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On pairs of planar polynomial foliations

Regilene D. S. Oliveira*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: regilene@icmc.usp.br

Marco Antonio Teixeira†

Departamento de Matemática, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083-970 Campinas, SP, Brazil
E-mail: teixeira@ime.unicamp.br

In this article we deal with pairs of polynomial planar foliations. The main results concern global and local structural stability as well as the finite determinacy for these pairs. These results can be applyed to study a special class of quadratic differential forms in the plane. October, 2004 ICMC-USP

1. INTRODUCTION

The geometric-qualitative study of flows and general dynamical systems on surfaces has been during many decades the object of growing interest in many branches of pure and applied mathematics. After the work of Poincaré, Lyapunov and Bendixson this has become a well-established subject in mathematics and the focus of considerable attention. Moreover, nowadays it is fairly accessible for a broad scientific audience. From different point of views attention has been paid to the concept of structural stability and to the classification of phase portraits of the systems up to $C^0-equivalence$.

In this paper topological aspects of pairs of foliations in the plane represented by polynomial 1-forms are considered. As a matter of fact we are concerned with the simultaneous behavior of such pairs.

Pairs of 1-forms appear naturally in several mathematical contexts. For example in quadratic differential forms (QDF) or Binary Equations [2], Differential Geometry (see [23])) and Partial differential equations (see [14]).

On the other hand, if $\alpha_i = A_i dx + B_i dy$ is a 1-form in the plane, $i = 1, 2$ then $\omega = \alpha_1\alpha_2$ is a positive quadratic differential form in the plane (See [12] for details).

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Guinez, in [11], introduced the set $\mathcal{F}_n$ of the planar polynomial positive QDF with degree less than or equal to a positive integer m. He characterized the structurally stable elements in $\mathcal{F}_n$. We emphasize that in [11] the case where the pair of foliations $f_i(i = 1$ and/or $2)$ associated to the form has no characteristic directions is not considered.

Recently many authors have worked in planar homogeneous polynomial vector fields; for example, Sibirsky [21], Cima and Llibre [15], Collins [5], Llibre, Del Rio and Rodrigues [4], among others. All of them treat the problem of the classification of homogeneous polynomial vector fields.

Concerning QDF, Guinez and Bruce-Tari worked on stability and normal forms of families of QDF. In [8]-[11] Guinez restricted the analysis to homogeneous systems of order m and related problems. In a recent paper, Gutierrez, Oliveira and Teixeira [12] classified the singularities of a special class of QDF via Newton diagrams. In this context the papers of Bruce-Fidal [3], Michel [16], Davydov [7], Teixeira [24] and Oliveira and Tari [19]-[20] should also be mentioned. In all these works the classification problem of pairs of foliations were treated but systems without characteristic orbits were avoided.

If a 1-form α is a planar homogeneous differential form of degree m we say that α belongs to the set H_m, while a pair in similar conditions belong to $A^k = H_m \times H_n$, where $k = m+n$. To each pair (α, β) we associate $Sing$ the set of the points where α and β are tangent.

We obtain the concept of structural stability in our class of pairs from the following definition: Two pairs (α_1, β_1) and (α_2, β_2) in A_k are equivalent if there exists a homeomorphism h that is a simultaneous equivalence between the pair of foliations and which takes the singular set of (α_1, β_1) to the singular set of (α_2, β_2).

In this paper we give a complete characterization of the structural stability in A^k. It is worthwhile to mention that the case where α and β are global foci is extensively studied here. Problems related with finite determinacy of pair of foliations are also considered.

In the remainder of this section we present basic definitions and results necessary in this paper and we state our main results. In Section 2 we study pairs of planar homogeneous foci. We exhibit a topological invariant for the structural stability of such pairs and give necessary and sufficient conditions for these pairs to be structurally stable. Local and global aspects are considered. We do the same for those pairs where α and/or β are not foci, in Section 3. We also present conditions on a planar polynomial pair of 1-forms to be finite determined in Section 4.

1.1. Setting the problem

A vector field $X = (P, Q) \in H_m$ in differential systems terminology is written as

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned} \tag{1}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the expressions above go over to

$$\begin{aligned} \dot{r} &= r^m f(\theta), \\ \dot{\theta} &= r^{m-1} g(\theta), \end{aligned}$$

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where

$$\begin{array}{rcl} f(\theta) & = & \cos\theta.P(\cos\theta,\sin\theta)+\sin\theta.Q(\cos\theta,\sin\theta), \\ g(\theta) & = & \cos\theta.Q(\cos\theta,\sin\theta)-\sin\theta.P(\cos\theta,\sin\theta). \end{array} \tag{2}$$

If s satisfies $\dot{s}=r^{m-1}$, then the system can be written as

$$\begin{array}{rcl} r' & = & rf(\theta), \\ \theta' & = & g(\theta), \end{array} \tag{3}$$

where r' and θ' denote the derivative of r and θ, respectively, with respect to s.

Throughout the paper we are assuming that $g(\theta)$ has only zeros with multiplicity $k=1$. We also assume that $f(\theta_0)\neq 0$ provided that $g(\theta_0)=0$. This means that we always get a linear system with hyperbolic critical points.

Finally, considering the change of variable $\rho=\dfrac{r}{1+r}$ we are able to study the system in a neighborhood of the infinity and we have:

$$\begin{array}{rcl} \rho' & = & \rho(1-\rho)f(\theta), \\ \theta' & = & g(\theta), \end{array} \tag{4}$$

when (ρ,θ) is taken in the open disk $D=\{(\rho,\theta):0\leq\rho<1\}$. Observe that this system is also defined for $\rho\geq 1$. We also observe that the boundary of D is an invariant circle under the flow of the system. This circle corresponds to the infinity of the first system. So the induced vector field $E(X)$ defined in a neighborhood U of $\overline{D}$ is an analytic extension of the vector field X at infinity.

