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Universal covering for control systems without drift vector field

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Let Σ_d be a control system without drift vector field on a manifold M and fix an initial condition $x_0 \in M$. Assume that the Lie algebra rank condition holds. We consider the manifold $\mathbf{\Gamma}(\Sigma, x_0)$ recently constructed in [1] for the particular case when $\Sigma = \Sigma_d$. The main result of the paper then reads as follows: The universal covering of control $\mathbf{\Gamma}(\Sigma_d, x_0)$ coincides with the universal covering manifold $\widetilde{M}$ of M. April, 2005 ICMC-USP

1. INTRODUCTION

The covering space construction for control systems first appeared in [1]. In this paper we study it for the particular case when the system is considered without drift. We keep the same notation in [1] and denote this space by $\mathbf{\Gamma}(\Sigma, x_0)$.

The covering space construction for control systems is done through the monotonic homotopy. This is a variant of usual homotopy in the set of trajectories of control systems. Homotopy properties of the space of trajectories for completely controllable systems, and in particular, for systems without drift was considered by Sarychev in [8], [9]. Also, in (Lie) semigroup theory the notion of monotonic homotopy was considered by Lawson in [6], [7].

In this work, we are mainly interested in systems without drift. Thus let Σ be a system without drift vector field on a finite dimensional connected smooth manifold M (the state space), and fix an initial point x_0 in M. Since we have controllability (under the Lie algebra rank condition), the accessible set $A(x_0)$ of Σ from $x_0 \in M$ is the whole manifold M. Hence, the simply connected universal covering $A(x_0)^{\sim}$ of $A(x_0)$ is the universal covering manifold $\widetilde{M}$ of M.

From standard homotopy theory, we can take $\widetilde{M}$ to be the set of homotopy classes of paths in M starting at x_0. We use first the universality property of $\mathbf{\Gamma}(\Sigma, x_0)$ as pointed out

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in [1], Section 8, and therefore consider, in particular, the universal covering manifold $\widetilde{M}$ with the lifted system $\widetilde{\Sigma}$ of Σ and the covering map $p : \widetilde{M} \to M$ that sends the homotopy class of a continuous path c in M to its end-point.

The universal property implies the existing of a (unique) lifting mapping $f : \mathbf{\Gamma}(\Sigma, x_0) \to \widetilde{M}$ which is a local diffeomorphism of $\mathbf{\Gamma}(\Sigma, x_0)$ into $\widetilde{M}$ that relates the corresponding systems on the manifolds $\mathbf{\Gamma}(\Sigma, x_0)$ and $\widetilde{M}$. Since, in this context, the lifted system $\widetilde{\Sigma}$ is also completely controllable it follows at once that the image imf is all $\widetilde{M}$, which does not occur in general, see [1].

Having a surjective local diffeomorphism at hand, it is sufficient to show its injectivity to establish a bijection between equivalence classes of monotonically homotopic trajectories and equivalence classes by usual homotopy. This is achieved by proving that the end-point mapping $\varepsilon : \mathbf{\Gamma}(\Sigma, x_0) \to M$ given by $\varepsilon([\alpha]) = \alpha(1)$ is a fibration (in Hurewicz sense). It then becomes clear that the two manifolds $\mathbf{\Gamma}(\Sigma, x_0)$ and $\widetilde{M}$ can be identified to each other.

Identifying $\widetilde{M}$ with the homotopy classes of paths in M starting at $x_{0,}$ the diffeomorphism $\mathbf{\Gamma}(\Sigma, x_0) \to \widetilde{M}$ may be viewed as sending the monotonic homotopy class of a regular trajectory to its homotopy class as a path. With this at hand we have shown that 1) The end-point mapping $\varepsilon : \mathbf{\Gamma}(\Sigma, x_0) \to M$ is a covering, and 2) Two trajectories homotopic as paths are also monotonically homotopic, which is a quite interesting fact.

We remark that none of these assertions is true for general control systems. We also give at the end of the paper a brief exposition about isomorphic coverings.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper we keep basically the main structure of the previous work, [1], and follow the same notation introduced there. Hence, let us take an n-dimensional connected smooth ($\mathcal{C}^\infty$) manifold M, and assume that M is given with a Riemannian metric which induces a distance function d_R.

