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Continuation of the connection matrix in singular perturbation problems

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In this paper, which is a sequel to our previous work (*On a general Conley index continuation principle for singular perturbation problems*, Ergodic Theory and Dynamical Systems **22** (2002), 729–755) we establish a general *Singular nested index filtration principle* which, in particular, implies a continuation principle for homology index braids and connection matrices in singular perturbation problems. May, 2005 ICMC-USP

1. INTRODUCTION

In singular perturbation problems, like those arising from evolution equations on thin domains or evolution equations with a small coefficient in the highest order term, one is usually given a family $(\pi_\varepsilon)_{\varepsilon\in]0,1]}$ of semiflows on the space X, a family $(\rho_\varepsilon)_{\varepsilon\in]0,1]}$ of metrics on X and a singular limit semiflow π_0 which is defined only on a subspace X_0 of X. Moreover, π_ε converges to π_0 only in some restricted sense with respect to the above family of metrics. For such problems, under an additional asymptotic compactness assumption, we have recently established a general *singular Conley index continuation principle*, which roughly states that every compact isolated invariant set S_0 of the limit semiflow π_0 continues to a family S_ε of compact isolated invariant sets of π_ε, $\varepsilon > 0$ small, having the same Conley index as S_0. (See [3, Theorem 4.1] and also [1] for some applications.) This result, like its usual counterpart for 'regular' perturbations, is useful in applications in that it permits, e.g. to prove existence of nonempty invariant sets for the perturbed semiflows from the existence of invariant sets of the limit semiflow with nonzero Conley index.

The main step in the proof of [3, Theorem 4.1] actually establishes a more general result, namely a construction of families $(N_{1,\varepsilon}, N_{2,\varepsilon})$ and $(\widetilde{N}_{1,\varepsilon}, \widetilde{N}_{2,\varepsilon})$ of index pairs for S_ε, $\varepsilon > 0$

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small, enjoying the following *singular nesting property*:

$$N_{i,\varepsilon} \subset N_{i,0} \subset \widetilde{N}_{i,\varepsilon} \subset \widetilde{N}_{i,0}, \quad i = 1, 2,$$

where $(N_{1,0}, N_{2,0})$ and $(\widetilde{N}_{1,0}, \widetilde{N}_{2,0})$ are properly 'inflated' index pairs for S_0. Once this *singular index pair principle* is proved, the singular Conley continuation principle easily follows.

Morse decompositions (see e.g. [6, 23, 21, 7]) are a useful tool in the analysis of flows or semiflows. Through the use of homology index braids and the ensuing Conley connection matrix theory, which is a refinement of the homological Conley index, Morse decompositions can be used to detect connections, i.e. heteroclinic orbits in dynamical systems.

The connection matrix theory for flows defined on locally compact spaces was developed by Franzosa in his thesis [7] and in subsequent papers [8, 9, 10]. Important contributions to the theory and applications were made in [13, 17, 18]. (Cf. also the recent volume [14] for various articles on connection and transition matrices and the references contained therein.) In [11] Franzosa and Mischaikow extended part of the theory of partially ordered Morse decompositions and connection matrices to the setting of Conley index theory, developed in [19, 20], for admissible local semiflows on (not necessarily locally compact) metric spaces.

One of the objectives of this paper is to show that a singular continuation principle also holds for homology index braids and the connection matrices. This is the contents of Theorem 3.21, which says that whenever S_0 is a (compact) isolated invariant set, $(P, \prec)$ is a finite, partially ordered set and $(M_{p,0})_{p\in P}$ is a Morse decomposition of S_0, relative to the limit semiflow π_0, then $(S_0, (M_{p,0})_{p\in P})$ continues to a family $(S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$, $\varepsilon > 0$ small, where S_ε is an isolated invariant set and $(M_{p,\varepsilon})_{p\in P}$ is a Morse decomposition of S_ε (relative to the semiflow π_ε), such that the homology index braids of $(S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and $(S_0, (M_{p,0})_{p\in P})$ are isomorphic and, therefore, determine the same set of connection matrices.

The proof of Theorem 3.21 is again a simple application of a more general result, the *singular nested index filtration principle* (Theorem 3.20), in which we construct (for all sufficiently small $\varepsilon > 0$) special index filtrations $\mathcal{N}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon$ for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ as well as properly 'inflated' index filtrations $\mathcal{N}_0$ and $\widetilde{\mathcal{N}}_0$ for $(\pi_0, S_0, (M_{p,0})_{p\in P})$ enjoying the following *singular nesting property*:

$$\mathcal{N}_\varepsilon \subset \mathcal{N}_0 \subset \widetilde{\mathcal{N}}_\varepsilon \subset \widetilde{\mathcal{N}}_0.$$

It should be remarked that the proof of Theorem 3.20, although very technical, is largely self-contained and does not make any use of homology index braid or connection matrix theory.

This paper is organized as follows. In Section 2 we recall some basic concepts from Conley index and homology index braid theory and establish some preliminary results. In particular, we state, in Theorem 2.10 below, an extension of an existence result for index filtrations proved in [11]. In Section 3 we state the main results of this paper, Theorems 3.20 and 3.21. We also briefly illustrate these results by applying them to a thin domain problem. More extensive applications will be given in a subsequent publication.

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The last three sections of the paper are devoted to the proof of Theorem 3.20. In Section 4 we use Theorem 2.10 to prove an abstract existence result (Theorem 4.24) for a family of index filtrations $\mathcal{N}_\varepsilon$ for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$, with $\varepsilon > 0$ small, such that $\mathcal{N}_\varepsilon$ is, in some singular sense, asymptotically included in a properly inflated index filtration $\mathcal{N}_0$ for $(\pi_0, S_0, (M_{p,0})_{p\in P})$. In Section 5 we construct families of index triples having special properties required for the application of Theorem 4.24 (see Theorems 5.27 and 5.32 below). In Section 6 we make two very specific applications of Theorems 5.27, 5.32 and 4.24 to obtain families of index filtrations which 'almost' satisfy the singular nesting property. By appropriately modifying these index filtrations, using Proposition 2.9, we finally achieve the full nesting property and thus complete the proof of Theorem 3.20.

2. PRELIMINARIES

The purpose of this section is to recall a few concepts from Conley index theory and to establish some preliminary results needed later in this paper. We assume the reader's familiarity with the (infinite-dimensional) Conley index theory, as expounded in [21], and with the papers [8] and [11].

In this section, unless otherwise specified, X is a metric space, π is a local semiflow on X and all concepts are defined *relative to* π.

Suppose that Y is a subset of X. By $\mathrm{Inv}^+_\pi(Y)$, resp. $\mathrm{Inv}^-_\pi(Y)$, resp. $\mathrm{Inv}_\pi(Y)$ we denote the largest positively invariant, resp. negatively invariant, resp. invariant subset of Y. Moreover, let the function $\rho_Y = \rho_{Y,\pi}\colon Y \to \mathbb{R}\cup\{\,\infty\,\}$ be given by

$$\rho_Y(x) := \sup\{\, t \geq 0 \mid x\pi t \text{ is defined and } x\pi[0,t] \subset Y\,\}.$$

Y is called *π-admissible* if Y is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \to \infty$ and $x_n\pi\,[0,t_n] \subset Y$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi t_n)_n$ has a convergent subsequence. We say that π *does not explode in* Y if whenever $x \in X$ and $x\pi t \in Y$ as long as $x\pi t$ is defined, then $x\pi t$ is defined for all $t \in [0,\infty[$. Y is called *strongly π-admissible* if Y is π-admissible and π does not explode in Y.

Let N and Y be subsets of X. The set Y is called *N-positively invariant* if whenever $x \in Y$, $t \geq 0$ are such that $x\pi\,[0,t] \subset N$, then $x\pi\,[0,t] \subset Y$.

Let N, Y_1 and Y_2 be subsets of X. The set Y_2 is called an *exit ramp for N within Y_1* if whenever $x \in Y_1$ and $x\pi t' \notin N$ for some $t' \in [0,\infty[$, then there exists a $t_0 \in [0,t']$ such that $x\pi\,[0,t_0] \subset N$ and $x\pi t_0 \in Y_2$.

If Y_1 and Y_2 are subsets of X then Y_2 is called an *exit ramp for Y_1* if Y_2 is an exit ramp for N within Y_1, where $N = Y_1$.

Definition 2.1. *Let $B \subset X$ be a closed set and $x \in \partial B$. The point x is called a strict egress (respectively strict ingress, respectively bounce-off) point of B, if for every solution $\sigma\colon [-\delta_1,\delta_2] \to X$ through x, with $\delta_1 \geq 0$ and $\delta_2 > 0$, the following properties hold:*

1. *There exists an $\varepsilon_2 \in\,]0,\delta_2[$ such that $\sigma(t) \notin B$ (respectively $\sigma(t) \in \mathrm{Int}(B)$, respectively $\sigma(t) \notin B$), for $t \in\,]0,\varepsilon_2]$.*
2. *If $\delta_1 > 0$, then there exists an $\varepsilon_1 \in\,]0,\delta_1[$ such that $\sigma(t) \in \mathrm{Int}(B)$ (respectively $\sigma(t) \notin B$, respectively $\sigma(t) \notin B$), for $t \in [-\varepsilon_1, 0[$.*

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The set of all strict egress (respectively strict ingress, respectively bounce-off) points of B is denoted by B^e (respectively B^i, respectively B^b). Moreover, we call $B^- := B^e \cup B^b$ the exit set of B and $B^+ := B^i \cup B^b$ the entrance set of B. B is called an isolating block, if $\partial B = B^e \cup B^i \cup B^b$ and B^- is closed. If B is also an isolating neighborhood of an invariant set S, then we say that B is an isolating block for S.

If B is an isolating block then (B, B^-) is an example of an index pair in B. More generally, we have the following definition.

Definition 2.2. *Let N be closed in X. A pair (N_1, N_2) is called an index pair in N if:*

1. *N_1 and N_2 are closed and N-positively invariant subsets of N;*
2. *N_2 is an exit ramp for N within N_1;*
3. *$\mathrm{Inv}_\pi(N)$ is closed and $\mathrm{Inv}_\pi(N) \subset \mathrm{Int}(N_1 \setminus N_2)$.*

The next definition introduces a more general concept.

Definition 2.3. *A pair (N_1, N_2) is called a Franzosa-Mischaikow-index pair (or FM-index pair) for S if:*

1. *N_1 and N_2 are closed subsets of X with $N_2 \subset N_1$ and N_2 is N_1-positively invariant;*
2. *N_2 is an exit ramp for N_1;*
3. *S is closed, $S \subset \mathrm{Int}(N_1 \setminus N_2)$ and S is the largest invariant set in $\mathrm{Cl}(N_1 \setminus N_2)$;*

Proposition 2.4. (cf. [11]) *Let (N_1, N_2) be a pair of closed subsets of X with $N_2 \subset N_1$.*

1. *If S is an isolated invariant set, N_1 is an isolating neighborhood of S and (N_1, N_2) is an index pair in N_1, then (N_1, N_2) is an FM-index pair for S.*
2. *If (N_1, N_2) is an FM-index pair for S and N is an isolating neighborhood of S with $N_1 \setminus N_2 \subset N$, then $N_1 \cap N$ is an isolating neighborhood of S and $(N_1 \cap N, N_2 \cap N)$ is an index pair in $N_1 \cap N$.*

Proposition 2.5. *Let (N_1, N_2) be an FM-index pair for S. Let Y be a closed set such that $S \subset \mathrm{Int}(Y)$ and such that N_2 is an exit ramp for Y. Then $(Y \cap N_1, Y \cap N_2)$ is an FM-index pair for S.*

Proof. It is clear that $Y \cap N_1$ and $Y \cap N_2$ are closed. If $x \in Y \cap N_2$ and $x\pi[0,t] \subset Y \cap N_1$ for some $t \geq 0$, then, since N_2 is N_1-positively invariant, we obtain $x\pi[0,t] \subset N_2$ and so $x\pi[0,t] \subset Y \cap N_2$. This proves that $Y \cap N_2$ is $(Y \cap N_1)$-positively invariant.

There exist open sets V and W such that $S \subset W \subset N_1 \setminus N_2$ and $S \subset V \subset Y$. Thus $S \subset V \cap W \subset Y \cap (N_1 \setminus N_2) \subset (Y \cap N_1) \setminus (Y \cap N_2)$ and so $S \subset \mathrm{Int}((Y \cap N_1) \setminus (Y \cap N_2))$ Thus $S \subset \mathrm{Inv}_\pi \mathrm{Cl}((Y \cap N_1) \setminus (Y \cap N_2)) = \mathrm{Inv}_\pi \mathrm{Cl}(Y \cap (N_1 \setminus N_2)) \subset \mathrm{Inv}_\pi \mathrm{Cl}(N_1 \setminus N_2) = S$.

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To complete the proof, let $x \in Y \cap N_1$ and assume that there exists a $t' \geq 0$ such that $x\pi t' \notin Y \cap N_1$. Suppose first that $\rho_Y(x) > \rho_{N_1}(x)$. Then $\rho_{N_1}(x) < \infty$ and so $x\pi[0, \rho_{N_1}(x)] \subset Y \cap N_1$ and $x\pi\rho_{N_1}(x) \in N_2$ since (N_1, N_2) is an FM-index pair for S. Thus, $x\pi\rho_{N_1}(x) \in Y \cap N_2$. Now assume that $\rho_Y(x) \leq \rho_{N_1}(x)$. Then $\rho_Y(x) < \infty$ and so $x\pi[0, \rho_Y(x)] \subset Y \cap N_1$. Moreover, since N_2 is an exit ramp for Y, we have $x\pi\rho_Y(x) \in N_2$ and so $x\pi\rho_Y(x) \in Y \cap N_2$. The proof is complete. ▌

Definition 2.6. *Let S be an isolated invariant set and (A, A^*) be an attractor-repeller pair in S. A pair (B_1, B_2) is called a block pair (for (π, S, A, A^*)) if B_1 is an isolating block for A^*, B_2 is an isolating block for A, $B := B_1 \cup B_2$ is an isolating block for S and $B_1 \cap B_2 \subset B_1^- \cap B_2^+$.*

If (B_1, B_2) is a block pair then $(B, B_2 \cup B^-, B^-)$ is an example of an FM-index triple:

Definition 2.7. *Let S be an isolated invariant set and (A, A^*) be an attractor-repeller pair in S. A triple (N_1, N_2, N_3) with $N_3 \subset N_2 \subset N_1$ is called an FM-index triple (for (π, S, A, A^*)) if (N_1, N_3) is an FM-index pair for S and (N_2, N_3) is an FM-index pair for A.*

Proposition 2.8. (cf. [11]) *If (N_1, N_2, N_3) is an FM-index triple for (π, S, A, A^*) then (N_1, N_2) is an FM-index pair for A^*.*

Given an isolated invariant set K having a strongly π-admissible isolating neighborhood we denote by $h(K) = h(\pi, K)$ the Conley-index of K and by $H(K) = H(\pi, K) = H(h(K))$ the homology Conley index, where H is the singular homology functor (with coefficients in some fixed module G over a PID).

If (A, A^*) is an attractor-repeller pair in S and (N_1, N_2, N_3) is an FM-index triple for (π, S, A, A^*) with N_1 strongly π-admissible, then the inclusion induced sequence

$$N_2/N_3 \xrightarrow{i} N_1/N_3 \xrightarrow{p} N_1/N_2$$

induces a long exact homology sequence

$$\longrightarrow H_q(N_2/N_3) \xrightarrow{i} H_q(N_1/N_3) \xrightarrow{p} H_q(N_1/N_2) \xrightarrow{\partial} H_{q-1}(N_2/N_3) \longrightarrow .$$

This sequence is independent of the choice of (N_1, N_2, N_3) and so there is a well-defined long exact sequence

$$\longrightarrow H_q(A) \xrightarrow{i} H_q(S) \xrightarrow{p} H_q(A^*) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow$$

called the *homology index sequence of* (π, S, A, A^*).

Recall that a *strict partial order* on a set P is a relation $\prec \subset P \times P$ which is irreflexive and transitive. As usual, we write $x \prec y$ instead of $(x, y) \in \prec$. The symbol $<$ will be reserved for the less-than-relation on $\mathbb{R}$.

For the rest of this paper, unless specified otherwise, let P be a fixed finite set and $\prec$ be a fixed strict partial order on P.

A set $I \subset P$ is called *a $\prec$-interval* if whenever i, j, $k \in P$, i, $k \in I$ and $i \prec j \prec k$, then $j \in I$. By $\mathcal{I}(\prec)$ we denote the set of all $\prec$-intervals in P. A set I is called *a $\prec$-attracting interval* if whenever i, $j \in P$, $j \in I$ and $i \prec j$, then $i \in I$. By $\mathcal{A}(\prec)$ we denote the set of all $\prec$-attracting intervals in P. Of course, $\mathcal{A}(\prec) \subset \mathcal{I}(\prec)$.

An *adjacent n-tuple of $\prec$-intervals* is a sequence $(I_j)_{j=1}^n$ of pairwise disjoint $\prec$-intervals whose union is a $\prec$-interval and such that, whenever $j < k$, $p \in I_j$ and $p' \in I_k$, then $p' \not\prec p$ (i.e. $p \prec p'$ or else p and p' are not related by $\prec$). By $\mathcal{I}_n(\prec)$ we denote the set of all adjacent n-tuples of $\prec$-intervals.

Let S be a compact invariant set. A family $(M_i)_{i \in P}$ of subsets of S is called a *$\prec$-ordered Morse decomposition of S* if the following properties hold:

1. The sets M_i, $i \in P$, are closed, π-invariant and pairwise disjoint.

2. For every full solution σ of π lying in S either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are k, $l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Let S be a compact invariant set and $(M_i)_{i \in P}$ be a $\prec$-ordered Morse decomposition of S. If A, $B \subset X$ then the (π, S)*-connection set* $\mathrm{CS}_{\pi,S}(A,B)$ *from A to B* is the set of all points $x \in X$ for which there is a solution $\sigma\colon \mathbb{R} \to S$ of π with $\sigma(0) = x$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$.

For an arbitrary $\prec$-interval I set

$$M(I) = M_{\pi,S}(I) = \bigcup_{(i,j) \in I \times I} \mathrm{CS}_{\pi,S}(M_i, M_j).$$

An *index filtration for* $(\pi, S, (M_p)_{p \in P})$ is a family $\mathcal{N} = (N(I))_{I \in \mathcal{A}(\prec)}$ of closed subsets of X such that

1. for each $I \in \mathcal{A}(\prec)$, the pair $(N(I), N(\emptyset))$ is an FM-index pair for $M(I)$,
2. for each I_1, $I_2 \in \mathcal{A}(\prec)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$.

$\mathcal{N}$ is called *strongly π-admissible* if $N(P)$ is strongly π-admissible.

The following result is an immediate consequence of Proposition 2.5.

Proposition 2.9. *Let Y be a closed set such that $S \subset \mathrm{Int}(Y)$ and such that $N(\emptyset)$ is an exit ramp for Y. Let $\mathcal{N} = (N(I))_{I \in \mathcal{A}(\prec)}$ be an index filtration for $(\pi, S, (M_p)_{p \in P})$. Then $(Y \cap N(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi, S, (M_p)_{p \in P})$.*

A special version of the following basic result was established in [11].

Theorem 2.10. (cf Theorem 3.5 in [11]) *Let N_1^i, N_2^i, N_I^i, $i = 2$, 4, $I \in \mathcal{A}(\prec)$, be sets such that, for each $I \in \mathcal{A}(\prec)$, (N_1^i, N_I^i, N_2^i), $i = 2$, 4, is an FM-index triple for $(\pi, S, M(I), M(P \setminus I))$. Suppose $N_1^2 \subset N_1^4$, $N_2^2 \subset N_2^4$ and $N_I^2 \subset N_I^4$, $I \in \mathcal{A}(\prec)$. For each*

$p \in P$ define the following sets:

$$D_p := \Big(\bigcap_{\substack{I\\ p\in I}} \operatorname{Int}(N_I^2 \setminus N_2^4)\Big) \cap \Big(\bigcap_{\substack{I\\ p\notin I}} \operatorname{Int}(N_1^2 \setminus N_I^4)\Big)$$

and

$$E_p := \{\, x \in N_1^2 \mid \text{there exists a } t \geq 0 \text{ such that } x\pi[0,t] \subset N_1^2 \text{ and } x\pi t \in D_p \,\}.$$

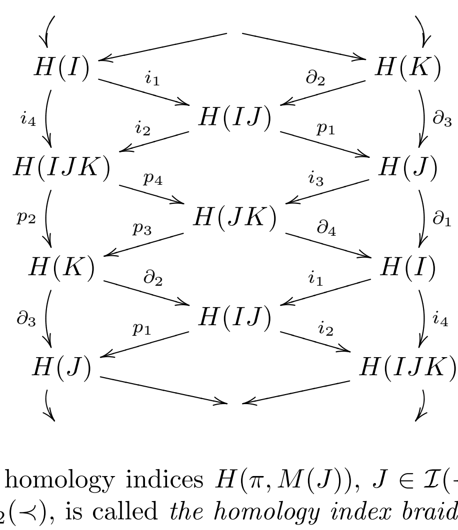
For each $I \in \mathcal{A}(\prec)$, define $N(I) := N_1^2 \setminus \bigcup_{p\in P\setminus I} E_p$. Then $(N(I))_{I\in\mathcal{A}(\prec)}$ is an index filtration for $(\pi, S, (M_p)_{p\in P})$. Moreover, $N_2^2 \subset N(\emptyset)$ and $N_I^2 \subset N(I)$ for all $I \in \mathcal{A}(\prec)$.

Proof. For $N_1^2 = N_1^4$, $N_2^2 = N_2^4$ and $N_I^2 = N_I^4$, $I \in \mathcal{A}(\prec)$, this is just Theorem 3.5 in [11], whose proof carries over almost verbatim to the present, more general case. Details are left to the reader. ∎

Let $\mathcal{N}$ be a strongly π-admissible index filtration for $(\pi, S, (M_p)_{p\in P})$. For $J \in \mathcal{I}(\prec)$ the set $M(J)$ is an isolated invariant set and we write $H(J) = H(\pi, J) := H(\pi, M(J))$. If $(I, J) \in \mathcal{I}_2(\prec)$, then $(M(I), M(J))$ is an attractor-repeller filtration in $M(IJ)$, where $IJ := I \cup J$. Hence there is the corresponding homology index sequence

$$\longrightarrow H_q(I) \xrightarrow{i_{I,J}} H_q(IJ) \xrightarrow{p_{I,J}} H_q(J) \xrightarrow{\partial_{I,J}} H_{q-1}(I) \longrightarrow$$

of $(\pi, M(IJ), M(I), M(J))$. Using the index filtration $\mathcal{N}$ one proves that for every triple $(I, J, K) \in \mathcal{I}_3(\prec)$ the following diagram, made up of the four homology index sequences defined by (I, J, K), commutes:



The collection of all the homology indices $H(\pi, M(J))$, $J \in \mathcal{I}(\prec)$, and all the maps $i_{I,J}$, $p_{I,J}$ and $\partial_{I,J}$, $(I, J) \in \mathcal{I}_2(\prec)$, is called *the homology index braid of* $(\pi, S, (M_p)_{p\in P})$ and is denoted by $\mathcal{H}(\pi, S, (M_p)_{p\in P})$.

For the rest of this section assume that, for $i = 1, 2$, π_i is a local semiflow on the metric space X_i, S_i is an isolated invariant set and $(M_{p,i})_{p\in P}$ is a $\prec$-ordered Morse decomposition of S_i, relative to π_i. Write $M_i(I) = M_{\pi_i,S_i}(I)$, $H_i(I) = H(\pi_i, M_i(I))$ and $\mathcal{H}_i := \mathcal{H}(\pi_i, S_i, (M_{p,i})_{p\in P})$, for $i = 1, 2$ and $I \in \mathcal{I}(\prec)$.

Suppose $[\Theta] := ([\Theta](J))_{J\in\mathcal{I}(\prec)}$ is a family $[\Theta](J)\colon H_1(J) \to H_2(J)$, $J \in \mathcal{I}(\prec)$, of maps such that, for all $(I, J) \in \mathcal{I}_2(\prec)$, the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H_{1,q}(I) & \xrightarrow{i_{I,J}} & H_{1,q}(IJ) & \xrightarrow{p_{I,J}} & H_{1,q}(J) & \xrightarrow{\partial_{I,J}} & H_{1,q-1}(I) & \longrightarrow \\
 & \Big\downarrow{\scriptstyle [\Theta](I)} & & \Big\downarrow{\scriptstyle [\Theta](IJ)} & & \Big\downarrow{\scriptstyle [\Theta](J)} & & \Big\downarrow{\scriptstyle [\Theta](I)} & \\
\longrightarrow & H_{2,q}(I) & \xrightarrow[i_{I,J}]{} & H_{2,q}(IJ) & \xrightarrow[p_{I,J}]{} & H_{2,q}(J) & \xrightarrow[\partial_{I,J}]{} & H_{2,q-1}(I) & \longrightarrow
\end{array} \tag{1}$$

commutes. Then we say that $[\Theta]$ is a *morphism from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ and we write $[\Theta]\colon \mathcal{H}_1 \to \mathcal{H}_2$. If each $[\Theta](J)$ is an isomorphism, then we say that $[\Theta]$ is an *isomorphism* and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* homology index braids.

REMARK 2.11. *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic homology index braids, then, by Proposition 1.5 in [10], $\mathcal{H}_1$ and $\mathcal{H}_2$ determine the same collection of connection matrices and the same collection of C-connection matrices.*

3. SINGULAR CONTINUATION OF HOMOLOGY INDEX BRAIDS

In this section we will state the main results of this paper: the *Singular nested index filtration theorem* and the *Singular continuation principle for homology index braids and connection matrices*. We shall also briefly discuss an application to a thin domain problem.

Let (X_0, d_0) be a metric space. Let $\varepsilon_0 \in \,]0,\infty[$ and for each $\varepsilon \in \,]0, \varepsilon_0]$ let $(Y_\varepsilon, d_\varepsilon)$ be a metric space and $\theta_\varepsilon \in Y_\varepsilon$ be a distinguished point of Y_ε. The open ball in Y_ε of center in v and radius $\beta > 0$ is denoted by $B_\varepsilon(v, \beta)$. Analogously, $B_\varepsilon[v, \beta]$ is the closed ball in Y_ε of center in $v \in Y_\varepsilon$ and radius $\beta > 0$.

For each $\varepsilon \in \,]0, \varepsilon_0]$ define the set $Z_\varepsilon := X_0 \times Y_\varepsilon$. Endow Z_ε with the metric

$$\Gamma_\varepsilon((u, v), (u', v')) := \max\{\, d_0(u, u'), d_\varepsilon(v, v') \,\} \text{ for } (u, v),\ (u', v') \in Z_\varepsilon.$$

Given a subset V of X_0, $\beta > 0$ and $\varepsilon \in \,]0, \varepsilon_0]$ define the 'inflated' subsets $]V[_{\varepsilon,\beta}$ and $[V]_{\varepsilon,\beta}$ of Z_ε as follows:

$$]V[_{\varepsilon,\beta} := \{\, (u, v) \in Z_\varepsilon \mid u \in V \text{ and } v \in B_\varepsilon(\theta_\varepsilon, \beta) \,\},$$
$$[V]_{\varepsilon,\beta} := \{\, (u, v) \in Z_\varepsilon \mid u \in V \text{ and } v \in \mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \beta)) \,\}.$$

Note the following simple result:

LEMMA 3.12. *If V is open in X_0, $\alpha \in \,]0,\infty[$, $u_0 \in V$, $\varepsilon_n \to 0$, and*

$$\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty,$$

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then $w_n \in \,]V[_{\varepsilon_n,\alpha}$ *for all* $n \in \mathbb{N}$ *large enough.*

We recall the following result proved in [3].