The concept of structural stability in $A^k=H_m\times H_n$ is the following

$(X,Y)\in A^k$ is *structurally stable* with respect to perturbations in $H_m\times H_n$ if there exists a neighborhood $U\times V$ of (X,Y) in A^k such that for all $(X',Y')\in U\times V$, (X,Y) and (X',Y') are topologically equivalent.

We shall say that two pairs of vector fields (X,Y) and (X',Y') are locally topologically equivalent at the origin (resp. at infinity) if there exist two neighborhoods U and V of the origin (resp. infinity) and a homeomorphism $h:U\to V$ that carries orbits of the pair of flows induced by (X,Y) onto orbits of the pair of flow induced by (X',Y'). As usual, we derive the concept of local equivalence between two pairs of vector fields at a point p.

Denote by $\Delta^k=\Sigma_0^m\times\Sigma_0^n$, where Σ_0^m is the set of all planar structurally stable homogeneous polynomial vector field of degree m with respect to perturbations in H_m and $m+n=k$.

1.2. Basic results

The structurally stable homogeneous polynomial vector fields in the plane have been studied in [4] and [15]. Next, we recall some basic results.

PROPOSITION 1.1. *Let $X\in H_m$. Assume that $E(X)$ has no critical points on ∂D and $I_X=\int_0^{2\pi}\frac{f(\theta)}{g(\theta)}d\theta\neq 0$, where f and g are given as in system (2). Then the phase portrait of $E(X)$ in D is a global focus.*

Proposition 1.2. *Let $X \in H_m$. Assume that $(0,0)$ is an isolated critical point of X and $E(X)$ has hyperbolic critical points on ∂D. Then $E(X)$ has no limit circles in D. If θ_0 is a zero of $g(\theta)$ then the straight line with slope $\tan\theta_0$ which passes through the origin is invariant under the flow induced by $E(X)$.*

The following proposition shows that there exists a duality between the flow of the induced vector field $E(X)$ in a neighborhood of the origin and in a neighborhood of the infinity.

Proposition 1.3. *Let $X \in H_m$ and suppose that $(0,0)$ is an isolated critical point of it.*
(i) Assume $E(X)$ has no critical point in $\rho = 1$. Then $\rho = 0$ is an isolated periodic orbit for the flow induced by the system if and only if $I_X \neq 0$.
(ii) Assume $E(X)$ has critical points in $\rho = 1$. Then $(1,\theta_0)$ is a hyperbolic critical point if and only if the critical point $(0,\theta_0)$ is also hyperbolic. Moreover, the critical points $(0,\theta_0)$ and $(1,\theta_0)$ are topologically different.

Theorem 1.1. *The vector field $X \in H_m$ is structurally stable with respect to perturbations in H_m if and only if, it satisfies one of the following conditions:*
(i) If $E(X)$ has no critical points on ∂D and $I_X \neq 0$.
(ii) If $E(X)$ has critical point on ∂D and all these points are hyperbolic.

Propositions 1.1 and 1.2 are proved in [4].
Proposition 1.3 and Theorem 1.1 are proved in [15].

Suppose that $X \in H_m$ and $Y \in H_n$, with $m > n$. Then the induced pair $(E(X), E(Y))$ is expressed as

$$E(X) = \begin{cases} r' = rf_1(\theta) \\ \theta' = g_1(\theta) \end{cases}$$
$$E(Y) = \begin{cases} r' = r^{m-n}f_2(\theta) \\ \theta' = r^{m-n-1}g_2(\theta) \end{cases}, \tag{5}$$

respectively, where f_i and g_i are defined as in the system (4), to $i = 1,2$.

The system above is simultaneously equivalent to the following system since that the multiplication of a vector field by a non-zero function leaves its phase portrait unchanged.

$$E(X) = \begin{cases} r' = rf_1(\theta) \\ \theta' = g_1(\theta), \end{cases}$$
$$E(Y) = \begin{cases} r' = rf_2(\theta) \\ \theta' = g_2(\theta) \end{cases}, \tag{6}$$

Then, by means of the change coordinates $\rho = r/r+1$ we get

$$\begin{cases} \rho' &= \rho(1-\rho)f_1(\theta), \\ \theta' &= g_1(\theta) \end{cases}$$
$$\begin{cases} \rho' &= \rho(1-\rho)f_2(\theta), \\ \theta' &= g_2(\theta), \end{cases} \tag{7}$$

where f_i and g_i are defined as in (2), $i = 1, 2$.

Note that the induced vector field $E(X)$ has no critical points on ∂D provided that $g_1(\theta) \neq 0$ for all θ in S^1. As g_1 is a homogeneous polynomial function of degree $m+1$, we conclude that $E(X)$ has no critical points on ∂D, provide that m is odd.

