_{a,b}$ by $\vec{\mathcal{E}}_a$ and $\vec{\mathcal{M}}_{a,b}$ by $\vec{\mathcal{M}}_a$. If R is some ring, we denote by $\langle g_1, \ldots, g_k \rangle_R$ the R-module generated by $\{g_i\}_{i=1}^k$.

2.1. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant map germs

The following is well-known (for instance [7]). The ring $\mathcal{E}_z^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ of smooth $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-invariant germs is generated by $u = x^2$ and $v = y^2$. The module $\vec{\mathcal{E}}_z^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ of smooth $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant maps $f : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ is generated over $\mathcal{E}_z^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ by $(x, 0)$ and $(0, y)$.

Hence, any bifurcation germ $g : (\mathbb{R}^2 \times \mathbb{R}^l, 0) \to \mathbb{R}^2$ (with parameters $\lambda \in (\mathbb{R}^l, 0)$) has the form

$$g(z, \lambda) = (p(u, v, \lambda)x, q(u, v, \lambda)y),$$

with $p, q \in \mathcal{E}_{(z,\lambda)}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$. We use the notation $g = [p, q]$.

The zero-set of $g = 0$ is a stratified set composed of four pieces, each depending on the isotropy of the solutions. With maximal isotropy we have $\mathcal{S}_0$ of solution $z = 0$. With each copy of Z_2, we have $(x, 0, \lambda) \in \mathcal{S}_x$ of equation $p(x^2, 0, \lambda) = 0$ and $(0, y, \lambda) \in \mathcal{S}_y$ of equation $q(0, y^2, \lambda) = 0$. Finally, with the trivial isotropy, we have $(z, \lambda) \in \mathcal{S}_z$ of equation $p(z, \lambda) = q(z, \lambda) = 0$.

2.2. $\mathcal{K}_\lambda^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$-equivalence

The classification of bifurcation problems is via contact equivalences preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetry. Let $f, g \in \vec{\mathcal{E}}_{(z,\lambda)}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$, f is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-*equivalent* to g if there exists a map $S : (\mathbb{R}^{2+l}, 0) \to GL_2(\mathbb{R})$ and a diffeomorphism $(z, \lambda) \mapsto (Z(z, \lambda), L(\lambda))$ such that

$$f(z, \lambda) = S(z, \lambda)\, g(Z(z, \lambda), L(\lambda))$$

where $(Z^o, L^o) = (0, 0)$, $\det(L_\lambda^o) > 0$, $Z \in \vec{\mathcal{E}}_{(z,\lambda)}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$, $L \in \vec{\mathcal{E}}_\lambda$ and

$$S((\epsilon, \delta) \cdot (x, y, \lambda)) \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix} S(x, y, \lambda). \tag{1}$$

In addition, we require S^o and Z_z^o to be (diagonal) matrices with positive entries. The set of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivalences (S, Z, L) has a semidirect product group structure by composition, and is denoted by $\mathcal{K}_\lambda^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$.

2.3. Unfolding theory

The perturbations of any $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ are described by k-parameter unfoldings of g which are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant map-germs $G : (\mathbb{R}^{2+l+k}, 0) \to \mathbb{R}^2$ such that $G(z, \lambda, 0) = g(z, \lambda)$. We extend in a straightforward manner the previous definitions to the k-parametrized version for unfoldings. The unfolding and finite determinacy theorems follow the general theory of Damon [3] as our group is a geometric subgroup. Let $G(z, \lambda, \alpha)$ and $F(z, \lambda, \beta)$ be unfoldings of a germ $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ with $\alpha \in (\mathbb{R}^k, 0)$ and $\beta \in (\mathbb{R}^s, 0)$. We say that F *maps into* G, or F *factors through* G, if there exist a β-unfolding (S, Z, L) of the identity in $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$ and a map $A : (\mathbb{R}^s, 0) \to (\mathbb{R}^k, 0)$ such that

$$F(z, \lambda, \beta) = S(z, \lambda, \beta)\, G(Z(z, \lambda, \beta), L(\lambda, \beta), A(\beta)).$$

The unfolding G is called *versal* if any unfolding F of g maps into G and *miniversal* if it has the minimal number of parameters necessary to be versal. That number is given by the $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-codimension $c(g)$ of g that is calculated as the real dimension of the extended normal space.
The *extended normal space* of $g = [p, q] \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ is defined by

$$\mathcal{N}_e\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}(g) = \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)} / \mathcal{T}_e\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}(g),$$

where

$$T_e\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}(g) = \langle [p, 0], [vq, 0], [0, up], [0, q], [up_u, uq_u], [vp_v, vq_v] \rangle_{\mathcal{E}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}} + \langle [p_\lambda, q_\lambda] \rangle_{\mathcal{E}_\lambda},$$

is the *extended tangent space* of g (cf. [7]) which is a module over the system of rings $\{\mathcal{E}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}, \mathcal{E}_\lambda\}$.

Moreover, if $\{d_i\}_{i=1}^{c(g)}$ is a basis for $\mathcal{N}_e\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}(g)$, then a miniversal unfolding of g is

$$G(z, \lambda, \alpha) = g(z, \lambda) + \sum_{i=1}^{c(g)} \alpha_i d_i(z, \lambda).$$

2.4. Determinacy and recognition theories

Let $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$. Another consequence of finite codimension is that g is *finitely determined*, that is, there exists an integer $k \geq 1$ such that every germ with the same k^{th}-jet as g is $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-equivalent to g. The *recognition problem* seeks conditions under which a germ $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ is $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-equivalent to a given normal form. To solve a particular recognition problem means to explicitly characterize a $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-equivalence class in terms of a finite number of polynomial equalities and inequalities to be satisfied by the Taylor coefficients of the elements of that class. For this we need further ideas and results. A subspace $M \subset \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ is *intrinsic* if it contains the $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-orbit of all its elements. If $V \subset \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$

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then the *intrinsic part* of V, denoted by $\mathrm{itr}\, V$, is the largest intrinsic subspace of $\vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ contained in V. In [10] it is shown that the the ideals generated by powers of $u = x^2$, $v = y^2$ or λ are intrinsic. Moreover a $\mathcal{E}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$-module $M = [\mathcal{I}, \mathcal{J}] = \{[p, q] \mid p \in \mathcal{I},\, q \in \mathcal{J}\}$ is intrinsic if and only if $\mathcal{I}, \mathcal{J}$ are intrinsic ideals such that $v\mathcal{J} \subset \mathcal{I}$ and $u\mathcal{I} \subset \mathcal{J}$. Let $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$, the "perturbation term" $w \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ is of *higher order* with respect to g if $f + w$ is $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-equivalent to g for every f in the $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$-orbit to g. By definition such a perturbation cannot enter into a solution of the recognition problem for g. We denote by $\mathcal{P}(g)$ the set of all higher order terms of g. Then, for each $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ the set $\mathcal{P}(g)$ is an intrinsic $\mathcal{E}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$-submodule of $\vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ (cf. [8]). To evaluate $\mathcal{P}(g)$ we introduce a subgroup $\mathcal{UK}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$ of $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}$ of *unipotent equivalences* represented by equivalences whose linear part is unipotent. The unipotent tangent space of $g = [p, q] \in \vec{\mathcal{E}}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}$ is given by

$$\mathcal{TUK}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}(g) = \langle [p,0],[0,q],[up_u, uq_u],[vp_v, vq_v]\rangle_{\mathcal{M}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}} + \langle [vq,0],[0,up]\rangle_{\mathcal{E}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{(z,\lambda)}} + \langle [p_\lambda, q_\lambda]\rangle_{\mathcal{M}^2_\lambda}.$$

Following the proof of Theorem 1.17 ([6], p.108) we know that $\mathrm{itr}\ \mathcal{TUK}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}_{\lambda}(g) \subset \mathcal{P}(g)$.