Lemma 3.13. *[3, Lemma 4.10] Let* $\varepsilon \in \,]0, \varepsilon_0]$ *be arbitrary and* $\eta \in \,]0, \infty[$ *be such that the set* $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \eta))$ *is contractible to* θ_ε*. Let* N_1 *and* N_2 *be closed subsets in* X_0 *such that* $N_2 \subset N_1$*. Let* $\iota_{\varepsilon,\eta} \colon N_1/N_2 \to [N_1]_{\varepsilon,\eta} \,/\, [N_2]_{\varepsilon,\eta}$ *be the map induced by the inclusion* $u \mapsto (u, \theta_\varepsilon)$ *and* $T_{\varepsilon,\eta} \colon \, [N_1]_{\varepsilon,\eta} \,/\, [N_2]_{\varepsilon,\eta} \to N_1/N_2$ *be the map induced by the projection onto the first factor.*

Then the maps $\iota_{\varepsilon,\eta}$ *and* $T_{\varepsilon,\eta}$ *are homotopy inverses to each other in the category of pointed spaces. In particular, the pointed spaces* N_1/N_2 *and* $[N_1]_{\varepsilon,\eta} \,/\, [N_2]_{\varepsilon,\eta}$ *have the same homotopy type.*

Let $\varepsilon \in \,]0, \varepsilon_0]$, π_ε (resp. π_0) be a local semiflow on Z_ε (resp. on X_0), S_ε (resp. S_0) be an isolated invariant set relative to π_ε (resp. π_0) and $(M_{p,\varepsilon})_{p\in P})$ (resp. $(M_{p,0})_{p\in P}$) be a Morse decomposition of S_ε (resp. S_0) relative to π_ε (resp. π_0). Let $\mathcal{N}_\varepsilon = (N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ be a strongly π_ε-admissible index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and $\mathcal{N}_0 = (N_0(I))_{I\in\mathcal{A}(\prec)}$ be a strongly π_0-admissible index filtration for $(\pi_0, S_0, (M_{p,0})_{p\in P})$.

Let $\eta \in \,]0, \infty[$ and suppose that the following *singular nesting property* holds:

$$N_\varepsilon(I) \subset [N_0(I)]_{\varepsilon,\eta} \text{ for all } I \in \mathcal{A}(\prec).$$

For $J \in \mathcal{I}(\prec)$ choose I, $K \in \mathcal{A}(\prec)$ with $(I, J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Then $(N_\varepsilon(K), N_\varepsilon(I))$ is an FM-index pair for $M_\varepsilon(J)$, relative to π_ε and $(N_0(K), N_0(I))$ is an FM-index pair for $M_0(J)$, relative to π_0. The composition of the inclusion induced map from $N_\varepsilon(K)/N_\varepsilon(I)$ to $[N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta}$ followed by the map from $[N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta}$ to $N_0(K)/N_0(I)$ induced by the projection onto the first factor induce a homomorphism

$$[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}(J) \colon H(\pi_\varepsilon, M_\varepsilon(J)) \to H(\pi_0, M_0(J)).$$

Of course, this homomorphism depends on the choice of $\varepsilon \in \,]0, \varepsilon_0]$, $\eta \in \,]0, \infty[$, $\mathcal{N}_\varepsilon$ and $\mathcal{N}_0$, but we claim that it is independent of the choice of I and K. In fact, if I' and $K' \in \mathcal{A}(\prec)$ are such that $(I', J) \in \mathcal{I}_2(\prec)$ and $K' = I'J$ then property (2) of index filtrations implies that $N_\varepsilon(K) \setminus N_\varepsilon(I) = N_\varepsilon(K') \setminus N_\varepsilon(I')$ and $N_0(K) \setminus N_0(I) = N_0(K') \setminus N_0(I')$.

Notice also that if $(u, v) \in [N_0(K)]_{\varepsilon,\eta} \setminus [N_0(I)]_{\varepsilon,\eta}$, then $u \in N_0(K) \setminus N_0(I)$ and so $u \in N_0(K') \setminus N_0(I')$. Hence $(u, v) \in [N_0(K')]_{\varepsilon,\eta} \setminus [N_0(I')]_{\varepsilon,\eta}$ and so there is an inclusion induced, hence commutative, diagram of pointed spaces

$$\begin{array}{ccc} N_\varepsilon(K)/N_\varepsilon(I) & \longrightarrow & N_\varepsilon(K')/N_\varepsilon(I') \\ \downarrow & & \downarrow \\ [N_0(K)]_{\varepsilon,\eta} \,/\, [N_0(I)]_{\varepsilon,\eta} & \longrightarrow & [N_0(K')]_{\varepsilon,\eta} \,/\, [N_0(I')]_{\varepsilon,\eta}\,. \end{array}$$

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Moreover, the diagram

$$\begin{array}{ccc}
[N_0(K)]_{\varepsilon,\eta}\,/\,[N_0(I)]_{\varepsilon,\eta} & \longrightarrow & [N_0(K')]_{\varepsilon,\eta}\,/\,[N_0(I')]_{\varepsilon,\eta} \\
\downarrow & & \downarrow \\
N_0(K)/N_0(I) & \longrightarrow & N_0(K')/N_0(I')
\end{array}$$

commutes, where the horizontal maps are inclusion induced and the vertical maps are projection induced. Composing these two diagrams we obtain a commutative diagram

$$\begin{array}{ccc}
N_\varepsilon(K)/N_\varepsilon(I) & \longrightarrow & N_\varepsilon(K')/N_\varepsilon(I') \\
\downarrow & & \downarrow \\
N_0(K)/N_0(I) & \longrightarrow & N_0(K')/N_0(I')
\end{array}$$

with inclusion induced rows. Thus the two vertical maps represent the same map from the (Categorial) homology Morse index of $(\pi_\varepsilon, M_\varepsilon(J))$ to the homology Morse index of $(\pi_0, M_0(J))$. This proves our claim. We write

$$[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0} = ([\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}(J))_{J\in\mathcal{I}(\prec)}.$$

We also claim that $[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0} \colon \mathcal{H}_\varepsilon \to \mathcal{H}_0$. In fact, let $(I,J) \in \mathcal{I}_2(\prec)$ and let B be the set of all $p \in P \setminus (IJ)$ for which there is a $p' \in IJ$ with $p \prec p'$. It follows that B, BI, $BIJ \in \mathcal{A}(\prec)$. Setting $N_{1,\varepsilon} = N_\varepsilon(BIJ)$, $N_{2,\varepsilon} = N_\varepsilon(BI)$ and $N_{3,\varepsilon} = N_\varepsilon(B)$ and $N_{1,0} = N_0(BIJ)$, $N_{2,0} = N_0(BI)$ and $N_{3,0} = N_0(B)$ we see that $(N_{1,\varepsilon}, N_{2,\varepsilon}, N_{3,\varepsilon})$ is an FM-index triple for $(\pi_\varepsilon, M_\varepsilon(IJ), M_\varepsilon(I), M_\varepsilon(J))$ and $(N_{1,0}, N_{2,0}, N_{3,0})$ is an FM-index triple for $(\pi_0, M_0(IJ), M_0(I), M_0(J))$. Moreover, composing the inclusion induced commutative diagram

$$\begin{array}{ccccc}
N_{2,\varepsilon}/N_{3,\varepsilon} & \xrightarrow{\;i\;} & N_{1,\varepsilon}/N_{3,\varepsilon} & \xrightarrow{\;p\;} & N_{1,\varepsilon}/N_{2,\varepsilon} \\
\downarrow & & \downarrow & & \downarrow \\
[N_{2,0}]_{\varepsilon,\eta}\,/\,[N_{3,0}]_{\varepsilon,\eta} & \xrightarrow{\;i\;} & [N_{1,0}]_{\varepsilon,\eta}\,/\,[N_{3,0}]_{\varepsilon,\eta} & \xrightarrow{\;p\;} & [N_{1,0}]_{\varepsilon,\eta}\,/\,[N_{2,0}]_{\varepsilon,\eta}
\end{array}$$

with the inclusion and projection induced commutative diagram

$$\begin{array}{ccccc}
[N_{2,0}]_{\varepsilon,\eta}\,/\,[N_{3,0}]_{\varepsilon,\eta} & \xrightarrow{\;i\;} & [N_{1,0}]_{\varepsilon,\eta}\,/\,[N_{3,0}]_{\varepsilon,\eta} & \xrightarrow{\;p\;} & [N_{1,0}]_{\varepsilon,\eta}\,/\,[N_{2,0}]_{\varepsilon,\eta} \\
\downarrow & & \downarrow & & \downarrow \\
N_{2,0}/N_{3,0} & \xrightarrow{\;i\;} & N_{1,0}/N_{3,0} & \xrightarrow{\;p\;} & N_{1,0}/N_{2,0}
\end{array}$$

yields a commutative diagram which implies the commutativity of diagram (1).

We call $[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}$ a *singular inclusion induced morphism from $\mathcal{H}_\varepsilon$ to $\mathcal{H}_0$*. Under certain conditions, $[\Theta]_{\varepsilon,\eta,\mathcal{N}_\varepsilon,\mathcal{N}_0}$ is an isomorphism:

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PROPOSITION 3.14. *Let ε, $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and $(\pi_0, S_0, (M_{p,0})_{p\in P})$ be as above. Suppose there are $\widetilde{\rho}$, $\widetilde{\eta} \in \,]0,\infty[$ are such that $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \widetilde{\rho}))$ and $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \widetilde{\eta}))$ are contractible to θ_ε. Let $\mathcal{N}_0 = (N_0(I))_{I\in\mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_0 = (\widetilde{N}_0(I))_{I\in\mathcal{A}(\prec)}$ be strongly π_0-admissible index filtrations for $(\pi_0, S_0, (M_{p,0})_{p\in P})$, $\mathcal{N}_\varepsilon = (N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_\varepsilon = (\widetilde{N}_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ be strongly π_ε-admissible index filtrations for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ and such that the following* singular nesting property *holds:*

$$N_\varepsilon(I) \subset [N_0(I)]_{\varepsilon,\widetilde{\rho}} \subset \widetilde{N}_\varepsilon(I) \subset [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}} \text{ for all } I \in \mathcal{A}(\prec). \tag{2}$$

Then $\mathcal{H}(\pi_0, S_0, (M_p)_{p\in P})$ and $\mathcal{H}(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P}))$ are isomorphic.

Proof. Formula (2) implies that

$$N_0(I) \subset \widetilde{N}_0(I), \quad I \in \mathcal{A}(\prec). \tag{3}$$

Let $J \in \mathcal{I}(\prec)$ and choose I, $K \in \mathcal{A}(\prec)$ with $(I,J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Hence there are inclusion induced maps

$$a : N_\varepsilon(K)/N_\varepsilon(I) \to [N_0(K)]_{\varepsilon,\widetilde{\rho}} \,/\, [N_0(I)]_{\varepsilon,\widetilde{\rho}}\,,$$

$$b : [N_0(K)]_{\varepsilon,\widetilde{\rho}} \,/\, [N_0(I)]_{\varepsilon,\widetilde{\rho}} \to \widetilde{N}_\varepsilon(K)/\widetilde{N}_\varepsilon(I),$$

and

$$c : \widetilde{N}_\varepsilon(K)/\widetilde{N}_\varepsilon(I) \to [N_0(K)]_{\varepsilon,\widetilde{\eta}} \,/\, [N_0(I)]_{\varepsilon,\widetilde{\eta}}$$

in the homotopy category of pointed spaces. Moreover, there is a commutative diagram

$$\begin{array}{ccc} [N_0(K)]_{\varepsilon,\widetilde{\rho}} \,/\, [N_0(I)]_{\varepsilon,\widetilde{\rho}} & \longrightarrow & [\widetilde{N}_0(K)]_{\varepsilon,\widetilde{\eta}} \,/\, [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}} \\ \big\downarrow & & \big\downarrow \\ N_0(K)/N_0(I) & \longrightarrow & \widetilde{N}_0(K)/\widetilde{N}_0(I) \end{array}$$

in the homotopy category of pointed spaces, with inclusion induced horizontal maps and projection induced vertical maps. By Lemma 3.13 the vertical maps are isomorphisms. The lower horizontal map is an isomorphism, lying in the Categorial Morse index of $(\pi_0, M_0(J))$. (See Theorem I.9.4 in [21].) This implies that the upper horizontal map is an isomorphism. In other words, $c\circ b$ is an isomorphism. Since the inclusion induced map $b\circ a$ lies the Categorial Morse index of $(\pi_\varepsilon, M_\varepsilon(J))$, it follows that $b\circ a$ is an isomorphism. Altogether we obtain that a, b and c are isomorphisms. Thus the map $T_{\varepsilon,\widetilde{\rho}} \circ a$ is an isomorphism, where $T_{\varepsilon,\widetilde{\rho}}$ is as in Lemma 3.13 with $(N_1, N_2) := (N_0(K), N_0(I))$. This shows that the map $[\Theta]_{\varepsilon,\tilde{\rho},\mathcal{N}_\varepsilon,\mathcal{N}_0}(J) : H(\pi_\varepsilon, M_\varepsilon(J)) \to H(\pi_0, M_0(J))$ induced by $T_{\varepsilon,\widetilde{\rho}} \circ a$ is an isomorphism. The proposition is proved. ∎

In the next two definitions, introduced in [3], $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ is a family such that, for every $\varepsilon \in \,]0, \varepsilon_0]$, π_ε is a local semiflow on Z_ε. Moreover, π_0 is a local semiflow on X_0.

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Definition 3.15. *With the notation introduced above, we say that the family* $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *converges singularly to* π_0 *if whenever* $(\varepsilon_n)_n$ *and* $(t_n)_n$ *are sequences of positive numbers such that* $\varepsilon_n \to 0$, $t_n \to t_0$ *as* $n \to \infty$, *for some* $t_0 \in [0,\infty[$ *and whenever* $u_0 \in X_0$ *and* $w_n \in Z_{\varepsilon_n}$ *are such that* $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n})) \to 0$ *as* $n \to \infty$ *and* $u_0\pi_0 t_0$ *is defined, then there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$, $w_n\pi_{\varepsilon_n}t_n$ *is defined and*

$$\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n,(u_0\pi_0 t_0,\theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

Definition 3.16. *Let* β *be a positive number and* N *be a closed subset of* X_0. *We say that* N *is a singularly strongly admissible set with respect to* β *and the family* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *if the following conditions are satisfied:*

1. N *is a strongly* π_0*-admissible set;*
2. *for each* $\varepsilon \in]0,\varepsilon_0]$ *the set* $[N]_{\varepsilon,\beta}$ *is strongly* π_ε*-admissible;*
3. *whenever* $(\varepsilon_n)_n$ *and* $(t_n)_n$ *are sequences of positive numbers such that* $\varepsilon_n \to 0$, $t_n \to \infty$ *as* $n \to \infty$ *and whenever* $w_n \in Z_{\varepsilon_n}$ *is such that* $w_n\pi_{\varepsilon_n}[0,t_n] \subset [N]_{\varepsilon_n,\beta}$, *then there exist a* $u_0 \in N$ *and a subsequence of the sequence* $(w_n\pi_{\varepsilon_n}t_n)_n$ *of endpoints, denoted again by* $(w_n\pi_{\varepsilon_n}t_n)_n$, *such that* $\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n,(u_0,\theta_{\varepsilon_n})) \to 0$ *as* $n \to \infty$.

The following singular continuation result for Morse decompositions was established in [5].

Theorem 3.17. [cf. Corollaries 4.14 and 4.15 in [5]] *Assume* $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *is a family of local semiflows that converges singularly to the local semiflow* π_0, $\beta \in]0,\infty[$ *and* $\widetilde{N}$ *is a singularly strongly admissible set with respect to* β *and* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$. *Moreover, suppose that* $S_0 := \mathrm{Inv}_{\pi_0}(\widetilde{N})$ *relative to* π_0 *and* $(M_{p,0})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* S_0 *relative to* π_0. *For each* $p \in P$, *let* $\Xi_p \subset \widetilde{N}$ *be closed in* X_0 *and such that* $M_{p,0} = \mathrm{Inv}_{\pi_0}(\Xi_p) \subset \mathrm{Int}_{X_0}(\Xi_p)$. (Such sets Ξ_p, $p \in P$, always exist.)

Let $\eta \in]0,\beta]$. *For* $\varepsilon \in]0,\varepsilon_0]$ *and* $p \in P$ *set* $S_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}([\widetilde{N}]_{\varepsilon,\eta})$ *and* $M_{p,\varepsilon} := \mathrm{Inv}_{\pi_\varepsilon}([\Xi_p]_{\varepsilon,\eta})$. *Then there is an* $\widetilde{\varepsilon} \in]0,\varepsilon_0]$ *such that for every* $\varepsilon \in]0,\widetilde{\varepsilon}]$ *and* $p \in P$, $S_\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}([\widetilde{N}]_{\varepsilon,\eta})$ *and the family* $(M_{p,\varepsilon})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* S_ε *relative to* π_ε.

Remark 3.18. *The family* $(S_\varepsilon,(M_{p,\varepsilon})_{p\in P})$ *is, asymptotically for* $\varepsilon \to 0$, *independent of the choice of* $\beta \in]0,\infty[$, $\eta \in]0,\infty]$, $\tilde{N}$ *and* Ξ_p, $p \in P$ *in the sense that if the hypotheses of Theorem 3.17 are also satisfied with respect to some* β', η', $\tilde{N}'$ *and* Ξ'_p, $p \in P$, *then, for some* $\varepsilon_1 \in]0,\varepsilon_0]$ *and all* $\varepsilon \in]0,\varepsilon_1]$, $\mathrm{Inv}_{\pi_\varepsilon}([\tilde{N}]_{\varepsilon,\eta}) = \mathrm{Inv}_{\pi_\varepsilon}([\tilde{N}']_{\varepsilon,\eta'})$ *and* $\mathrm{Inv}_{\pi_\varepsilon}([\Xi]_{\varepsilon,\eta}) = \mathrm{Inv}_{\pi_\varepsilon}([\Xi']_{\varepsilon,\eta'})$, $p \in P$. *This is a consequence of following result whose proof follows from Corollary 4.7 in [5] and the proof of Lemma 4.5 in [3].*

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Proposition 3.19. *Suppose that* $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *is a family of local semiflows that converges singularly to the local semiflow* π_0, $\beta\in\,]0,\infty[$, *and* Y_1 *and* Y_2 *are two (not necessarily distinct) closed sets which are singularly strongly admissible set with respect to* β *and* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$. *Suppose that* $\mathrm{Inv}_{\pi_0}(Y_1)=\mathrm{Inv}_{\pi_0}(Y_2)\subset\mathrm{Int}_{X_0}(Y_1)\cap\mathrm{Int}_{X_0}(Y_2)$. *Let* $0<\alpha<\eta<\beta$. *Then there exists an* $\overline{\varepsilon}\in\,]0,\varepsilon_0]$ *such that for all* $\varepsilon\in\,]0,\overline{\varepsilon}]$, $\mathrm{Inv}_{\pi_\varepsilon}([Y_1]_{\varepsilon,\eta})=\mathrm{Inv}_{\pi_\varepsilon}([Y_2]_{\varepsilon,\alpha})\subset\mathrm{Int}_{Z_\varepsilon}([Y_1]_{\varepsilon,\eta})\cap\mathrm{Int}_{Z_\varepsilon}([Y_2]_{\varepsilon,\alpha})$.

We will use Proposition 3.19 tacitly in the sequel.

We can now state the first main result of the paper, the *Singular nested index filtration principle*.

Theorem 3.20. *Assume the hypotheses (and thus also the conclusions) of Theorem 3.17 and let* $\widetilde{\varepsilon}>0$ *be as in that theorem. Let* $\widetilde{\beta}_0\in\,]0,\beta[$ *be fixed. Then there are* $\widetilde{\rho}$, $\widetilde{\eta}\in\,]0,\widetilde{\beta}_0]$ *and an* $\varepsilon_{\mathrm{c}}\in\,]0,\widetilde{\varepsilon}]$ *such that for every* $\varepsilon\in[0,\varepsilon_{\mathrm{c}}]$ *there exist strongly* π_ε*-admissible index filtrations* $\mathcal{N}_\varepsilon=(N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ *and* $\widetilde{\mathcal{N}}_\varepsilon=(\widetilde{N}_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ *for* $(\pi_\varepsilon,S_\varepsilon,(M_{p,\varepsilon})_{p\in P})$ *such that the following* singular nesting property *holds:*

$$N_\varepsilon(I)\subset[N_0(I)]_{\varepsilon,\widetilde{\rho}}\subset\widetilde{N}_\varepsilon(I)\subset[\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}}\ \textit{for all}\ I\in\mathcal{A}(\prec)\ \textit{and}\ \varepsilon\in\,]0,\varepsilon_{\mathrm{c}}]. \tag{4}$$

Theorem 3.20, Proposition 3.14 and Remark 2.11 immediately imply the following *Singular continuation principle for homology index braids and connection matrices*.

Theorem 3.21. *Assume the hypotheses of Theorem 3.20. Suppose that there exists an* $\beta_0>0$ *such that for all* $\varepsilon\in\,]0,\varepsilon_0]$ *and all* $\eta\in\,]0,\beta_0]$ *the set* $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\eta))$ *is contractible. Then there exists an* $\varepsilon_{\mathrm{c}}\in\,]0,\widetilde{\varepsilon}]$ *such that the homology index braids* $\mathcal{H}(\pi_0,S_0,(M_p)_{p\in P})$ *and* $\mathcal{H}(\pi_\varepsilon,S_\varepsilon,(M_{p,\varepsilon})_{p\in P}))$, $\varepsilon\in\,]0,\varepsilon_{\mathrm{c}}]$, *are isomorphic and determine the same collection of connection matrices and the same collection of* C*-connection matrices.*

Proof. Let $\widetilde{\beta}_0<\min\{\beta_0,\beta\}$. Theorem 3.20 implies that there are $\widetilde{\rho}$, $\widetilde{\eta}\in\,]0,\widetilde{\beta}_0]$ and an $\varepsilon_{\mathrm{c}}\in\,]0,\widetilde{\varepsilon}]$ such that for every $\varepsilon\in[0,\varepsilon_{\mathrm{c}}]$ there exist strongly π_ε-admissible index filtrations $\mathcal{N}_\varepsilon=(N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_\varepsilon=(\widetilde{N}_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ for $(\pi_\varepsilon,S_\varepsilon,(M_{p,\varepsilon})_{p\in P})$ such that the singular nesting property (4) holds. Since $\widetilde{\eta}$, $\widetilde{\rho}\in\,]0,\beta_0[$, Proposition 3.14 and Remark 2.11 imply the result. ▌

Theorem 3.21 refines the corresponding singular Conley index continuation principle established in [3].

We will now briefly illustrate Theorem 3.21 by applying it to the thin domain problem considered in [15] and [1]. More extensive applications will appear in a subsequent publication. We assume the reader's familiarity with [15] and [1] and only recall some of the relevant notations and definitions.

Let M and N be positive integers. Write (x,y) for a generic point of $\mathbb{R}^M\times\mathbb{R}^N$. Let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^M\times\mathbb{R}^N$ with Lipschitz boundary and let

$\varepsilon > 0$ be arbitrary. Define the symmetric bilinear form $a_\epsilon \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a_\epsilon(u,v) := \int_\Omega \left(\nabla_x u \cdot \nabla_x v + \frac{1}{\epsilon^2} \nabla_y u \cdot \nabla_y v \right) dx\, dy$$

and let b be the scalar product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$. Let $A_\varepsilon \colon D(A_\varepsilon) \subset H^1(\Omega) \to L^2(\Omega)$ be the linear operator generated by the pair (a_ε, b). We define on $H^1(\Omega)$ the scalar product

$$(u,v)_\varepsilon := a_\varepsilon(u,v) + b(u,v), \quad u, v \in H^1(\Omega)$$

and the corresponding norm

$$|u|_\epsilon := \left(a_\epsilon(u,u) + |u|^2_{L^2(\Omega)} \right)^{1/2}, \quad u \in H^1(\Omega)$$

which is equivalent to the usual norm on $H^1(\Omega)$.

We also define the "limit" space $H^1_s(\Omega)$ by

$$H^1_s(\Omega) = \{\, u \in H^1(\Omega) \mid \nabla_y u = 0 \,\}.$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of $H^1(\Omega)$ so $H^1_s(\Omega)$ is a Hilbert space under the usual scalar product of $H^1(\Omega)$.

Furthermore, define the space $L^2_s(\Omega)$ to be the closure of the set $H^1_s(\Omega)$ in $L^2(\Omega)$. It follows that $L^2_s(\Omega)$ is a Hilbert space under the scalar product of $L^2(\Omega)$.

Now let $a_0 \colon H^1_s(\Omega) \times H^1_s(\Omega) \to \mathbb{R}$ be the "limit" bilinear form defined by

$$a_0(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx\, dy = \int_\Omega \nabla_x u \cdot \nabla_x v \, dx\, dy.$$

Finally, let b_0 be the restriction of the scalar product b to $L^2_s(\Omega) \times L^2_s(\Omega)$. Denote by A_0 the operator generated by the pair (a_0, b_0).

Now let $\varepsilon_0 \in\,]0,1]$ be arbitrary and $(f_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ be a family satisfying hypothesis (A1) introduced in Definition 2.6 in [1]. For $\varepsilon \in\,]0, \varepsilon_0]$ let π_ε be the local semiflow on $H^1(\Omega)$ generated by the solutions of the evolution equation

$$\dot{u} = A_\varepsilon u + f_\varepsilon(u).$$

Moreover, let π_0 be the local semiflow on $H^1_s(\Omega)$ generated by the solutions of the evolution equation

$$\dot{u} = A_0 u + f_0(u).$$

We will need the following singular convergence result proved in [1].

PROPOSITION 3.22. (Cf. Corollary 2.15 in [1] and its proof) *Let $(\varepsilon_n)_n$ be an arbitrary sequence of positive numbers convergent to zero. Moreover let $t \in [0, \infty[$ and $(t_n)_n$ be a*

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sequence in $[0,\infty[$ *converging to* t. *Finally, let* $u_0 \in H^1_s(\Omega)$ *and* $(u_n)_n$ *be a sequence in* $H^1(\Omega)$ *such that* $|u_n - u_0|_{\varepsilon_n} \to 0$ $n \to \infty$. *Assume that* $u_0\pi_0 t$ *is defined. Then, for all* $n \in \mathbb{N}$ *large enough,* $u_n\pi_n t_n$ *is defined and*

$$\left|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t\right|_{\varepsilon_n} \to 0 \quad \text{as } n \to \infty.$$

For all $\varepsilon \in]0,\varepsilon_0]$ set $\theta_\varepsilon := 0 \in H^1(\Omega)$ and let $Q_\varepsilon\colon H^1(\Omega) \to H^1(\Omega)$ be the orthogonal projector onto $H^1_s(\Omega)$ with respect to the scalar product $(\cdot,\cdot)_\varepsilon$. Let $X_0 := H^1_s(\Omega)$ be endowed with the usual norm of $H^1(\Omega)$ and d_0 be the corresponding metric on X_0. Moreover, let $Y_\varepsilon := (I - Q_\varepsilon)(H^1(\Omega))$ be endowed with the norm $|\cdot|_\varepsilon$ and let d_ε be the corresponding metric on Y_ε. Set $Z_\varepsilon := X_0 \times Y_\varepsilon \cong H^1(\Omega)$ and note that the norm

$$\|(u,v)\|_\varepsilon := \max\{|u|_{H^1(\Omega)}, |v|_\varepsilon\}, \quad (u,v) \in X_0 \times Y_\varepsilon,$$

is equivalent to the norm $|\cdot|_\varepsilon$ on $H^1(\Omega)$ with constants independent of $\varepsilon \in]0,\varepsilon_0]$. Let Γ_ε be the metric on Z_ε generated by the norm $\|\cdot\|_\varepsilon$.