Given vector fields $X = (P_1, Q_1)$ and $Y = (P_2, Q_2)$ in H_m and H_n, respectively, we define the tangent set $S_{(X,Y)}$ as the set of all points p in the plane where $X(p)$ and $Y(p)$ are linearly dependent. Then

$$S_{(X,Y)} = \{(x,y) : (P_1Q_2 - P_2Q_1)(x,y) = 0\}.$$

The tangent set of the induced pair $(E(X), E(Y))$ will be denoted by $E(S)$. Let us check what happens with this set in coordinates (ρ, θ). We have

$$\begin{aligned} E(S) &= \{(\rho,\theta) : \rho.(\rho-1).(f_1g_2 - f_2g_1)(\cos\theta, \sin\theta) = 0\} \\ &= \{\rho = 0\} \cup \{\rho = 1\} \cup \{\lambda : (P_1Q_2 - Q_2P_1)(1,\lambda) = 0\}, \end{aligned}$$

where $\lambda = \tan\theta$. As $\{\rho = 0\}$ and $\{\rho = 1\}$ are invariant sets of both vector fields, they belong to the tangent set (common orbits). If (X,Y) has no tangency outside the origin and the infinity then $E(S) = \{\rho = 0\} \cup \{\rho = 1\}$. Otherwise the tangent set is given by $\{\rho = 0\} \cup \{\rho = 1\}$ plus the points where X and Y are tangent.

We denote by $S_{(X,Y)}$ the set of the zeros of the homogeneous polynomial function of degree $m+n$,

$$\delta(x,y) = (P_1Q_2 - P_2Q_1)(x,y).$$

If δ has zeros with multiplicity $k > 1$, then for small perturbation of (X,Y) in $H_m \times H_n$, the perturbed tangent set does not have the same number of zeros. This shows that the pair is not stable. Then, if we want to study pairs (X,Y) that are structurally stable, we must impose that δ either has no zeros or has only simple zeros, i.e., $\delta'(\lambda_0) \neq 0$ provided that $\delta(\lambda_0) = 0$. In this case we will say that δ is simple.

We consider the following subclasses in the space of pair of planar polynomial differential forms in the plane:

1. Non-simple set:

$$\Lambda^k_{ff} = \{(\alpha,\beta) \in \Delta^k : \text{both } E(\alpha) \text{ and } E(\beta) \text{ have no critical points in } \partial D\}$$

2. Simple set:

$$\Lambda^k = \Delta^k / \Lambda^k_{ff}$$

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1.3. Statement of main results

We denote by Δ_0^k the set of all structurally stable QDF in Δ^k.

Theorem A: *The pair $(\alpha, \beta) \in \Delta^k$ belongs to Δ_0^k if and only if it satisfies the following conditions:*
i) (α, β) belongs to Λ^k;
ii) The critical points of $E(\alpha)$ and $E(\beta)$ at ∂D are distinct.
iii) δ is simple.

About the finite determinacy of an analytical pair of differential forms we have the following conclusion:

Theorem B: *Let $(\alpha, \beta) = (\sum_{i \geq m} \alpha_i, \sum_{i \geq m} \beta_i)$ be a germ of a pair of analytical form in $(\mathbb{R}^2, 0)$, where α_i and β_i are homogeneous polynomial differential 1-forms of degree i. The pair (α, β) is m-determined, provided that one of the following conditions is satisfied:*
(i) (α_m, β_m) satisfies the conditions of Theorem A;
(ii) $(\alpha_m, \beta_m) \in \Lambda_{ff}^m$ where δ has no zeros.

2. THE NON-SIMPLE SET

First we consider $(X, Y) \in \Lambda_{ff}^k$. Assuming that δ is simple, two situations must be considered: (i) δ has no zeros; (ii) δ has zeros.

2.1. Global approach

Consider the case where (X, Y) is a pair of global foci where δ has only simple zeros. Then there exist transversal sections T_i associated to $E(X)$ and $E(Y)$ contained in $S_{(X,Y)}$. Fix a transversal section T_i. Let (ϕ_X, ϕ_Y) be the respective associated returning maps.

Let $(\widetilde{X}, Y)$ be a homogeneous polynomial perturbations of (X, Y). Any equivalence h between them induces an simultaneous equivalence $\widetilde{h}$ between the returning maps. As a consequence of the construction, $\widetilde{h}$ is a simultaneous conjugacy between the returning maps.

We also note that departing from a point $p \in T_i$ we have several itineraries walking alternately on pieces of orbits of X and Y. This means that there exist many ways to return to T_i through the orbits of X and Y. This fact give us a suspicious that the equivalence $\widetilde{h}$ can not exist. This is proved in the next result.

Proposition 2.1. *Let $(X, Y) \in \Delta^k$. Assume that $E(X)$ and $E(Y)$ has no critical points at ∂D and δ has at least a zero. Then the pair (X, Y) is not structurally stable under perturbations in $H_m \times H_n$.*

To prove this Theorem a topological invariant associated to pairs of returning maps is exhibited. Let us construct this invariant.

Suppose that X and Y are global foci and δ has at least one zero at $\theta = \theta_0$. Let ρ_1 (resp. ρ_2) be a returning map of the vector field X (resp. Y) defined in a transversal

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section $T_0 = \{(r,\theta) : \theta = \theta_0\}$. So $\rho_i(t) = \lambda_i t$, where $\lambda_i = \int_0^{2\pi} \frac{f_i(\theta)}{g_i(\theta)} d\theta, i = 1,2$ (thanks to homogeneous condition).

We can suppose, without lost of generality that $\lambda_i \in (0,1)$, $i = 1,2$.

Given an arbitrary point $p_0 \in T_0$ in a neighborhood of $r = 1$ (infinity), take the fundamental domain $[\rho_1(p_0), p_0] \subset T_0$, so associated to each point q in $(0, \rho_1(p_0))$ there exists an integer l such that $q = \rho_1^l(p)$, where $p \in (\rho_1(p_0), p_0)$. So if $q = \rho_2^k(p_0)$ we have

$$p = \rho_1^{-l} \circ \rho_2^k(p_0) = \frac{\lambda_2^k}{\lambda_1^l} p_0.$$

This implies that

$$\lambda_1 p_0 \leq \frac{\lambda_2^k}{\lambda_1^l} p_0 \leq p_0 \quad \Leftrightarrow \quad \lambda_1 \leq \frac{\lambda_2^k}{\lambda_1^l} \leq 1 \tag{8}$$

to $l > k$.