Let E denote a finite dimensional vector subspace of $\mathcal{X}^\infty(M)$, the (real) vector space of all smooth vector fields on M. In order to have a topology on E (and, on the corresponding function spaces) we assume that E is endowed with an inner product $\langle \cdot, \cdot \rangle$.

We consider a control system Σ on M as a generating convex cone of (complete) vector fields in E. Also, we assume throughout the paper that Σ satisfies so-called *Lie algebra rank condition*, i.e, $\mathcal{L}(x_0) = T_{x_0}M$ for all $x_0 \in M$ where $\mathcal{L}$ denotes the smallest Lie algebra of vector fields containing Σ.

By a trajectory of Σ we mean an absolutely continuous curve α in M such that $\alpha^{'}(t) \in \Sigma(\alpha(t))$. In general, the domain of a trajectory is understood as an interval $[0, T]$, for some $T > 0$. We are especially interested in geometrical properties of trajectories. Hence, we reparametrize trajectories and define them in the unit interval $[0, 1]$ due to the fact that Σ is a cone.

Denote by $\mathcal{E}$ the Banach space of bounded and measureble functions $u : [0, 1] \to E$ endowed with the ess sup-norm $\| \cdot \|_\infty$, where the norm on E is induced by the inner

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product. Let $\mathcal{U}$ be the convex cone

$$\{u \in \mathcal{E} : u(t) \in \Sigma, \quad 0 \leq t \leq 1\}.$$

Since the system Σ is assumed to be a generating cone, we have that the interior $\mathrm{int}_{\mathcal{E}}\, \mathcal{U}$ of $\mathcal{U}$ in $\mathcal{E}$ (w.r.t. the sup-norm) is non-empty. In addition to the norm topology on E it will, sometimes, be useful to consider $\mathcal{E}$ with the weak* topology (See, Colonious-Kliemann [2]).

Also we denote by $T(\Sigma)$ the set of trajectories of Σ and the space of trajectories of Σ starting at x_0 and ending at y_0 by $T(\Sigma, x_0, y_0)$. The set of regular trajectories at x_0 will be denoted by $R(\Sigma, x_0)$ while the set of regular controls at x_0 will be denoted by $\mathcal{R}_\Sigma(x_0)$. Note that by a regular trajectory we mean a trajectory that corresponds to a regular control, See [1] for a more detailed exposition.

In the sequel we deal with control systems without drift and consider its universal coverings. From now on, we fix the notation Σ_d for such a system and call from time to time as time-reversible or symmetric system.

We also recall that there is a well known controllability result for this particular class of control systems, the Chow-Rashevskiĭ theorem which ensures that the set $T(\Sigma, x_0, y_0)$ is nonempty since x_0 can be joined to y_0 via a trajectory of Σ. On the other hand, By Proposition 3.6 in [1], every point attanaible from x_0 is actually regularly attanaible. Hence, the set $R(\Sigma, x_0, y_0)$ is also nonempty.

3. COVERINGS OF CONTROL SYSTEMS

Let M be an n-dimensional connected manifold and fix $x_0 \in M$. It is well known that the simply connected covering of M is obtained from standard homotopy theory. Roughly speaking, let $\widetilde{M}$ denote the set of homotopy classes of paths $c : [0,1] \to M, \quad c(0) = x_0$, keeping the end-points fixed. Define $p : \widetilde{M} \to M$ by $p([c]) = c(1)$, where $[c]$ is the homotopy class of c. It is well known that the space $\widetilde{M}$ is an n-dimensional connected and simply connected manifold and the map p is a covering. Furthermore, $\widetilde{M}$ is locally diffeomorphic to M.

In [1] we have been used the same ideia (in a slight different setting, of course) for the covering space construction of control systems. More precisely, we have been considered monotonic homotopy, as a modification of the usual homotopy, in the set of (arbitrary) trajectories of a given control system Σ on M instead of paths in M, and actually restricted our attention to the context of regular trajectories.