The remarks just made imply that, for every $\varepsilon \in]0,\varepsilon_0]$, π_ε is a local semiflow on Z_ε and π_0 is a local semiflow on X_0, while Proposition 3.22 just says that $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ singularly converges to π_0.

Now an application of Lemma 2.21 in [1] shows that whenever $\beta > 0$ and N is closed and bounded in X_0 then N is singularly admissible with respect to β and the family $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$.

It is clear that for all $\varepsilon \in]0,\varepsilon_0]$ and all $\beta \in]0,\infty[$ the set $\mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\beta))$ is contractible to θ_ε.

We thus obtain the following corollary of Theorems 3.17 and 3.20.

Theorem 3.23. *Let* β *be a positive number and* $N \subset H^1_s(\Omega)$ *be closed and bounded. Suppose that* $(M_p)_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* $S_0 := \mathrm{Inv}_{\pi_0}(N)$ *relative to* π_0. *For each* $p \in P$, *let* $\Xi_p \subset N$ *be closed in* X_0 *and such that* $M_p = \mathrm{Inv}_{\pi_0}(\Xi_p) \subset \mathrm{Int}_{X_0}(\Xi_p)$. *Moreover, for every* $I \in I(\prec)$, *let* $\Xi_I \subset N$ *be closed in* X_0 *and such that* $M_{\pi_0,S_0}(I) = \mathrm{Inv}_{\pi_0}(\Xi_I) \subset \mathrm{Int}_{X_0}(\Xi_I)$. *For* $\varepsilon \in]0,\varepsilon_0]$ *and* $p \in P$ *set* $M_p(\varepsilon) := \mathrm{Inv}_{\pi_\varepsilon}([\Xi_p]_{\varepsilon,\beta})$. *Then there is an* $\widetilde{\varepsilon} \in]0,\varepsilon_0]$ *such that for every* $\varepsilon \in]0,\widetilde{\varepsilon}]$ *the family* $(M_p(\varepsilon))_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* $S_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}([N]_{\varepsilon,\beta})$ *relative to* π_ε. *Moreover,*

$$M_I(\varepsilon) := M_{\pi_\varepsilon,S_\varepsilon} = \mathrm{Inv}_{\pi_\varepsilon}([\Xi_I]_{\varepsilon,\beta}) \subset \mathrm{Int}_{Z_\varepsilon}([\Xi_I]_{\varepsilon,\beta}), \quad I \in \mathcal{I}(\prec).$$

Finally, the homology index braids $\mathcal{H}(\pi_0, S_0, (M_p)_{p\in P})$ *and* $\mathcal{H}(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P}))$ *are isomorphic and determine the same collection of connection matrices and the same collection of* C*-connection matrices.*

4. SEQUENCES OF INDEX FILTRATIONS

This and the next two sections of this paper are devoted to the proof of Theorem 3.20. Therefore, for the rest of the paper, assume the hypotheses of Theorem 3.17 with $\eta := \beta$ and let $\widetilde{\varepsilon} > 0$ be as in that theorem. For $I \in \mathcal{I}(\prec)$ and $\varepsilon \in [0, \widetilde{\varepsilon}]$, let $M_\varepsilon(I) := M_{\pi_\varepsilon, S_\varepsilon}(I)$.

For each $\varepsilon \in]0, \varepsilon_0]$, given $w \in Z_\varepsilon$ and $t \geq 0$ such that $w\pi_\varepsilon t$ is defined, the components of $w\pi_\varepsilon t$ are denoted by $(w\phi_\varepsilon t, w\psi_\varepsilon t)$, where $w\phi_\varepsilon t \in X_0$ and $w\psi_\varepsilon t \in Y_\varepsilon$.

If $S_0 = \emptyset$, then, by Proposition 4.4 in [3], $S_\varepsilon = \emptyset$ for all ε small enough, so we may choose $N_\varepsilon(I) := \widetilde{N}_\varepsilon(I) := \emptyset$, $I \in \mathcal{A}(\prec)$, $\varepsilon \in [0, \widetilde{\varepsilon}]$. Hence Theorem 3.20 holds in this case.

Therefore we may assume that $S_0 \neq \emptyset$. Taking the sets $\widetilde{N}$ and Ξ_p, $p \in P$, smaller if necessary, we may assume from now on that $\widetilde{N}$ is an isolating block relative to π_0 (cf. [19] or [21]). Let $\widetilde{U} = \mathrm{Int}_{X_0}(\widetilde{N})$.

In this section, starting with sequences of FM-index triples satisfying certain inclusions, we will construct index filtrations with a “singular asymptotic” nesting property. This is the crucial abstract step in the proof of Theorem 3.20.

Theorem 4.24. *Let N_1^i, N_2^i, N_I^i, $i = 2, 4$, $I \in \mathcal{A}(\prec)$, be sets such that, for each $I \in \mathcal{A}(\prec)$, (N_1^i, N_I^i, N_2^i) is an FM-index triple for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$, $i = 2, 4$. Moreover let $\widetilde{\varepsilon}_0 \in]0, \widetilde{\varepsilon}]$ and for each $\varepsilon \in]0, \widetilde{\varepsilon}_0]$, let $N_{1,\varepsilon}$, $N_{2,\varepsilon}$, $N_{I,\varepsilon}$, $I \in \mathcal{A}(\prec)$, be sets such that, for each $I \in \mathcal{A}(\prec)$, $(N_{1,\varepsilon}, N_{I,\varepsilon}, N_{2,\varepsilon})$ is an FM-index triple for $(\pi_\varepsilon, S_\varepsilon, M_\varepsilon(I), M_\varepsilon(P \setminus I))$. For each $p \in P$ and $\varepsilon \in]0, \widetilde{\varepsilon}_0]$ define the following sets:*

$$D_{p,\varepsilon} := \Big(\bigcap_{\substack{I \\ p \in I}} \mathrm{Int}_{Z_\varepsilon}(N_{I,\varepsilon} \setminus N_{2,\varepsilon})\Big) \cap \Big(\bigcap_{\substack{I \\ p \notin I}} \mathrm{Int}_{Z_\varepsilon}(N_{1,\varepsilon} \setminus N_{I,\varepsilon})\Big),$$

$$D_{p,0} := \Big(\bigcap_{\substack{I \\ p \in I}} \mathrm{Int}_{X_0}(N_I^2 \setminus N_2^4)\Big) \cap \Big(\bigcap_{\substack{I \\ p \notin I}} \mathrm{Int}_{X_0}(N_1^2 \setminus N_I^4)\Big),$$

$$E_{p,\varepsilon} := \{\, w \in N_{1,\varepsilon} \mid \text{there is a } t \geq 0 \text{ such that } w\pi_\varepsilon[0,t] \subset N_{1,\varepsilon} \text{ and } w\pi_\varepsilon t \in D_{p,\varepsilon} \,\}$$

and

$$E_{p,0} := \{\, u \in N_1^2 \mid \text{there exists a } t \geq 0 \text{ such that } u\pi_0[0,t] \subset N_1^2 \text{ and } u\pi_0 t \in D_{p,0} \,\}.$$

Let μ, α and η be positive numbers with $\mu < \alpha < \eta$ and V_1 and $V_{I,i}$, $i = 3, 4$, $I \in \mathcal{A}(\prec)$, be open sets in X_0 such that for all $\varepsilon \in]0, \widetilde{\varepsilon}_0]$ and $I \in \mathcal{A}(\prec)$ the following inclusions hold:

$$N_1^2 \subset \widetilde{U}, \quad [N_1^2]_{\varepsilon,\mu} \subset]V_1[_{\varepsilon,\alpha} \subset N_{1,\varepsilon} \subset [N_1^4]_{\varepsilon,\eta}, \quad N_2^2 \subset N_2^4, \quad N_I^2 \subset N_I^4,$$
$$\mathrm{Cl}_{Z_\varepsilon}([N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta}) \subset]V_{I,3}[_{\varepsilon,\alpha} \subset \mathrm{Int}_{Z_\varepsilon}(N_{I,\varepsilon} \setminus N_{2,\varepsilon}),$$
$$\mathrm{Cl}_{Z_\varepsilon}([N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta}) \subset]V_{I,4}[_{\varepsilon,\alpha} \subset \mathrm{Int}_{Z_\varepsilon}(N_{1,\varepsilon} \setminus N_{I,\varepsilon}).$$

For each $I \in \mathcal{A}(\prec)$ define

$$N_0(I) := N_1^2 \setminus \bigcup_{p \in P \setminus I} E_{p,0},$$
$$N_\varepsilon(I) := N_{1,\varepsilon} \setminus \bigcup_{p \in P \setminus I} E_{p,\varepsilon} \text{ for all } \in]0, \widetilde{\varepsilon}_0].$$

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Then $(N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ is an index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ for all $\varepsilon \in [0, \widetilde{\varepsilon}_0]$. Moreover,

$$N_2^2 \subset N_0(\emptyset) \text{ and } N_I^2 \subset N_0(I) \subset N_1^2 \text{ for all } I \in \mathcal{A}(\prec), \tag{5}$$

and

$$N_{2,\varepsilon} \subset N_\varepsilon(\emptyset) \text{ and } N_{I,\varepsilon} \subset N_\varepsilon(I) \subset N_{1,\varepsilon} \text{ for all } \varepsilon \in \,]0, \widetilde{\varepsilon}_0] \text{ and for all } I \in \mathcal{A}(\prec). \tag{6}$$

Furthermore, whenever $I \in \mathcal{A}(\prec)$, $(\varepsilon_n)_n$ is a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$ and $(w_n)_n$, with $w_n = (u_n, v_n)$ for $n \in \mathbb{N}$, is a sequence such that $w_n \in [N_1^2]_{\varepsilon_n,\mu} \cap N_{\varepsilon_n}(I)$ for all $n \in \mathbb{N}$ and $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$, for some $u_0 \in X_0$, then there exists an $n_0 \in \mathbb{N}$ such that $u_n \in N_0(I)$ for all $n \geq n_0$.

Proof. Theorem 2.10 immediately implies that $(N_0(I))_{I\in\mathcal{A}(\prec)}$ is an index filtration for $(\pi_0, S_0, (M_{p,0})_{p\in P})$ satisfying (5). Moreover, the same theorem with $N_j^i = N_{j,\varepsilon}$ and $N_I^i = N_{I,\varepsilon}$ for $i = 2$, 4 and $j = 1$, 2 implies that for all $\varepsilon \in \,]0, \widetilde{\varepsilon}_0]$, $(N_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ is an index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ satisfying (6).

Suppose the second part of the theorem does not hold. Then there exist a $J \in \mathcal{A}(\prec)$, a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence $(w_n)_n$, with $w_n = (u_n, v_n)$ for $n \in \mathbb{N}$, such that $w_n \in [N_1^2]_{\varepsilon_n,\mu} \cap N_{\varepsilon_n}(J)$ for all $n \in \mathbb{N}$, $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$, for some $u_0 \in X_0$ and $u_n \notin N_0(J)$ for all $n \in \mathbb{N}$.

The definitions of the sets $N_{\varepsilon_n}(J)$ and $N_0(J)$ imply that

$$w_n \in N_{1,\varepsilon_n} \setminus \bigcup_{p\in P\setminus J} E_{p,\varepsilon_n} \text{ and } u_n \notin N_1^2 \setminus \bigcup_{p\in P\setminus J} E_{p,0} \text{ for all } n \in \mathbb{N}.$$

Since $w_n \in [N_1^2]_{\varepsilon_n,\mu} \cap N_{\varepsilon_n}(I)$ for all $n \in \mathbb{N}$, it follows that $u_n \in N_1^2$ for all $n \in \mathbb{N}$ and so $u_n \in \bigcup_{p\in P\setminus J} E_{p,0}$ for all $n \in \mathbb{N}$. Thus, taking further subsequences if necessary, we may assume that there exists a $q \in P \setminus J$ such that for all $n \in \mathbb{N}$, $u_n \in E_{q,0}$. So, for each $n \in \mathbb{N}$, there exists a $t_n \geq 0$ such that $u_n\pi_0[0, t_n] \subset N_1^2$ and $u_n\pi_0 t_n \in D_{q,0}$. Set

$$\tilde{V}_p := (\bigcap_{\substack{I \\ p\in I}} V_{I,3}) \cap (\bigcap_{\substack{I \\ p\notin I}} V_{I,4}), \quad p \in P.$$

We claim that

$$[\mathrm{Cl}_{X_0}(D_{p,0})]_{\varepsilon,\mu} \subset \,]\widetilde{V}_p[_{\varepsilon,\alpha} \subset D_{p,\varepsilon}, \quad p \in P,\, \varepsilon \in \,]0, \widetilde{\varepsilon}_0]\,. \tag{7}$$

In fact, fix $p \in P$ and $\varepsilon \in \,]0, \widetilde{\varepsilon}_0]$. It follows that

$$\mathrm{Cl}_{X_0}(D_{p,0}) \subset (\bigcap_{\substack{I \\ p\in I}} \mathrm{Cl}_{X_0}(N_I^2 \setminus N_2^4)) \cap (\bigcap_{\substack{I \\ p\notin I}} \mathrm{Cl}_{X_0}(N_1^2 \setminus N_I^4)).$$

Let $w = (u, v) \in [\mathrm{Cl}_{X_0}(D_{p,0})]_{\varepsilon,\mu}$. Then, $u \in \mathrm{Cl}_{X_0}(D_{p,0})$ and $v \in \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon, \mu))$. Let $I \in \mathcal{A}(\prec)$. If $p \in I$, then $u \in \mathrm{Cl}_{X_0}(N_I^2 \setminus N_2^4)$ and so there exists a sequence $(\tilde{u}_n)_n$ in $N_I^2 \setminus N_2^4$ such that $d_0(\tilde{u}_n, u) \to 0$ as $n \to \infty$. Hence $(\tilde{u}_n, v) \in [N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta}$, for all $n \in \mathbb{N}$ and $\Gamma_\varepsilon((\tilde{u}_n, v), (u, v)) \to 0$ as $n \to \infty$. This implies that $w \in \mathrm{Cl}_{Z_\varepsilon}([N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta})$. Similar arguments prove that $w \in \mathrm{Cl}_{Z_\varepsilon}([N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta})$ for all $p \notin I$. Hence

$$\begin{aligned}
[\mathrm{Cl}_{X_0}(D_{p,0})]_{\varepsilon,\mu} &\subset \Big(\bigcap_{\substack{I\\ p\in I}} \mathrm{Cl}_{Z_\varepsilon}([N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta})\Big) \cap \Big(\bigcap_{\substack{I\\ p\notin I}} \mathrm{Cl}_{Z_\varepsilon}([N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta})\Big) \\
&\subset \Big(\bigcap_{\substack{I\\ p\in I}}]V_{I,3}[_{\varepsilon,\alpha}\Big) \cap \Big(\bigcap_{\substack{I\\ p\notin I}}]V_{I,4}[_{\varepsilon,\alpha}\Big) = \Big]\Big(\bigcap_{\substack{I\\ p\in I}} V_{I,3}\Big) \cap \Big(\bigcap_{\substack{I\\ p\notin I}} V_{I,4}\Big)\Big[_{\varepsilon,\alpha} =]\tilde{V}_p[_{\varepsilon,\alpha} \\
&\subset \Big(\bigcap_{\substack{I\\ p\in I}} \mathrm{Int}_{Z_\varepsilon}(N_{I,\varepsilon} \setminus N_{2,\varepsilon})\Big) \cap \Big(\bigcap_{\substack{I\\ p\notin I}} \mathrm{Int}_{Z_\varepsilon}(N_{1,\varepsilon} \setminus N_{I,\varepsilon})\Big) = D_{p,\varepsilon}.
\end{aligned}$$

This proves (7). From (7) obtain that

$$\mathrm{Cl}_{X_0} D_{p,0} \subset \tilde{V}_p, \quad p \in P. \tag{8}$$

In fact, if $u \in \mathrm{Cl}_{X_0} D_{p,0}$ then $(u, \theta_\varepsilon) \in [\mathrm{Cl}_{X_0}(D_{p,0})]_{\varepsilon,\mu} \subset]\tilde{V}_p[_{\varepsilon,\alpha}$ so $u \in \tilde{V}_p$.

To complete the proof of the theorem we will consider two cases.

Case 1. Suppose that $(t_n)_n$ is a bounded sequence. We can assume, taking subsequences if necessary, that there exists a $t \in [0, \infty[$ such that $t_n \to t$ as $n \to \infty$. Since $N_1^2 \subset \widetilde{U} \subset \widetilde{N}$, it follows that π_0 does not explode in N_1^2 and so

$$u_0\pi_0[0, t] \subset N_1^2 \subset V_1. \tag{9}$$

Recall that the family $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ converges singularly to the local semiflow π_0, $\varepsilon_n \to 0$ and $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$, thus, it follows that

$$\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n, (u_0\pi_0 t, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty. \tag{10}$$

Moreover, since $u_n\pi_0 t_n \in D_{q,0}$, $n \in \mathbb{N}$, it follows that $u_0\pi_0 t \in \mathrm{Cl}_{X_0} D_{q,0}$, so, by (8), $u_0\pi_0 t \in \tilde{V}_q$. Formulas (9) and (10) together with Lemma 3.12 imply that there is an $n_1 \in \mathbb{N}$ such that $w_n\pi_{\varepsilon_n}t_n \in]\widetilde{V}_q[_{\varepsilon_n,\alpha} \subset D_{q,\varepsilon_n}$ for all $n \geq n_1$. Thus $w_n \in E_{q,\varepsilon_n}$ for all $n \geq n_1$ with $q \in P \setminus J$. However, this is a contradiction to our choice of the sequence $(w_n)_n$.

Case 2. Suppose that $(t_n)_n$ is an unbounded sequence. We can assume, taking subsequences if necessary, that $t_n \to \infty$ as $n \to \infty$. Since $u_n\pi_0[0, t_n] \subset N_1^2$ for all $n \in \mathbb{N}$, and $d_0(u_n, u_0) \to 0$ as $n \to \infty$, it follows that $u_0\pi_0\,[0, \infty[\subset N_1^2$.

Note that if $n \in \mathbb{N}$ and $0 \leq t \leq t_n$, since $u_n\pi_0[0, t_n] \subset N_1^2$ and $u_n\pi_0 t_n \in D_{q,0}$, it follows that $(u_n\pi_0 t)\pi_0[0, t_n - t] \subset N_1^2$ and $(u_n\pi_0 t)\pi_0(t_n - t) = u_n\pi_0 t_n \in D_{q,0}$. In other words, $u_n\pi_0 t \in E_{q,0}$. Thus, for every $t \in [0, \infty[$ and for all $n \geq n_t$, for some $n_t \in \mathbb{N}$, we have $u_n\pi_0 t \in E_{q,0}$. Since $E_{q,0} \subset X_0 \setminus N_0(J)$, we conclude that

$$u_0\pi_0 t \in \mathrm{Cl}_{X_0}(E_{q,0}) \subset X_0 \setminus \mathrm{Int}_{X_0}(N_0(J)) \text{ for all } t \geq 0. \tag{11}$$

Let $(s_k)_k$ be a sequence of positive numbers such that $s_k \to \infty$ as $k \to \infty$. By admissibility, there exist a subsequence of $(u_0\pi_0 s_k)_k$ which will be denoted again by $(u_0\pi_0 s_k)_k$ and a $u \in S_0$ such that $u_0\pi_0 s_k \to u$ as $k \to \infty$. Formula (11) implies that $u\pi_0\,[0,\infty[\, \subset X_0 \setminus \mathrm{Int}_{X_0}(N_0(J))$. The properties of Morse decompositions imply that the ω-limit set $\omega(u)$ of u relative to π_0 is included in $M_{r,0}$, for some $r \in P$. Since $M_{r,0} \subset M_0(J) \subset \mathrm{Int}_{X_0}(N_0(J))$ for all $r \in J$, it follows that $r \in P \setminus J$. Since $M_{r,0} \subset D_{r,0}$ and $D_{r,0}$ is an open set, we have that there exists a $t \geq 0$ such that $u\pi_0 t \in D_{r,0}$ and so, for some $k \in \mathbb{N}$, $u_0\pi_0(s_k+t) \in D_{r,0} \subset \tilde{V}_r$. Since $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n})) \to 0$ as $n \to \infty$, it follows that

$$\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}(s_k+t),(u_0\pi_0(s_k+t),\theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty. \tag{12}$$

Lemma 3.12 now implies that

$$w_n\pi_{\varepsilon_n}s \in D_{r,\varepsilon_n} \text{ for all } n \text{ large enough,} \tag{13}$$

where $s := s_k + t$. Moreover, the fact that $u_0\pi_0[0,s] \subset N_1^2 \subset V_1$, the definition of singular convergence and Lemma 3.12 imply that $w_n\pi_{\varepsilon_n}[0,s] \subset \,]V_1[_{\varepsilon_n,\alpha} \subset N_{1,\varepsilon_n}$ for all n large enough. This, together with (13), shows that $w_n \in E_{r,\varepsilon_n}$ for all n large enough, a contradiction as $r \in P \setminus J$. The theorem is proved. ▮

5. INDEX TRIPLE CONSTRUCTIONS

In this section we will prove the existence of index triples (relative to the approximating semiflows π_ε) with special properties. We use some arguments from the proof of existence of isolating blocks from [21]. Define the function $F\colon X_0 \to [0,1]$ by $F(u) := \min\{\,1, d_0(u, \mathrm{Inv}^-_{\pi_0}(\widetilde{N}))\,\}$. Furthermore, let

$$s_0^+ := \rho_{\widetilde{N},\pi_0}, \quad t_0^+ := \rho_{\widetilde{U},\pi_0}$$

and

$$t^+_{\varepsilon,\nu} := \rho_{]\widetilde{U}[_{\varepsilon,\nu},\pi_\varepsilon}, \quad \varepsilon \in \,]0,\varepsilon_0]\,,\ \nu > 0.$$

Define the function $g^-\colon \widetilde{N} \to \mathbb{R}$ by

$$g^-(u) := \sup\{\,\alpha(t)F(u\pi_0 t) \mid t \in [0, s_0^+(u)], \text{ if } s_0^+(u) < \infty \text{ and } t \in [0,\infty[, \text{ if } s_0^+(u) = \infty\,\},$$

where $\alpha\colon\ [0,\infty[\, \to [1,2[$ is a monotone increasing C^∞-diffeomorphism. Given $\delta > 0$ and $b > 0$, define

$$B_{\delta,b} := \mathrm{Cl}_{X_0}(\{\,u \in \widetilde{U} \mid g^-(u) < \delta \text{ and } t_0^+(u) > b\,\}).$$

We require the following result.

Lemma 5.25. *There exist* δ_0, $\bar{b} \in \,]0,\infty[$ *such that for all* $\delta \in \,]0,\delta_0]$ *and* $b \in [\bar{b},\infty[$, $B_{\delta,b} \subset \widetilde{U}$ *and* $B_{\delta,b}$ *is an isolating block for* S_0 *relative to* π_0 *with exit set* $B^-_{\delta,b} = \{ u \in \partial B_{\delta,b} \mid g^-(u) \leq \delta \text{ and } t_0^+(u) = b \}$.

Proof. Suppose there exist sequences $(\delta_n)_n$ and $(b_n)_n$ of positive numbers and $(u_n)_n$ such that $\delta_n \to 0$, $b_n \to \infty$ and $u_n \in B_{\delta_n,b_n} \cap (X_0 \setminus \widetilde{U})$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists a $\tilde{u}_n \in \widetilde{U}$ with $g^-(\tilde{u}_n) < \delta_n$ and $t_0^+(\tilde{u}_n) > b_n$ such that $d(u_n, \tilde{u}_n) < 2^{-n}$. By admissibility, we may assume, taking subsequences if necessary, that there exists an $u_0 \in S_0$ such that $\tilde{u}_n \to u_0$ and so $u_n \to u_0$. Therefore, $u_0 \in S_0 \cap (X_0 \setminus \widetilde{U})$ which is a contradiction to the definition of the open set $\widetilde{U}$. Thus, there exist a $\delta_0 \in \,]0,\infty[$ and a $\bar{b} \in \,]0,\infty[$ such that $B_{\delta,b} \subset \widetilde{U}$ for all $\delta \in \,]0,\delta_0]$ and $b \in [\bar{b},\infty[$. Moreover, $\mathrm{Inv}_{\pi_0}(B_{\delta,b}) \subset \mathrm{Inv}_{\pi_0}(\mathrm{Cl}_{X_0}(\widetilde{U})) \subset \mathrm{Inv}_{\pi_0}(\widetilde{N}) = S_0$. On the other hand, if $u \in S_0$, then $u \in \widetilde{U}$, $g^-(u) = 0$ and $t_0^+(u) = \infty$ so $u \in B_{\delta,b}$. Thus $S_0 \subset \mathrm{Inv}_{\pi_0}(B_{\delta,b})$. Therefore, $S_0 = \mathrm{Inv}_{\pi_0}(B_{\delta,b}) \subset \{ u \in \widetilde{U} \mid g^-(u) < \delta \text{ and } t_0^+(u) > b \} \subset \mathrm{Int}_{X_0}(B_{\delta,b})$. The remaining assertions follow immediately since the functions g^- and t_0^+ decrease along solutions of π_0 and g^- is upper-semicontinuous while t_0^+ is lower-semicontinuous. ▌

Given δ, $b \in \,]0,\infty[$ and $G \subset X_0$ define the sets

$$B_{1,\delta,b,G} := B_{\delta,b} \cap G, \quad B_{2,\delta,b,G} := \mathrm{Cl}_{X_0}(B_{\delta,b} \setminus G). \tag{14}$$

Given $I \in \mathcal{A}(\prec)$ with $M_0(P \setminus I) \neq \emptyset$ let $U_{P\setminus I} \subset \widetilde{N}$ be an arbitrary open neighborhood of $M_0(P \setminus I)$ with $\mathrm{Cl}_{X_0}(U_{P\setminus I}) \cap M_0(I) = \emptyset$. Let $g^+_{P\setminus I} \colon U_{P\setminus I} \to [0,\infty[$ be the map given by

$$g^+_{P\setminus I}(u) := \inf\{ (1+t)^{-1} G(u\pi_0 t) \mid 0 \leq t < \rho_{U_{P\setminus I},\pi_0}(u) \},$$

where $G(u) := d_0(u, M_0(P \setminus I))/(d_0(u, M_0(P \setminus I)) + d_0(u, X_0 \setminus \mathrm{Cl}_{X_0}(U_{P\setminus I})))$, $u \in U_{P\setminus I}$.