So

$$\frac{k}{l} \leq \frac{log\lambda_2}{log\lambda_1} \leq \frac{k}{l+1}. \tag{9}$$

When k goes to infinity, $l > k$ goes to infinity and $\frac{k}{l} - \frac{k}{l+1}$ goes to zero. From this we can conclude that $\alpha_{(X,Y)} = \frac{log\lambda_2}{log\lambda_1}$ is a number associated to the pair (X,Y).

Lemma 2.1. *Let $(X,Y) \in \Delta^k$ be a pair of vector fields such that both $E(X)$ and $E(Y)$ have no critical points on ∂D and δ has at least one zero. Then the number $\alpha_{(X,Y)}$ is a topological invariant for (X,Y) in $H_m \times H_n$.*

Proof. If there exists an equivalence $h : (\widetilde{X}, \widetilde{Y}) \simeq (X,Y)$ then the same h must send $S_{(X,Y)}$ to $S_{(\widetilde{X},\widetilde{Y})}$. Therefore if T_0 denote a branch of $\Delta_{(\widetilde{X},\widetilde{Y})}$ then $h(T_0)$ is also a branch of $\Delta_{(X,Y)}$.

Observe that T_0 (resp. $T_1 = h(T_0)$) is a transversal section to X and Y (resp. $\widetilde{X}$ and $\widetilde{Y}$), hence in T_0 (resp. T_1) we obtain returning maps associated to (X,Y) (resp. to $(\widetilde{X},\widetilde{Y})$). Moreover h induces a simultaneous equivalence between (ρ_1,ρ_2) and $(\widetilde{\rho}_1,\widetilde{\rho}_2)$. This implies that $\alpha(X,Y) = \alpha(\widetilde{X},\widetilde{Y})$. ▌

Proof of Proposition 2.1

We can suppose that an arbitrary perturbation of (X,Y) has the form $(X,\widetilde{Y})$, where $\widetilde{Y}$ is an arbitrary perturbation of Y in H_n. This observation plus Lemma 2.1 shows that if two pairs are equivalent then $\lambda_i = \widetilde{\lambda}_i$, $i = 1,2$. So (X,Y), under our assumptions, it can not be structurally stable under perturbations in $H_m \times H_n$. □

2.2. Local approach

Theorem 2.1. *Let $(X,Y) \in \Delta^k$. Assume that $E(X)$ and $E(Y)$ has no critical points on ∂D. Then the pair (X,Y) is C^0 locally structurally stable at origin under perturbations in $H_m \times H_n$ if and only if δ has no zeros.*

Proof. Suppose that δ has at least one zero. We follow the ideas of Theorem 2.1 to show the non-local structurally stability of the pair (X, Y).

We need to show that there exists a local equivalence between the pair (X, Y) and any small perturbation $(X, \widetilde{Y})$ of it provided that δ has no zeros.

First, consider the lift of the pair (X, Y) to the cylinder $S^1 \times [0,1]$ and denote by $\phi_X^t(p)$ and $\varphi_Y^t(p)$ the orbits of X and Y through p in the cylinder, respectively. We need to consider the cases:

A) $sign(I_X) = sign(I_Y)$. In this case there exists a unique intersection point between $\phi_X^t(p)$ and $\varphi_Y^t(p)$ to each p fixed. Fix a circle $r = r_0$ in the cylinder and a point p in the disc D_0 centered at the origin and radius $r = r_0$. The orbit $\phi_X^t(p)$ and $\varphi_Y^t(p)$ through the point p will intercept $r = r_0$ in points p_X and p_Y, respectively. Denote the orbit of $\widetilde{Y}$ through p_Y by $\psi_Y^t(p_Y)$. This orbit will intercept $\phi_X^t(p)$ at a unique point q. Finally we define a map $H : D_0 \to D_0$ such that $H(p) = q$.

B) $sign(I_X) = -sign(I_Y)$. In this case, the orbit of X and Y through a point p in D_0 will intercept each other infinitively many times. Then, given a point $p_0 = (r_0, \theta)$, take $\phi_X^t(p_0)$ and $\varphi_Y^t(p_0)$, orbit of X and Y through p_0. Denote by p_1 the first point where $\phi_X^t(p_0)$ intercept $\varphi_Y^t(p_0)$. The region limited by $\phi_X^t(p_0)$ and $\varphi_Y^t(p_0)$ between p_0 and p_1 is diffeomorph to a disk D_1 with center at origin. To define the local equivalence H, between (X, Y) and $(X, \widetilde{Y})$, in a point q in D_1, we proceed as in the above case. We shall find two points q_X and q_Y in the boundary ∂D_1 of D_1 and a number k that represents the number of intersections that occur between the orbits until they meet ∂D_1. Through q_Y we will consider the orbit of $\widetilde{Y}$. Then we define $H(q) = \tilde{q}$, where $\tilde{q}$ is the k^{th} intersection of the orbit of X through q_X and the orbit of $\widetilde{Y}$ through q_Y.