Hence the formal definition of monotonic homotopy is as follows :

Definition 3.3.1. *Two regular trajectories α and β are said to be monotonically homotopic (write $\alpha \simeq_m \beta$) if their extremal points are equal, that is, for some x_0, y_0 in M, $\alpha, \beta \in R(\Sigma, x_0, y_0)$ and α and β belong to the same path component of $R(\Sigma, x_0, y_0)$.*

The set of trajectories is topologized with the $\mathcal{C}^1$-topology which is a metric space given by the distance

$$d_1(\alpha,\beta) = \sup_{t\in[0,1]} d_R(\alpha(t),\beta(t)) + \text{ess} \sup_{t\in[0,1]} |\alpha'(t) - \beta'(t)|.$$

By the above definition it is clear that the relation of being monotonically homotopic is an equivalence relation. Hence, for a fixed initial condition x_0 in M we have the set $\mathbf{\Gamma}(\Sigma, x_0)$ of monotonic homotopy classes of trajectories in $R(\Sigma, x_0)$, and the canonical projection

$$\pi : R(\Sigma, x_0) \to \mathbf{\Gamma}(\Sigma, x_0) = R(\Sigma, x_0)/\simeq_m$$

that associates to α its monotonic homotopy class $[\alpha]_m$.

Given a system Σ on M and $x_0 \in M$ it was constructed in [1] a smooth manifold structure of dimension $n = \dim(M)$ on $\mathbf{\Gamma}(\Sigma, x_0)$, and the end point mapping

$$\varepsilon_{x_0} : \mathbf{\Gamma}(\Sigma, x_0) \longrightarrow \mathcal{A}_R(\Sigma, x_0) \subset M, \quad [\alpha]_m \longmapsto \alpha(1), \tag{1}$$

which is a local diffeomorphism in the sense that its differential is an isomorphism at every point of $\mathbf{\Gamma}(\Sigma, x_0)$.

Moreover, the image of ε_{x_0} (or simply ε) is contained in the interior $\text{int}\mathcal{A}(x_0)$ of accessible set from x_0, and is in fact $\text{int}\mathcal{A}(x_0)$ if the Lie algebra rank condition holds. Since the mapping ε_{x_0} is a local diffeomorphism, the restriction of Σ to $\mathcal{A}_R(\Sigma, x_0)$ can be lifted to $\mathbf{\Gamma}(\Sigma, x_0)$. Hence, we get a new control system $\widehat{\Sigma}$ on $\mathbf{\Gamma}(\Sigma, x_0)$ from Σ so that both Σ and $\widehat{\Sigma}$ are in a bijection.

Remark: In view of this bijection, the control functions of Σ are also control functions of $\widehat{\Sigma}$. In the sequel we always use the same control space $\mathcal{U}$ for systems related by local diffeomorphisms. Also it follows by the bijection between Σ and $\widehat{\Sigma}$ that the Lie algebra rank condition holds for $\widehat{\Sigma}$, that is,

$$\dim \mathcal{L}(\widehat{\Sigma})(z) = \dim \mathbf{\Gamma}(\Sigma, x_0) = \dim M, \quad \forall z \in \mathbf{\Gamma}(\Sigma, x_0).$$

The set $\mathbf{\Gamma}(\Sigma, x_0)$ is a covering space contructed in control context and is simply connected in the same sense as the simply connected universal covering. For this reason, we will call it *universal covering of control*.

3.1. Universal property of $\mathbf{\Gamma}(\Sigma, x_0)$

Let N be a smooth manifold endowed with a system $\widetilde{\Sigma}$ which is assumed to be forward complete as Σ, and let π be a local diffeomorphism of N onto $\mathcal{A}_R(\Sigma, x_0)$. It was constructed in Theorem 8.8, [1], a local diffeomorphism

$$f : \mathbf{\Gamma}(\Sigma, x_0) \longrightarrow N \tag{2}$$

such that $\pi \circ f = \varepsilon$ and $df(\widehat{\Sigma}) = \widetilde{\Sigma}$.

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This construction is the analogue of the classical one that gives the covering space from the simply connected covering. We notice however that f may fail to be surjective contrary to the classical case. This is because of controllability of $\widetilde{\Sigma}$ on N. We call such a map *control mapping* and say that a control mapping is a *control covering* if it is surjective. Hence, f in general is not a control covering.