Choose open sets $V_{P\setminus I}$ and $W_{P\setminus I}$ such that $M_0(P \setminus I) \subset V_{P\setminus I} \subset \mathrm{Cl}_{X_0}(V_{P\setminus I}) \subset W_{P\setminus I} \subset \mathrm{Cl}_{X_0}(W_{P\setminus I}) \subset U_{P\setminus I}$ and $g^+_{P\setminus I}|\mathrm{Cl}_{X_0}(W_{P\setminus I})$ is continuous. This is possible by Proposition I.5.2 in [21].

Now, for arbitrary $I \in \mathcal{A}(\prec)$ and $\gamma \in \,]0,\infty[$ define

$$G_{P\setminus I,\gamma} := \begin{cases} \mathrm{Cl}_{X_0}(\{ u \in V_{P\setminus I} \mid g^+_{P\setminus I}(u) < \gamma \}) & \text{if } M_0(P \setminus I) \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

and set

$$B_{P\setminus I,\delta,b,\gamma} := B_{1,\delta,b,G_{P\setminus I,\gamma}},\ B_{I,\delta,b,\gamma} := B_{2,\delta,b,G_{P\setminus I,\gamma}},\ I \in \mathcal{A}(\prec),\ \gamma \in \,]0,\infty[\,.$$

The next lemma is fundamental for what follows.

Lemma 5.26. *Let* $\delta_0 \in \,]0,\infty[$ *and* $\bar{b} \in \,]0,\infty[$ *as in Lemma 5.25. Then there exist a* $\bar{\delta} \in \,]0,\delta_0]$ *and a* $\bar{\gamma} \in \,]0,\infty[$ *such that the following properties hold:*

1. *For all* $\delta \in \,]0, \overline{\delta}]$, $\gamma \in \,]0, \overline{\gamma}]$, $b \in \left[\overline{b}, \infty\right[$ *and for all* $I \in \mathcal{A}(\prec)$ *with* $M_0(P \setminus I) \neq \emptyset$, $B_{P\setminus I,\delta,b,\gamma} = B_{\delta,b} \cap G_{P\setminus I,\gamma} \subset V_{P\setminus I}$.

2. *For every* $\delta \in \,]0, \overline{\delta}]$, $\gamma \in \,]0, \overline{\gamma}]$, $b \in \left[\overline{b}, \infty\right[$ *and for every* $I \in \mathcal{A}(\prec)$, *the pair* $(B_{P\setminus I,\delta,b,\gamma}, B_{I,\delta,b,\gamma})$ *is a block pair for* $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$.

3. *Let* $I \in \mathcal{A}(\prec)$ *and* b_2, b_3, γ_2, γ_3, δ *and* δ_2 *be positive numbers such that* $\overline{b} \le b_2 < b_3$, $\gamma_2 < \gamma_3 \le \overline{\gamma}$ *and* $\delta < \delta_2 \le \overline{\delta}$. *Then* $B_{I,\delta,b_3,\gamma_3} \subset \operatorname{Int}_{X_0}(B_{I,\delta_2,b_2,\gamma_2})$.

Proof. If part (1) is not true, then there exist an $I \in \mathcal{A}(\prec)$ with $M_0(P \setminus I) \neq \emptyset$, sequences $(\delta_n)_n$ and $(\gamma_n)_n$ converging to zero, a sequence $(b_n)_n$ in $\left[\overline{b}, \infty\right[$ and $(u_n)_n$ such that

$$u_n \in B_{\delta_n,b_n} \cap G_{P\setminus I,\gamma_n} \cap (X_0 \setminus V_{P\setminus I}) \text{ for all } n \in \mathbb{N}.$$

Since $u_n \in B_{\delta_n,b_n}$, it follows that $g^-(u_n) \le \delta_n \to 0$ as $n \to \infty$. By admissibility, there exist a subsequence of $(u_n)_n$, denoted again by $(u_n)_n$, and an $u_0 \in \operatorname{Inv}^-_{\pi_0}(\widetilde{N})$ such that $u_n \to u_0$ as $n \to \infty$. The definition of the set $G_{P\setminus I,\gamma_n}$ implies that $u_0 \in \operatorname{Cl}_{X_0}(V_{P\setminus I}) \subset \operatorname{Cl}_{X_0}(W_{P\setminus I})$. The continuity of $g^+_{P\setminus I}|_{\operatorname{Cl}_{X_0}(W_{P\setminus I})}$ implies that $g^+_{P\setminus I}|_{\operatorname{Cl}_{X_0}(W_{P\setminus I})}(u_0) = 0$. It follows from Proposition I.5.2 in [21] that $u_0 \in \operatorname{Inv}^+_{\pi_0}(\operatorname{Cl}_{X_0}(U_{P\setminus I})) \subset \operatorname{Inv}^+_{\pi_0}(\widetilde{N})$ and so $u_0 \in S_0$. Since $N_{P\setminus I} \cap M_0(I) = \emptyset$ and $(M_0(I), M_0(P \setminus I))$ is an attractor-repeller pair in S_0, relative to π_0, we see that the ω-limit set $\omega(u_0)$ of u_0 relative to π_0 is a subset of $M_0(P \setminus I)$. Theorem III.1.4 in [21] implies that $u_0 \in M_0(P \setminus I)$ and so $u_0 \in M_0(P \setminus I) \cap (X_0 \setminus V_{P\setminus I})$ which contradicts our choice of the open set $V_{P\setminus I}$. This proves part (1).

Part (2) follows from part (1) and the proof of Theorem III.2.4 in [21].

To prove part (3) let $I \in \mathcal{A}(\prec)$ be arbitrary. If $M_0(P \setminus I) = \emptyset$ then the result is clear. Therefore, let $M_0(P \setminus I) \neq \emptyset$. We claim that $B_{I,\delta,b_3\gamma_3} \subset \operatorname{Int}_{X_0}(B_{\delta_2,b_2}) \setminus G_{P\setminus I,\gamma_2}$. Let $u \in B_{I,\delta,b_3\gamma_3}$ be arbitrary. Then, there exists a sequence $(u_n)_n$ such that $u_n \in B_{\delta,b_3} \setminus G_{P\setminus I,\gamma_3}$ for all n and $u_n \to u$. In particular, $u \in \widetilde{U}$ and $g^-(u) \le \delta < \delta_2$ and $t_0^+(u) \ge b_3 > b_2$. Thus $u \in \{\, u \in \widetilde{U} \mid g^-(u) < \delta_2 \text{ and } t_0^+(u) > b_2 \,\} \subset \operatorname{Int}_{X_0}(B_{\delta_2,b_2})$. Suppose $u \in G_{P\setminus I,\gamma_2}$. It follows from part (1) that $u \in B_{\delta_2,b_2} \cap G_{P\setminus I,\gamma_2} \subset V_{P\setminus I}$. So $u \in V_{P\setminus I}$ and $g^+_{P\setminus I}(u) \le \gamma_2 < \gamma_3$. Then for all n large enough, $u_n \in V_{P\setminus I}$ and $g^+_{P\setminus I}(u_n) < \gamma_3$. Thus $u_n \in G_{P\setminus I,\gamma_3}$ for all n large enough which is a contradiction. This proves the claim and completes the proof. ∎

We now can state and prove the main result of this section.

Theorem 5.27. *Let* $\overline{b}$ *be as in Lemma 5.25 and* $\overline{\delta}$ *and* $\overline{\gamma}$ *be as in Lemma 5.26. Fix positive numbers* b, b_2, b_3, γ_2, γ_3 *and* δ_2 *with* $\overline{b} < b_2 < b_3 < b$, $\gamma_3 < \gamma_2 < \overline{\gamma}$ *and* $\delta_2 < \overline{\delta}$. *For all* $I \in \mathcal{A}(\prec)$ *and* $\delta \in \,]0, \delta_2]$, *let* $\Lambda_{I,\delta}$ *be a closed subset of* X_0 *with*

$$M_0(I) \subset \operatorname{Int}_{X_0}(\Lambda_{I,\delta}) \subset \Lambda_{I,\delta} \subset B_{I,\delta,b_3,\gamma_3}$$

and such that whenever $I \in \mathcal{A}(\prec)$ *and* $\delta < \delta'$, $\Lambda_{I,\delta} \subset \Lambda_{I,\delta'}$.

Assume also that whenever $I \in \mathcal{A}(\prec)$ *and* $(\delta_n)_n$ *is a decreasing sequence converging to zero and* $u_n \in \Lambda_{I,\delta_n}$ *for all* $n \in \mathbb{N}$, *then the sequence* $(u_n)_n$ *has a convergent subsequence.*

For $\alpha > 0$, $\nu > 0$, $\eta > 0$, $\eta' > 0$, $\beta' > 0$, *with* $\eta' < \beta' < \eta < \nu < \beta$, *for all* $I \in \mathcal{A}(\prec)$, $\delta \in]0, \delta_2]$ *and* $\varepsilon \in]0, \widetilde{\varepsilon}]$, *define*

$$N_{1,\varepsilon}(\alpha,\delta) := [B_{\delta_2,b_2}]_{\varepsilon,\eta} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{there exist a } \overline{w} \in [B_{\delta,b_3}]_{\varepsilon,\alpha} \text{ and a } t \geq 0 \text{ such that } \overline{w}\pi_n[0,t] \subset]\widetilde{U}[_{\varepsilon,\beta} \text{ and } w = \overline{w}\pi_\varepsilon t \,\}),$$

$$N_{2,\varepsilon}(\alpha,\delta) := N_{1,\varepsilon}(\alpha,\delta) \cap \{\, w \in]\widetilde{U}[_{\varepsilon,\nu} \mid t^+_{\varepsilon,\nu}(w) \leq b \,\}$$

and

$$N_{I,\varepsilon}(\alpha,\delta) := ([B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{ there exist a } \overline{w} \in [\Lambda_{I,\delta}]_{\varepsilon,\alpha} \text{ and a } t \geq 0 \text{ such that } \overline{w}\pi_\varepsilon[0,t] \subset]\widetilde{U}[_{\varepsilon,\beta'} \text{ and } w = \overline{w}\pi_\varepsilon t \,\})) \cup N_{2,\varepsilon}(\alpha,\delta).$$

Let $\nu > 0$, $\eta > 0$, $\eta' > 0$, $\beta' > 0$, *with* $\eta' < \beta' < \eta < \nu < \beta$, *be fixed. Under these assumptions, there exist a* $\delta_3 \in]0, \delta_2]$ *and an* $\alpha_0 \in]0, \eta']$ *such that for all* $\delta \in]0, \delta_3]$ *and for all* $\alpha \in]0, \alpha_0]$, *there exists an* $\widetilde{\varepsilon}_0 = \widetilde{\varepsilon}_0(\alpha,\delta) \in]0, \widetilde{\varepsilon}]$ *such that for all* $\varepsilon \in]0, \widetilde{\varepsilon}_0]$ *and for all* $I \in \mathcal{A}(\prec)$ *the triple* $(N_{1,\varepsilon}(\alpha,\delta), N_{I,\varepsilon}(\alpha,\delta), N_{2,\varepsilon}(\alpha,\delta))$ *is an FM-index triple for* $(\pi_\varepsilon, S_\varepsilon, M_\varepsilon(I), M_\varepsilon(P \setminus I))$.

Proof. It is clear that for all $I \in \mathcal{A}(\prec)$, $\varepsilon \in]0, \widetilde{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$, the sets $N_{1,\varepsilon}(\alpha,\delta)$, $N_{I,\varepsilon}(\alpha,\delta)$ and $N_{2,\varepsilon}(\alpha,\delta)$ are closed in Z_ε and $N_{2,\varepsilon}(\alpha,\delta) \subset N_{I,\varepsilon}(\alpha,\delta)$.

Since $\beta' < \beta$ and $\eta' < \eta$, we have $B_{I,\delta_2,b_2,\gamma_2} \subset B_{\delta_2,b_2}$, $]\widetilde{U}[_{\varepsilon,\beta'} \subset]\widetilde{U}[_{\varepsilon,\beta}$ and $[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}$ is included in $[B_{\delta_2,b_2}]_{\varepsilon,\eta}$ for all $\varepsilon \in]0, \widetilde{\varepsilon}]$. Moreover, since $\Lambda_{I,\delta} \subset B_{I,\delta,b_3,\gamma_3} \subset B_{\delta,b_3}$, it follows that $[\Lambda_{I,\delta}]_{\varepsilon,\alpha} \subset [B_{\delta,b_3}]_{\varepsilon,\alpha}$ for all $\varepsilon \in]0, \widetilde{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$. This proves that for all $\varepsilon \in]0, \widetilde{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$,

$$N_{I,\varepsilon}(\alpha,\delta) \subset N_{1,\varepsilon}(\alpha,\delta). \tag{15}$$

Lemmas 5.30 and 5.31 below complete the proof of Theorem 5.27. ∎

Remark 5.28. *Notice that if* $\alpha > 0$, $\nu > 0$, $\eta > 0$, $\eta' > 0$ *and* $\beta' > 0$ *are such that* $\alpha < \eta' < \beta' < \eta < \nu < \beta$, *then for all* $\delta \in]0, \delta_2]$, $\varepsilon \in]0, \widetilde{\varepsilon}]$ *and for all* $I \in \mathcal{A}(\prec)$, *we have* $[B_{\delta,b_3}]_{\varepsilon,\alpha} \subset N_{1,\varepsilon}(\alpha,\delta) \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta}$ *and* $[\Lambda_{I,\delta}]_{\varepsilon,\alpha} \subset N_{I,\varepsilon}(\alpha,\delta) \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'} \cup N_{2,\varepsilon}(\alpha,\delta)$.

To prove Lemmas 5.30 and 5.31 we will need the following result that follows from the proof of Proposition 4.9 in [3].

Lemma 5.29. *Let* $b > 0$. *Let* $(\zeta_n)_n$, $(\alpha_n)_n$ *and* $(\varepsilon_n)_n$ *be sequences in* $]0, \infty[$ *converging to zero. For each* $n \in \mathbb{N}$, *let* $\overline{w}_n = (\overline{u}_n, \overline{v}_n) \in [B_{\zeta_n,b}]_{\varepsilon_n,\alpha_n}$ *and* $t_n \geq 0$ *such that* $\overline{w}_n\pi_{\varepsilon_n}[0,t_n] \subset [\widetilde{N}]_{\varepsilon_n,\beta}$. *Then there is a subsequence of* $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$ *which we will denote again by* $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$ *and there is a* $\tilde{u}_0 \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ *such that*

$$\Gamma_{\varepsilon_n}(\overline{w}_n\pi_{\varepsilon_n}t_n, (\tilde{u}_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

Proof. For each $n \in \mathbb{N}$, $\overline{w}_n = (\overline{u}_n, \overline{v}_n) \in [B_{\zeta_n,b}]_{\varepsilon_n,\alpha_n}$. It follows that $g^-(\overline{u}_n) \leq \zeta_n$ and $d_{\varepsilon_n}(\overline{v}_n, \theta_{\varepsilon_n}) \leq \alpha_n$. Since $\zeta_n \to 0$ as $n \to \infty$, there exists a subsequence of $(\overline{u}_n)_n$, denoted again by $(\overline{u}_n)_n$, and a $u_0 \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ such that $\overline{u}_n \to u_0$ as $n \to \infty$ and so

$$\Gamma_{\varepsilon_n}(\overline{w}_n, (u_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty. \tag{16}$$

To complete the proof of the lemma, we will need to consider two cases.

First assume that the sequence $(t_n)_n$ is bounded. By taking subsequence, if necessary, we may assume that there exists a $t \in [0, \infty[$ such that $t_n \to t$ as $n \to \infty$. Since $(\pi_\varepsilon)_{\varepsilon \in]0,\varepsilon_0]}$ converges singularly to π_0, formula (16) implies that

$$\Gamma_{\varepsilon_n}(\overline{w}_n \pi_{\varepsilon_n} t_n, (u_0 \pi_0 t, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

Since $\mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ is an $\widetilde{N}$-positively invariant set relative to π_0, it follows that

$$\tilde{u}_0 := u_0 \pi_0 t \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N}).$$

Suppose that $(t_n)_n$ is an unbounded sequence. Then we may assume that $t_n \to \infty$ as $n \to \infty$. Since $\overline{w}_n \pi_{\varepsilon_n} [0, t_n] \subset [\widetilde{N}]_{\varepsilon_n,\beta}$ and $\widetilde{N}$ is singularly strongly admissible with respect to β and $(\pi_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$, condition (3) of Definition 3.16 implies that there exist a subsequence of $(\overline{w}_n \pi_{\varepsilon_n} t_n)_n$, denoted again by $(\overline{w}_n \pi_{\varepsilon_n} t_n)_n$, and there is a $\tilde{u}_0 \in \widetilde{N}$ such that

$$\Gamma_{\varepsilon_n}(\overline{w}_n \pi_{\varepsilon_n} t_n, (\tilde{u}_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

Lemma 3.5.(1) in [3] implies that $\tilde{u}_0 \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N})$. The proof is concluded. ∎

Lemma 5.30. *There exist a $\delta' \in]0, \delta_2]$ and an $\alpha' \in]0, \eta']$ such that for all $\delta \in]0, \delta']$ and for all $\alpha \in]0, \alpha']$, there exist an $\varepsilon_1 = \varepsilon_1(\alpha, \delta) \in]0, \widehat{\varepsilon}]$ such that for all $\varepsilon \in]0, \varepsilon_1]$ the pair $(N_{1,\varepsilon}(\alpha, \delta), N_{2,\varepsilon}(\alpha, \delta))$ is an FM-index pair for S_ε.*

Proof. We claim that for all $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$, the set $N_{2,\varepsilon}(\alpha, \delta)$ is $N_{1,\varepsilon}(\alpha, \delta)$-positively invariant relative to π_ε.

In fact, let $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$ be fixed. Let $w \in N_{2,\varepsilon}(\alpha, \delta)$ and $t \geq 0$ such that $w\pi_\varepsilon[0, t] \subset N_{1,\varepsilon}(\alpha, \delta)$. Hence, $w\pi_\varepsilon[0, t] \subset N_{1,\varepsilon}(\alpha, \delta) \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta}$. Let $s \in [0, t]$. Then

$$w\phi_\varepsilon s \in B_{\delta_2,b_2} \subset \widetilde{U} \text{ and } d_\varepsilon(w\psi_\varepsilon s, \theta_\varepsilon) \leq \eta < \nu.$$

Hence, $w\pi_\varepsilon s \in]\widetilde{U}[_{\varepsilon,\nu}$ for all $s \in [0, t]$. Since $t^+_{\varepsilon,\nu}(w\pi_\varepsilon s) \leq t^+_{\varepsilon,\nu}(w) \leq b$, it follows that $w\pi_\varepsilon s \in \{ z \in]\widetilde{U}[_{\varepsilon,\nu} \mid t^+_{\varepsilon,\nu}(z) \leq b \}$ and so $w\pi_\varepsilon s \in N_{2,\varepsilon}(\alpha, \delta)$ for all $s \in [0, t]$. This concludes the proof of our claim.

The condition (1) of the Definition 2.3 is verified for all $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$.

Let $\delta \in]0, \delta_2]$ and $\alpha \in]0, \eta']$. Lemma 5.25 implies that B_{δ_2,b_2} and B_{δ,b_2} are isolating blocks for S_0. Proposition 3.19 implies that there exists an $\widetilde{\varepsilon}_1 = \widetilde{\varepsilon}_1(\alpha, \delta) \in]0, \widetilde{\varepsilon}]$ such that for all $\varepsilon \in]0, \widetilde{\varepsilon}_1]$ the sets $[B_{\delta_2,b_2}]_{\varepsilon,\eta}$ and $[B_{\delta,b_3}]_{\varepsilon,\alpha}$ are isolating neighborhoods for S_ε. Hence

$$S_\varepsilon \subset N_{1,\varepsilon}(\alpha, \delta) \text{ for all } \varepsilon \in]0, \widetilde{\varepsilon}_1].$$

Moreover, if $w \in S_\varepsilon$, $\varepsilon \in]0, \widetilde{\varepsilon}_1]$, then $w \in [B_{\delta_2,b_2}]_{\varepsilon,\eta} \subset]\widetilde{U}[_{\varepsilon,\nu}$ and $t^+_{\varepsilon,\nu}(w) = \infty$ and so $w \notin N_{2,\varepsilon}(\alpha, \delta)$. Thus

$$S_\varepsilon \subset N_{1,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta) \text{ for all } \varepsilon \in]0, \widetilde{\varepsilon}_1].$$

We claim that there exist a $\delta'_3 \in]0, \delta_2]$ and an $\alpha'_0 \in]0, \eta']$ such that for all $\delta \in]0, \delta'_3]$ and for all $\alpha \in]0, \alpha'_0]$, there exists an $\varepsilon'_1 = \varepsilon'_1(\alpha, \delta) \in]0, \widetilde{\varepsilon}_1(\alpha, \delta)]$ such that $]\mathrm{Int}_{X_0}(B_{\delta,b_3})[_{\varepsilon,\alpha} \subset N_{1,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta)$ for all $\varepsilon \in]0, \varepsilon'_1]$.

Notice that Remark 5.28 implies that $]\mathrm{Int}_{X_0}(B_{\delta,b_3})[_{\varepsilon,\alpha} \subset N_{1,\varepsilon}(\alpha, \delta)$ for all $\varepsilon \in]0, \widetilde{\varepsilon}]$, $\alpha \in]0, \eta']$ and $\delta \in]0, \delta_2]$. Suppose that our claim is not true. Then there are sequences $(\zeta_n)_n$, with $\zeta_n < \delta_2$ for all $n \in \mathbb{N}$, $(\alpha_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $\zeta_n \to 0$, $\alpha_n \to 0$, $\varepsilon_n \to 0$ as $n \to \infty$ and

$$w_n = (u_n, v_n) \in]\mathrm{Int}_{X_0}(B_{\zeta_n,b_3})[_{\varepsilon_n,\alpha_n} \cap N_{2,\varepsilon_n}(\alpha_n, \zeta_n) \text{ for all } n \in \mathbb{N}.$$

Thus $u_n \in \mathrm{Int}_{X_0}(B_{\zeta_n,b_3})$, $v_n \in B(\theta_{\varepsilon_n}, \alpha_n)$, $w_n \in]\widetilde{U}[_{\varepsilon_n,\nu}$ and $t^+_{\varepsilon_n,\nu}(w_n) \le b$ for all $n \in \mathbb{N}$.

Since $u_n \in \mathrm{Int}_{X_0}(B_{\zeta_n,b_3})$, it follows that $g^-(u_n) < \zeta_n$. Hence, there exists a subsequence of $(u_n)_n$, denoted again by $(u_n)_n$, and a $u_0 \in \widetilde{N}$ such that $u_n \to u_0$ as $n \to \infty$. Moreover, since for all $n \in \mathbb{N}$, $\mathrm{Int}_{X_0}(B_{\zeta_n,b_3}) \subset B_{\delta_2,b_3}$, it follows that $u_0 \in B_{\delta_2,b_3} \subset \widetilde{U}$. The continuity of t^+_0 implies that $t^+_0(u_0) \ge b_3 > b$.

Since $v_n \in B(\theta_{\varepsilon_n}, \alpha_n)$, we have $d_{\varepsilon_n}(v_n, \theta_{\varepsilon_n}) < \alpha_n$ and so $d_{\varepsilon_n}(v_n, \theta_{\varepsilon_n}) \to 0$ as $n \to \infty$. Hence

$$\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty$$

and Lemma 4.6 in [3] implies that $t^+_{\varepsilon_n,\nu}(w_n) \to t^+_0(u_0)$ as $n \to \infty$ and so $t^+_{\varepsilon_n,\nu}(w_n) > b$ for all n large enough. However, this is a contradiction to our choice of the sequence $(w_n)_n$. The claim is proved which, in turn, implies that for all $\delta \in]0, \delta'_3]$ and for all $\alpha \in]0, \alpha'_0]$, there exists an $\varepsilon'_1 = \varepsilon'_1(\alpha, \delta) \in]0, \widetilde{\varepsilon}_1(\alpha, \delta)]$ such that

$$S_\varepsilon \subset]\mathrm{Int}_{X_0}(B_{\delta,b_3})[_{\varepsilon,\alpha} \subset N_{1,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta) \text{ for all } \varepsilon \in]0, \varepsilon'_1].$$

Therefore $S_\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}(N_{1,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta))$ so $S_\varepsilon \subset \mathrm{Inv}_{\pi_\varepsilon}(N_{1,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta))$ for all $\varepsilon \in]0, \varepsilon'_1]$. Since $N_{1,\varepsilon}(\alpha, \delta) \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta} \subset [\widetilde{N}]_{\varepsilon,\beta}$ and $S_\varepsilon = \mathrm{Inv}_{\pi_\varepsilon}([\widetilde{N}]_{\varepsilon,\beta})$, it follows that $S_\varepsilon = \mathrm{Inv}_{\pi_\varepsilon} \mathrm{Cl}_{Z_\varepsilon}(N_{1,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta))$ for all $\varepsilon \in]0, \varepsilon'_1]$. Hence, the condition (3) of Definition 2.3 is verified for all $\varepsilon \in]0, \varepsilon'_1]$.

The next claim is that there exist a $\delta' \in]0, \delta'_3]$ and an $\alpha' \in]0, \alpha'_0]$ such that for all $\delta \in]0, \delta']$ and for all $\alpha \in]0, \alpha']$, there exists an $\varepsilon_1 = \varepsilon_1(\alpha, \delta) \in]0, \varepsilon'_1(\alpha, \delta)]$ such that for all $\varepsilon \in]0, \varepsilon_1]$, the set $N_{2,\varepsilon}(\alpha, \delta)$ is an exit ramp for $N_{1,\varepsilon}(\alpha, \delta)$.

First notice that for all $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$, the set $N_{1,\varepsilon}(\alpha, \delta)$ is $[B_{\delta_2,b_2}]_{\varepsilon,\eta}$-positively invariant relative to π_ε. In fact, let $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$ be fixed. Let $w \in N_{1,\varepsilon}(\alpha, \delta)$ and $s \geq 0$ such that $w\pi_\varepsilon [0, s] \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta}$. Hence there exist a sequence $(w_n)_n$ in $[B_{\delta,b_3}]_{\varepsilon,\alpha}$ and a sequence $(t_n)_n$ of nonnegative numbers such that

$$w_n\pi_\varepsilon [0, t_n] \subset]\widetilde{U}[_{\varepsilon,\beta} \tag{17}$$

and $\overline{w}_n := w_n\pi_\varepsilon t_n$ is such that $\Gamma_\varepsilon (\overline{w}_n, w) \to 0$ as $n \to \infty$. We will show that $\overline{w}_n\pi_\varepsilon [0, s] \subset]\widetilde{U}[_{\varepsilon,\beta}$ for all n large enough. Since $w\pi_\varepsilon [0, s] \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta}$, it follows that for all $\tau \in [0, s]$

$$w\phi_\varepsilon\tau \in B_{\delta_2,b_2} \subset \widetilde{U} \text{ and } w\psi_\varepsilon\tau \in \mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \eta)).$$

Since $\eta < \beta$, it follows that $w\psi_\varepsilon\tau \in B_\varepsilon(\theta_\varepsilon, \beta)$. Thus $w\pi_\varepsilon [0, s] \subset]\widetilde{U}[_{\varepsilon,\beta}$. Since π_ε is continuous, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\overline{w}_n\pi_\varepsilon [0, s] \subset]\widetilde{U}[_{\varepsilon,\beta}$. Therefore

$$w_n\pi_\varepsilon [0, t_n + s] \subset]\widetilde{U}[_{\varepsilon,\beta} \text{ for all } n \geq n_0. \tag{18}$$

Let $\tau \in [0, s]$ be arbitrary. Then

$$\Gamma_\varepsilon (w_n\pi_\varepsilon(t_n + \tau), w\pi_\varepsilon\tau) = \Gamma_\varepsilon (\overline{w}_n\pi_\varepsilon\tau, w\pi_\varepsilon\tau) \to 0 \text{ as } n \to \infty. \tag{19}$$

It follows from (18) and (19) that

$$w\pi_\varepsilon\tau \in \mathrm{Cl}_{Z_\varepsilon}(\{\, \overline{z} \mid \text{there exist a } z \in [B_{\delta,b_3}]_{\varepsilon,\alpha} \text{ and a } t \geq 0 \text{ such that } z\pi_\varepsilon[0, t] \subset]\widetilde{U}[_{\varepsilon,\beta} \text{ and } z\pi_\varepsilon t = \overline{z}\,\}).$$

Therefore $w\pi_\varepsilon [0, s] \subset N_{1,\varepsilon}(\alpha, \delta)$.