In both cases, the map H defines an equivalence between (X, Y) and $(X, \widetilde{Y})$ in a neighborhood of the origin. See Figure 1. ∎

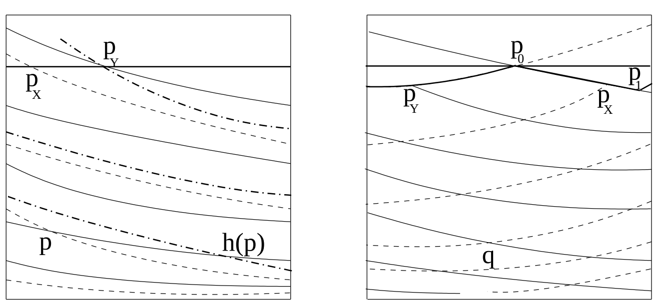

FIG. 1. Lifting of the pair (X, Y) into the cylinder

Remark 2. 1. The map H is not a local equivalence in a neighborhood of the infinity (H is not necessarily continuous in a neighborhood of the infinity).

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It follows from the proof of the above Theorem some conclusion about the local structural stability of pairs (satisfying the above assumption) at origin, with respect to polynomial perturbations:

PROPOSITION 2.2. *Let (X,Y) be a pair of germs of analytical vector fields at origin in the plane. Suppose that $(X,Y) = (\sum_{k\geq n} X_k, \sum_{k\geq n} Y_k)$, where X_k and Y_k are homogeneous polynomial vector fields of degree k. Assume (X_m, Y_n) satisfies the assumptions of Theorem 2.1. Then the phase portrait of (X,Y) at origin is locally equivalent to the phase portrait of (X_m, Y_n) at origin.*

In the later Proposition we also guarantee that these pairs are finitely determined. Finite determinacy is also a natural question for pairs satisfying the conditions of Theorem 2.1. The next example is related to this situation.

EXAMPLE 2.1. Suppose that $(\varphi_1(t), \varphi_2(t)) = (\frac{1}{2}t, \frac{1}{3}t)$ is a pair of returning maps associated to the pair of vector fields (X_0, Y_0) in the conditions of Theorem 2.1. Then

$$\varphi_1^n(t_0) > \varphi_2^n(t_0), \forall t_0 \text{ and } \forall n \in \mathbb{N}^*. \tag{10}$$

Suppose that a small perturbation of (X_0, Y_0) by higher terms gives the following pair of returning maps

$$\psi_1(t) = \frac{1}{2}t - at^2, \quad \psi_2(t) = \varphi_2(t) = \frac{1}{3}t.$$

We know that if there exists an equivalence h between (X_0, Y_0) and $X = (X_0, Y_0) + h(x,y)$, the non-homogeneous polynomial perturbation of it, then h induces an equivalence $\widetilde{h}$ between the pair of returning maps. But we shall show that such equivalence can not exist. In fact, from (10), we have

$$\widetilde{h}(\varphi_1^n(t_0)) = \psi_1^n(\widetilde{h}(t_0)) > \psi_2^n(\widetilde{h}(t_0)) = \widetilde{h}(\varphi_2^n(t_0)), \forall t_0 \text{ and } \forall n \in \mathbb{N}^*.$$

It is straightforward to derive that

$$\varphi^n(\widetilde{h}(t_0)) = \frac{1}{2^n}\widetilde{h}(t_0) - \frac{a}{2^{n-2}}.(\frac{1}{2} + ... + \frac{1}{2^n}).\widetilde{h}(t_0)^2 + ...$$

Take a positive integer $n_0 > \frac{log(1-4.h(t_0).a)}{log2-log3}$. This implies that

$$\psi_1{}^{n_0}(\widetilde{h}(t_0)) - \psi_2{}^{n_0}(\widetilde{h}(t_0)) < 0.$$

PROPOSITION 2.3. *Let $(X,Y) \in \Delta^k$. Assume that both $E(X)$ and $E(Y)$ have no critical points on ∂D. Then the pair (X,Y) is not finite determined provided that δ has a simple zero.*

Proof. Let (ρ_1, ρ_2) be the pair of returning maps associated to (X, Y), with $\rho_i(t) = \lambda_i t$, where λ_i is given as in Proposition 2.1, $i = 1, 2$. Assume without lost of generality that $0 < \lambda_2 < \lambda_1 < 1$ and write $\lambda_1 = \frac{1}{\alpha}$ and $\lambda_2 = \frac{1}{\beta}$, where α and β are greater than 1. Then $\lambda_1^n \geq \lambda_2^n$ for all positive integer n.

Note that

$$\varphi_1^n(p_0) - \varphi_2^n(p_0) = \lambda_1^n p_0 - \lambda_2^n p_0 > 0$$

and goes to the origin when n goes to infinity.

Consider a small perturbation of (X, Y) with respect to non-homogeneous terms of higher degree and suppose that there exists an equivalence h between (X, Y) and its perturbation. Then $\widetilde{h}(\varphi_1^n(p_0)) - \widetilde{h}(\varphi_2^n(p_0)) = \psi_1(\widetilde{h}(p_0)) - \psi_2(\widetilde{h}(p_0)) > 0$.

It is clear that any such perturbation can be chosen as $(\widetilde{X}, Y)$. Moreover, the returning map associated to the perturbation $\widetilde{X}$ can be written as $\psi_1(t) = \lambda_1 t - R_1(t)$, where $R_1(t) = at^k + ...$, with $a > 0$ and k is the first non-zero jet of R_1.