We finalize this section with two theorems that present the normal forms that classify the $Z_2 \oplus Z_2$-equivariant bifurcation problems using classical techniques of singularity theory.

THEOREM 2.1. *([8], Th. XIV 4.2) Let $g = [p, q] \in \vec{\mathcal{E}}_{x,y,\lambda}(Z_2 \oplus Z_2)$ a bifurcation problem with $\lambda \in (\mathbb{R}, 0)$. If*

$$p^o_u, \quad q^o_v, \quad p^o_\lambda, \quad q^o_\lambda, \quad p^o_u q^o_v - p^o_v q^o_u, \quad p^o_u q^o_\lambda - p^o_\lambda q^o_u, \quad q^o_v p^o_\lambda - p^o_v q^o_\lambda$$

are all nonzero at the origin, then g is $Z_2 \oplus Z_2$-equivalent to

$$h_1 = [\varepsilon_1 u + m\, v + \varepsilon_2 \lambda, \eta u + \varepsilon_3 v + \varepsilon_4 \lambda],$$

$\epsilon_1 = sign(p^o_u)$, $\epsilon_2 = sign(p^o_\lambda)$, $\epsilon_3 = sign(q^o_v)$, $\epsilon_4 = sign(q^o_\lambda)$ and modal parameters

$$m = \left|\frac{q^o_\lambda}{q^o_v p^o_\lambda}\right| p^o_v, \quad \eta = \left|\frac{p^o_\lambda}{p^o_u q^o_\lambda}\right| q^o_u.$$

Moreover, the moduli μ and η satisfy the conditions

$$m \neq \epsilon_2\epsilon_3\epsilon_4, \quad \eta \neq \epsilon_1\epsilon_2\epsilon_4, \quad m\eta \neq \epsilon_1\epsilon_3.$$

Also, it is shown in [7] that the normal form h_1 has $c(h_1) = 3$ and miniversal unfolding

$$H_1(x, y, \lambda, \tilde{m}, \tilde{\eta}, \sigma) = [\varepsilon_1 u + \tilde{m} v + \varepsilon_2 \lambda, \tilde{\eta} + u + \varepsilon_3 v + \varepsilon_4(\lambda - \sigma)]$$

where $(\tilde{m}, \tilde{\eta}, \sigma)$ varies on a neighborhood of $(m, \eta, 0)$.

The table with the other normal forms that complete the classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems is given by the theorem to follow:

THEOREM 2.2. *([10], Th. 4.1) If a germ $g = [p,q] \in \vec{\mathcal{E}}_{x,y,\lambda}(Z_2 \oplus Z_2)$, $\lambda \in (\mathbb{R}, 0)$, satisfies the recognition conditions in following table, then g is $Z_2 \oplus Z_2$-equivalent to h_j, $j = 2, \ldots, 8$.*

Normal Forms

$$\begin{aligned}
h_2 &= [\varepsilon_1 u + \varepsilon_4 v + \varepsilon_2 \lambda + \varepsilon_5 u^2, \varepsilon_1\varepsilon_3\varepsilon_4 u + \varepsilon_3 v + \kappa\lambda] \\
h_3 &= [\varepsilon_1 u + \mu v + \varepsilon_2 \lambda, \varepsilon_1\varepsilon_2\varepsilon_4 u + \varepsilon_3 v + \varepsilon_4\lambda + \varepsilon_5\lambda^2] \\
h_4 &= [\varepsilon_1 u + \varepsilon_2\varepsilon_3\varepsilon_4 v + \varepsilon_2\lambda, \eta u + \varepsilon_3 v + \varepsilon_4\lambda + \varepsilon_5\lambda^2] \\
h_5 &= [\varepsilon_1 u^2 + \mu v + \varepsilon_2\lambda, \varepsilon_5 u + \varepsilon_3 v + \varepsilon_4\lambda] \\
h_6 &= [\varepsilon_1 u + \varepsilon_5 v + \varepsilon_2\lambda, \eta u + \varepsilon_3 v^2 + \varepsilon_4\lambda] \\
h_7 &= [\varepsilon_1 u + \varepsilon_5 v + \varepsilon_2\lambda^2, \eta u + \varepsilon_3 v + \varepsilon_4\lambda] \\
h_8 &= [\varepsilon_1 u + \mu v + \varepsilon_2\lambda, \varepsilon_5 u + \varepsilon_3 v + \varepsilon_4\lambda^2]
\end{aligned}$$

Normal Form	*Recognition Conditions*	ε_1	ε_2	ε_3	ε_4	ε_5	*Modal Parameter*	*Unfolding Terms*
h_2	$p_u^o q_v^o - p_v^o q_u^o = 0$ $p_u^o, q_v^o, p_v^o, p_\lambda^o, q_\lambda^o, \rho_2$, $p_u^o q_\lambda^o - p_\lambda^o q_u^o \neq 0$	p_u^o	p_λ^o	q_v^o	p_v^o	$\rho_2 q_v^o$	$\kappa = \frac{p_v^o}{p_\lambda^o q_v^o}\, q_\lambda^o$	$[u,0]$ $[0,\lambda]$ $[0,1]$
h_3	$p_\lambda^o q_u^o - p_u^o q_\lambda^o = 0$ $p_u^o, q_v^o, p_\lambda^o, q_\lambda^o, \rho_3$, $p_u^o q_v^o - p_v^o q_u^o \neq 0$	p_u^o	p_λ^o	q_v^o	q_λ^o	$\rho_3 p_\lambda^o$	$\mu = \frac{q_\lambda^o}{p_\lambda^o q_v^o}\, p_v^o$	$[v,0]$ $[0,u]$ $[0,1]$
h_4	$p_v^o q_\lambda^o - p_\lambda^o q_v^o = 0$ $p_u^o, q_v^o, p_\lambda^o, q_\lambda^o, \rho_4$, $p_u^o q_v^o - p_v^o q_u^o \neq 0$	p_u^o	p_λ^o	q_v^o	q_λ^o	$\rho_4 p_\lambda^o$	$\eta = \frac{p_\lambda^o}{p_u^o q_\lambda^o}\, q_u^o$	$[v,0]$ $[0,u]$ $[0,1]$
h_5	$p_u^o = 0$ $p_\lambda^o, q_v^o, q_\lambda^o, p_{uu}^o$, $p_v^o q_\lambda^o - p_\lambda^o q_v^o \neq 0$	p_{uu}^o	p_λ^o	q_v^o	q_λ^o	q_u^o	$\mu = \frac{q_\lambda^o}{p_\lambda^o q_v^o}\, p_v^o$	$[v,0]$ $[u,0]$ $[0,1]$
h_6	$q_v^o = 0$ $p_u^o, p_\lambda^o, q_\lambda^o, p_v^o, q_u^o, q_{vv}^o$, $p_u^o q_\lambda^o - p_\lambda^o q_u^o \neq 0$	p_u^o	p_λ^o	q_{vv}^o	q_λ^o	p_v^o	$\eta = \frac{p_\lambda^o}{p_u^o q_\lambda^o}\, q_u^o$	$[0,v]$ $[0,u]$ $[1,0]$
h_7	$p_\lambda^o = 0$ $p_u^o, p_v^o, q_v^o, q_\lambda^o, p_{\lambda\lambda}^o$, $p_u^o q_v^o - p_v^o q_u^o \neq 0$	p_u^o	$p_{\lambda\lambda}^o$	q_v^o	q_λ^o	p_v^o	$\eta = \frac{p_v^o}{p_u^o q_v^o}\, q_u^o$	$[\lambda,0]$ $[0,u]$ $[1,0]$