Suppose that our claim is not true. Since for all $\varepsilon \in]0, \widetilde{\varepsilon}]$, $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$, the set $N_{1,\varepsilon}(\alpha, \delta)$ is $[B_{\delta_2,b_2}]_{\varepsilon,\eta}$-positively invariant relative to π_ε, it follows that there are sequences $(\zeta_n)_n$, $(\alpha_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $\zeta_n \to 0$, $\alpha_n \to 0$, $\varepsilon_n \to 0$ as $n \to \infty$ and

$$w_n = (u_n, v_n) \in (N_{1,\varepsilon_n}(\alpha_n, \zeta_n) \cap \partial\, [B_{\delta_2,b_2}]_{\varepsilon_n,\eta}) \setminus N_{2,\varepsilon_n}(\alpha_n, \zeta_n) \text{ for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, there exists a $\overline{w}_n = (\overline{u}_n, \overline{v}_n) \in [B_{\zeta_n,b_3}]_{\varepsilon_n,\alpha_n}$ and a $t_n \geq 0$ such that $\overline{w}_n\pi_{\varepsilon_n}[0, t_n] \subset]\widetilde{U}[_{\varepsilon_n,\beta}$ and $\Gamma_{\varepsilon_n}(w_n, \overline{w}_n\pi_{\varepsilon_n}t_n) < 2^{-n}$.

Since $]\widetilde{U}[_{\varepsilon_n,\beta} \subset [\widetilde{N}]_{\varepsilon_n,\beta}$, the hypotheses of Lemma 5.29 are verified and so there exists a subsequence of $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$, denoted again by $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$, and there is a $\tilde{u}_0 \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ such that $\Gamma_{\varepsilon_n} (\overline{w}_n\pi_{\varepsilon_n}t_n, (\tilde{u}_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$ which, in turn, implies that

$$\Gamma_{\varepsilon_n} (w_n, (\tilde{u}_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty. \tag{20}$$

Therefore, there exists an $n_0 \in \mathbb{N}$ such that $d_{\varepsilon_n}(v_n, \theta_{\varepsilon_n}) < \eta$ for all $n \geq n_0$. Since $(u_n, v_n) \in \partial\, [B_{\delta_2,b_2}]_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$, we have $u_n \in \partial B_{\delta_2,b_2}$ for all $n \geq n_0$ and so $\tilde{u}_0 \in \partial B_{\delta_2,b_2} \subset \widetilde{U}$. This fact, formula (20) and Lemma 4.6 in [3] imply that

$$t^+_{\varepsilon_n,\nu}(w_n) \to t^+_0(\tilde{u}_0) \text{ as } n \to \infty. \tag{21}$$

Since B_{δ_2,b_2} is an isolating block for S_0 relative to π_0, $g^-(\tilde{u}_0) = 0$ and $\tilde{u}_0 \in \partial B_{\delta_2,b_2}$, it follows that $t_0^+(\tilde{u}_0) = b_2$.

On the other hand, since $w_n \notin N_{2,\varepsilon_n}(\alpha_n, \zeta_n)$ for all $n \in \mathbb{N}$, we have $t^+_{\varepsilon_n,\nu}(w_n) > b$ for all $n \in \mathbb{N}$. This fact, together with formula (21), implies that $b_2 = t_0^+(\tilde{u}_0) \geq b$ which is a contradiction. Hence there exist a $\delta' \in]0, \delta'_3]$ and an $\alpha' \in]0, \alpha'_0]$ such that for all $\delta \in]0, \delta']$ and for all $\alpha \in]0, \alpha']$, there exists an $\varepsilon_1 = \varepsilon_1(\alpha, \delta) \in]0, \varepsilon'_1]$ such that for all $\varepsilon \in]0, \varepsilon_1]$, the set $N_{2,\varepsilon}(\alpha, \delta)$ is an exit ramp for $N_{1,\varepsilon}(\alpha, \delta)$ and the proof of the lemma is complete. ▮

Lemma 5.31. *Let δ' and α' as in Lemma 5.30. Given $\delta \in]0, \delta']$ and $\alpha \in]0, \alpha']$, let $\varepsilon_1(\alpha, \delta) \in]0, \widehat{\varepsilon}]$ as in that lemma. Then there exist a $\delta'' \in]0, \delta']$ and an $\alpha'' \in]0, \alpha']$ such that for all $\delta \in]0, \delta'']$ and for all $\alpha \in]0, \alpha'']$, there exist an $\varepsilon_2 = \varepsilon_2(\alpha, \delta) \in]0, \varepsilon_1(\alpha, \delta)]$ such that for all $\varepsilon \in]0, \varepsilon_2]$ and for all $I \in \mathcal{A}(\prec)$, the pair $(N_{I,\varepsilon}(\alpha, \delta), N_{2,\varepsilon}(\alpha, \delta))$ is an FM-index pair for $M_\varepsilon(I)$.*

Proof. It is clear that for all $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$, the set $N_{2,\varepsilon}(\alpha, \delta)$ is $N_{I,\varepsilon}(\alpha, \delta)$-positively invariant relative to π_ε. Indeed, let $w \in N_{2,\varepsilon}(\alpha, \delta)$ and $t \geq 0$ such that $w\pi_\varepsilon[0, t] \subset N_{I,\varepsilon}(\alpha, \delta)$. Formula (15) implies that $w\pi_\varepsilon[0, t] \subset N_{1,\varepsilon}(\alpha, \delta)$. It follows from the proof of Lemma 5.30 that $N_{2,\varepsilon}(\alpha, \delta)$ is $N_{1,\varepsilon}(\alpha, \delta)$-positively invariant relative to π_ε and so $w\pi_\varepsilon[0, t] \subset N_{2,\varepsilon}(\alpha, \delta)$. Hence, the condition (1) of the Definition 2.3 is verified for all $\varepsilon \in]0, \widehat{\varepsilon}]$, $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$.

We claim that given $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$ there exists an $\widetilde{\varepsilon}_2 = \widetilde{\varepsilon}_2(\alpha, \delta) \in]0, \varepsilon_1(\alpha, \delta)]$ such that for all $\varepsilon \in]0, \widetilde{\varepsilon}_2]$,

$$M_\varepsilon(I) = \mathrm{Inv}_{\pi_\varepsilon}(\mathrm{Cl}_{Z_\varepsilon}(N_{I,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta))) \subset \mathrm{Int}_{Z_\varepsilon}(N_{I,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta)).$$

Indeed, let $\alpha \in]0, \alpha']$ and $\delta \in]0, \delta']$. Lemma 5.26.(2), Theorem 3.17 and Proposition 3.19 imply that there exists an $\widetilde{\varepsilon}_2 = \widetilde{\varepsilon}_2(\alpha, \delta) \in]0, \varepsilon_1(\alpha, \delta)]$ such that for all $\varepsilon \in]0, \widetilde{\varepsilon}_2]$, $M_\varepsilon(I) = \mathrm{Inv}_{\pi_\varepsilon}([B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'})$ and $M_\varepsilon(I) = \mathrm{Inv}_{\pi_\varepsilon}([\Lambda_{I,\delta}]_{\varepsilon,\alpha}) \subset \mathrm{Int}_{Z_\varepsilon}([\Lambda_{I,\delta}]_{\varepsilon,\alpha})$.

Let $\varepsilon \in]0, \widetilde{\varepsilon}_2]$. Since $\alpha < \beta'$, Remark 5.28 implies that $[\Lambda_{I,\delta}]_{\varepsilon,\alpha} \subset N_{I,\varepsilon}(\alpha, \delta)$. Thus $M_\varepsilon(I) \subset N_{I,\varepsilon}(\alpha, \delta) \subset N_{1,\varepsilon}(\alpha, \delta)$ and $M_\varepsilon(I) \subset [\Lambda_{I,\delta}]_{\varepsilon,\alpha} \subset [B_{I,\delta,b_3,\gamma_3}]_{\varepsilon,\eta'} \subset]\widetilde{U}[_{\varepsilon,\nu}$. On the other hand, if $w \in M_\varepsilon(I)$, then $t^+_{\varepsilon,\nu}(w) = \infty$ and so $w \notin N_{2,\varepsilon}(\alpha, \delta)$. Hence

$$M_\varepsilon(I) \subset \mathrm{Int}_{Z_\varepsilon}([\Lambda_{I,\delta}]_{\varepsilon,\alpha}) \cap \{\, w \in]\widetilde{U}[_{\varepsilon,\nu} \mid t^+_{\varepsilon,\nu}(w) > b \,\} \subset N_{I,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta). \tag{22}$$

This also implies that

$$M_\varepsilon(I) \subset \mathrm{Inv}_{\pi_\varepsilon}(\mathrm{Cl}_{Z_\varepsilon}(N_{I,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta))). \tag{23}$$

Since $N_{I,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta) \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}$, it follows that

$$\mathrm{Inv}_{\pi_\varepsilon}(\mathrm{Cl}_{Z_\varepsilon}(N_{I,\varepsilon}(\alpha, \delta) \setminus N_{2,\varepsilon}(\alpha, \delta))) \subset \mathrm{Inv}_{\pi_\varepsilon}([B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}) = M_\varepsilon(I).$$

This and inclusions (22) and (23) imply the claim.

Our next claim it that for all $\varepsilon \in \,]0,\widetilde{\varepsilon}]$, $\alpha \in \,]0,\alpha']$ and $\delta \in \,]0,\delta']$, the set

$$N^c_{I,\varepsilon}(\alpha,\delta) := [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{ there exist an } \overline{w} \in [\Lambda_{I,\delta}]_{\varepsilon,\alpha} \text{ and } \\ \text{a } t \geq 0 \text{ such that } \overline{w}\pi_\varepsilon[0,t] \subset \,]\widetilde{U}[_{\varepsilon,\beta'} \text{ and } w = \overline{w}\pi_\varepsilon t \,\})$$

is $[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}$-positively invariant relative to π_ε.

Let $\varepsilon \in \,]0,\widetilde{\varepsilon}]$, $\alpha \in \,]0,\alpha']$ and $\delta \in \,]0,\delta']$ be fixed. Let $w \in N^c_{I,\varepsilon}(\alpha,\delta)$ and $s \geq 0$ such that $w\pi_\varepsilon\,[0,s] \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}$. Hence there exist a sequence $(\overline{w}_n)_n$ in $[\Lambda_{I,\delta}]_{\varepsilon,\alpha}$ and a sequence $(t_n)_n$ of positive numbers such that

$$\overline{w}_n\pi_\varepsilon\,[0,t_n] \subset \,]\widetilde{U}[_{\varepsilon,\beta'} \tag{24}$$

and $w_n := \overline{w}_n\pi_\varepsilon t_n$ is such that $\Gamma_\varepsilon\,(w_n,w) \to 0$ as $n \to \infty$. We will show that $w_n\pi_\varepsilon\,[0,s] \subset \,]\widetilde{U}[_{\varepsilon,\beta'}$ for all n large enough. Since $w\pi_\varepsilon\,[0,s] \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}$, it follows that for all $\tau \in [0,s]$

$$w\phi_\varepsilon\tau \in B_{I,\delta_2,b_2,\gamma_2} \subset \widetilde{U} \text{ and } w\psi_\varepsilon\tau \in \mathrm{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon,\eta')).$$

Since $\eta' < \beta'$, it follows that $w\psi_\varepsilon\tau \in B_\varepsilon(\theta_\varepsilon,\beta')$. Thus $w\pi_\varepsilon\,[0,s] \subset \,]\widetilde{U}[_{\varepsilon,\beta'}$. Since π_ε is continuous, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $w_n\pi_\varepsilon\,[0,s] \subset \,]\widetilde{U}[_{\varepsilon,\beta'}$. Therefore

$$\overline{w}_n\pi_\varepsilon\,[0,t_n+s] \subset \,]\widetilde{U}[_{\varepsilon,\beta'} \text{ for all } n \geq n_0. \tag{25}$$

Let $\tau \in [0,s]$ be arbitrary. Then

$$\Gamma_\varepsilon\,(\overline{w}_n\pi_\varepsilon(t_n+\tau), w\pi_\varepsilon\tau) = \Gamma_\varepsilon\,(w_n\pi_\varepsilon\tau, w\pi_\varepsilon\tau) \to 0 \text{ as } n \to \infty. \tag{26}$$

It follows from (25) and (26) that

$$w\pi_\varepsilon\tau \in \mathrm{Cl}_{Z_\varepsilon}(\{\, z \mid \text{there exist a } \overline{z} \in [\Lambda_{I,\delta}]_{\varepsilon,\alpha} \text{ and a } t \geq 0 \text{ such that} \\ \overline{z}\pi_\varepsilon[0,t] \subset \,]\widetilde{U}[_{\varepsilon,\beta'} \text{ and } \overline{z}\pi_\varepsilon t = z\,\}).$$

Therefore $w\pi_\varepsilon\,[0,s] \subset N^c_{I,\varepsilon}(\alpha,\delta)$. The proof of our claim is complete.

Since $N^c_{I,\varepsilon}(\alpha,\delta)$ is $[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'}$-positively π_ε-invariant, to prove that $N_{2,\varepsilon}(\alpha,\delta)$ is an exit ramp for $N_{I,\varepsilon}(\alpha,\delta)$ we need to show that there exist a $\delta'' \in \,]0,\delta']$ and an $\alpha'' \in \,]0,\alpha']$ such that for all $\delta \in \,]0,\delta'']$ and for all $\alpha \in \,]0,\alpha'']$, there exist an $\varepsilon_2 = \varepsilon_2(\alpha,\delta) \in \,]0,\widetilde{\varepsilon}_2(\alpha,\delta)]$ such that for all $\varepsilon \in \,]0,\varepsilon_2]$ and for all $I \in \mathcal{A}(\prec)$

$$\partial\,[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta'} \cap N_{I,\varepsilon}(\alpha,\delta) \subset N_{2,\varepsilon}(\alpha,\delta).$$

Suppose this does not hold. Then there exist an $I \in \mathcal{A}(\prec)$ and sequences $(\alpha_n)_n$, $(\zeta_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $(\zeta_n)_n$ is decreasing, with $\zeta_n < \delta_2$ for all $n \in \mathbb{N}$, $\alpha_n \to 0$, $\zeta_n \to 0$, $\varepsilon_n \to 0$ as $n \to \infty$ and

$$w_n = (u_n,v_n) \in \partial\,[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'} \cap N_{I,\varepsilon_n}(\alpha_n,\zeta_n) \setminus N_{2,\varepsilon_n}(\alpha_n,\zeta_n), \text{ for all } n \in \mathbb{N}. \tag{27}$$

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Hence for all $n \in \mathbb{N}$, $w_n \in N_{1,\varepsilon_n}(\alpha_n, \zeta_n) \subset [B_{\delta_2,b_2}]_{\varepsilon_n,\eta} \subset]\widetilde{U}[_{\varepsilon_n,\nu}$ and so $t^+_{\varepsilon_n,\nu}(w_n) > b$. Moreover, for each $n \in \mathbb{N}$, there exist a $\overline{w}_n = (\overline{u}_n, \overline{v}_n) \in [\Lambda_{I,\zeta_n}]_{\varepsilon_n,\alpha_n}$ and a $t_n \geq 0$ such that $\overline{w}_n\pi_{\varepsilon_n}[0,t_n] \subset]\widetilde{U}[_{\varepsilon_n,\beta'}$, and $\Gamma_{\varepsilon_n}(w_n, \overline{w}_n\pi_{\varepsilon_n}t_n) < 2^{-n}$.

Since $(\overline{u}_n, \overline{v}_n) \in [\Lambda_{I,\zeta_n}]_{\varepsilon_n,\alpha_n}$, it follows that $\overline{u}_n \in \Lambda_{I,\zeta_n} \subset B_{I,\zeta_n,b_3,\gamma_3} \subset B_{\zeta_n,b_3}$, and so $g^-(\overline{u}_n) \leq \zeta_n$, and $d_{\varepsilon_n}(\overline{v}_n, \theta_{\varepsilon_n}) \leq \alpha_n$. Since $\zeta_n \to 0$ as $n \to \infty$, there exists a subsequence of $(\overline{u}_n)_n$, denoted again by $(\overline{u}_n)_n$, and a $u_0 \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ such that $\overline{u}_n \to u_0$ as $n \to \infty$ and so

$$\Gamma_{\varepsilon_n}(\overline{w}_n, (u_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty. \tag{28}$$

For all $n \in \mathbb{N}$, we have $\zeta_n \leq \zeta_1 < \delta_2$. Hence Lemma 5.26 implies that

$$\overline{u}_n \in \Lambda_{I,\zeta_n} \subset B_{I,\zeta_n,b_3,\gamma_3} \subset B_{I,\zeta_1,b_3,\gamma_3} \subset \mathrm{Int}_{X_0}(B_{I,\delta_2,b_2,\gamma_2})$$

which, in turn, implies that $u_0 \in B_{I,\zeta_1,b_3,\gamma_3} \subset \mathrm{Int}_{X_0}(B_{I,\delta_2,b_2,\gamma_2})$.

Notice that $\overline{w}_n \in [\Lambda_{I,\zeta_n}]_{\varepsilon_n,\alpha_n} \subset [B_{\zeta_n,b_3}]_{\varepsilon_n,\alpha_n}$ and $\overline{w}_n\pi_{\varepsilon_n}[0,t_n] \subset]\widetilde{U}[_{\varepsilon_n,\beta'} \subset [\widetilde{N}]_{\varepsilon_n,\beta}$. Hence, Lemma 5.29 implies that there is a subsequence of $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$ which we will denote again by $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$ and there is a $\tilde{u}_0 \in \mathrm{Inv}^-_{\pi_0}(\widetilde{N})$ such that

$$\Gamma_{\varepsilon_n}(\overline{w}_n\pi_{\varepsilon_n}t_n, (\tilde{u}_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty \tag{29}$$

and so $\Gamma_{\varepsilon_n}(w_n, (\tilde{u}_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$. Thus $d_0(u_n, \tilde{u}_0) \to 0$ as $n \to \infty$. Since $w_n \in [B_{\delta_2,b_2}]_{\varepsilon_n,\eta'} \subset [\widetilde{U}]_{\varepsilon_n,\nu}$, it follows that $u_n \in B_{\delta_2,b_2}$ and so $\tilde{u}_0 \in B_{\delta_2,b_2} \subset \widetilde{U}$. Formula (29) and Lemma 4.6 in [3] imply that

$$t^+_{\varepsilon_n,\nu}(\overline{w}_n\pi_{\varepsilon_n}t_n) \to t^+_0(\tilde{u}_0) \text{ as } n \to \infty. \tag{30}$$

Let $b' > 0$ be such that $b_2 < b' < b$. Since $t^+_{\varepsilon_n,\nu}(w_n) > b$ for all $n \in \mathbb{N}$, formula (30) implies that $t^+_0(\tilde{u}_0) \geq b > b'$ and, again, by formula (30) we have

$$t^+_{\varepsilon_n,\nu}(\overline{w}_n\pi_{\varepsilon_n}t_n) > b' \text{ for all } n \text{ large enough.} \tag{31}$$

We claim that

$$\overline{w}_n\pi_{\varepsilon_n}[0,t_n] \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'} \text{ for all } n \in \mathbb{N}. \tag{32}$$

Suppose this claim is not true. Then for each $n \in \mathbb{N}$, there exists a $\tau_n \in [0,t_n]$ such that $\overline{w}_n\pi_{\varepsilon_n}[0,\tau_n] \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'}$ and $\overline{w}_n\pi_{\varepsilon_n}\tau_n \in \partial\,[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'}$.

We will show that there exists a subsequence of $(\tau_n)_n$, denoted again by $(\tau_n)_n$, and an $r > 0$ such that $\tau_n > r$ for all $n \in \mathbb{N}$. In fact, suppose that $\tau_n \to 0$ as $n \to \infty$. Since $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ converges singularly to π_0, formula (28) implies that

$$\Gamma_{\varepsilon_n}(\overline{w}_n\pi_{\varepsilon_n}\tau_n, (u_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

Thus, there exists an $n_0 \in \mathbb{N}$ such that $\Gamma_{\varepsilon_n}(\overline{w}_n\pi_{\varepsilon_n}\tau_n, (u_0, \theta_{\varepsilon_n})) < \eta'$ for all $n \geq n_0$ which implies that $d_{\varepsilon_n}(\overline{w}_n\psi_{\varepsilon_n}\tau_n, \theta_{\varepsilon_n}) < \eta'$ for all $n \geq n_0$. Since $\overline{w}_n\pi_{\varepsilon_n}\tau_n \in \partial\,[B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'}$ for all $n \in \mathbb{N}$, we have $\overline{w}_n \phi_{\varepsilon_n} \tau_n \in \partial B_{I,\delta_2,b_2,\gamma_2}$ for all $n \geq n_0$. Therefore, $u_0 \in \partial B_{I,\delta_2,b_2,\gamma_2} \cap \mathrm{Int}_{X_0}(B_{I,\delta_2,b_2,\gamma_2})$ which is a contradiction. Thus, there exists a subsequence of $(\tau_n)_n$, denoted again by $(\tau_n)_n$, and an $r > 0$ such that $\tau_n > r$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, define $z_n := \overline{w}_n \pi_{\varepsilon_n}(\tau_n - r)$. Lemma 5.29 implies that there exists a subsequence of $(z_n)_n$ which will be denoted again by $(z_n)_n$ and a $z_0 \in \widetilde{N}$ such that

$$\Gamma_{\varepsilon_n}(z_n, (z_0, \theta_n)) \to 0 \text{ as } n \to \infty. \tag{33}$$

Notice that $z_n \pi_{\varepsilon_n}[0, r] = \overline{w}_n \pi_{\varepsilon_n}[\tau_n - r, \tau_n] \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'}$ and so

$$z_n \phi_{\varepsilon_n} s \in B_{I,\delta_2,b_2,\gamma_2} \text{ for all } s \in [0, r]. \tag{34}$$

Since $(\pi_\varepsilon)_{\varepsilon \in]0,\varepsilon_0]}$ converges singularly to π_0, formula (33) implies that for all $s \in [0, r]$

$$\Gamma_{\varepsilon_n}(z_n \pi_{\varepsilon_n} s, (z_0 \pi_0 s, \theta_n)) \to 0 \text{ as } n \to \infty. \tag{35}$$

This fact, together with formula (34), implies that $z_0 \pi_0[0, r]$ is included in $B_{I,\delta_2,b_2,\gamma_2}$ and $d_{\varepsilon_n}(z_n \psi_{\varepsilon_n} r, \theta_{\varepsilon_n}) < \eta'$ for all n large enough. Thus, $z_n \phi_{\varepsilon_n} r = \overline{w}_n \phi_{\varepsilon_n} \tau_n \in \partial B_{I,\delta_2,b_2,\gamma_2}$ and formula (35) implies that $z_0 \pi_0 r \in \partial B_{I,\delta_2,b_2,\gamma_2}$. Hence, $z_0 \pi_0 r \notin B^+_{I,\delta_2,b_2,\gamma_2}$.

Since $(B_{I,\delta_2,b_2,\gamma_2}, B_{P\setminus I,\delta_2,b_2,\gamma_2})$ is a block pair for $(M_0(I), M_0(P \setminus I))$, it follows that $z_0 \pi_0 r \notin B_{P\setminus I,\delta_2,b_2,\gamma_2}$. Indeed, otherwise, $z_0 \pi_0 r \in B_{I,\delta_2,b_2,\gamma_2} \cap B_{P\setminus I,\delta_2,b_2,\gamma_2} \subset B^+_{I,\delta_2,b_2,\gamma_2} \cap B^-_{P\setminus I,\delta_2,b_2,\gamma_2}$ which is a contradiction. Hence

$$z_0 \pi_0 r \in \partial B_{I,\delta_2,b_2,\gamma_2} \setminus B_{P\setminus I,\delta_2,b_2,\gamma_2} \subset \partial B_{\delta_2,b_2}.$$

Moreover, $z_0 \pi_0[0, r] \subset B_{\delta_2,b_2}$. Hence $z_0 \pi_0 r \in B^-_{\delta_2,b_2}$ and so, by Lemma 5.25, $t^+(z_0 \pi_0 r) = b_2$. On the other hand, since $\tau_n \leq t_n$ for all $n \in \mathbb{N}$, it follows that $t^+_{\varepsilon_n,\nu}(\overline{w}_n \pi_{\varepsilon_n} \tau_n) \geq t^+_{\varepsilon_n,\nu}(\overline{w}_n \pi_{\varepsilon_n} t_n) > b'$ for all n large enough, by (31). Since $t^+_{\varepsilon_n,\nu}(\overline{w}_n \pi_{\varepsilon_n} \tau_n) \to t^+_0(z_0 \pi_0 r)$, we have $t^+_0(z_0 \pi_0 r) \geq b' > b_2$ which is a contradiction. Hence, our claim (32) holds.