Also, recall that the construction of f requires trajectories belonging to $\mathcal{A}_R(\Sigma, x_0)$ which is in general not true. By controllability, the accessible set $\mathcal{A}_R(\Sigma, x_0)$ is the whole manifold M and thus we are allowed to lift entire trajectories starting at x_0 to trajectories of $\widetilde{\Sigma}$ starting at a prescribed point above x_0.

As a particular case we substitute N in (2) by $\mathcal{A}_R(\Sigma, x_0)^\sim$ the simply connected universal covering of $\mathcal{A}_R(\Sigma, x_0)$. By controllability it is equal to the universal covering manifold of M so that the control covering $\pi : N \to \mathcal{A}(x_0)$ for a forward complete system $\widetilde{\Sigma}$ on N is taken to be the eventual covering projection $\pi : \widetilde{M} \to M$.

Proposition 3.3.2. *Pick a control system $\Sigma = \Sigma_d$ on M and assume that it is forward complete and that satisfies the Lie algebra rank condition at any point. Then there is a (unique) control mapping $f : \mathbf{\Gamma}(\Sigma, x_0) \to \widetilde{M}$ which is onto, i.e. f is a control covering.*

Proof: Let $w_0 \in \widetilde{M}$ be a point such that it projects down to x_0 in M. As said above, the Chow-Rashevskiĭ theorem guarants the controllability property of Σ on M. This enables us to lift entire trajectories starting at x_0 to trajectories of $\widetilde{\Sigma}$ starting at w_0. By the very definition of f, the image imf of f becomes the accessible set $\mathcal{A}_R(\widetilde{\Sigma}, w_0)$ which is in general a proper subset of $\mathcal{A}_R(\Sigma, x_0)^\sim = \widetilde{M}$.

By the remark stated before, the lifted system $\widetilde{\Sigma}$ on $\widetilde{M}$ -as a time-reversible system- satisfies the Lie algebra rank condition and hence $\widetilde{\Sigma}$ is also controllable on $\widetilde{M}$. Consequently, the image of f is the whole manifold $\widetilde{M}$.

Uniqueness of f follows from the condition $x_0 \in \mathrm{int}\mathcal{A}(x_0)$ which is implied by the assumption $\dim \mathcal{L}(\Sigma)(x_0) = n$ since the system Σ is symmetric. ▮

Having this proposition proved the next step is to determine if f is also injective which would imply immediately a bijection of $\mathbf{\Gamma}(\Sigma, x_0)$ onto $\widetilde{M}$. This is analysed in the following section.

4. FIBRATION

Throughout this section we shall consider a given map

$$p : E \to B$$

of a space E called *total space* into a space B called the *base space.*

4.1. Liftings of paths and homotopies

Definition 4.4.1. *The map $p : E \to B$ is said to have the path lifting property (abbrev.* PLP*) if, for each $e \in E$ and each path $c : [0, 1] \to B$ with $p(e) = c(0)$, there exists a* *(unique) path $\widetilde{c}:[0,1]\to E$ such that $\widetilde{c}(0)=e$ and $p\circ\widetilde{c}=c$, and that the lifting $\widetilde{c}$ depends continuously on e and c, [4].*

Also, a map $p:E\to B$ is said to have the local path lifting property (abbrev. LPLP) if, for each $b\in B$ there exists an open neigborhood U of b in B such that the map $p\big|_{p^{-1}(U)}:p^{-1}(U)\to U$ has the PLP.

Definition 4.4.2. *The map $p:E\to B$ is said to be a fibration (in Hurewicz sense) if it has the covering homotopy property (abbrev.* CHP*) for any space X. That is, let X be a given space. If for every map $f^*:X\to E$ and every homotopy $f_t:X\to B$, $0\le t\le 1$, of the map $f=p\circ f^*:X\to B$, there exists a homotopy $f_t^*:X\to E$, $0\le t\le 1$, of f^* which covers the homotopy f_t.*

It is known [5] that in particular if the base space B of p is paracompact then the covering homotopy property can be verified locally, that is,

$$\text{LPLP}\Longrightarrow\text{PLP}\Longleftrightarrow\text{CHP}. \tag{3}$$

We give below a well known lemma for the future references.