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	$q^o_\lambda = 0$							$[0, \lambda]$
h_8	$p^o_u, p^o_\lambda, q^o_u, q^o_v, q^o_{\lambda\lambda}$	p^o_u	p^o_λ	q^o_v	$q^o_{\lambda\lambda}$,	q^o_u	$\mu = \frac{q^o_u}{p^o_u q^o_v} p^o_v$	$[v, 0]$
	$p^o_u q^o_v - p^o_v q^o_u \neq 0$							$[1, 0]$

$$\begin{aligned}
\rho_2 = &\quad q^o_{uv} p^o_u (p^o_v)^2 - q^o_{vv} (p^o_u)^2 p^o_v - q^o_{uu} (p^o_v)^3 - p^o_{uv} q^o_v p^o_u p^o_v + p^o_{vv} q^o_v (p^o_u)^2 + p^o_{uu} q^o_v (p^o_v)^2 \\
\rho_3 = &\quad q^o_{uu} (p^o_\lambda)^3 - p^o_{uu} (p^o_\lambda)^2 q^o_\lambda - q^o_{u\lambda} (p^o_\lambda)^2 p^o_u + q^o_{\lambda\lambda} p^o_\lambda (p^o_u)^2 + p^o_{u\lambda} p^o_\lambda q^o_\lambda p^o_u - p^o_{\lambda\lambda} (p^o_u)^2 q^o_\lambda \\
\rho_4 = &\quad q^o_{\lambda\lambda} p^o_\lambda (q^o_v)^2 - p^o_{vv} p^o_\lambda (q^o_\lambda)^2 - q^o_{v\lambda} p^o_\lambda q^o_\lambda q^o_v - p^o_{vv} (q^o_\lambda)^3 - p^o_{\lambda\lambda} (q^o_v)^2 q^o_\lambda + p^o_{v\lambda} (q^o_\lambda)^2 q^o_v
\end{aligned}$$

3. PATH FORMULATION

This section is divided into two parts. First we explain how we associate a path to each bifurcation problem. Second we recall the equivalence on paths that corresponds to the contact equivalence for the bifurcation diagrams.

3.1. Cores and paths

Let $g \in \vec{\mathcal{E}}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}_{(z,\lambda)}$. The germ $g_0 \in \vec{\mathcal{E}}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}_{z}$ defined by $g_0(z) = g(z, 0)$ is called the *core* of g. When $\lambda = 0$, the group $\mathcal{K}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}_{\lambda}$ simplifies to $\mathcal{K}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$, the classical group of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivalences without parameters. A germ g is of *finite core* if g_0 is of finite $\mathcal{K}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$-codimension, say k. Consider now g as an unfolding of g_0 with l parameters. From the $\mathcal{K}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$-theory of unfoldings, if G_0 is a miniversal unfolding of g_0 (of codimension k, say), then g factors through G_0. That is, there exists changes of coordinates S, Z such that

$$g(z, \lambda) = S(z, \lambda)\, G_0(Z(z, \lambda), \bar{\alpha}(\lambda)), \tag{2}$$

where $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$. We say that $\bar{\alpha}$ is the *path* associated to g. Thus $\bar{\alpha}$ induces a new bifurcation problem defined by

$$\bar{\alpha}^\star G_0(z, \lambda) = G_0(z, \bar{\alpha}(\lambda)).$$

The *space of paths* will be denoted by $\vec{\mathcal{P}}_\lambda = \{\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)\}$.

From (2), g and the pull-back $\bar{\alpha}^\star G_0$ are $\mathcal{K}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$-equivalent (with (S, Z, I_l) providing the equivalence).

3.2. Path equivalence and its tangent spaces

We can now define an equivalence between two paths with the same core. That is, we say that $\bar{\alpha}, \bar{\beta} : (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$ are *path equivalent* if

$$\bar{\alpha}(\lambda) = H(\lambda, \bar{\beta}(L(\lambda))) \tag{3}$$

where $L : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ is an orientation-preserving diffeomorphism and $H : (\mathbb{R}^{l+k}, 0) \to (\mathbb{R}^k, 0)$ is a λ-parametrized family of local orientation-preserving diffeomorphism on $(\mathbb{R}^k, 0)$

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that lifts to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant diffeomorphism on $G_0^{-1}(0)$. More precisely, there exists a λ-family of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant diffeomorphisms $\Phi : (\mathbb{R}^{2+k+l}, 0) \to (\mathbb{R}^{2+k}, 0)$ preserving $G_0^{-1}(0)$ such that $H \circ \pi_{G_0} = \pi_{G_0} \circ \Phi$ on $G_0^{-1}(0)$ where $\pi_{G_0} : G_0^{-1}(0) \to (\mathbb{R}^k, 0)$ is the restriction to $G_0^{-1}(0)$ of the natural projection $(\mathbb{R}^{2+k}, 0) \to (\mathbb{R}^k, 0)$. In this case we shall see in Section 5 that this definition is equivalent to the definition in [12].

For a fixed core g_0, the group of path equivalences, denoted by $\mathcal{K}_{\Delta^{G_0}}$, is a geometric subgroup which acts on the space of paths, hence the general theory of [3] applies. Note that we cannot in general simplify H in (3) to a λ-parametrized matrix like with the usual contact-equivalence. An explicit description of the diffeomorphisms H is in general very hard, if not impossible. But the tangent space of $\bar{\alpha}$ can be determined explicitly. Let $\mathrm{Derlog}^\star(\Delta^{G_0})$ be the module of *liftable vector fields* ξ satisfying

$$S(z,\alpha)\, G_0(z,\alpha) = (\mathrm{d}G_0)_z(z,\alpha)\, Z(z,\alpha) + (\mathrm{d}G_0)_\alpha(z,\alpha)\, \xi(\alpha), \qquad \alpha \in \mathbb{R}^k. \tag{4}$$

The tangent space at a path $\bar{\alpha}$ is the $\mathcal{E}_\lambda$-module of $\vec{\mathcal{P}}_\lambda$ given by

$$\mathcal{T}_e\mathcal{K}_{\Delta^{G_0}}(\bar{\alpha}) = \langle \bar{\alpha}_\lambda \rangle_{\mathcal{E}_\lambda} + \bar{\alpha}^\star \mathrm{Derlog}^\star(\Delta^{G_0}). \tag{5}$$

Also we define the codimension of path $\bar{\alpha}$ by $\mathrm{cod}_{\mathcal{K}_{\Delta^{G_0}}}(\bar{\alpha}) = \dim_{\mathbb{R}} \vec{\mathcal{P}}_\lambda / \mathcal{T}_e\mathcal{K}_{\Delta^{G_0}}(\bar{\alpha})$.