Then

$$\psi_1^n(h(p_0)) = \frac{1}{\alpha^n} h(t_0) - \frac{a}{\alpha^{n-2}}.\left(\frac{1}{\alpha} + \frac{1}{\alpha^k} + ... + \frac{1}{\alpha^{(n-1)k-(n-2)}}\right).h(t_0)^k + ...$$

As in the above example, we can find a positive integer n_0 such that the iteration order changes after some n_0, showing that such equivalence can not exist. ▮

As consequence of Theorem 2.1 we can get some conclusions about the behavior of special positive quadratic differential forms. Let ω be a positive quadratic differential form given by the product of two 1-forms in the plane satisfying the assumptions of Proposition 2.2. Since ω is a homogeneous polynomial form, the Newton Diagram associated to ω has a unique face and by Proposition 2.2 we have that ω is locally equivalent to ω_Δ (See [12] for details). We observe that here both differential 1-forms have no characteristic directions. Re-writing:

Corollary 2.1. *Let $\omega = \alpha.\beta$ be a positive quadratic differential form in the plane, where α and β are planar polynomial differential 1-forms satisfying the assumptions of Proposition 2.2. Then ω is locally equivalent to ω_Δ at origin.*

Now we shall consider a non usual global equivalence between pairs of vector fields, introduced in [24].

Definition 2.1. Two pairs (X, Y) and (X_0, Y_0) in A_k are mild-equivalent if there exists a homeomorphism $h : \overline{D} \to \overline{D}$ such that (X_0, Y_0) at p is germ equivalent to (X, Y) at $h(p)$.

Let $(X, Y) \in \Lambda_{ff}^k \cap \Delta^k$, where δ has only simple zeros. The above equivalence sends transversal section of X and Y belongs to $S_{(X,Y)}$ to transversal section of X_0 and Y_0

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in $S_{(X_0,Y_0)}$. Moreover, the number of open regions in $\overline{D} - \Delta_{(X,Y)}$ is the same that in $h(\overline{D}) - \Delta_{(X_0,Y_0)}$. Then, given an arbitrary point $p \in \overline{D}$ we can describe the local phase portrait of the pair (X,Y) in a neighborhood of p. Lets consider each situation:

1. If $p \in D - \Delta_{(X,Y)}$. Then X and Y are regular and transversal. Then (X,Y) is locally topologically equivalent to $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.
2. If $p \in \Delta_{(X,Y)}$ and $\Delta_{(X,Y)}$ has only non degenerated singularities (for all θ_0 such that $\Delta(\theta_0) = 0$ we have $\Delta'(\theta_0) \neq 0$). Then (X,Y) is locally topologically equivalent to $(\frac{\partial}{\partial x}, \frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y})$ (See [19]).
3. If $p \in \partial D - \Delta_{(X,Y)}$. In this case we have two regular vector fields without contact except at $y = 0$ (common leaf). Then, the pair is locally topologically equivalent to $(\frac{\partial}{\partial x}, (1+x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, to $y \geq 0$ (See [2]).
4. If $p \in \partial D \cap \Delta_{(X,Y)}$. The X and Y are regular vector fields with discriminant set given by $y = 0$ (common leaf) plus a regular curve. Then the pair (X,Y) is locally topologically equivalent to $(\frac{\partial}{\partial x}, (1+x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y})$, to $y \geq 0$ (See [19].

3. THE SIMPLE SET

Here we consider $(X,Y) \in \Delta^k$, where either $E(X)$ or $E(Y)$ has critical points at ∂D, say $E(Y)$.

We know that if $E(X)$ has no critical points at ∂D then m is odd and to each critical point $\theta_0 \in \partial D$ of $E(Y)$, we have associated a curve (See Proposition 1.2). As in [12], we called it of separating curve (it is called a separatrix if θ_0 is a saddle point and a pseudo-separatrix, if θ_0 is a node point). This curve is an invariant manifold of θ_0. Then we call sector of $E(Y)$ in D to each region between two consecutive separating curves of $E(Y)$ (See [12] for details).

Moreover, if $E(Y)$ is a vector field with critical points at ∂D and n is also odd then we get some immediate conclusions.

Lemma 3.1. *Let $Y \in H_m$ be a vector field with n odd. Suppose that Y has critical points on ∂D, then:*

1.If k is the number of zeros of $g_2(\theta)$, then $k = 4j$, where j is a non zero positive integer.
2.The phase portrait of $E(X)$ has an even number of singular sectors (elliptic, parabolic and hyperbolic).

Proof. 1. From (2) we observe that $g_2(\theta)$ is a homogeneous polynomial of degree $n+1$ (even). As $E(Y)$ has only hyperbolic critical points on ∂D, $g_2(\theta)$ has only simple zeros. Then g_1 has an even number of zeros $\lambda = \tan\theta$. Moreover, if θ is a root of g_2, i.e., $\lambda_2 = \tan\theta$ is a zero, then $\theta mod(k\pi)$ is also a zero. As the zeros are simple, we get a number $4j$ of zeros of g_2, where j is a positive integer greater than 1. This fact implies that we get a even number of sectors in D, where each separating curve is associated to a hyperbolic singular point of $E(Y)$.

Moreover, as the topological type of a critical point p on ∂D is characterized by the sign of $g_2'(\theta).f_2(\theta)$ and the sign is the same for θ and $\theta+\pi$ ($\tan\theta = \tan(\theta+\pi)$), we conclude that all symmetric critical points have the same topological type. ▌

Consequences of the later Lemma:

1. There exists a symmetry in D, with respect to the involution $\sigma(x,y) = (-x,-y)$.

2. To know the phase portrait of X is enough to know the sequence of the $2j$ critical points on ∂D. The others $2j$ critical points will have the same behavior.

Example: If $m = 1$ (linear vector field), we get 4 critical points. From remark before we can conclude that we can find 3 distinct classes the equivalence to Y if $n = 1$: all critical points are of the node type, all critical points are of the saddle type, the sequence is node, saddle, node, saddle (or saddle,node,saddle,node).