Suppose that $t_n \to 0$ as $n \to \infty$. Since $(\pi_\varepsilon)_{\varepsilon \in]0,\varepsilon_0]}$ converges singularly to π_0 and $\Gamma_{\varepsilon_n}(\overline{w}_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$, we have $\Gamma_{\varepsilon_n}(\overline{w}_n \pi_{\varepsilon_n} t_n, (u_0, \theta_{\varepsilon_n})) \to 0$ and

$$\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

This implies that $d_{\varepsilon_n}(v_n, \theta_{\varepsilon_n}) < \eta'$ for all n large enough. Recall from formula (27) that $w_n \in \partial\, [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'}$. Thus, $u_n \in \partial B_{I,\delta_2,b_2,\gamma_2}$ for all n large enough and so $u_0 \in \partial B_{I,\delta_2,b_2,\gamma_2}$ but this is a contradiction. Hence there exists a subsequence of $(t_n)_n$, denoted again by $(t_n)_n$, and an $s > 0$ such that $\tau_n > s$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, define $\tilde{z}_n := \overline{w}_n \pi_{\varepsilon_n}(t_n - s)$. Lemma 5.29 implies that there exists a subsequence of $(\tilde{z}_n)_n$ which will be denoted again by $(\tilde{z}_n)_n$, and a $\tilde{z}_0 \in \widetilde{N}$ such that

$$\Gamma_{\varepsilon_n}(\tilde{z}_n, (\tilde{z}_0, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty. \tag{36}$$

Notice that $\tilde{z}_n \pi_{\varepsilon_n}[0, s] = \overline{w}_n \pi_{\varepsilon_n}[t_n - s, t_n] \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon_n,\eta'}$ and so, by (32)

$$\tilde{z}_n \phi_{\varepsilon_n} \tau \in B_{I,\delta_2,b_2,\gamma_2} \text{ for all } \tau \in [0, s]. \tag{37}$$

Since $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ converges singularly to π_0, formula (36) implies that for each $\tau \in [0, s]$

$$\Gamma_{\varepsilon_n}(\tilde{z}_n\pi_{\varepsilon_n}\tau, (\tilde{z}_0\pi_0\tau, \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty \tag{38}$$

and so, for each $\tau \in [0, s]$, $d_0(\tilde{z}_n\phi_{\varepsilon_n}\tau, \tilde{z}_0\pi_0\tau) \to 0$ as $n \to \infty$. This and formula (37) imply that $\tilde{z}_0\pi_0[0, s] \subset B_{I,\delta_2,b_2,\gamma_2}$. Since $\tilde{z}_n\pi_{\varepsilon_n}s = \overline{w}_n\pi_{\varepsilon_n}t_n$, it follows that

$$\Gamma_{\varepsilon_n}(w_n, (\tilde{z}_0\pi_0 s, \theta_n)) \to 0 \text{ as } n \to \infty.$$

Thus $d_\varepsilon(v_n, \theta_{\varepsilon_n}) < \eta'$ for all n large enough. This, together with (27), implies that $u_n \in \partial B_{I,\delta_2,b_2,\gamma_2}$ for all n large enough and so $\tilde{z}_0\pi_0 s \in \partial B_{I,\delta_2,b_2,\gamma_2}$. Thus, $\tilde{z}_0\pi_0 s \notin B^+_{I,\delta_2,b_2,\gamma_2}$ and so $\tilde{z}_0\pi_0 s \in \partial B_{I,\delta_2,b_2,\gamma_2} \setminus B_{P\setminus I,\delta_2,b_2,\gamma_2} \subset \partial B_{\delta_2,b_2}$. This, together with $\tilde{z}_0\pi_0[0, s] \subset B_{\delta_2,b_2}$, implies that $\tilde{z}_0\pi_0 s \in B^-_{\delta_2,b_2}$. By Lemma 5.25, we have $t^+_0(z_0\pi_0 r) = b_2$. On the other hand, since $t^+_{\varepsilon_n,\nu}(\overline{w}_n\pi_{\varepsilon_n}t_n) > b'$ for all n large enough, by formula (31), and $t^+_{\varepsilon_n,\nu}(\overline{w}_n\pi_{\varepsilon_n}t_n) \to t^+_0(\tilde{z}_0\pi_0 s)$, we have $t^+_0(\tilde{z}_0\pi_0 s) \geq b' > b_2$ which is a contradiction. The proof of the lemma is complete. ∎

Theorem 5.32. *Let $\overline{b}$ be as in Lemma 5.25 and $\overline{\delta}$ be as in Lemma 5.26. Let the positive numbers b, b_2, b_3, b'_2, b''_1, δ_2, ν, η, β' be such that $\overline{b} < b_2 < b_3 < b'_2 < b < b''_1$, $\delta_2 \in]0, \overline{\delta}[$ and $\beta' < \eta < \nu < \beta$. Then the following holds:*

1. *there exist a $\delta'_4 \in]0, \delta_2]$ and an $\mu_0 \in]0, \beta']$ such that for all $\delta \in]0, \delta'_4]$ and for all $\mu \in]0, \mu_0]$, there exists an $\varepsilon'_2 = \varepsilon'_2(\mu, \delta) \in]0, \widehat{\varepsilon}]$ such that for all $\varepsilon \in]0, \varepsilon'_2]$,*

$$[B_{\delta,b_3} \cap \{\, u \in \widetilde{U} \mid t^+_0(u) \leq b'_2 \,\}]_{\varepsilon,\mu} \subset [B_{\delta,b_3}]_{\varepsilon,\mu} \cap \{\, w \in]\widetilde{U}[_{\varepsilon,\nu} \mid t^+_{\varepsilon,\nu}(w) \leq b \,\}.$$

2. *there exist a $\delta''_4 \in]0, \delta_2]$ and an $\alpha''_1 \in]0, \beta']$ such that for all $\delta \in]0, \delta''_4]$ and for all $\alpha \in]0, \alpha''_1]$, there exists an $\varepsilon''_2 = \varepsilon''_2(\alpha, \delta) \in]0, \widehat{\varepsilon}]$ such that for all $\varepsilon \in]0, \varepsilon''_2]$,*

$$N_{2,\varepsilon}(\alpha, \delta) \subset [B_{\delta_2,b_2} \cap \{\, u \in \widetilde{U} \mid t^+_0(u) \leq b''_1 \,\}]_{\varepsilon,\eta}\,.$$

Proof. Suppose the conclusion of (1) does not hold. Then there exist sequences $(\zeta_n)_n$, $(\mu_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $\zeta_n \to 0$, $\mu_n \to 0$, $\varepsilon_n \to 0$ and

$$w_n = (u_n, v_n) \in [B_{\zeta_n,b_3} \cap \{\, u \in \widetilde{U} \mid t^+_0(u) \leq b'_2 \,\}]_{\varepsilon_n,\mu_n} \text{ for all } n \in \mathbb{N} \tag{39}$$

and

$$w_n = (u_n, v_n) \notin [B_{\zeta_n,b_3}]_{\varepsilon_n,\mu_n} \cap \{\, w \in]\widetilde{U}[_{\varepsilon_n,\nu} \mid t^+_{\varepsilon_n,\nu}(w) \leq b \,\} \text{ for all } n \in \mathbb{N}. \tag{40}$$

(39) implies that $u_n \in B_{\zeta_n,b_3} \subset B_{\delta_3,b_3} \subset B_{\delta_2,b_3} \subset \widetilde{U}$ and $v_n \in \mathrm{Cl}_{Y_{\varepsilon_n}}(B(\theta_{\varepsilon_n}, \mu_n))$ for all $n \in \mathbb{N}$. Hence, $w_n \in [B_{\zeta_n,b_3}]_{\varepsilon_n,\mu_n}$ for all $n \in \mathbb{N}$. Since $\mu_n \to 0$ as $n \to \infty$, it follows

that $\mu_n < \nu$ for all n large enough and so $w_n \in]\widetilde{U}[_{\varepsilon_n,\nu}$ for all n large enough. Hence, formula (40) implies that

$$t^+_{\varepsilon_n,\nu}(w_n) > b \text{ for all } n \text{ large enough.} \tag{41}$$

Since $u_n \in B_{\zeta_n,b_3}$ and $\zeta_n \to 0$ as $n \to \infty$, admissibility implies that there exist a $u_0 \in \widetilde{N}$ and a subsequence of $(u_n)_n$, denoted again by $(u_n)_n$, such that $u_n \to u_0$ as $n \to \infty$. Hence, $u_0 \in \widetilde{U}$ and so $t^+_0(u_n) \to t^+_0(u_0)$ as $n \to \infty$. Formula (39) implies that $t^+_0(u_0) \le b'_2 < b$.

Since $v_n \in \mathrm{Cl}_{Y_{\varepsilon_n}}(B(\theta_{\varepsilon_n},\mu_n))$ for all $n \in \mathbb{N}$ and $\mu_n \to 0$ as $n \to \infty$, it follows that $d_{\varepsilon_n}(v_n,\theta_{\varepsilon_n}) \to 0$ as $n \to \infty$ and so $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n})) \to 0$ as $n \to \infty$. Hence, Lemma 4.6 in [3] implies that $t^+_{\varepsilon_n,\nu}(w_n) \to t^+_0(u_0)$ as $n \to \infty$ and so $t^+_{\varepsilon_n,\nu}(w_n) < b$ for all n large enough. But this contradicts formula (41).

Suppose the conclusion of (2) does not hold. Then there exist sequences $(\zeta_n)_n$, $(\alpha_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $\zeta_n \to 0$, $\alpha_n \to 0$, $\varepsilon_n \to 0$ and

$$w_n = (u_n,v_n) \in N_{2,\varepsilon_n}(\alpha_n,\zeta_n) \setminus [B_{\delta_2,b_2} \cap \{\, u \in \widetilde{U} \mid t^+_0(u) \le b''_1 \,\}]_{\varepsilon_n,\eta}\,. \tag{42}$$

Since $w_n \in N_{2,\varepsilon_n}(\alpha_n,\zeta_n)$, it follows that $w_n \in]\widetilde{U}[_{\varepsilon_n,\nu}$ and $t^+_{\varepsilon_n,\nu}(w_n) \le b$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, $w_n \in [B_{\delta_2,b_2}]_{\varepsilon_n,\eta}$ and there exist an $\overline{w}_n \in [B_{\zeta_n,b_2}]_{\varepsilon_n,\alpha_n}$ and a $t_n \ge 0$ such that $\overline{w}_n\pi_{\varepsilon_n}[0,t_n] \subset]\widetilde{U}[_{\varepsilon_n,\beta}$ and $\Gamma_{\varepsilon_n}(w_n,\overline{w}_n\pi_{\varepsilon_n}t_n) < 2^{-n}$.

Hence, $u_n \in \widetilde{U}$ and $v_n \in \mathrm{Cl}_{Y_{\varepsilon_n}}(B(\theta_{\varepsilon_n},\eta))$ for all $n \in \mathbb{N}$ and so, by formula (42), we have $t^+_0(u_n) > b''_1$ for all $n \in \mathbb{N}$.

Lemma 5.29 implies that there exist a subsequence of $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$ which we will denote again by $(\overline{w}_n\pi_{\varepsilon_n}t_n)_n$ and a $u_0 \in \widetilde{N}$ such that $\Gamma_{\varepsilon_n}(\overline{w}_n\pi_{\varepsilon_n}t_n,(u_0,\theta_{\varepsilon_n})) \to 0$ as $n \to \infty$. Hence, $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n})) \to 0$ as $n \to \infty$. Since $u_n \in B_{\delta_2,b_2}$ for all $n \in \mathbb{N}$, it follows that $u_0 \in B_{\delta_2,b_2} \subset \widetilde{U}$ and so $t^+_0(u_n) \to t^+_0(u_0)$. Thus, $t^+_0(u_0) \ge b''_1$. On the other hand, Lemma 4.6 in [3] implies that $t^+_{\varepsilon_n,\nu}(w_n) \to t^+_0(u_0) \to 0$ as $n \to \infty$ and so $t^+_0(u_0) \le b < b''_1$ which is a contradiction. ▌

6. THE PROOF OF THE SINGULAR NESTED INDEX FILTRATION THEOREM

Let $\overline{b}$ be as in Lemma 5.25 and $\overline{\delta}$ and $\overline{\gamma}$ be as in Lemma 5.26. Fix real numbers γ_i, b_i, $i = 1,\dots, 5$, δ_i, $i = 1, 2$, b, b'_i, b''_i, $i = 1, 2$ such that

$$0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_5 < \overline{\gamma},\ 0 < \delta_2 < \delta_1 < \overline{\delta} \tag{43}$$

and

$$\overline{b} < b_1 < b_2 < b_3 < b_4 < b_5 < b'_1 < b'_2 < b < b''_1 < b''_2. \tag{44}$$

For all $\delta \in]0,\delta_2]$ and $I \in \mathcal{A}(\prec)$ define $\Lambda_{I,\delta} := B_{I,\delta,b_3,\gamma_3}$. It is immediately checked that the family $\Lambda_{I,\delta}$ of sets satisfies the hypotheses of Theorem 5.27.

Let $\nu > 0$, $\eta > 0$, $\beta' > 0$, with $\beta' < \eta < \nu < \widetilde{\beta}_0 < \beta$, be fixed.

Let $\delta_4' \in]0, \delta_2]$ and $\mu_0 \in]0, \beta']$ be as in Theorem 5.32.(1). For all $\delta \in]0, \delta_4']$ and for all $\mu \in]0, \mu_0]$, let $\varepsilon_2'(\mu, \delta) \in]0, \widetilde{\varepsilon}]$ be such that the conclusions of Theorem 5.32.(1) hold for all $\varepsilon \in]0, \varepsilon_2'(\mu, \delta)]$.

Fix $\eta_0 \in]0, \mu_0[$. Hence, $\eta_0 < \beta' < \eta < \nu < \widetilde{\beta}_0 < \beta$.

For all $I \in \mathcal{A}(\prec)$, $\delta \in]0, \delta_2]$, $\alpha \in]0, \eta_0]$, and $\varepsilon \in]0, \widetilde{\varepsilon}]$, let the sets $N_{1,\varepsilon}(\alpha, \delta)$, $N_{I,\varepsilon}(\alpha, \delta)$ and $N_{2,\varepsilon}(\alpha, \delta)$ be as in Theorem 5.27 with respect to the set $\Lambda_{I,\delta} = B_{I,\delta,b_3,\gamma_3}$ and $\eta' = \eta_0$. More explicitly,

$$N_{1,\varepsilon}(\alpha, \delta) := [B_{\delta_2,b_2}]_{\varepsilon,\eta} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{there exist a } \overline{w} \in [B_{\delta,b_3}]_{\varepsilon,\alpha} \text{ and a } t \geq 0 \\ \text{such that } \overline{w}\pi_n[0,t] \subset]\widetilde{U}[_{\varepsilon,\beta} \text{ and } w = \overline{w}\pi_\varepsilon t \,\}),$$

$$N_{2,\varepsilon}(\alpha, \delta) := N_{1,\varepsilon}(\alpha, \delta) \cap \{\, w \in]\widetilde{U}[_{\varepsilon,\nu} \mid t^+_{\varepsilon,\nu}(w) \leq b \,\}$$

and

$$N_{I,\varepsilon}(\alpha, \delta) := [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta_0} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{ there exist a } \overline{w} \in [B_{I,\delta,b_3,\gamma_3}]_{\varepsilon,\alpha} \text{ and} \\ \text{a } t \geq 0 \text{ such that } \overline{w}\pi_\varepsilon[0,t] \subset]\widetilde{U}[_{\varepsilon,\beta'} \text{ and } w = \overline{w}\pi_\varepsilon t \,\}) \cup N_{2,\varepsilon}(\alpha, \delta).$$

An application of Theorem 5.27 shows that there exist a $\delta_3 \in]0, \delta_2[$ and an $\alpha_0 \in]0, \eta_0]$ such that for all $\delta \in]0, \delta_3]$ and for all $\alpha \in]0, \alpha_0]$, there exists an $\widetilde{\varepsilon}_0(\alpha, \delta) \in]0, \widetilde{\varepsilon}]$ such that for all $\varepsilon \in]0, \widetilde{\varepsilon}_0(\alpha, \delta)]$ and for all $I \in \mathcal{A}(\prec)$ the triple

$$(N_{1,\varepsilon}(\alpha, \delta), N_{I,\varepsilon}(\alpha, \delta), N_{2,\varepsilon}(\alpha, \delta))$$

is an FM-index triple for $(\pi_\varepsilon, S_\varepsilon, M_\varepsilon(I), M_\varepsilon(P \setminus I))$.

Moreover, let $\delta_4'' \in]0, \delta_2]$ and $\alpha_1'' \in]0, \beta']$ be as in Theorem 5.32.(2). For each $\delta \in]0, \delta_4'']$ and for each $\alpha \in]0, \alpha_1'']$, let $\varepsilon_2''(\alpha, \delta) \in]0, \varepsilon_0]$ be such that the conclusions of Theorem 5.32.(2) hold for all $\varepsilon \in]0, \varepsilon_2''(\alpha, \delta)]$. Choose positive numbers δ_4 and α_1 such that $\delta_4 < \min\{\, \delta_3, \delta_4', \delta_4'' \,\}$ and $\alpha_1 < \min\{\, \alpha_0, \alpha_1'', \eta_0 \,\}$.

Define $\varepsilon_1 := \min\{\, \widetilde{\varepsilon}_0(\alpha_1, \delta_4), \varepsilon_2'(\eta_0, \delta_4), \varepsilon_2''(\alpha_1, \delta_4) \,\}$. Choose

$$\mu \in]0, \alpha_1[\text{ and } \delta_5 \in]0, \delta_4[. \tag{45}$$

For $\varepsilon \in]0, \varepsilon_1]$ and $I \in \mathcal{A}(\prec)$, define the following sets:

$$\begin{aligned}
&N_1^2 := B_{\delta_5,b_4}, \quad N_2^2 := B_{\delta_5,b_4} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \leq b_1' \,\},\\
&N_1^4 := B_{\delta_1,b_1}, \quad N_2^4 := B_{\delta_1,b_1} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \leq b_2'' \,\},\\
&N_{1,\varepsilon} := N_{1,\varepsilon}(\alpha_1, \delta_4), \quad N_{2,\varepsilon} := N_{2,\varepsilon}(\alpha_1, \delta_4),\\
&N_I^2 := B_{I,\delta_5,b_4,\gamma_4} \cup N_2^2, \quad N_I^4 := B_{I,\delta_1,b_1,\gamma_1} \cup N_2^4, \quad N_{I,\varepsilon} := N_{I,\varepsilon}(\alpha_1, \delta_4),\\
&V_1 := \mathrm{Int}_{X_0}(B_{\delta_4,b_3}),\ V_{I,3} := \mathrm{Int}_{X_0}(B_{I,\delta_4,b_3,\gamma_3}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\},\\
&V_{I,4} := \mathrm{Int}_{X_0}(B_{\delta_4,b_3}) \cap (X_0 \setminus B_{I,\delta_2,b_2,\gamma_2}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\}.
\end{aligned}$$

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It is clear that for each $I \in \mathcal{A}(\prec)$ the triple (N_1^i, N_I^i, N_2^i) is an FM-index triple for $(\pi_0, S_0, M_0(I), M_0(P\setminus I))$, $i = 2, 4$. We need to show that these sets satisfy the inclusions of Theorem 4.24. It is easy to prove that

$$N_1^2 \subset \widetilde{U}, \quad N_2^2 \subset N_2^4 \tag{46}$$

and

$$N_I^2 \subset N_I^4 \text{ for all } I \in \mathcal{A}(\prec).$$

Lemma 6.33. *For all $\varepsilon \in \,]0, \varepsilon_1]$, the following inclusions hold*

$$[N_1^2]_{\varepsilon,\mu} \subset \,]V_1[_{\varepsilon,\alpha_1} \subset N_{1,\varepsilon} \subset [N_1^4]_{\varepsilon,\eta}\,.$$

Proof. Let $\varepsilon \in \,]0, \varepsilon_1]$ be fixed. It is straightforward to prove that $N_1^2 \subset V_1$. Since $\mu < \alpha_1$, it follows that $[N_1^2]_{\varepsilon,\mu} \subset \,]V_1[_{\varepsilon,\alpha_1}$.

Since $\delta_4 < \delta_2 < \overline{\delta}$, $b_3 > b_2 > \overline{b}$ and $\alpha_1 < \eta < \beta$, it follows that $[B_{\delta_4,b_3}]_{\varepsilon,\alpha_1} \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta}$ and $[B_{\delta_4,b_3}]_{\varepsilon,\alpha_1} \subset \,]\widetilde{U}[_{\varepsilon,\beta}$ and so $[B_{\delta_4,b_3}]_{\varepsilon,\alpha_1} \subset N_{1,\varepsilon}(\alpha_1,\delta_4) \subset [B_{\delta_2,b_2}]_{\varepsilon,\eta}$ (see also Remark 5.28). Hence $]V_1[_{\varepsilon,\alpha_1} \subset [B_{\delta_4,b_3}]_{\varepsilon,\alpha_1} \subset N_{1,\varepsilon}(\alpha_1,\delta_4) \subset [B_{\delta_1,b_1}]_{\varepsilon,\eta} = [N_1^4]_{\varepsilon,\eta}$ as $\delta_2 < \delta_1$ and $b_2 > b_1$. The lemma is proved. ∎

Lemma 6.34. *For all $\varepsilon \in \,]0, \varepsilon_1]$ and for all $I \in \mathcal{A}(\prec)$, the following inclusions hold*

$$\mathrm{Cl}_{Z_\varepsilon}([N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta}) \subset \,]V_{I,3}[_{\varepsilon,\alpha_1} \subset \mathrm{Int}_{Z_\varepsilon}(N_{I,\varepsilon} \setminus N_{2,\varepsilon}).$$

Proof. Let $\varepsilon \in \,]0, \varepsilon_1]$ and $I \in \mathcal{A}(\prec)$ be fixed. Let $w = (u, v) \in [N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta}$. Hence $u \in N_I^2 = B_{I,\delta_5,b_4,\gamma_4} \cup N_2^2$ and $v \in \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon,\mu)) \subset \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon,\eta))$. Thus, $u \notin N_2^4$ and so $u \in B_{I,\delta_5,b_4,\gamma_4} \cup N_2^2 \setminus N_2^4$. The second inclusion in (46) implies that $u \in B_{I,\delta_5,b_4,\gamma_4} \setminus N_2^4$. Hence

$$\begin{aligned}\mathrm{Cl}_{Z_\varepsilon}([N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta}) &\subset \mathrm{Cl}_{Z_\varepsilon}([B_{I,\delta_5,b_4,\gamma_4} \setminus N_2^4]_{\varepsilon,\mu}) = [\mathrm{Cl}_{X_0}(B_{I,\delta_5,b_4,\gamma_4} \setminus N_2^4)]_{\varepsilon,\mu}\\ &\subset [B_{I,\delta_5,b_4,\gamma_4}]_{\varepsilon,\mu}\,.\end{aligned}$$

It is clear that $B_{I,\delta_5,b_4,\gamma_4} \subset B_{\delta_5,b_4} \subset B_{\delta_1,b_1} \subset \widetilde{U}$ and so $B_{I,\delta_5,b_4,\gamma_4} \setminus N_2^4 \subset B_{I,\delta_5,b_4,\gamma_4} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \ge b_2'' \,\}$. Moreover, the inequality $b_2'' > b_1''$ and Lemma 5.26 imply that $B_{I,\delta_5,b_4,\gamma_4} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \ge b_2'' \,\} \subset \mathrm{Int}_{X_0}(B_{I,\delta_4,b_3,\gamma_3}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\} = V_{I,3}$. Hence

$$\mathrm{Cl}_{Z_\varepsilon}([N_I^2]_{\varepsilon,\mu} \setminus [N_2^4]_{\varepsilon,\eta}) \subset \,]V_{I,3}[_{\varepsilon,\alpha_1}\,, \tag{47}$$

as $\mu < \alpha_1$. It follows from Remark 5.28 that

$$[B_{I,\delta_4,b_3,\gamma_3}]_{\varepsilon,\alpha_1} \subset N_{I,\varepsilon}(\alpha_1,\delta_4) \subset [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta_0} \cup N_{2,\varepsilon}(\alpha_1,\delta_4) \;\subset [B_{\delta_2,b_2}]_{\varepsilon,\eta_0} \cup N_{2,\varepsilon}(\alpha_1,\delta_4)$$

and so

$$]V_{I,3}[_{\varepsilon,\alpha_1} \subset N_{I,\varepsilon}(\alpha_1,\delta_4) \cap]\{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\}[_{\varepsilon,\alpha_1} =: T$$

If $(u,v) \in T$ then $t_0^+(u) > b_1''$ so, by Theorem 5.32, $(u,v) \notin N_{2,\varepsilon}(\alpha_1,\delta_4)$. It follows that $T \subset N_{I,\varepsilon}(\alpha_1,\delta_4) \setminus N_{2,\varepsilon}(\alpha_1,\delta_4)$ and this, together with (47), concludes the proof of the lemma. ∎

LEMMA 6.35. *For all $\varepsilon \in\,]0,\varepsilon_1]$ and for all $I \in \mathcal{A}(\prec)$, the following inclusions hold*

$$\mathrm{Cl}_{Z_\varepsilon}([N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta}) \subset\,]V_{I,4}[_{\varepsilon,\alpha_1} \subset \mathrm{Int}_{Z_\varepsilon}(N_{1,\varepsilon} \setminus N_{I,\varepsilon}).$$

Proof. Let $\varepsilon \in\,]0,\varepsilon_1]$ and $I \in \mathcal{A}(\prec)$ be fixed. Let $w = (u,v) \in [N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta}$. Hence $u \in N_1^2 = B_{\delta_5,b_4}$ and $v \in \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon,\mu)) \subset \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon,\eta))$. Thus, $u \notin N_I^4$ and so $u \in B_{\delta_5,b_4} \setminus (B_{I,\delta_1,b_1,\gamma_1} \cup N_2^4)$. Hence

$$\begin{aligned}\mathrm{Cl}_{Z_\varepsilon}([N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta}) &\subset \mathrm{Cl}_{Z_\varepsilon}([B_{\delta_5,b_4} \setminus (B_{I,\delta_1,b_1,\gamma_1} \cup N_2^4)]_{\varepsilon,\mu}) \\ &= [\mathrm{Cl}_{X_0}(B_{\delta_5,b_4} \setminus (B_{I,\delta_1,b_1,\gamma_1} \cup N_2^4))]_{\varepsilon,\mu} \\ &\subset [B_{\delta_5,b_4} \setminus \mathrm{Int}_{X_0}(B_{I,\delta_1,b_1,\gamma_1} \cup N_2^4)]_{\varepsilon,\mu}\,.\end{aligned}$$

A simple computation shows that $B_{\delta_5,b_4} \setminus \mathrm{Int}_{X_0}(B_{I,\delta_1,b_1,\gamma_1} \cup N_2^4) \subset \mathrm{Int}_{X_0}(B_{\delta_4,b_3}) \cap (X_0 \setminus B_{I,\delta_2,b_2,\gamma_2}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\} = V_{I,4}$. Since $\mu < \alpha_1$, it follows that

$$\mathrm{Cl}_{Z_\varepsilon}([N_1^2]_{\varepsilon,\mu} \setminus [N_I^4]_{\varepsilon,\eta}) \subset\,]V_{I,4}[_{\varepsilon,\alpha_1}\,.$$

We claim that $]V_{I,4}[_{\varepsilon,\alpha_1} \subset N_{1,\varepsilon} \setminus N_{I,\varepsilon}$. First notice that $]V_{I,4}[_{\varepsilon,\alpha_1} \subset [B_{\delta_4,b_3}]_{\varepsilon,\alpha_1} \subset N_{1,\varepsilon}(\alpha_1,\delta_4)$. Suppose that there exists a $(u,v) \in\,]V_{I,4}[_{\varepsilon,\alpha_1} \cap N_{I,\varepsilon}(\alpha_1,\delta_4)$. Hence $(u,v) \in [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta_0} \cup N_{2,\varepsilon}(\alpha_1,\delta_4)$.

If $(u,v) \in [B_{I,\delta_2,b_2,\gamma_2}]_{\varepsilon,\eta_0}$, then $u \in B_{I,\delta_2,b_2,\gamma_2}$, but this contradicts the definition of $V_{I,4}$. If $(u,v) \in N_{2,\varepsilon}(\alpha_1,\delta_4)$, Theorem 5.32.(2) implies that

$$(u,v) \in [B_{\delta_2,b_2} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \le b_1'' \,\}]_{\varepsilon,\eta}$$

and so $u \in B_{\delta_2,b_2} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \le b_1'' \,\}$ which is a contradiction to the definition of the set $V_{I,4}$. Therefore our claim is proved and this concludes the proof of the lemma. ∎

The above results mean that all assumptions of Theorem 4.24 are satisfied. That theorem, therefore, implies the following.