In Section 5 we discuss the geometric interpretation of the path equivalence showing that H is also exactly the discriminant preserving contact equivalences, hence the notation with Δ^{G_0}. In the mean time one has the following result about path and contact equivalences.

Theorem 3.1.

1. Let $g \in \vec{\mathcal{E}}_{(z,\lambda)}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$ be a finite $\mathcal{K}_\lambda^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$-codimension. If g has a core of finite $\mathcal{K}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$-codimension, then there exists a path $\bar{\alpha}$ such that g is $\mathcal{K}_\lambda^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$-equivalent to $\bar{\alpha}^\star G_0$, where G_0 is a miniversal unfolding of the core g_0 of g.

2. The $\mathcal{K}_{\Delta^{G_0}}$-codimension of $\bar{\alpha}$ is finite if and only if the $\mathcal{K}_\lambda^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$-codimension of $\bar{\alpha}^\star G_0$ is finite, as correspondents normal spaces are isomorphics.

3. Let $\bar{\alpha}$, $\bar{\beta}$ be two paths in $\vec{\mathcal{P}}_\lambda$. Then, $\bar{\alpha}$ is $\mathcal{K}_{\Delta^{G_0}}$-equivalent to $\bar{\beta}$ if and only if $\bar{\alpha}^\star G_0$ is $\mathcal{K}_\lambda^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$-equivalent to $\bar{\beta}^\star G_0$ for finite codimension problems.

The proof may be adapted to the correspondent theorem in [5].

4. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-EQUIVARIANT CORES

In this section we discuss the cores and their universal unfolding of low codimension. The Theorem 4.1 and Theorem 4.2 classify the normal forms for the cores. We observe the presence of only one modal parameter in the generic core h_1^c. It differs from the classical classification where the core has two modal parameters (see the core of h_1 in the Theorem 2.1). To simplify the description of their recognition problem we define the following quantities: $\epsilon_1 = \mathrm{sign}(p_u^o)$, $\epsilon_2 = \mathrm{sign}(p_v^o)$, $\epsilon_3 = \mathrm{sign}(q_u^o)$, $\epsilon_4 = \mathrm{sign}(q_v^o)$, $\epsilon_5 = \mathrm{sign}(p_{uu}^o)$ and $\epsilon_6 = \mathrm{sign}(q_{vv}^o)$.

The generic core and its miniversal unfolding is given in the following result whose proof follows from a simple re-scaling and a straightforward calculation of the unipotent tangent space.

THEOREM 4.1. *Let $g \in \vec{\mathcal{E}}_z^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ be given by $g(x,y) = (p(u,v)x, q(u,v)y) = [p,q]$. If ϵ_i, $1 \le i \le 4$, and $p_u^o q_v^o - p_v^o q_u^o$ are all nonzero at the origin, then g is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivalent to*

$$h_1^c = [\epsilon_1 u + mv, \epsilon_3 u + \epsilon_4 v]$$

with the modal parameter $m = \left|\frac{q_u^o}{p_u^o q_v^o}\right| p_v^o$ $(m \neq \epsilon_1\epsilon_3\epsilon_4)$. A miniversal unfolding of the core h_1^c is

$$H_1^c = [\epsilon_1 u + (m + \alpha_3)v + \alpha_1, \epsilon_3 u + \epsilon_4 v + \alpha_2].$$

Proof. Consider the change of coordinates given by $S(z)g(Z(z)) = [\bar{p}, \bar{q}]$, $Z = [a,b] \in \vec{\mathcal{E}}_z^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$, $S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$ satisfying (1) (without λ). It follows

$$\bar{p}_u^0 = p_u^0 a_0^3 s_1^0, \quad \bar{p}_v^0 = p_v^0 a_0 b_0^2 s_1^0, \quad \bar{q}_u^0 = q_u^0 a_0^2 b_0 s_4^0, \quad \bar{q}_v^0 = q_v^0 b_0^3 s_4^0,$$

where $a_0 > 0$, $b_0 > 0$, $s_1^0 > 0$, $s_4^0 > 0$ are the coefficients in the Taylor expansion of the functions a, b, s_1 and s_4, respectively.

Normalizing these coefficients we get the formula to h_1^c and the respective parameters ϵ_i's and m.

Now, to find the miniversal unfolding, first we calculate that the quadratic terms are in the unipotent tangent space of h_1^c to justify that they can be ignored. Then we use the normal space to find the unfolding terms. If $m \neq \epsilon_1\epsilon_3\epsilon_4$, then $d_1 = [1,0]$, $d_2 = [0,1]$ and $d_3 = [v,0]$ are generators of the extended normal space of h_1^c, thus H_1^c is a versal unfolding of h_1^c. ∎

The generic core is of smooth codimension 3 but topological codimension 2. The remaining cores of codimension 3 are given by degenerating the previous conditions in Theorem 4.1. There are 5 of them but they can be grouped in 3 types by interchanging x and y (as well then as p and q). Define

$$\rho_2 = q_{uv}^o p_u^o (p_v^o)^2 - q_{vv}^o (p_u^o)^2 p_v^o - q_{uu}^o (p_v^o)^3 - p_{uv}^o q_v^o p_u^o p_v^o + p_{vv}^o q_v^o (p_u^o)^2 + p_{uu}^o q_v^o (p_v^o)^2.$$

THEOREM 4.2. *Let $g \in \vec{\mathcal{E}}_z^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ be given by $g = [p,q]$.*

(a) If $p_u^o q_v^o - p_v^o q_u^o = 0$ with ϵ_i, $1 \le i \le 4$, and ρ_2 all nonzero, then g is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivalent to

$$h_2^c = [\epsilon_1 u + \epsilon_2 v + \hat{\epsilon}_5 u^2, \epsilon_1\epsilon_2\epsilon_4 u + \epsilon_4 v],$$

with miniversal unfolding $H_2^c = [(\epsilon_1 + \alpha_3)u + \epsilon_2 v + \hat{\epsilon}_5 u^2 + \alpha_1, \epsilon_1\epsilon_2\epsilon_4 u + \epsilon_4 v + \alpha_2]$ *and* $\hat{\epsilon}_5 = sign(q_v^o \rho_2)$.

(b) If $p_u^o = 0$ *with* ϵ_i, $2 \leq i \leq 5$, *all nonzero, then* g *is* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$*-equivalent to*

$$h_5^c = [\epsilon_5 u^2 + \epsilon_2 v, \epsilon_3 u + \epsilon_4 v]$$

with miniversal unfolding $H_5^c = [\epsilon_5 u^2 + \alpha_3 u + \epsilon_2 v + \alpha_1, \epsilon_3 u + \epsilon_4 v + \alpha_2]$.