3. Two vector fields X and Y in Δ_k with different sequence of critical points on ∂D are non-equivalents (See Proposition 4.10 in [12]).

3.1. Global approach

First, consider $(X,Y) \in \Delta^k$, where $E(X)$ has no critical points on ∂D. Assume that θ_0 and θ_1 are two consecutive critical points of $E(Y)$ on ∂D with the same topological type. The correspondent sector S of $E(Y)$ will be an elliptic sector (if both critical points are of the nodal type) or hyperbolic type (if both critical points are of the saddle type). Recall that $E(X)|_S$ is equivalent to the vector field $\frac{\partial}{\partial x}$. Then near ∂D in S, each integral curve of $E(Y)$ will intercept a integral curve of $E(X)$ in two distinct points (See Figure 2). This shows that there is a curve of tangency between the integral curves of $E(Y)$ and $E(X)$ in hyperbolic and elliptic sectors. Assuming now that θ_0 and θ_1 are two consecutive critical points of $E(Y)$ with distinct topological type (node and saddle). The associated sector of $E(Y)$ is parabolic and there is no curve of contact in S (See Figure 2), provided that δ has only simple zeros.

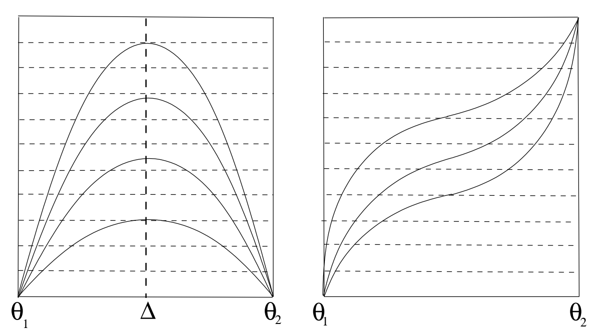

FIG. 2. Contact in a hyperbolic (or in an elliptic) sector and in a parabolic sector

THEOREM 3.1. *Let (X,Y) be in Δ^k. Assume that $E(X)$ has no critical points on ∂D. If $E(Y)$ has critical points on ∂D, then (X,Y) is C^0 structurally stable under perturbations in $H_m \times H_n$ if only if δ has only simple zeros.*

Proof. As observed later, if δ has one zero with multiplicity ≥ 2, (X, Y) is not structurally stable. To show the required stability, under conditions, is enough to exhibit an equivalence that maintain the integral curves of X and takes the integral curves of Y onto the integral curves of $\widetilde{Y}$, where $\widetilde{Y}$ is an arbitrary perturbation of Y in H_n.

In D we have finite many number of hyperbolic, elliptic and parabolic sectors of $E(Y)$. Let T be an arbitrary separating curve of a sector of $E(Y)$. Then T is also a transversal section for $E(X)$ and in T we have defined a returning map ρ associated to $E(X)$. The same holds for a small perturbation of $E(Y)$ in H_n. Since an equivalence takes a separating curve of $E(Y)$ in a separating curve of $E(\widetilde{Y})$, we have an induced returning map in $h(T)$ satisfying the following relation $\rho \circ h = h \circ \rho'$, where ρ' is the returning map associated to $E(X)$ in $h(T)$.

For any T_i, another separating curve of $E(Y)$, we have a homeomorphism $l_i : T \to T_i$ obtained by sliding along the integral curve of $E(X)$ from T to T_i.

Let $\wp$ (resp. $\wp'$) be a sector of $E(Y)$ (resp. $E(\widetilde{Y})$) with separating curves T_1 and T_2 (resp. T'_1 and T'_2). Denote by Δ_S (resp. Δ'_S) the curve of tangency between $E(X)$ and $E(Y)$ in $\wp$ (resp. $E(X)$ and $E(\widetilde{Y})$ in $\wp'$), if there exist and by R_i (resp. R'_i) the open region between T_i and Δ_S (resp. T'_i and Δ'_S), $i = 1, 2$.

As before, there exist induced mappings $l_1 : T \to T_1$, $l_2 : T \to T_2$ and $l_{\Delta_S} : T \to \Delta_S$. For each $i = 1, 2$, we derive homeomorphisms $k_i : T_i \to T'_i$, defined by $k_i = (l'_i)^{-1} \circ h \circ l_i$, $i = 1, 2$ and $k_\Delta : \Delta_S \to \Delta'_S$ given by $k_\Delta = l_\Delta^{-1} \circ h \circ l_\Delta$. These homemorphism there exist since we must have a homeomorphism $h : T \to T'$ satisfying $h \circ \rho = \rho' \circ h$.

Let us see how to extend h to each sector S of $E(X)$ onto the corresponding sector S' of $E(\widetilde{X})$.

- In one elliptic sector. Consider $p_0 \in R_1$. Denote by σ_X and σ_Y the integral curves of $E(X)$ and $E(Y)$ passing through p_0. These integral curves intercept Δ_S in p_X and p_Y, respectively. Consider the points $k_\Delta(p_X)$ and $k_\Delta(p_Y)$ in Δ'_S and the integral curves of $E(X)$ and $E(\widetilde{Y})$ through these points, respectively. These curves meet each other in $q_0 \in R'_i$. Then define $h(p_0) = q_0$. The map h is a homeomorphism from R_1 to R'_1. Now in R_2 proceed in the same way and the extension to S is well done because H agree with k_Δ in Δ.

Now it is possible to extend h to ∂D in a natural way.