Lemma 4.4.3. *Let X and Y be two smooth manifolds and $f:X\to Y$ a surjective local diffeomorphism which has the PLP. If X is connected and Y is simply connected, then f is a diffeomorphism of X onto Y.*

Proof: It is enough to show that f is one-to-one. For it, let x_0 and x_1 belong to X such that $f(x_0)=f(x_1)$. Take a path $\widetilde{c}:[0,1]\to X$ starting at x_0 and ending at x_1. The path $c=f\circ\widetilde{c}$ is closed in Y, and so is homotopic to a constant path. It then follows from the path-lifting property of f that the lifting $\widetilde{c}$ of c is also closed which implies $x_0=x_1$. ∎

We recall that the path-lifting property of f in the preceding lemma is essential. For example, consider the function $f:\mathbb{C}\setminus\{\pm1\}\to\mathbb{C}$ given by $f(z)=z^3-3z$. It is easy to see that f is an onto local diffeomorphism with simply connected image but fails to be injective since a cubic polynomial has three roots in $\mathbb{C}$. Of course, it it is not possible to lift under f homotopies in $\mathbb{C}$ to $\mathbb{C}\setminus\{\pm1\}$.

Consider now the end-point mapping ε_{x_0} given in (1) which is a local diffeomorphism of $\mathbf{\Gamma}(\Sigma,x_0)$ into M. By controllability $\mathcal{A}_R(\Sigma,x_0)=M$ and hence it is surjective.

In view of Lemma 4.4.3 we assume without loss of generality that the state space M of Σ is simply connected and show in the sequel that the end-point mapping $\varepsilon_{x_0}:\mathbf{\Gamma}(\Sigma,x_0)\to M$ is a Hurewicz fibration. With this we would achieve our main result, namely the problem of establishing a diffeomorphism between the universal covering of control $\mathbf{\Gamma}(\Sigma,x_0)$ and the universal covering of M.

By the implication stated in (3) it suffices to verify locally path-lifting property of the end-point mapping. More precisely, for any point x in M there exists a neighborhood U of

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x in M such that any path $c : [0,1] \to U$ lifts to a unique path $\widehat{c} : [0,1] \to \varepsilon_{x_0}^{-1}(U)$, which is continuously dependent on c and the initial point $\widehat{c}(0)$.

The locally path-lifting property of the end-point mapping is then established in the following lemma which is the central step in proving our main result.

LEMMA 4.4.4. *Let Σ be a system without drift vector field and assume the Lie algebra rank condition holds. Then the end-point mapping $\varepsilon : \mathbf{\Gamma}(\Sigma, x_0) \to M$ is an onto local diffeomorphism which is at the same time a Hurewicz fibration.*

Proof: We must check that ε_{x_0} satisfies the LPLP. The points of M are normally accessible from x_0, so that given $x \in M$, there exists a mapping $\rho_{x_0} : \mathbb{R}^k \to M$ such that $\rho_{x_0}(\tau) = x$, $\tau = (t_1 \ldots, t_k)$, and ρ_{x_0} has full rank at τ. By the implicit function theorem there exists a neighborhood U of x in M and a slice of $V \subset \mathbb{R}^k$ of ρ_{x_0} such that V and U are diffeomorphic under ρ_{x_0}. Now, let $c : [0,1] \to U$ be a path in U. For $s \in [0,1]$ let $\tau(s) \in V$ be such that $\rho_{x_0}(\tau(s)) = c(s)$. Of course, $\rho_{x_0}(\tau(s))$ is the end-point of a trajectory starting at x_0 defined by a piecewise constant control, say u_s. Denote this trajectory by $\alpha_s = \mathrm{trj}_{x_0}(u_s)$. The set of controls u_s gives the desired lifting of c. ∎

Now, we state our main result.