When $q_v^0 = 0$, *interchanging* x *and* y, *we obtain* $h_6^c = [\epsilon_1 u + \epsilon_2 v, \epsilon_3 u + \epsilon_6 v^2]$ *of miniversal unfolding* $H_6^c = [\epsilon_1 u + \epsilon_2 v + \alpha_1, \epsilon_3 u + \epsilon_6 v^2 + \alpha_3 v + \alpha_2]$.

(c) If $q_u^o = 0$ *with* $\epsilon_1, \epsilon_3, \epsilon_4$ *all nonzero, then* g *is* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$*-equivalent to*

$$h_9^c = [\epsilon_1 u + \epsilon_2 v, \epsilon_4 v]$$

with miniversal unfolding $H_9^c = [\epsilon_1 u + \epsilon_2 v + \alpha_1, \alpha_3 u + \epsilon_4 v + \alpha_2]$.

When $p_v^0 = 0$, *interchanging* x *and* y, *we obtain* $h_{10}^c = [\epsilon_1 u, \epsilon_3 u + \epsilon_4 v]$ *with miniversal* $H_{10}^c = [\epsilon_1 u + \alpha_3 v + \alpha_1, \epsilon_3 u + \epsilon_4 v + \alpha_2]$.

Proof. We proceed as before with more complicated calculations. This is done with details in [2]. ■

5. DERLOGS

In this section we calculate here the Derlogs of the cores of Theorems 4.1 and 4.2. First we discuss an important geometric notion linked with the liftable vector fields.

5.1. Discriminants and Derlogs

The *discriminant* Δ^{G_0} of G_0 is the local bifurcation set of G_0, that is, the set of $\alpha \in (\mathbb{R}^k, 0)$ where G_0 is singular. Here the liftability condition on H can be replaced by preserving the discriminant Δ^{G_0} of G_0 in the sense that $H(\lambda, \Delta^{G_0}) \subset \Delta^{G_0}$ for all $\lambda \in (\mathbb{R}^l, 0)$. This is a weaker condition because any liftable vector field must be tangent to the discriminant. In the non equivariant case it is well known that both notion coincide (cf. [9]). Here both modules are also equal and also free. The proof is similar to the one in [5].

The discriminants of the miniversal unfoldings of the cores are formed of the following local bifurcation varieties:

(a) $\mathcal{P}_x, \mathcal{P}_y$ of equations $q(0,0,\alpha) = 0$, resp. $p(0,0,\alpha) = 0$, representing the bifurcations of the branches $\mathcal{S}_x$, resp. $\mathcal{S}_y$, from the trivial branch,

(b) $\mathcal{P}_{y,z}$ of equation $p(0,v,\alpha) = q(0,v,\alpha) = 0$ representing the bifurcation of the branches $\mathcal{S}_z$ from $\mathcal{S}_y$, $\mathcal{B}_y$ of equation $p(0,v,\alpha) = p_u(0,0,\alpha) = 0$ representing turning points on the $\mathcal{S}_y$-branches. In a similar fashion we define $\mathcal{B}_x$ and $\mathcal{P}_{x,z}$.

(c) $\mathcal{B}_z$ of equation $p = q = p_u q_v - p_v q_u = 0$ representing fold points in $\mathcal{S}_z$.

To fully exploit methods from algebraic geometry we need to complexify our situation. Nothing will be lost in finite codimension because we can work with germs equivalent to polynomials and we take care to preserve the real and complex algebras. Our results are valid for real germs viewed as real slices of the holomorphic objects. Let $G_0^{\mathbb{C}}$ be the complexification of the miniversal unfolding G_0 (chosen as a polynomial from finite determinacy). The discriminant $\Delta^{G_0^{\mathbb{C}}}$ of $G_0^{\mathbb{C}}$ is the set of singular values of the projection $\pi_{G_0^{\mathbb{C}}} : (G_0^{\mathbb{C}})^{-1}(0) \to \mathbb{C}^k$. The real slice of $\Delta^{G_0^{\mathbb{C}}}$ defines the discriminant Δ^{G_0} instead of the equivalent formula for G_0. We actually define $\mathrm{Derlog}^\star(\Delta^{G_0})$ as the submodule of the real vector fields of the module $\mathrm{Derlog}^\star(\Delta^{G_0^{\mathbb{C}}})$ of liftable vector fields. Let $I(\Delta^{G_0^{\mathbb{C}}})$ be the ideal of germs vanishing on $\Delta^{G_0^{\mathbb{C}}}$. The module of vector fields tangent to $\Delta^{G_0^{\mathbb{C}}}$, called $\mathrm{Derlog}(\Delta^{G_0^{\mathbb{C}}})$, is given by

$$\mathrm{Derlog}(\Delta^{G_0^{\mathbb{C}}}) = \{\xi : (\mathbb{C}^k, 0) \to \mathbb{C}^k \mid \xi \cdot g_\alpha \in I(\Delta^{G_0^{\mathbb{C}}}),\ \forall g \in I(\Delta^{G_0^{\mathbb{C}}})\}.$$

5.2. Liftable vector fields

The discriminant of H_1^c is

$$\Delta^{H_1^c} = \alpha_1\alpha_2(\epsilon_1\alpha_1 - \epsilon_3\alpha_2)(\alpha_1 - \epsilon_4\alpha_2(m + \alpha_3)).$$

To find the vectors of $\mathrm{Derlog}^*(\Delta^{H_1^c})$ we solve the expression (4). The coordinates of $\mathrm{Derlog}^*(\Delta^{H_1^c})$, with respect to the basis $[1,0], [0,1]$ and $[v,0]$ of unfolding terms, are

$$\xi_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix},\ \xi_2 = \begin{pmatrix} -\epsilon_1\alpha_1^2 \\ -\epsilon_3\alpha_2^2 \\ (-\epsilon_1\alpha_1 + \epsilon_3\alpha_2)(m + \alpha_3) \end{pmatrix},\ \xi_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha_1 - \epsilon_4\alpha_2(m + \alpha_3) \end{pmatrix}.$$

Note that the determinant of the matrix formed by these vectors is equal to the discriminant $\Delta^{H_1^c}$ which shows that $\mathrm{Derlog}^*(\Delta^{H_1^c})$ is free and equal to $\mathrm{Derlog}(\Delta^{H_1^c})$ using Saito's criterion (cf. [13]).

The discriminant of H_{10}^c is

$$\Delta^{H_{10}^c} = \alpha_1\alpha_2(\epsilon_1\alpha_1 - \epsilon_3\alpha_2)(\alpha_1 - \epsilon_4\alpha_2\alpha_3).$$

The generators of $\mathrm{Derlog}^*(\Delta^{H_{10}^c})$ are

$$\xi_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix},\ \xi_2 = \begin{pmatrix} \epsilon_3\alpha_1\alpha_2 - \epsilon_1\alpha_1^2 \\ 0 \\ -\epsilon_1\alpha_1\alpha_3 + \epsilon_3\alpha_2\alpha_3 \end{pmatrix},\ \xi_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha_1 - \epsilon_4\alpha_2\alpha_3 \end{pmatrix},$$

and the module is still free.