THEOREM 6.36. *With the notation introduced above, there exists an $\varepsilon_1 \in\,]0,\widetilde{\varepsilon}]$ such that for every $\varepsilon \in [0,\varepsilon_1]$ there exists an index filtration $(N_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$ for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ such that for all $I \in \mathcal{A}(\prec)$ the following inclusions hold:*

1. $B_{I,\delta_5,b_4,\gamma_4} \cup N_2^2 = N_I^2 \subset N_0(I) \subset N_1^2 = B_{\delta_5,b_4}$,

2. $N_2^2 \subset N_0(\emptyset)$,
3. $N_{I,\varepsilon} \subset N_\varepsilon(I) \subset N_{1,\varepsilon}$ *for* $\varepsilon \in \,]0, \varepsilon_1]$ *and*
4. $N_{2,\varepsilon} \subset N_\varepsilon(\emptyset)$ *for* $\varepsilon \in \,]0, \varepsilon_1]$.

Furthermore, whenever $I \in \mathcal{A}(\prec)$, $(\varepsilon_n)_n$ *is a sequence such that* $\varepsilon_n \to 0$ *as* $n \to \infty$ *and* $(w_n)_n$*, with* $w_n = (u_n, v_n)$ *for* $n \in \mathbb{N}$*, is a sequence such that* $w_n \in [N_1^2]_{\varepsilon_n,\mu} \cap N_{\varepsilon_n}(I)$ *for all* $n \in \mathbb{N}$ *and* $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ *as* $n \to \infty$ *for some* $u_0 \in X_0$*, then there exists an* $n_0 \in \mathbb{N}$ *such that* $u_n \in N_0(I)$ *for all* $n \geq n_0$.

Let $\varepsilon_1 > 0$ and $(N_0(I))_{I \in \mathcal{A}(\prec)}$ be as in Theorem 6.36. For $I \in \mathcal{A}(\prec)$ we will now define a new family $\widetilde{G}_{P\setminus I,\gamma}$, $\gamma \in \,]0, \infty[$, of neighborhoods of $M_0(P \setminus I)$ to which the results of the preceding section, in particular Theorems 5.27 and 5.32, can be applied.

Given $I \in \mathcal{A}(\prec)$ with $M_0(P \setminus I) \neq \emptyset$ let

$$\widetilde{U}_{P\setminus I} := \operatorname{Int}_{X_0}(N_0(P) \setminus N_0(I)). \tag{48}$$

$\widetilde{U}_{P\setminus I}$ is an open neighborhood of $M_0(P\setminus I)$ with $\operatorname{Cl}_{X_0}(\widetilde{U}_{P\setminus I}) \cap M_0(I) = \emptyset$. Let $\widetilde{g}^+_{P\setminus I} \colon \widetilde{U}_{P\setminus I} \to [0, \infty[$ be the map given by

$$\widetilde{g}^+_{P\setminus I}(u) := \inf\{\, (1+t)^{-1}\widetilde{G}(u\pi_0 t) \mid 0 \leq t < \rho_{\widetilde{U}_{P\setminus I},\pi_0}(u) \,\},$$

where $\widetilde{G}(u) := d_0(u, M_0(P \setminus I))/(d_0(u, M_0(P \setminus I)) + d_0(u, X_0 \setminus \operatorname{Cl}_{X_0}(\widetilde{U}_{P\setminus I})))$, $u \in \widetilde{U}_{P\setminus I}$.

Choose open sets $\widetilde{V}_{P\setminus I}$ and $\widetilde{W}_{P\setminus I}$ such that $M_0(P \setminus I) \subset \widetilde{V}_{P\setminus I} \subset \operatorname{Cl}_{X_0}(\widetilde{V}_{P\setminus I}) \subset \widetilde{W}_{P\setminus I} \subset \operatorname{Cl}_{X_0}(\widetilde{W}_{P\setminus I}) \subset \widetilde{U}_{P\setminus I}$ and $g^+_{P\setminus I}|_{\operatorname{Cl}_{X_0}(\widetilde{W}_{P\setminus I})}$ is continuous. This is possible by Proposition I.5.2 in [21].

Now, for arbitrary $I \in \mathcal{A}(\prec)$ and $\gamma \in \,]0, \infty[$ define

$$\widetilde{G}_{P\setminus I,\gamma} := \begin{cases} \operatorname{Cl}_{X_0}(\{\, u \in \widetilde{V}_{P\setminus I} \mid \widetilde{g}^+_{P\setminus I}(u) < \gamma \,\}) & \text{if } M_0(P \setminus I) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

and set

$$\widetilde{B}_{P\setminus I,\delta,b,\gamma} := B_{1,\delta,b,\widetilde{G}_{P\setminus I,\gamma}}, \; \widetilde{B}_{I,\delta,b,\gamma} := B_{2,\delta,b,\widetilde{G}_{P\setminus I,\gamma}}, \; I \in \mathcal{A}(\prec),\, \gamma \in \,]0, \infty[\,.$$

(cf. (14).)

Lemma 5.26 implies that there exist a $\widetilde{\delta}_0 \in \,]0, \delta_5[$ (with δ_5 as in (45)) and a $\widetilde{\gamma}_0 \in \,]0, \overline{\gamma}[$ such that for all $\widetilde{\delta} \in \,]0, \widetilde{\delta}_0]$, $\widetilde{\gamma} \in \,]0, \widetilde{\gamma}_0]$, $b \in \left[\overline{b}, \infty\right[$ and for all $I \in \mathcal{A}(\prec)$, the pair $(\widetilde{B}_{P\setminus I,\delta,b,\gamma}, \widetilde{B}_{I,\delta,b,\gamma})$ is a block pair for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$. Moreover, if $0 < \widetilde{\gamma}_3 < \widetilde{\gamma}_2 \leq \widetilde{\gamma}_0$, $0 < \widetilde{\delta}_3 < \widetilde{\delta}_2 \leq \widetilde{\delta}_0$ and $\widetilde{b}_2 < \widetilde{b}_3 \leq \overline{b}$, then

$$\widetilde{B}_{I,\widetilde{\delta}_3,\widetilde{b}_3,\widetilde{\gamma}_3} \subset \operatorname{Int}_{X_0}(\widetilde{B}_{I,\widetilde{\delta}_2,\widetilde{b}_2,\widetilde{\gamma}_2}). \tag{49}$$

Formula (48) implies that

$$N_0(I)\cap B_{\widetilde{\delta},b}\subset \widetilde{B}_{I,\widetilde{\delta},b,\widetilde{\gamma}},\quad \widetilde{\delta}\in\left]0,\widetilde{\delta}_0\right],\ \widetilde{\gamma}\in\left]0,\widetilde{\gamma}_0\right],\ b\in\left[\overline{b},\infty\right[,\ I\in\mathcal{A}(\prec). \tag{50}$$

Let b_i, $i=1,\ldots,5$, b, b_i', b_i'', $i=1,2$ be the positive numbers fixed as in (44). Let $\widetilde{\gamma}_3\in\left]0,\widetilde{\gamma}_0\right[$. It follows easily that for every $\widetilde{\delta}\in\left]0,\widetilde{\delta}_0\right]$ and $I\in\mathcal{A}(\prec)$ the set $\Lambda_{I,\widetilde{\delta}}:=N_0(I)\cap B_{\widetilde{\delta},b_3}$ is a closed subset of X_0 and satisfies the conditions on Theorem 4.24, i.e.

$$M_0(I)\subset \operatorname{Int}_{X_0}(\Lambda_{I,\widetilde{\delta}})\subset\Lambda_{I,\widetilde{\delta}}\subset\widetilde{B}_{I,\widetilde{\delta},b_3,\widetilde{\gamma}_3},$$

whenever $\widetilde{\delta}<\widetilde{\delta}'$, then $\Lambda_{I,\widetilde{\delta}}\subset\Lambda_{I,\widetilde{\delta}'}$ and whenever $(\widetilde{\delta}_n)_n$ is a decreasing sequence converging to zero and $u_n\in\Lambda_{I,\widetilde{\delta}_n}$ for all $n\in\mathbb{N}$, then the sequence $(u_n)_n$ has a convergent subsequence.

For all $\widetilde{\alpha}>0$, $\widetilde{\nu}>0$, $\widetilde{\eta}>0$, $\widetilde{\eta}'>0$, $\widetilde{\beta}'>0$, $\widetilde{\beta}>0$ with $\widetilde{\eta}'<\widetilde{\beta}'<\widetilde{\eta}<\widetilde{\nu}<\widetilde{\beta}$, for all $I\in\mathcal{A}(\prec)$, $\widetilde{\delta}\in\left]0,\widetilde{\delta}_2\right]$ and $\varepsilon\in\left]0,\varepsilon_1\right]$, define

$$\widetilde{N}_{1,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}):=[B_{\widetilde{\delta}_2,b_2}]_{\varepsilon,\widetilde{\eta}}\cap\operatorname{Cl}_{Z_\varepsilon}(\{\,w\mid \text{there exist a } \overline{w}\in[B_{\widetilde{\delta},b_3}]_{\varepsilon,\widetilde{\alpha}} \text{ and a } t\geq 0 \text{ such that } \overline{w}\pi_n[0,t]\subset\,]\widetilde{U}[_{\varepsilon,\widetilde{\beta}} \text{ and } w=\overline{w}\pi_\varepsilon t\,\}),$$

$$\widetilde{N}_{2,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}):=\widetilde{N}_{1,\varepsilon}(\widetilde{\alpha},\widetilde{\delta})\cap\{\,w\in\,]\widetilde{U}[_{\varepsilon,\widetilde{\nu}}\mid t^+_{\varepsilon,\widetilde{\nu}}(w)\leq b\,\}$$

and

$$\widetilde{N}_{I,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}):=([\widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\gamma}_2}]_{\varepsilon,\widetilde{\eta}'}\cap\operatorname{Cl}_{Z_\varepsilon}(\{\,w\mid \text{ there exist a } \overline{w}\in[N_0(I)\cap B_{\widetilde{\delta},b_3}]_{\varepsilon,\widetilde{\alpha}} \text{ and a } t\geq 0 \text{ such that } \overline{w}\pi_\varepsilon[0,t]\subset\,]\widetilde{U}[_{\varepsilon,\widetilde{\beta}'} \text{ and } w=\overline{w}\pi_\varepsilon t\,\})\cup\widetilde{N}_{2,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}).$$

Let $\mu>0$ be as in (45). Let $\widetilde{\nu}>0$, $\widetilde{\eta}>0$, $\widetilde{\beta}'>0$, $\widetilde{\beta}>0$, with $\widetilde{\beta}'<\widetilde{\eta}<\widetilde{\nu}<\widetilde{\beta}<\mu$, be fixed.

Let $\widetilde{\delta}_4'\in\left]0,\widetilde{\delta}_2\right]$ and $\widetilde{\mu}_0\in\left]0,\widetilde{\beta}'\right]$ such that for all $\widetilde{\delta}\in\left]0,\widetilde{\delta}_4'\right]$ and for all $\widetilde{\mu}\in\left]0,\widetilde{\mu}_0\right]$, there exists an $\widetilde{\varepsilon}_2'(\widetilde{\mu},\widetilde{\delta})\in\left]0,\widetilde{\varepsilon}\right]$ such that the conclusions of Theorem 5.32.(1) hold for all $\varepsilon\in\left]0,\widetilde{\varepsilon}_2'(\widetilde{\mu},\widetilde{\delta})\right]$, i.e.

$$[B_{\widetilde{\delta},b_3}\cap\{\,u\in\widetilde{U}\mid t_0^+(u)\leq b_2'\,\}]_{\varepsilon,\widetilde{\mu}}\subset[B_{\widetilde{\delta},b_3}]_{\varepsilon,\widetilde{\mu}}\cap\{\,w\in\,]\widetilde{U}[_{\varepsilon,\widetilde{\nu}}\mid t^+_{\varepsilon,\widetilde{\nu}}(w)\leq b\,\}. \tag{51}$$

Fix $\widetilde{\eta}_0\in\left]0,\widetilde{\mu}_0\right[$. Hence, $\widetilde{\eta}_0<\widetilde{\beta}'<\widetilde{\eta}<\widetilde{\nu}<\widetilde{\beta}$. For all $I\in\mathcal{A}(\prec)$, $\widetilde{\delta}\in\left]0,\widetilde{\delta}_2\right]$, $\widetilde{\alpha}>0$ and $\varepsilon\in\left]0,\widetilde{\varepsilon}\right]$, consider the set $\widetilde{N}_{I,\varepsilon}(\widetilde{\alpha},\widetilde{\delta})$ with $\widetilde{\eta}':=\widetilde{\eta}_0$.

An application of Theorem 5.27 shows that there exist a $\widetilde{\delta}_3\in\left]0,\widetilde{\delta}_2\right]$ and an $\widetilde{\alpha}_0\in\left]0,\widetilde{\eta}_0\right]$ such that for all $\widetilde{\delta}\in\left]0,\widetilde{\delta}_3\right]$ and for all $\widetilde{\alpha}\in\left]0,\widetilde{\alpha}_0\right]$, there exists an $\widetilde{\varepsilon}_0'(\widetilde{\alpha},\widetilde{\delta})\in\left]0,\widetilde{\varepsilon}\right]$ such that for all $\varepsilon\in\left]0,\widetilde{\varepsilon}_0'(\widetilde{\alpha},\widetilde{\delta})\right]$ and for all $I\in\mathcal{A}(\prec)$ the triple $(\widetilde{N}_{1,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}),\widetilde{N}_{I,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}),\widetilde{N}_{2,\varepsilon}(\widetilde{\alpha},\widetilde{\delta}))$ is an FM-index triple for $(\pi_\varepsilon,S_\varepsilon,M_\varepsilon(I),M_\varepsilon(P\setminus I))$.

Moreover, let $\widetilde{\delta}_4''\in\left]0,\widetilde{\delta}_2\right]$ and $\widetilde{\alpha}_1''\in\left]0,\widetilde{\beta}'\right]$ be as in Theorem 5.32.(2). For all $\widetilde{\delta}\in\left]0,\widetilde{\delta}_4''\right]$ and for all $\widetilde{\alpha}\in\left]0,\widetilde{\alpha}_1''\right]$, let $\widetilde{\varepsilon}_2''(\widetilde{\alpha},\widetilde{\delta})\in\left]0,\widetilde{\varepsilon}\right]$ be such that for all $\varepsilon\in\left]0,\widetilde{\varepsilon}_2''(\widetilde{\alpha},\widetilde{\delta})\right]$,

$$\widetilde{N}_{2,\varepsilon}(\widetilde{\alpha},\widetilde{\delta})\subset[B_{\widetilde{\delta}_2,b_2}\cap\{\,u\in\widetilde{U}\mid t_0^+(u)\leq b_1''\,\}]_{\varepsilon,\widetilde{\eta}}\,. \tag{52}$$

Choose $\widetilde{\delta}_4 > 0$ and $\widetilde{\alpha}_1 > 0$ such that $\widetilde{\delta}_4 < \min\{\,\widetilde{\delta}_3, \widetilde{\delta}'_4, \widetilde{\delta}''_4\,\}$ and $\widetilde{\alpha}_1 < \min\{\,\widetilde{\alpha}_0, \widetilde{\alpha}''_1, \widetilde{\eta}_0\,\}$.

Define $\widetilde{\varepsilon}_1 := \min\{\,\widetilde{\varepsilon}'_0(\alpha_1, \delta_4), \widetilde{\varepsilon}'_2(\widetilde{\eta}_0, \delta_4), \widetilde{\varepsilon}''_2(\alpha_1, \delta_4)\,\}$. Choose

$$\widetilde{\mu} \in]0, \widetilde{\alpha}_1[$$

Fix positive numbers $\widetilde{\gamma}_i$, $i = 1, \ldots, 4$, $\widetilde{\delta}_1$ and $\widetilde{\delta}_5$ such that

$$0 < \widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \widetilde{\gamma}_3 < \widetilde{\gamma}_4 < \widetilde{\gamma}_0 \text{ and } 0 < \widetilde{\delta}_5 < \widetilde{\delta}_4 < \widetilde{\delta}_3 < \widetilde{\delta}_2 < \widetilde{\delta}_1 < \widetilde{\delta}_0. \tag{53}$$

For $\varepsilon \in]0, \widetilde{\varepsilon}_1]$ and $I \in \mathcal{A}(\prec)$, define the following sets:

$$\begin{aligned}
&\widetilde{N}_1^2 := B_{\widetilde{\delta}_5, b_5}, \quad \widetilde{N}_2^2 := B_{\widetilde{\delta}_5, b_5} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \le b'_1\,\},\\
&\widetilde{N}_1^4 := B_{\widetilde{\delta}_1, b_1}, \quad \widetilde{N}_2^4 := B_{\widetilde{\delta}_1, b_1} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \le b''_2\,\},\\
&\widetilde{N}_{1,\varepsilon} := \widetilde{N}_{1,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4), \quad \widetilde{N}_{2,\varepsilon} := \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4),\\
&\widetilde{N}_I^2 := B_{I, \widetilde{\delta}_5, b_5, \gamma_5} \cup \widetilde{N}_2^2, \quad \widetilde{N}_I^4 := \widetilde{B}_{I, \widetilde{\delta}_1, b_1, \widetilde{\gamma}_1} \cup \widetilde{N}_2^4, \quad \widetilde{N}_{I,\varepsilon} := \widetilde{N}_{I,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4),\\
&\widetilde{V}_1 := \operatorname{Int}_{X_0}(B_{\widetilde{\delta}_4, b_3}),\ \widetilde{V}_{I,3} := \operatorname{Int}_{X_0}(B_{I, \widetilde{\delta}_4, b_4, \gamma_4}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b''_1\,\},\\
&\widetilde{V}_{I,4} := \operatorname{Int}_{X_0}(B_{\widetilde{\delta}_4, b_3}) \cap (X_0 \setminus \widetilde{B}_{I, \widetilde{\delta}_2, b_2, \widetilde{\gamma}_2}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b''_1\,\}.
\end{aligned}$$

Note that there is no tilde $\widetilde{}$ over the letter 'B' in the definitions of $\widetilde{N}_I^2$ and $\widetilde{V}_{I,3}$.

It is clear that for each $I \in \mathcal{A}(\prec)$ the triple $(\widetilde{N}_1^i, \widetilde{N}_I^i, \widetilde{N}_2^i)$ is an FM-index triple for $(\pi_0, S_0, M_0(I), M_0(P \setminus I))$, $i = 2, 4$. It is easy to prove that the following inclusions hold:

$$\widetilde{N}_1^2 \subset \widetilde{U}, \quad \widetilde{N}_2^2 \subset \widetilde{N}_2^4$$

and

$$\widetilde{N}_I^2 \subset \widetilde{N}_I^4 \text{ for all } I \in \mathcal{A}(\prec).$$

The next lemmas complete the proof that all assumptions of Theorem 4.24 are satisfied for this new choice of FM-index triples.

Lemma 6.37. *For all $\varepsilon \in]0, \widetilde{\varepsilon}_1]$, the following inclusions hold*

$$[\widetilde{N}_1^2]_{\varepsilon, \widetilde{\mu}} \subset]\widetilde{V}_1[_{\varepsilon, \widetilde{\alpha}_1} \subset \widetilde{N}_{1,\varepsilon} \subset [\widetilde{N}_1^4]_{\varepsilon, \widetilde{\eta}}\,.$$

Proof. Let $\varepsilon \in]0, \widetilde{\varepsilon}_1]$ be fixed. It is straightforward to prove that $\widetilde{N}_1^2 \subset \widetilde{V}_1$. Since $\widetilde{\mu} < \widetilde{\alpha}_1$, it follows that $[\widetilde{N}_1^2]_{\varepsilon, \widetilde{\mu}} \subset]\widetilde{V}_1[_{\varepsilon, \widetilde{\alpha}_1}$.

Since $\widetilde{\delta}_4 < \widetilde{\delta}_2 < \overline{\delta}$, $b_3 > b_2 > \overline{b}$ and $\widetilde{\alpha}_1 < \widetilde{\eta} < \widetilde{\beta}$, it follows that $[B_{\widetilde{\delta}_4, b_3}]_{\varepsilon, \widetilde{\alpha}_1} \subset [B_{\widetilde{\delta}_2, b_2}]_{\varepsilon, \widetilde{\eta}}$ and $[B_{\widetilde{\delta}_4, b_3}]_{\varepsilon, \widetilde{\alpha}_1} \subset]\widetilde{U}[_{\varepsilon, \widetilde{\beta}}$ and so $[B_{\widetilde{\delta}_4, b_3}]_{\varepsilon, \widetilde{\alpha}_1} \subset \widetilde{N}_{1,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \subset [B_{\widetilde{\delta}_2, b_2}]_{\varepsilon, \widetilde{\eta}}$ (see also Remark 5.28). Hence, $]\widetilde{V}_1[_{\varepsilon, \widetilde{\alpha}_1} \subset [B_{\widetilde{\delta}_4, b_3}]_{\varepsilon, \widetilde{\alpha}_1} \subset \widetilde{N}_{1,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \subset [B_{\widetilde{\delta}_1, b_1}]_{\varepsilon, \widetilde{\eta}} = [\widetilde{N}_1^4]_{\varepsilon, \widetilde{\eta}}$ as $\widetilde{\delta}_2 < \widetilde{\delta}_1$ and $b_2 > b_1$. The lemma is proved. ∎

Lemma 6.38. *For all $\varepsilon \in \,]0, \widetilde{\varepsilon}_1]$ and for all $I \in \mathcal{A}(\prec)$, the following inclusions hold*

$$\mathrm{Cl}_{Z_\varepsilon}([\widetilde{N}_I^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_2^4]_{\varepsilon,\widetilde{\eta}}) \subset \,]\widetilde{V}_{I,3}[_{\varepsilon,\widetilde{\alpha}_1} \subset \mathrm{Int}_{Z_\varepsilon}(\widetilde{N}_{I,\varepsilon} \setminus \widetilde{N}_{2,\varepsilon}).$$

Proof. Let $\varepsilon \in \,]0, \widetilde{\varepsilon}_1]$ and $I \in \mathcal{A}(\prec)$ be fixed. Let $w = (u, v) \in [\widetilde{N}_I^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_2^4]_{\varepsilon,\widetilde{\eta}}$. Hence $u \in \widetilde{N}_I^2 = B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \cup \widetilde{N}_2^2$ and $v \in \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon, \widetilde{\mu})) \subset \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon, \widetilde{\eta}))$. Thus, $u \notin \widetilde{N}_2^4$ and so $u \in B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \cup \widetilde{N}_2^2 \setminus \widetilde{N}_2^4$. Since $\widetilde{N}_2^2 \subset \widetilde{N}_2^4$ we have $u \in B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \setminus \widetilde{N}_2^4$. Hence

$$\begin{aligned}\mathrm{Cl}_{Z_\varepsilon}([\widetilde{N}_I^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_2^4]_{\varepsilon,\widetilde{\eta}}) &\subset \mathrm{Cl}_{Z_\varepsilon}([B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \setminus \widetilde{N}_2^4]_{\varepsilon,\widetilde{\mu}}) = [\mathrm{Cl}_{X_0}(B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \setminus \widetilde{N}_2^4)]_{\varepsilon,\widetilde{\mu}} \\ &\subset [B_{I,\widetilde{\delta}_5,b_5,\gamma_5}]_{\varepsilon,\widetilde{\mu}}\,.\end{aligned}$$

It is clear that $B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \subset B_{\widetilde{\delta}_5,b_4} \subset B_{\widetilde{\delta}_1,b_1} \subset \widetilde{U}$ and so $B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \setminus \widetilde{N}_2^4 \subset B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \geq b_2'' \,\}$. Moreover, it follows from $b_2'' > b_1''$ and formula (49), that $B_{I,\widetilde{\delta}_5,b_5,\gamma_5} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \geq b_2'' \,\} \subset \mathrm{Int}_{X_0}(B_{I,\widetilde{\delta}_4,b_4,\gamma_4}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\} = \widetilde{V}_{I,3}$. Since $\widetilde{\mu} < \widetilde{\alpha}_1$, we obtain

$$\mathrm{Cl}_{Z_\varepsilon}([\widetilde{N}_I^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_2^4]_{\varepsilon,\widetilde{\eta}}) \subset \,]\widetilde{V}_{I,3}[_{\varepsilon,\widetilde{\alpha}_1}\,. \tag{54}$$

To complete the proof we need to show that $]\widetilde{V}_{I,3}[_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{I,\varepsilon} \setminus \widetilde{N}_{2,\varepsilon}$. Note that $\widetilde{V}_{I,3} \subset N_0(I) \cap B_{\widetilde{\delta}_4,b_3}$. Indeed, $\widetilde{V}_{I,3} \subset \mathrm{Int}_{X_0}(B_{I,\widetilde{\delta}_4,b_4,\gamma_4}) \subset B_{\widetilde{\delta}_4,b_4} \subset B_{\widetilde{\delta}_4,b_3}$ and

$$\widetilde{V}_{I,3} \subset \mathrm{Int}_{X_0}(B_{I,\widetilde{\delta}_4,b_4,\gamma_4}) \subset B_{I,\delta_5,b_4,\gamma_4} \subset B_{I,\delta_5,b_4,\gamma_4} \cup N_2^2 = N_I^2 \subset N_0(I),$$

where the last inclusion follows from Theorem 6.36. Moreover, it follows from Remark 5.28 that $[N_0(I) \cap B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{I,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \subset [\widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\gamma}_2}]_{\varepsilon,\widetilde{\eta}_0} \cup \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \subset [B_{\widetilde{\delta}_2,b_2}]_{\varepsilon,\widetilde{\eta}_0} \cup \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4)$. Hence

$$]\widetilde{V}_{I,3}[_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{I,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \cap \,]\{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\}[_{\varepsilon,\widetilde{\alpha}_1} =: T$$

If $(u, v) \in T$, then $t_0^+(u) > b_1''$ so, by (52), $(u, v) \notin \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4)$. Thus $T \subset \widetilde{N}_{I,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \setminus \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4)$ and this, together with (54), concludes the proof of the lemma. ▮

Lemma 6.39. *For all $\varepsilon \in \,]0, \widetilde{\varepsilon}_1]$ and for all $I \in \mathcal{A}(\prec)$, the following inclusions hold*

$$\mathrm{Cl}_{Z_\varepsilon}([\widetilde{N}_1^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_I^4]_{\varepsilon,\widetilde{\eta}}) \subset \,]\widetilde{V}_{I,4}[_{\varepsilon,\widetilde{\alpha}_1} \subset \mathrm{Int}_{Z_\varepsilon}(\widetilde{N}_{1,\varepsilon} \setminus \widetilde{N}_{I,\varepsilon}).$$

Proof. Let $\varepsilon \in \,]0, \widetilde{\varepsilon}_1]$ and $I \in \mathcal{A}(\prec)$ be fixed. Let $w = (u, v) \in [\widetilde{N}_1^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_I^4]_{\varepsilon,\widetilde{\eta}}$. Hence $u \in \widetilde{N}_1^2 = B_{\widetilde{\delta}_5,b_5}$ and $v \in \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon, \widetilde{\mu})) \subset \mathrm{Cl}_{Y_\varepsilon}(B(\theta_\varepsilon, \widetilde{\eta}))$. Thus, $u \notin \widetilde{N}_I^4$ and so

$u \in B_{\widetilde{\delta}_5,b_5} \setminus (\widetilde{B}_{I,\widetilde{\delta}_1,b_1,\widetilde{\gamma}_1} \cup \widetilde{N}_2^4)$. Hence

$$\begin{aligned}\mathrm{Cl}_{Z_\varepsilon}([\widetilde{N}_1^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_I^4]_{\varepsilon,\widetilde{\eta}}) &\subset \mathrm{Cl}_{Z_\varepsilon}([B_{\widetilde{\delta}_5,b_5} \setminus (\widetilde{B}_{I,\widetilde{\delta}_1,b_1,\widetilde{\gamma}_1} \cup \widetilde{N}_2^4)]_{\varepsilon,\widetilde{\mu}}) \\ &= [\mathrm{Cl}_{X_0}(B_{\widetilde{\delta}_5,b_5} \setminus (\widetilde{B}_{I,\widetilde{\delta}_1,b_1,\widetilde{\gamma}_1} \cup \widetilde{N}_2^4))]_{\varepsilon,\widetilde{\mu}} \\ &\subset [B_{\widetilde{\delta}_5,b_5} \setminus \mathrm{Int}_{X_0}(\widetilde{B}_{I,\widetilde{\delta}_1,b_1,\widetilde{\gamma}_1} \cup \widetilde{N}_2^4)]_{\varepsilon,\widetilde{\mu}}\,.\end{aligned}$$

A simple computation shows that $B_{\widetilde{\delta}_5,b_5} \setminus \mathrm{Int}_{X_0}(\widetilde{B}_{I,\widetilde{\delta}_1,b_1,\widetilde{\gamma}_1} \cup \widetilde{N}_2^4) \subset \mathrm{Int}_{X_0}(B_{\widetilde{\delta}_4,b_3}) \cap (X_0 \setminus \widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\gamma}_2}) \cap \{\, u \in \widetilde{U} \mid t_0^+(u) > b_1'' \,\} = \widetilde{V}_{I,4}$. Since $\widetilde{\mu} < \widetilde{\alpha}_1$, it follows that

$$\mathrm{Cl}_{Z_\varepsilon}([\widetilde{N}_1^2]_{\varepsilon,\widetilde{\mu}} \setminus [\widetilde{N}_I^4]_{\varepsilon,\widetilde{\eta}}) \subset\,]\widetilde{V}_{I,4}[_{\varepsilon,\widetilde{\alpha}_1}\,.$$

We claim that $]\widetilde{V}_{I,4}[_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{1,\varepsilon} \setminus \widetilde{N}_{I,\varepsilon}$. First notice that $]\widetilde{V}_{I,4}[_{\varepsilon,\widetilde{\alpha}_1} \subset [B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{1,\varepsilon}(\widetilde{\alpha}_1,\widetilde{\delta}_4)$. Suppose that there exists a $(u,v) \in\,]\widetilde{V}_{I,4}[_{\varepsilon,\widetilde{\alpha}_1} \cap \widetilde{N}_{I,\varepsilon}(\widetilde{\alpha}_1,\widetilde{\delta}_4)$. Hence $(u,v) \in [\widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\gamma}_2}]_{\varepsilon,\widetilde{\eta}_0} \cup \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1,\widetilde{\delta}_4)$.