THEOREM 4.4.5. *Let $\Sigma = \Sigma_d$ satisfy the Lie algebra rank condition and assume that M is simply connected. Fix $x_0 \in M$. Then, the universal covering $\mathbf{\Gamma}(\Sigma, x_0)$ of Σ is diffeomorphic to the universal covering $\widetilde{M}$ of M. In other words, we can reach the universal covering manifold of the state space M using the control theoretic setting.*

Proof: By Lemma 4.4.4, the end-point mapping ε_{x_0} admits the path-lifting property and hence the result follows at once from the Lemma 4.4.3. ∎

COROLLARY 4.4.6. *The two systems $\widehat{\Sigma}$ and $\widetilde{\Sigma}$ on $\mathbf{\Gamma}(\Sigma, x_0)$ and $\widetilde{M}$, respectively, are (globally) state-space equivalent and the manifold $\mathbf{\Gamma}(\Sigma, x_0)$ is connected.*

COROLLARY 4.4.7. *Two trajectories homotopic in M are also monotonically homotopic in M.*

Proof: By Lemma 7.7 in [1], two trajectories in $\mathbf{\Gamma}(\Sigma, x_0)$ having the same initial and end point are monotonically homotopic. On the other hand, two trajectories in M with the same extremal points are homotopic as paths.

The lemmas 4.4.3 and 4.4.4 together imply that the end-point mapping $\varepsilon_{x_0} : \mathbf{\Gamma}(\Sigma, x_0) \to M$ given by $\varepsilon_{x_0}([\gamma]) = \gamma(1)$ is injective. This means that homotopy implies monotonic homotopy. ∎

COROLLARY 4.4.8. *The end-point mapping* $\varepsilon_{x_0} : \mathbf{\Gamma}(\Sigma, x_0) \to \mathcal{A}_R(\Sigma, x_0) \subset M$ *is a covering and hence the system* Σ *is geometric at* x_0.

Proof: The first assertion is clear since by the above theorem f is a diffeomorphism. That Σ is geometric at x_0 follows from the fact that any covering satisfies CHP for $[0,1]$. ∎

We end this section by considering the notion of the fundamental semigroup formed by closed trajectories (i.e., loops) of Σ and give a simple result about it. Hence, let Σ be a symmetric control system that satisfies the Lie algebra rank condition.We recall that since the condition $\dim \mathcal{L}(\Sigma_d)(x_0) = n$ implies that $x_0 \in \text{int}\mathcal{A}(x_0)$, or equivalently $x_0 \in \mathcal{A}_R(\Sigma_d, x_0)$ there are closed regular trajectories based at x_0. This permits us to introduce a fundamental semigroup based at x_0, analogous to the fundamental group of a topological space (cf. [3]).

Let us fix $x_0 \in \text{int}\mathcal{A}(x_0)$. The standard concatenation yields a semigroup structure on $R(\Sigma_d, x_0, x_0)$. Then the fundamental semigroup based at x_0 is defined as the quotient space

$$\Lambda(\Sigma_d, x_0) = R(\Sigma_d, x_0, x_0)/ \simeq_m .$$

By Corollary 4.4.8 we know that the end-point mapping ε is a covering which implies that the system is geometric and moreover, the fundamental semigroup coincides with the fundamental group. As the fundamental group does not depend on the choice of a base point for path-connected topological spaces, we give below the analogue of this situation for fundamental semigroups with different base points.

PROPOSITION 4.4.9. *With the assumptions and notations as above, let* $\Lambda(\Sigma_d, x_0)$ *and* $\Lambda(\Sigma_d, x_1)$ *denote the fundamental semigroups based at* x_0 *and* x_1*, respectively. Then* $\Lambda(\Sigma_d, x_0)$ *and* $\Lambda(\Sigma_d, x_1)$ *are isomorphic.*

Proof: By Corollary 4.4.8, the fundamental semigroup $\Lambda(\Sigma, x_0)$ coincides with the fundamental group $\pi_1(M, x_0)$. ∎

5. ISOMORPHIC COVERINGS

We consider here the relation between the coverings of different control system and, in particular, the independency of the covering construction from the choice of initial point.

First we show below that the covering $\mathbf{\Gamma}(\Sigma, x_0)$ constructed starting from a control system Σ on M and a fixed $x_0 \in M$ does not depend on initial condition x_0 in the sense that coverings with different initial conditions are isomorphic.

PROPOSITION 5.5.1. *Keep the notations and assumptions as before. Then* $\mathbf{\Gamma}(\Sigma_d, \cdot)$ *does not depend on a choice of initial condition.*

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Proof: Let $x_0 \in \mathcal{A}_R(\Sigma, z_0)$. We know by the constructions done in [1] that $\mathbf{\Gamma}(\Sigma_d, x_0)$ is diffeomorphic to the accessible set $\mathcal{A}_R(\widehat{\Sigma}_d, y_0)$ in $\mathbf{\Gamma}(\Sigma_d, z_0)$ for any $y_0 \in \varepsilon^{-1}(z_0)$, and hence it is an open submanifold of $\mathbf{\Gamma}(\Sigma_d, z_0)$, in general.