The discriminant of H_2^c is $\Delta^{H_2^c} = \alpha_1\alpha_2(\epsilon_4\alpha_1 - \epsilon_2\alpha_2)(\epsilon_1\epsilon_2\epsilon_4\alpha_2\alpha_3^3 + 3\epsilon_2\epsilon_4\alpha_2\alpha_3^2 + 3\epsilon_1\epsilon_2\epsilon_4\alpha_2\alpha_3 + \epsilon_2\epsilon_4\alpha_2 - \alpha_1\alpha_3^2 - 2\epsilon_1\alpha_1\alpha_3 - \alpha_1 - \epsilon_5\alpha_2^2\alpha_3^2 - 2\epsilon_1\epsilon_5\alpha_2^2\alpha_3 - \epsilon_5\alpha_2^2 - 4\epsilon_1\epsilon_2\epsilon_4\epsilon_5\alpha_1\alpha_2\alpha_3 + 4\epsilon_5\alpha_1^2 - 4\epsilon_2\epsilon_4\epsilon_5\alpha_1\alpha_2 + 4\alpha_1\alpha_2^2)(\epsilon_2\epsilon_4\epsilon_5\alpha_3^2 - 4\epsilon_2\epsilon_4\alpha_1 + 4\alpha_2)$. The coordinates of $\mathrm{Derlog}^*(\Delta^{H_2^c})$, with respect to the basis $[1,0], [0,1]$ and $[u,0]$ of unfolding terms, are

$$\xi_1 = \begin{pmatrix} 4\alpha_1(\epsilon_4\alpha_1 - \epsilon_2\alpha_2) + A\alpha_3 \\ 2\alpha_2(\epsilon_4\alpha_1 - \epsilon_2\alpha_2) + B\alpha_3 \\ 2\epsilon_1(\epsilon_4\alpha_1 - \epsilon_2\alpha_2) + C\alpha_3 \end{pmatrix},$$

with $A = \epsilon_1\epsilon_4\epsilon_5\alpha_1 + \epsilon_4\epsilon_5\alpha_1\alpha_3 - 2\epsilon_1\epsilon_2\alpha_1\alpha_2$, $B = \epsilon_1\epsilon_4\epsilon_5\alpha_2 + \epsilon_4\epsilon_5\alpha_2\alpha_3 - 2\epsilon_1\epsilon_2\alpha_2^2$, $C = -3\epsilon_2\alpha_2 + 4\epsilon_4\alpha_1 - \epsilon_1\epsilon_2\alpha_2\alpha_3$,

$$\xi_2 = \begin{pmatrix} 2\epsilon_1\alpha_1(\epsilon_4\epsilon_5\alpha_1 - \epsilon_2\epsilon_5\alpha_2 - 2\epsilon_2\alpha_1\alpha_2 + 2\epsilon_4\alpha_2^2) + D\alpha_3 \\ 2\epsilon_1\alpha_2(\epsilon_4\epsilon_5\alpha_1 - \epsilon_2\epsilon_5\alpha_2 - 2\epsilon_2\alpha_1\alpha_2 + 2\epsilon_4\alpha_2^2) + E\alpha_3 \\ -6\epsilon_2\alpha_1\alpha_2 + 4\epsilon_4\alpha_1^2 + 2\epsilon_4\alpha_2^2 + F\alpha_3 \end{pmatrix},$$

with $D = -5\epsilon_2\epsilon_5\alpha_1\alpha_2 + 4\epsilon_4\epsilon_5\alpha_1^2 - 3\epsilon_1\epsilon_2\epsilon_5\alpha_1\alpha_2\alpha_3 + 4\epsilon_4\alpha_1\alpha_2^2$, $E = -5\epsilon_2\epsilon_5\alpha_2^2 + 4\epsilon_4\epsilon_5\alpha_1\alpha_2 - 3\epsilon_1\epsilon_2\epsilon_5\alpha_2^2\alpha_3 + 4\epsilon_4\alpha_2^3$, $F = -4\epsilon_1\epsilon_2\alpha_1\alpha_2 - 2\epsilon_2\epsilon_5\alpha_2\alpha_3 + 4\epsilon_1\epsilon_4\alpha_2^2 + \epsilon_1\epsilon_4\epsilon_5\alpha_1 + \epsilon_4\epsilon_5\alpha_1\alpha_3 - \epsilon_1\epsilon_2\epsilon_5\alpha_2 - \epsilon_1\epsilon_2\epsilon_5\alpha_2\alpha_3^2 + 2\epsilon_4\alpha_2^2\alpha_3$, and

$$\xi_3 = \begin{pmatrix} 2\epsilon_1\alpha_1(3\alpha_2 - 2\epsilon_2\epsilon_4\alpha_1 - 4\epsilon_2\epsilon_4\epsilon_5\alpha_2^2) + G\alpha_3 \\ 2\epsilon_1\alpha_2(3\alpha_2 - 2\epsilon_2\epsilon_4\alpha_1 - 4\epsilon_2\epsilon_4\epsilon_5\alpha_2^2) + H\alpha_3 \\ -2\epsilon_2\epsilon_5(-\epsilon_2\epsilon_5\alpha_2 + \epsilon_4\epsilon_5\alpha_1 - 2\epsilon_2\alpha_1\alpha_2 + 2\epsilon_4\alpha_2^2) + I\alpha_3 \end{pmatrix}$$

with $G = -\epsilon_2\epsilon_4\epsilon_5\alpha_1 - \epsilon_1\epsilon_2\epsilon_4\epsilon_5\alpha_1\alpha_3 + 8\alpha_1\alpha_2$, $H = -\epsilon_2\epsilon_4\epsilon_5\alpha_2 - \epsilon_1\epsilon_2\epsilon_4\epsilon_5\alpha_2\alpha_3 + 8\alpha_2^2$ and $I = -4\epsilon_1\epsilon_2\epsilon_4\alpha_1 + 5\epsilon_1\alpha_2 + 3\alpha_2\alpha_3 - 4\epsilon_1\epsilon_2\epsilon_4\epsilon_5\alpha_2^2$. The module is free.

The discriminant of H_5^c is $\Delta^{H_5^c} = \alpha_1\alpha_2(\epsilon_2\alpha_1 - \epsilon_4\alpha_2)(-\epsilon_1 - \epsilon_1\alpha_3^2 - 4\epsilon_2\epsilon_4\alpha_2 + 2\epsilon_2\epsilon_3\epsilon_4\epsilon_1\alpha_3 + 4\alpha_1)(\epsilon_3\alpha_2\alpha_3^3 - \alpha_1\alpha_3^2 - \epsilon_1\alpha_2^2\alpha_3^2 - 4\epsilon_3\epsilon_1\alpha_1\alpha_2\alpha_3 + 4\epsilon_1\alpha_1^2 + 4\alpha_1\alpha_2^2)$. The generators of $\mathrm{Derlog}^*(\Delta^{H_5^c})$ are listed below