If $(u,v) \in [\widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\gamma}_2}]_{\varepsilon,\widetilde{\eta}_0}$, then $u \in \widetilde{B}_{I,\widetilde{\delta}_2,b_2,\widetilde{\gamma}_2}$, but this contradicts the definition of $\widetilde{V}_{I,4}$. Suppose that $(u,v) \in \widetilde{N}_{2,\varepsilon}(\widetilde{\alpha}_1,\widetilde{\delta}_4)$. Then formula (52) implies that $(u,v) \in [B_{\widetilde{\delta}_2,b_2} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \le b_1'' \,\}]_{\varepsilon,\widetilde{\eta}}$ and so $u \in B_{\widetilde{\delta}_2,b_2} \cap \{\, u \in \widetilde{U} \mid t_0^+(u) \le b_1'' \,\}$ which is a contradiction to the definition of the set $\widetilde{V}_{I,4}$. Therefore our claim is proved and this concludes the proof of the lemma. ▌

We have proved the following result.

Theorem 6.40. *With the notation introduced above, there exists an $\widetilde{\varepsilon}_1 \in\,]0,\widetilde{\varepsilon}]$ such that for every $\varepsilon \in [0,\widetilde{\varepsilon}_1]$ there exists an index filtration $(\widetilde{N}_\varepsilon(I))_{I\in\mathcal{A}(\prec)}$ for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ such that for all $I \in \mathcal{A}(\prec)$ the following inclusions hold:*

1. $\widetilde{N}_I^2 \subset \widetilde{N}_0(I) \subset \widetilde{N}_1^2 = B_{\widetilde{\delta}_5,b_5}$,
2. $\widetilde{N}_2^2 \subset \widetilde{N}_0(\emptyset)$,
3. $[N_0(I) \cap B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{I,\varepsilon} \subset \widetilde{N}_\varepsilon(I) \subset \widetilde{N}_{1,\varepsilon}$ *for* $\varepsilon \in\,]0,\widetilde{\varepsilon}_1]$ *and*
4. $\widetilde{N}_{2,\varepsilon} \subset \widetilde{N}_\varepsilon(\emptyset)$ *for* $\varepsilon \in\,]0,\widetilde{\varepsilon}_1]$.

Furthermore, whenever $I \in \mathcal{A}(\prec)$, $(\varepsilon_n)_n$ is a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$ and $(w_n)_n$, with $w_n = (u_n,v_n)$ for $n \in \mathbb{N}$, is a sequence such that $w_n \in [\widetilde{N}_1^2]_{\varepsilon_n,\widetilde{\mu}} \cap \widetilde{N}_{\varepsilon_n}(I)$ for all $n \in \mathbb{N}$ and $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n})) \to 0$ as $n \to \infty$ for some $u_0 \in X_0$, then there exists an $n_0 \in \mathbb{N}$ such that $u_n \in \widetilde{N}_0(I)$ for all $n \ge n_0$.

The index filtrations that we have obtained from Theorem 6.36 and Theorem 6.40 do not as yet form a singular nested sequence as described in Theorem 3.20. However, after intersecting these index filtrations with appropriate sets and using Proposition 2.9 we will obtain new index filtrations which do satisfy the singular nesting property. This will complete the proof of Theorem 3.20.

For $\xi \in \,]0, \widetilde{\delta}_5[$, $\rho \in \,]0, \widetilde{\mu}[$ and $\varepsilon \in \,]0, \widetilde{\varepsilon}]$ define the sets

$$\begin{aligned}\widetilde{Y}_{1,\varepsilon}(\rho,\xi) &:= [B_{\widetilde{\delta}_5,b_5}]_{\varepsilon,\widetilde{\mu}} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{there exist an } \overline{w} \in [B_{\xi,b_1'}]_{\varepsilon,\rho} \text{ and a } t \geq 0 \\ &\qquad\qquad \text{such that } \overline{w}\pi_\varepsilon[0,t] \subset \,]\widetilde{U}[_{\varepsilon,\widetilde{\beta}} \text{ and } w = \overline{w}\pi_\varepsilon t \,\}), \\ \widetilde{Y}_{2,\varepsilon}(\rho,\xi) &:= \widetilde{Y}_{1,\varepsilon}(\rho,\xi) \cap \{\, w \in \,]\widetilde{U}[_{\varepsilon,\widetilde{\nu}} \mid t^+_{\varepsilon,\widetilde{\nu}}(w) \leq b_1' \,\}.\end{aligned}$$

Then there exist a $\widetilde{\xi}_0 \in \,]0, \widetilde{\delta}_5[$ and $\widetilde{\rho}_0 \in \,]0, \widetilde{\mu}[$ such that for all $\xi \in \,]0, \widetilde{\xi}_0]$ and $\rho \in \,]0, \widetilde{\rho}_0]$, there exists an $\widetilde{\varepsilon}_2(\rho,\xi) \in \,]0, \widetilde{\varepsilon}]$ such that for all $\varepsilon \in \,]0, \widetilde{\varepsilon}_2(\rho,\xi)]$, $S_\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}(\widetilde{Y}_{1,\varepsilon}(\rho,\xi))$ and the pair $(\widetilde{Y}_{1,\varepsilon}(\rho,\xi), \widetilde{Y}_{2,\varepsilon}(\rho,\xi))$ is an FM-index pair for S_ε relative to π_ε. (cf. Lemma 5.30).

Given such an $\varepsilon \in \,]0, \widetilde{\varepsilon}_2]$, since $\widetilde{\rho}_0 < \widetilde{\mu} < \widetilde{\alpha}_1 < \widetilde{\eta}$, $\widetilde{\xi}_0 < \widetilde{\delta}_5$ we see, using (44), (53) and Theorem 6.40 that

$$\widetilde{Y}_{1,\varepsilon}(\rho,\xi) \subset [B_{\widetilde{\delta}_5,b_5}]_{\varepsilon,\widetilde{\mu}} \subset [B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\alpha}_1} \subset \widetilde{N}_{1,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4)$$

and so

$$\widetilde{Y}_{2,\varepsilon}(\rho,\xi) \subset \widetilde{N}_{1,\varepsilon}(\widetilde{\alpha}_1, \widetilde{\delta}_4) \cap \{\, w \in \,]\widetilde{U}[_{\varepsilon,\widetilde{\nu}} \mid t^+_{\varepsilon,\widetilde{\nu}}(w) \leq b \,\} = \widetilde{N}_{2,\varepsilon} \subset \widetilde{N}_\varepsilon(\emptyset).$$

Since $\widetilde{Y}_{2,\varepsilon}(\rho,\xi)$ is an exit ramp for $\widetilde{Y}_{1,\varepsilon}(\rho,\xi)$ relative to π_ε, it thus follows that $\widetilde{N}_\varepsilon(\emptyset)$ is an exit ramp for $\widetilde{Y}_{1,\varepsilon}(\rho,\xi)$ relative to π_ε.

An application of Proposition 2.9 now implies the following result.

Lemma 6.41. *For every $\xi \in \,]0, \widetilde{\xi}_0]$ and $\rho \in \,]0, \widetilde{\rho}_0]$ and for all $\varepsilon \in \,]0, \widetilde{\varepsilon}_2(\rho,\xi)]$, $(\widetilde{Y}_{1,\varepsilon}(\rho,\xi) \cap \widetilde{N}_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$.*

We also have the following

Lemma 6.42. *There exist a $\widetilde{\xi}_1 \in \,]0, \widetilde{\xi}_0]$ and a $\widetilde{\rho}_1 \in \,]0, \widetilde{\rho}_0]$ such that for all $\xi \in \,]0, \widetilde{\xi}_1]$ and all $\rho \in \,]0, \widetilde{\rho}_1]$, there exists an $\widetilde{\varepsilon}_3(\rho,\xi) \in \,]0, \widetilde{\varepsilon}_2(\rho,\xi)]$ such that $\widetilde{Y}_{1,\varepsilon}(\rho,\xi) \cap \widetilde{N}_\varepsilon(I) \subset [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}}$ for all $\varepsilon \in \,]0, \widetilde{\varepsilon}_3(\rho,\xi)]$ and all $I \in \mathcal{A}(\prec)$.*

Proof. Suppose the conclusion of the lemma is not true. Then for some $I \in \mathcal{A}(\prec)$ there exist sequences $(\xi_n)_n$, $(\rho_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $\xi_n \to 0$, $\rho_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$ and

$$w_n = (u_n, v_n) \in (\widetilde{Y}_{1,\varepsilon_n}(\rho_n,\xi_n) \cap \widetilde{N}_{\varepsilon_n}(I)) \setminus [\widetilde{N}_0(I)]_{\varepsilon_n,\widetilde{\eta}} \text{ for all } n \in \mathbb{N}.$$

Notice that for all $n \in \mathbb{N}$, $(u_n, v_n) \in \widetilde{Y}_{1,\varepsilon_n}(\rho_n,\xi_n) \subset [B_{\widetilde{\delta}_5,b_5}]_{\varepsilon_n,\widetilde{\mu}} = [\widetilde{N}_1^2]_{\varepsilon_n,\widetilde{\mu}}$. Moreover, for each $n \in \mathbb{N}$, there exist an $\overline{w}_n \in [B_{\xi_n,b_1'}]_{\varepsilon_n,\rho_n}$ and a $t_n \geq 0$ such that $\overline{w}_n\pi_{\varepsilon_n}[0,t_n] \subset \,]\widetilde{U}[_{\varepsilon_n,\widetilde{\beta}} \subset [\widetilde{N}]_{\varepsilon_n,\beta}$ and $\Gamma_{\varepsilon_n}(w_n, \overline{w}_n\pi_{\varepsilon_n}t_n) < 2^{-n}$.

Lemma 5.29 implies that there is a $u_0 \in \widetilde{N}$ such that $\Gamma_{\varepsilon_n}(\overline{w}_n\pi_{\varepsilon_n}t_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$ and so $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$. In particular, there exists an $n_0 \in \mathbb{N}$ such that $d_{\varepsilon_n}(v_n, \theta_{\varepsilon_n}) < \widetilde{\eta}$ for all $n \geq n_0$.

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Since $w_n \notin [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}}$ for all $n \in \mathbb{N}$, it follows that $u_n \notin \widetilde{N}_0(I)$ for all $n \geq n_0$. However, Theorem 6.40 implies that $u_n \in \widetilde{N}_0(I)$ for all n large enough, which is a contradiction. ▮

Let $\widetilde{\xi} \in]0, \widetilde{\xi}_1]$ and $\widetilde{\rho} \in]0, \widetilde{\rho}_1]$ be fixed. Hence, setting $\widetilde{\varepsilon}_3 = \widetilde{\varepsilon}_3(\widetilde{\rho}, \widetilde{\xi})$ we have the following corollary.

Corollary 6.43. *There exist a $\widetilde{\xi} \in]0, \widetilde{\delta}_5]$, a $\widetilde{\rho} \in]0, \widetilde{\mu}]$ and an $\widetilde{\varepsilon}_3 \in]0, \widetilde{\varepsilon}]$ such that, whenever $\varepsilon \in]0, \widetilde{\varepsilon}_3]$, then $(\widetilde{Y}_{1,\varepsilon}(\widetilde{\rho}, \widetilde{\xi}) \cap \widetilde{N}_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ and $\widetilde{Y}_{1,\varepsilon}(\widetilde{\rho}, \widetilde{\xi}) \cap \widetilde{N}_\varepsilon(I) \subset [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}}$ for all $I \in \mathcal{A}(\prec)$.*

We claim that

$$B^-_{\widetilde{\xi},b'_1} \subset N^2_2 \subset N_0(\emptyset) \tag{55}$$

Here, $B^-_{\widetilde{\xi},b'_1}$ is the exit set of the isolating block $B_{\widetilde{\xi},b'_1}$ relative to π_0.

In fact, the inequalities $\widetilde{\xi} \leq \widetilde{\delta}_5 < \delta_5$ and $b'_1 > b_5$ imply that $B_{\widetilde{\xi},b'_1} \subset B_{\widetilde{\delta}_5,b_5}$ Moreover, if $u \in B^-_{\widetilde{\xi},b'_1}$, then $u \in \widetilde{U}$ and $t^+_0(u) = b'_1$. It follows that $B^-_{\widetilde{\xi},b'_1} \subset B_{\delta_5,b_4} \cap \{\, u \in \widetilde{U} \mid t^+(u) \leq b'_1 \,\} = N^2_2$. The last inclusion in (55) follows from Theorem 6.36 and our claim is proved.

Since $B^-_{\widetilde{\xi},b'_1}$ is an exit ramp for $B_{\widetilde{\xi},b'_1}$ relative to π_0, it follows from (55) that $N_0(\emptyset)$ is an exit ramp for $B_{\widetilde{\xi},b'_1}$ relative to π_0. The inclusion $S_0 \subset \mathrm{Int}_{X_0}(B_{\widetilde{\xi},b'_1})$ and Proposition 2.9 imply the following result.

Lemma 6.44. *$(N_0(I) \cap B_{\widetilde{\xi},b'_1})_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_0, S_0, (M_{p,0})_{p \in P})$.*

Since $\widetilde{\xi} \leq \widetilde{\delta}_5 < \widetilde{\delta}_4$ and $b_3 < b'_1$, it follows that $B_{\widetilde{\xi},b'_1} \subset B_{\widetilde{\delta}_4,b_3}$ and $B_{\widetilde{\xi},b'_1} = B_{\widetilde{\xi},b'_1} \cap B_{\widetilde{\delta}_4,b_3}$. Hence $N_0(I) \cap B_{\widetilde{\xi},b'_1} = N_0(I) \cap (B_{\widetilde{\xi},b'_1} \cap B_{\widetilde{\delta}_4,b_3})$. Since $\widetilde{\rho} < \widetilde{\mu} < \widetilde{\alpha}_1$, we have

$$\begin{aligned}
[N_0(I) \cap B_{\widetilde{\xi},b'_1}]_{\varepsilon,\widetilde{\rho}} &= [N_0(I) \cap (B_{\widetilde{\xi},b'_1} \cap B_{\widetilde{\delta}_4,b_3})]_{\varepsilon,\widetilde{\rho}} = [N_0(I) \cap B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\rho}} \cap [B_{\widetilde{\xi},b'_1}]_{\varepsilon,\widetilde{\rho}} \\
&\subset [N_0(I) \cap B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\alpha}_1} \cap [B_{\widetilde{\xi},b'_1}]_{\varepsilon,\widetilde{\rho}} \\
&\subset [N_0(I) \cap B_{\widetilde{\delta}_4,b_3}]_{\varepsilon,\widetilde{\alpha}_1} \cap \widetilde{Y}_{1,\varepsilon}(\widetilde{\rho}, \widetilde{\xi}).
\end{aligned}$$

This fact, together with Corollary 6.43 and Theorem 6.40, implies that, setting $\varepsilon'_0 := \min\{\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_3\}$, for all $\varepsilon \in]0, \varepsilon'_0]$ and for all $I \in \mathcal{A}(\prec)$

$$[N_0(I) \cap B_{\widetilde{\xi},b'_1}]_{\varepsilon,\widetilde{\rho}} \subset \widetilde{N}_\varepsilon(I) \cap \widetilde{Y}_{1,\varepsilon}(\widetilde{\rho}, \widetilde{\xi}) \subset [\widetilde{N}_0(I)]_{\varepsilon,\widetilde{\eta}}\,. \tag{56}$$

Let $\delta' := \min\{\widetilde{\xi}, \delta_5\}$ and $\mu' \in]0, \widetilde{\rho}[$. For $\xi \in]0, \infty[$, $\rho \in]0, \mu'[$ and $\varepsilon \in]0, \widetilde{\varepsilon}]$ define the sets

$$\begin{aligned}
Y_{1,\varepsilon}(\rho, \xi) &:= [B_{\delta',b'_1}]_{\varepsilon,\mu'} \cap \mathrm{Cl}_{Z_\varepsilon}(\{\, w \mid \text{ there exist an } \overline{w} \in [B_{\xi,b}]_{\varepsilon,\rho} \text{ and a } t \geq 0 \text{ such} \\
&\qquad\qquad \text{that } \overline{w}\pi_\varepsilon[0,t] \subset]\widetilde{U}[_{\varepsilon,\beta} \text{ and } w = \overline{w}\pi_\varepsilon t \,\}), \\
Y_{2,\varepsilon}(\rho, \xi) &:= Y_{1,\varepsilon}(\rho, \xi) \cap \{\, w \in]\widetilde{U}[_{\varepsilon,\nu} \mid t^+_{\varepsilon,\nu}(w) \leq b \,\}.
\end{aligned}$$

Then there exist a $\xi_0 \in]0, \delta_5[$ and $\rho_0 \in]0, \mu'[$ such that for all $\xi \in]0, \xi_0]$ and $\rho \in]0, \rho_0]$, there exists an $\varepsilon_4(\rho, \xi) \in]0, \widetilde{\varepsilon}]$ such that for all $\varepsilon \in]0, \varepsilon_4(\rho, \xi)]$, $S_\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}(Y_{1,\varepsilon}(\rho, \xi))$ and the pair $(Y_{1,\varepsilon}(\rho, \xi), Y_{2,\varepsilon}(\rho, \xi))$ is an FM-index pair for S_ε relative to π_ε. (cf. Lemma 5.30).

Given such an $\varepsilon \in]0, \varepsilon_4(\rho, \xi)]$, since $\rho_0 < \mu' < \alpha_1 < \eta$, $\delta' < \delta_5 < \delta_4$ we see, using (44) and Theorem 6.36, that

$$Y_{1,\varepsilon}(\rho, \xi) \subset [B_{\delta', b_1'}]_{\varepsilon,\mu'} \subset [B_{\delta_4, b_3}]_{\varepsilon,\alpha_1} \subset N_{1,\varepsilon}(\alpha_1, \delta_4)$$

and so $Y_{2,\varepsilon}(\rho, \xi) \subset N_{2,\varepsilon} \subset N_\varepsilon(\emptyset)$. Since $Y_{2,\varepsilon}(\rho, \xi)$ is an exit ramp for $Y_{1,\varepsilon}(\rho, \xi)$ relative to π_ε, it thus follows that $N_\varepsilon(\emptyset)$ is an exit ramp for $Y_{1,\varepsilon}(\rho, \xi)$ relative to π_ε.

An application of Proposition 2.9 now implies the following result.

Lemma 6.45. *For all $\xi \in]0, \xi_0]$ and $\rho \in]0, \rho_0]$ and for all $\varepsilon \in]0, \varepsilon_4(\rho, \xi)]$, $(Y_{1,\varepsilon}(\rho, \xi) \cap N_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$.*

We also have the following

Lemma 6.46. *There exist a $\xi_1 \in]0, \xi_0]$ and a $\rho_1 \in]0, \rho_0]$ such that for all $\xi \in]0, \xi_1]$ and all $\rho \in]0, \rho_1]$, there exists an $\varepsilon_5(\rho, \xi) \in]0, \varepsilon_4(\rho, \xi)]$ such that $Y_{1,\varepsilon}(\rho, \xi) \cap N_\varepsilon(I) \subset [N_0(I) \cap B_{\widetilde{\xi}, b_1'}]_{\varepsilon,\widetilde{\rho}}$ for all $\varepsilon \in]0, \varepsilon_5(\rho, \xi)]$ and all $I \in \mathcal{A}(\prec)$.*

Proof. Suppose the conclusion of the lemma is not true. Then for some $I \in \mathcal{A}(\prec)$ there exist sequences $(\xi_n)_n$, $(\rho_n)_n$, $(\varepsilon_n)_n$ and $(w_n)_n$ such that $\xi_n \to 0$, $\rho_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$ and

$$w_n = (u_n, v_n) \in (Y_{1,\varepsilon_n}(\rho_n, \xi_n) \cap N_{\varepsilon_n}(I)) \setminus [N_0(I) \cap B_{\widetilde{\xi}, b_1'}]_{\varepsilon_n,\widetilde{\rho}} \text{ for all } n \in \mathbb{N}.$$

Notice that $(u_n, v_n) \in Y_{1,\varepsilon_n}(\rho_n, \xi_n) \subset [B_{\delta', b_1'}]_{\varepsilon_n,\mu'} \subset [B_{\delta_5, b_4}]_{\varepsilon_n,\mu} = [N_1^2]_{\varepsilon_n,\mu}$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, there exist an $\overline{w}_n \in [B_{\xi_n, b}]_{\varepsilon_n,\rho_n}$ and a $t_n \geq 0$ such that $\overline{w}_n \pi_{\varepsilon_n}[0, t_n] \subset]\widetilde{U}[_{\varepsilon_n,\beta} \subset [\widetilde{N}]_{\varepsilon_n,\beta}$ and $\Gamma_{\varepsilon_n}(w_n, \overline{w}_n \pi_{\varepsilon_n} t_n) < 2^{-n}$.

Lemma 5.29 implies that there is a $u_0 \in \widetilde{N}$ such that $\Gamma_{\varepsilon_n}(\overline{w}_n \pi_{\varepsilon_n} t_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$ and so $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \to 0$ as $n \to \infty$. In particular, there exists an $n_0 \in \mathbb{N}$ such that $\Gamma_{\varepsilon_n}(v_n, \theta_{\varepsilon_n}) < \widetilde{\rho}$ for all $n \geq n_0$.

Since $u_n \in B_{\widetilde{\xi}, b_1'}$ and $w_n \notin [N_0(I) \cap B_{\widetilde{\xi}, b_1'}]_{\varepsilon_n,\widetilde{\rho}}$ for all $n \in \mathbb{N}$, it follows that $u_n \notin N_0(I)$ for all $n \geq n_0$. However, Theorem 6.36 implies that $u_n \in N_0(I)$ for all n large enough, which is a contradiction. ∎

Let $\xi \in]0, \xi_1]$ and $\rho \in]0, \rho_1]$ be fixed. Hence, setting $\varepsilon_5 = \widetilde{\varepsilon}_5(\rho, \xi)$ we have the following corollary.

Corollary 6.47. *There is a $\xi \in]0, \delta_5]$, $\rho \in]0, \mu']$ and an $\varepsilon_5 \in]0, \widetilde{\varepsilon}]$ such that, for all $\varepsilon \in]0, \varepsilon_5]$, $(Y_{1,\varepsilon}(\rho, \xi) \cap N_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$ is an index filtration for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ and $Y_{1,\varepsilon}(\rho, \xi) \cap N_\varepsilon(I) \subset [N_0(I) \cap B_{\widetilde{\xi}, b_1'}]_{\varepsilon,\widetilde{\rho}}$ for all $I \in \mathcal{A}(\prec)$.*

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Proof (Proof of Theorem 3.20). Let $\varepsilon_c := \min\{\varepsilon_0', \varepsilon_5\}$ and $\varepsilon \in]0, \varepsilon_c]$ be arbitrary. Define $\mathcal{N}_0 := (N_0(I) \cap B_{\widetilde{\gamma}, b_1'})_{I \in \mathcal{A}(\prec)}$, $\widetilde{\mathcal{N}_0} := (\widetilde{N}_0(I))_{I \in \mathcal{A}(\prec)}$, $\mathcal{N}_\varepsilon := (Y_{1,\varepsilon}(\rho, \xi) \cap N_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$ and $\widetilde{\mathcal{N}}_\varepsilon := (\widetilde{Y}_{1,\varepsilon}(\widetilde{\rho}, \widetilde{\xi}) \cap \widetilde{N}_\varepsilon(I))_{I \in \mathcal{A}(\prec)}$. Theorem 6.40, Lemma 6.44 and Corollaries 6.43 and 6.47 imply that, for each $\varepsilon \in [0, \varepsilon_c]$, $\mathcal{N}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon$ are index filtrations for $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$. Formula (56) and Corollary 6.47 imply that for all $\varepsilon \in]0, \varepsilon_c]$ and for all $I \in \mathcal{A}(\prec)$

$$Y_{1,\varepsilon}(\rho, \xi) \cap N_\varepsilon(I) \subset [N_0(I) \cap B_{\widetilde{\xi}, b_1'}]_{\varepsilon, \widetilde{\rho}} \subset \widetilde{N}_\varepsilon(I) \cap \widetilde{Y}_{1,\varepsilon}(\widetilde{\rho}, \widetilde{\xi}) \subset [\widetilde{N}_0(I)]_{\varepsilon, \widetilde{\eta}}\,.$$

i.e. the singular nesting property (4) holds. Since $\widetilde{N}_0(P) \subset \widetilde{U} \subset \widetilde{N}$ and $\widetilde{N}$ is singular strongly admissible, it also follows that for each $\varepsilon \in]0, \varepsilon_c]$, $\mathcal{N}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon$ are strongly π_ε-admissible. This completes the proof of Theorem 3.20. ▮

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