By Theorem 4.4.5 $\mathbf{\Gamma}(\Sigma_d, z_0)$ is connected. Since the conditions of Chow-Rashevskiĭ theorem are fulfilled for the lifted system $\widehat{\Sigma}_d$ on $\mathbf{\Gamma}(\Sigma_d, z_0)$ it follows that $\widehat{\Sigma}_d$ is globally controllable. In particular, $\widehat{\Sigma}_d$ is controllable from any y. This way we get that

$$\mathbf{\Gamma}(\Sigma_d, x_0) \cong \mathcal{A}_R(\widehat{\Sigma}_d, y_0) = \mathbf{\Gamma}(\Sigma_d, z_0) \quad \text{for any } y_0 \in \varepsilon^{-1}(z_0).$$

The same proof applies to the case if $z_0 \in \mathcal{A}_R(\Sigma, x_0)$. Hence, for any two points x_0, $z_0 \in M$ the coverings $\mathbf{\Gamma}(\Sigma_d, x_0)$ and $\mathbf{\Gamma}(\Sigma_d, z_0)$ are diffeomorphic. ∎

The following proposition consider the relation between coverings for different control systems.

Proposition 5.5.2. *Let Σ_1 and Σ_2 be two control systems on the manifols N_1 and N_2 respectively. Let $f : N_1 \to N_2$ be a control mapping such that $f(x_1) = x_2$ for some $x_1 \in N_1$ and $x_2 \in N_2$. Then, there exists a control mapping $\mathbf{\Gamma}(\Sigma_1, x_1) \to \mathbf{\Gamma}(\Sigma_2, x_2)$ between the liftings of Σ_1 and Σ_2.*

In particular, if f is a diffeomorphism then $\mathbf{\Gamma}(\Sigma_1, x_1)$ and $\mathbf{\Gamma}(\Sigma_2, x_2)$ are isomorphic coverings.

Proof: Consider the end-point mapping $\varepsilon_{x_i} : \mathbf{\Gamma}(\Sigma_i, x_i) \to \mathcal{A}_R(\Sigma_i, x_i)$, $i = 1, 2$, and the diagram below

$$\begin{array}{lcl} \mathbf{\Gamma}(\Sigma_1, x_1) & \cdots\triangleright & \mathbf{\Gamma}(\Sigma_2, x_2) \\ \downarrow \varepsilon_{x_1} & & \varepsilon_{x_2} \downarrow \\ \mathcal{A}_R(\Sigma_1, x_1) & \xrightarrow{f} & \mathcal{A}_R(\Sigma_2, x_2) \end{array}.$$

The composition map $\widehat{f} =: \varepsilon_{x_2}^{-1} \circ f \circ \varepsilon_{x_1}$ where $\varepsilon_{x_2}^{-1}$ is a local inverse of ε_{x_2} establishes the desired local diffeomorphism. Since, $d\varepsilon_{x_i}$, $i = 1, 2$, is an isomorphsim and $df(\Sigma_1) = \Sigma_2$ we see that $d\widehat{f}(\widehat{\Sigma}_1) = \widehat{\Sigma}_2$, that is, the map $\widehat{f} : \mathbf{\Gamma}(\Sigma_1, x_1) \to \mathbf{\Gamma}(\Sigma_2, x_2)$ is by definition a control mapping.

As a particular instant, if the control mapping f is a diffeomorphism we then have isomorphic coverings. In other words, equivalent systems produce isomorphic coverings. In fact, the induced mapping

$$f_* : \mathbf{\Gamma}(\Sigma_1, x_1) \longrightarrow \mathbf{\Gamma}(\Sigma_2, x_2), \qquad [\gamma]_m \longmapsto [f \circ \gamma]_m,$$

is a diffeomorphism of $\mathbf{\Gamma}(\Sigma_1, x_1)$ onto $\mathbf{\Gamma}(\Sigma_2, x_2)$. ∎

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