$$\xi_1 = \begin{pmatrix} -\epsilon_2\epsilon_3\epsilon_1\alpha_1\alpha_3 + \epsilon_4\epsilon_1\alpha_1\alpha_3^2 - 2\epsilon_3\epsilon_4\alpha_1\alpha_2\alpha_3 + 4\epsilon_4\alpha_1^2 - 2\epsilon_2\alpha_1\alpha_2 \\ -\epsilon_2\epsilon_3\epsilon_1\alpha_2\alpha_3 + \epsilon_4\epsilon_1\alpha_2\alpha_3^2 - 2\epsilon_3\epsilon_4\alpha_2^2\alpha_3 + 2\epsilon_4\alpha_1\alpha_2 \\ -2\epsilon_2\epsilon_3\alpha_1 + 4\epsilon_4\alpha_1\alpha_3 - \epsilon_3\epsilon_4\alpha_2\alpha_3^2 - \epsilon_2\alpha_2\alpha_3 \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} -4\epsilon_3\epsilon_4\alpha_1^2\alpha_2 - 2\epsilon_2\epsilon_3\epsilon_1\alpha_1^2 + A_1\alpha_3 \\ -4\epsilon_3\epsilon_4\alpha_1\alpha_2^2 - 2\epsilon_2\epsilon_3\epsilon_1\alpha_1\alpha_2 + A_2\alpha_3 \\ 4\epsilon_4\alpha_1^2 - 2\epsilon_2\alpha_1\alpha_2 + A_3\alpha_3 \end{pmatrix},$$

with $A_1 = \epsilon_2\epsilon_1\alpha_1\alpha_2 - 3\epsilon_3\epsilon_4\epsilon_1\alpha_1\alpha_2\alpha_3 + 4\epsilon_4\alpha_1\alpha_2^2 + 4\epsilon_4\epsilon_1\alpha_1^2$, $A_2 = \epsilon_2\epsilon_1\alpha_2^2 - 3\epsilon_3\epsilon_4\epsilon_1\alpha_2^2\alpha_3 + 4\epsilon_4\alpha_2^3 + 4\epsilon_4\epsilon_1\alpha_1\alpha_2$, $A_3 = -\epsilon_2\epsilon_3\epsilon_1\alpha_1 + \epsilon_4\epsilon_1\alpha_1\alpha_3 - 4\epsilon_3\epsilon_4\alpha_1\alpha_2 + \epsilon_2\epsilon_1\alpha_2\alpha_3 - \epsilon_3\epsilon_4\epsilon_1\alpha_2\alpha_3^2 + 2\epsilon_4\alpha_2^2\alpha_3$, and

$$\xi_3 = \begin{pmatrix} -4\epsilon_3\alpha_1^2 - \epsilon_3\epsilon_1\alpha_1\alpha_3^2 + \epsilon_2\epsilon_4\epsilon_1\alpha_1\alpha_3 + 8\alpha_1\alpha_2\alpha_3 - 8\epsilon_3\epsilon_1\alpha_1\alpha_2^2 - 2\epsilon_2\epsilon_3\epsilon_4\alpha_1\alpha_2 \\ -4\epsilon_3\alpha_1\alpha_2 - \epsilon_3\epsilon_1\alpha_2\alpha_3^2 + \epsilon_2\epsilon_4\epsilon_1\alpha_2\alpha_3 + 8\alpha_2^2\alpha_3 - 8\epsilon_3\epsilon_1\alpha_2^3 - 2\epsilon_2\epsilon_3\epsilon_4\alpha_2^2 \\ -4\epsilon_3\alpha_1\alpha_3 + 2\epsilon_2\epsilon_4\alpha_1 + 3\alpha_2\alpha_3^2 + 4\epsilon_1\alpha_1\alpha_2 - 4\epsilon_3\epsilon_1\alpha_2^2\alpha_3 - \epsilon_2\epsilon_3\epsilon_4\alpha_2\alpha_3 \end{pmatrix}.$$

5.3. Modal spaces

We observe that there is a modal parameter in our classification of the generic core instead of two modal parameters in the core of bifurcation problem in Theorem 2.1. Because of the equivalence between contact and path equivalence we expect the values of the modal space to be left *pointwise* invariant by path equivalence. This means that the liftable vector fields must vanish along the modal space. In general, this is a mechanism by which $\mathrm{Derlog}^*(\Delta)$ may be strictly smaller than $\mathrm{Derlog}(\Delta)$ because vector fields in the latest only need to be tangent to the discriminant, so we need to select those that actually vanish on the modal space. Such examples occur for the corank two representation of the dihedral group $\mathbb{D}_4$ (cf. [5]). Surprisingly, in the present case, all vector fields tangent to the discriminant also vanish on the modal space. In all our cases the modal space is the same, given by $m = \alpha_3$, $\alpha_1 = \alpha_2 = 0$. By inspection, each generator of the Derlogs vanishes on that space confirming that both Derlogs are the same.

6. CLASSIFICATION OF $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-EQUIVARIANT BIFURCATION PROBLEMS VIA PATH FORMULATION

The classification of the paths up to topological codimension 2 is at follows. First we define a few quantities (when they exist): $\delta_1 = \text{sign}(p^o_\lambda)$, $\delta_2 = \text{sign}(q^o_\lambda)$, $\delta_3 = \text{sign}(p^o_{\lambda\lambda})$, $\delta_4 = \text{sign}(q^o_{\lambda\lambda})$, $\delta_{51} = \text{sign}(p^o_\lambda \rho_3)$ and $\delta_{52} = \text{sign}(p^o_\lambda \rho_4)$ with

$$\rho_3 = q^o_{uu}(p^o_\lambda)^3 - p^o_{uu}(p^o_\lambda)^2 q^o_\lambda - q^o_{u\lambda}(p^o_\lambda)^2 p^o_u + q^o_{\lambda\lambda} p^o_\lambda (p^o_u)^2 + p^o_{u\lambda} p^o_\lambda q^o_\lambda p^o_u - p^o_{\lambda\lambda}(p^o_u)^2 q^o_\lambda,$$

and

$$\rho_4 = q^o_{\lambda\lambda} p^o_\lambda (q^o_v)^2 - p^o_{vv} p^o_\lambda (q^o_\lambda)^2 - q^o_{v\lambda} p^o_\lambda q^o_\lambda q^-_v (p^o_{vv} q^o_\lambda)^3 - p^o_{\lambda\lambda}(q^o_v)^2 q^o_\lambda + p^o_{v\lambda}(q^o_\lambda)^2 q^o_v.$$

Finally, the modal parameters are $\chi = \left|\frac{p^o_u}{q^o_u p^o_\lambda}\right| q^o_\lambda$, $\kappa = \left|\frac{p^o_v}{q^o_v p^o_\lambda}\right| q^o_\lambda$.

Second we define the following paths α_i's (with their miniversal unfoldings A_i's of unfolding parameters ν_j, $j = 1, 2, 3$):

- $\bar{\alpha}_1(\lambda) = (\delta_1\lambda, \chi\lambda, 0)$ with $A_1(\lambda, \nu) = (\delta_1\lambda, (\chi+\nu_2)\lambda+\nu_1, \nu_3)$ of topological codimension 1,
- $\bar{\alpha}_2(\lambda) = (\delta_1\lambda, \kappa\lambda, 0)$ with $A_2(\lambda, \nu) = (\delta_1\lambda, (\kappa+\nu_2)\lambda+\nu_1, \nu_3)$ of topological codimension 1,
- $\bar{\alpha}_3(\lambda) = (\delta_3\lambda^2, \delta_2\lambda, 0)$ with $A_3(\lambda, \nu) = (\delta_3\lambda^2 + \nu_1 + \nu_2\lambda, \delta_2\lambda, \nu_3)$ of topological codimension 2,
- $\bar{\alpha}_4(\lambda) = (\delta_1\lambda, \delta_4\lambda^2, 0)$ with $A_4(\lambda, \nu) = (\delta_1\lambda, \delta_4\lambda^2 + \nu_1 + \nu_2\lambda, \nu_3)$ of topological codimension 2 and
- $\bar{\alpha}_{5i}(\lambda) = (\delta_1\lambda, \delta_2\lambda + \delta_{5i}\lambda^2, 0)$, $i = 1, 2$, with $A_{5i}(\lambda, \nu) = (\delta_1\lambda, (\delta_2+\nu_2)\lambda + \delta_{5i}\lambda^2 + \nu_1, \nu_3)$ of topological codimension 2.