- In one hyperbolic sector we proceed in the same way as above.
- In one parabolic sector. Fix one leaf C_1 of $E(X)$ and notice that C_1 connects two points $p_1 \in T_1$ with $p_2 \in T_2$. Let C_2 be the leaf of $E(X)$ connecting $k_1(p_1)$ and $k_2(p_2)$. Define one homeomorphism h between C_1 and C_2. Given a point $q_0 \in S$. There is a unique point $q_Y \in C_1$ and $q_X \in T_1$, where the orbits of $E(X)$ and $E(Y)$ through q_0 intercept C_1 and T_1, respectively.

Applying $k_1 : T_1 \to T'_1$ and $h : C_1 \to C_2$ we obtain $h(q_Y) \in C_2$ and $k_1(q_X) \in T'_1$, respectively. Through these points there exists a unique integral curve of $E(\widetilde{Y})$ and $E(X)$, respectively, and they meet each other in a unique point $q \in S'$. Then extent h to S onto S' defining $h(q_0) = q$.

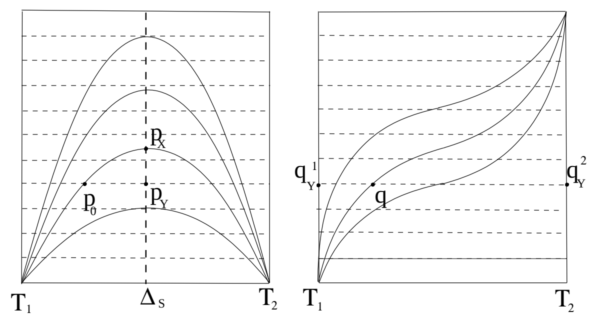

FIG. 3. Phase portrait in some sectors of X

Now we extend this homeomorphism to ∂D in a natural way. Given a point $q_0 \in \partial D$, consider a sequence (q_n) in S converging to q_0. Define $h(q_0) = \lim_{n\to\infty} h(q_n)$. Observe that the limit does not depend on the sequence.

The next step is to extend the homeomorphism constructed on each sector to all space. We need guarantee that the gluing process is well defined. But this is ensured due the fact that the homeomorphism h restricted to the separating curves agrees with the induced returning maps. ∎

Now, consider $(X, Y) \in \Delta^k$, where $E(X)$ and $E(Y)$ have distinct critical points on ∂D. As in the case studied later, we have finite many sectors of each vector field in $\overline{D}$.

Theorem 3.2. *Let (X, Y) be in Δ^k. Assume that $E(X)$ and $E(Y)$ have distinct critical points on ∂D. Then (X, Y) is C^0 structurally stable under perturbations in $H_m \times H_n$ if only if δ either has only simple zeros or no zeros.*

The proof here follows the same ideas used in the proof of Theorem 3.1. As before we just need to distinguish each sector and separating curves and the branches of the tangent set. See more details of the proof (to the local case) when δ has no zeros (Proposition 4.10 in [12]).

3.2. Local approach

As in the case where X and Y has no critical points in ∂D, we can get some conclusions about pairs of vector fields which can be written as a sum of homogeneous vector fields in the plane:

Proposition 3.1. *Let (X, Y) be a pair of analytical vector fields in the plane. Suppose that $(X, Y) = (\sum_{k\geq m} X_k, \sum_{k\geq n} Y_k)$, where X_i and Y_i are homogeneous polynomial vector fields of degree i. Either (X_m, Y_n) satisfies the assumptions of Theorem 3.1 or of Theorem 3.2. Then the phase portrait of (X, Y) at origin is locally equivalent to the phase portrait of (X_m, Y_n) at origin.*

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This Proposition can be proved in the same way as Theorem C in [4] using the technics of Theorems 3.1 and Theorem 3.2. Note that as a consequence, we have finite determinacy of the pairs (X, Y) satisfying the above conditions.

We have a dual result about local equivalence in a neighborhood of the infinity. Here we say "phase portrait of (X, Y) at infinity" instead of "phase portrait of (X, Y) in a neighborhood of the infinity".

PROPOSITION 3.2. *Let (X, Y) be a pair of analytical vector fields in the plane. Suppose that $(X, Y) = (\sum_{k=1}^{m} X_k, \sum_{k=1}^{n} Y_k)$, where X_k and Y_k are homogeneous polynomial vector fields of degree k. Assume that (X_m, Y_n) satisfies the assumptions either of Theorem 3.1 or of Theorem 3.2. Then the phase portrait of (X, Y) at infinity is locally equivalent to the phase portrait of (X_m, Y_n) at infinity.*

Applying the conclusions of this section to a special class of positive quadratic differential forms we have

COROLLARY 3.1. *Let $\omega = \alpha.\beta$ be a positive quadratic differential form in the plane, where α and β are planar polynomial differential 1-forms satisfying the assumptions of Proposition 3.1. Then ω is locally equivalent to ω_Δ at origin.*

4. PROOF OF THE MAIN RESULTS

Now we can joint the conclusions presented here to conclude the proof of the main results:

Proof of Theorem A: It follows from Theorems 2.1, 3.1 and 3.2.

Proof of Theorem B: If follows from Theorems 2.2 and 3.1.

Remark 4. 1. Observe that Λ^k_{ff} is a open in $\Sigma_0 \times \Sigma_0$ then Δ^k_0 is never dense in Δ^k.

Remark 4. 2. We note that similar conclusions can be obtained to positive quadratic differential forms in the plane that can be written as a product of pairs of polynomials 1-forms under some conditions.

Remark 4. 3. In the same way, under some conditions, the results obtained in this paper can be applied to study curvature lines in a smooth manifold M; for example, if the curvature lines in M are given by homogeneous QDF and they can decompose as a product of two homogeneous direction fields.

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