THEOREM 6.1. *(Classification of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant bifurcation problems using path formulation up to topological codimension two and one bifurcation parameter)*

(a) With the core H^c_1, the paths are $\bar{\alpha}_1$ (when $\chi \neq \delta_1\epsilon_1\epsilon_3$), $\bar{\alpha}_3$ (when $\delta_1 = 0$), $\bar{\alpha}_4$ (when $\delta_2 = 0$) and $\bar{\alpha}_{51}, \bar{\alpha}_{52}$ (when $\chi = \delta_1\epsilon_1\epsilon_3$).

(b) With the core H^c_{10} the paths are $\bar{\alpha}_1$ (when $\chi \neq \delta_1\epsilon_1\epsilon_3$), $\bar{\alpha}_4$ (when $\delta_2 = 0$) and $\bar{\alpha}_{51}$ (when $\chi = \delta_1\epsilon_1\epsilon_3$). With the core H^c_9 the path are $\bar{\alpha}_2$ (when $\kappa \neq \delta_1\epsilon_2\epsilon_4$), $\bar{\alpha}_3$ (when $\delta_1 = 0$), and $\bar{\alpha}_{52}$ (when $\kappa = \delta_1\epsilon_2\epsilon_4$).

(c) With the core H^c_6 the path is $\bar{\alpha}_1 = (\delta_1\lambda, \chi\lambda, 0)$ (when $\chi \neq \delta_1\epsilon_1\epsilon_3$), and with the core H^c_5 the path is $\bar{\alpha}_2 = (\delta_1\lambda, \kappa\lambda, 0)$ (when $\kappa \neq \delta_1\epsilon_2\epsilon_4$).

(d) With the core H_2^c the path is $\bar{\alpha}_2 = (\delta_1\lambda, \kappa\lambda, 0)$ when $\kappa \neq \delta_1\epsilon_2\epsilon_4$.

Proof. The proof of each case follows the same pattern. We show the first case with some details. Let $\bar{\alpha} : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ be a path such that $\bar{\alpha}(\lambda) = (\delta_1\lambda, \chi\lambda, 0) + \vec{\mathcal{M}}_\lambda^2$. From (5), the tangent space of $\bar{\alpha}$ is

$$T_e\mathcal{K}_{\Delta^{H_1^c}}(\bar{\alpha}) = \langle \bar{\alpha}_\lambda \rangle_{\mathcal{E}_\lambda} + \bar{\alpha}^\star \mathrm{Derlog}(\Delta^{H_1^c}).$$

At their lowest order the generators of $T_e\mathcal{K}_{\Delta^{H_1^c}}(\bar{\alpha})$ are:

$$v_1 = \begin{pmatrix} \delta_1 \\ \chi \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} \delta_1\lambda \\ \chi\lambda \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} -\epsilon_1\lambda^2 \\ -\epsilon_3\chi^2\lambda^2 \\ (-\epsilon_1\delta_1 + \epsilon_3\chi)m\lambda \end{pmatrix}, \; v_4 = \begin{pmatrix} 0 \\ 0 \\ (\delta_1 - \epsilon_4\chi m)\lambda \end{pmatrix}.$$

If $\delta_1 - \epsilon_4\chi m \neq 0$ the generator $(0,0,\lambda)$ is in the tangent space $T_e\mathcal{K}_{\Delta^{H_1^c}}(\bar{\alpha})$. If $\chi \neq \epsilon_1\epsilon_2\epsilon_3$ and $\chi \neq 0$, we get the generator $(0, \lambda^2, 0)$. We find that the normal space $N_e\mathcal{K}_{\Delta^{H_1^c}}(\bar{\alpha})$ is generated by $[0,1]$, $[v,0]$ and $[0,\lambda]$, or in vector notation $(0,1,0)$, $(0,0,1)$ and $(0,\lambda,0)$. In fact, consider the following table.

generators	$[1,0]$	$[\lambda,0]$	$[v,0]$	$[0,1]$	$[0,\lambda]$
v_1	δ_1	0	0	χ	0
v_2	0	δ_1	0	0	χ
$[0,1]$	0	0	0	1	0
$[v,0]$	0	0	1	0	0
$[0,\lambda]$	0	0	0	0	1

Since this 5×5 matrix has maximum rank, we obtain the generators of $N_e\mathcal{K}_{\Delta^{H_1^c}}(\bar{\alpha})$. The miniversal unfolding $H_1^\star$ of $h_1^\star$ is given by

$$H_1^\star = [\epsilon_1 u + \tilde{m}v + \delta_1\lambda, \epsilon_3 u + \epsilon_4 v + \tilde{\chi}\lambda + \nu_1]$$

where $\tilde{m} = m + \nu_3$ and $\tilde{\chi} = \chi + \nu_2$. Hence the topological codimension is 1 but the smooth codimension is 3.

For the other cases we proceed similarly. ∎

6.1. Comparison with the classical theory

The generic bifurcation problem in [8, 10] is $h_1 = [\epsilon_1 u + \mu v + \delta_1\lambda, \eta u + \epsilon_4 v + \delta_2\lambda]$. As expected it has two modal parameters but they live in the core. Our analysis shows that actually only one modal parameter is associated with the core, the other is linked with the path. More explicitly, the link between the two sets of modal parameters are $\mu = \delta_1\chi m$, $\eta = \delta_1\epsilon_3\chi^{-1}$.

Another point to make is that the classification of [10] contains 8 cases they denote h_1 to h_8. In our classification theorem we have 14 cases. Actually a more detailed comparison indicates that some of the cases in [10] correspond to several of our cases. Explicitly,

• h_1 corresponds to 3 pull-backs: $\bar{\alpha}_1^* H_1^c$, $\bar{\alpha}_2^* H_9^c$ or $\bar{\alpha}_1^* H_{10}^c$,

• the next 4 to two pull-backs: $h_{3,8}$ to $\bar{\alpha}_{51,4}^* H_1^c$ or $\bar{\alpha}_{51,4}^* H_{10}^c$ and $h_{4,7}$ to $\bar{\alpha}_{52,3}^* H_1^c$ or $\bar{\alpha}_{51,4}^* H_9^c$, and

• the final 3 to only one pull-back: h_2 to $\bar{\alpha}_2^* H_2^c$, h_5 to $\bar{\alpha}_2^* H_5^c$ and h_6 to $\bar{\alpha}_1^* H_6^c$.

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